Some Properties of Laurent Polynomial Matrices

Sylvie Icart, Pierre Comon

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Abstract

In the context of multivariate signal processing, factorizations involving so-called para-unitary matrices are relevant as well demonstrated in the book of Vaidyanathan [11], or [4, 1] and more recently in a series of papers by McWhirter and co-authors [5, 6]. However, known factorizations of matrix polynomials, such as the Smith form [10], involve unimodular matrices but usual factorizations such as QR, eigenvalue or singular value decompositions, have not been proved to exist for polynomial matrices, if defined with para-unitary matrices, except for very restrictive matrices [2]. It is clear that Cholesky factorization requires square roots, and that EVD and SVD require roots of higher degree polynomials. But one can ask oneself whether the closure of the field of polynomial coefficients is enough or not. It turns out that it is not. Nevertheless, density arguments allow to approximate any polynomial matrix by SVD factorization involving paraunitary polynomial matrices. With that goal, we define the appropriate framework for Laurent polynomial matrices, that is, polynomial matrices with both positive and negative powers in a single variable, particularly the notion of ordrer and degree. We introduce a Smith form for these matrices involving "L-unimodular" matrices which are matrices with a monomial non-zero determinant. The 'Elementary Polynomial Givens Rotations' of [6] are of that kind.

1 Notations and Definitions

Throughout the paper, vectors and matrices are denoted with underline lowercase and bold uppercase letters respectively. $\boldsymbol{I}$ and $\boldsymbol{0}$ denote identity and zero matrices. The entries of a matrix $\boldsymbol{M}$ are denoted m_{ij}, where subscript ij denotes the i-th row and the j-th column of $\boldsymbol{M}$. $(^*)$ stands for complex conjugation, $(^H)$ for conjugate transposition. Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ the subset of positive integers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ and $\mathcal{C}$ be the unit circle. $\mathcal{C}(A)$ denotes the set of continuous functions from $\mathcal{C}$ to A.

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Let $\mathbb{C}[z]$ be the ring of polynomials with complex coefficients: $p(z) = \sum_{i=0}^{n} p_i z^i$ with $n \in \mathbb{N}, p_i \in \mathbb{C}$. If $p_n \neq 0$, then the polynomial degree of p is $\deg(p) = n$, if $p_n = 1, p$ is said to be monic. Moreover, if $p_n = 1$ and $p_0 \neq 0$, p is said to be L-monic. Let $\mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials: $p(z) = \sum_{i=m}^{n} p_i z^i$ with $m, n \in \mathbb{Z}, m \leq n, p_i \in \mathbb{C}$. Let us suppose that $p_m p_n \neq 0$, then one can always factorize $p(z) = p_n z^m \pi(z)$ with $\pi \in \mathbb{C}[z]$ and L-monic. The L-degree of p is defined as the degree of π and is denoted as $d(p) = n - m$. According to this L-degree, $\mathbb{C}[z, z^{-1}]$ is an Euclidean ring, so a Principal Ideal Domain (PID). The invertible elements of $\mathbb{C}[z, z^{-1}]$ are non-zero L-monomials: $p(z) = az^\alpha$ with $a \in \mathbb{C}^*$ and $\alpha \in \mathbb{Z}$. As greatest common divisors (gcd) are defined up to an invertible element, one can set for uniqueness purposes, the gcd to be a L-monic polynomial of $\mathbb{C}[z]$.

Let $\mathbb{C}^{n\times n}[z^{-1}]$ be the ring of $n \times n$ matrices corresponding to Finite Impulse Response (FIR) sytems: $\boldsymbol{M}(z) = \sum_{k=0}^{l} \boldsymbol{M}_k z^{-k}$. The order of $\boldsymbol{M}$ is the greatest index such that $\boldsymbol{M}_k \neq \boldsymbol{0}$. Let $\mathbb{C}^{n\times n}[z]$ be the ring of $n \times n$ matrices with polynomials entries, the order is defined in the same way. The units of $\mathbb{C}^{n\times n}[z]$ are matrices whose determinant is constant and are called unimodular matrices [10, 11]. Let $\mathbb{C}^{n\times n}[z, z^{-1}]$ be the ring of matrices with L-polynomials entries, also called L-polynomial matrices. The units of $\mathbb{C}^{n\times n}[z, z^{-1}]$ are matrices whose determinant is a non-zero L-monomial and are called L-unimodular matrices. Let tilde denote the paraconjugation: $\widetilde{\boldsymbol{M}}(z) = \boldsymbol{M}^H(\frac{1}{z^*}), \forall z \in \mathbb{C}^*$. On the unit circle, one has $\widetilde{\boldsymbol{M}} = \boldsymbol{M}^H$. One says $\boldsymbol{H}$ is parahermitian iff $\boldsymbol{H} = \widetilde{\boldsymbol{H}}$, and $\boldsymbol{U}$ is paraunitary iff $\boldsymbol{U}\widetilde{\boldsymbol{U}} = \boldsymbol{I}$.

Let us define a sesquilinear form according to paraconjugation for two vectors of $\mathbb{C}^{n\times 1}[z, z^{-1}]$ by $\langle \underline{u}, \underline{v} \rangle = \widetilde{\underline{u}}\, \underline{v}$. As paraconjugation is an involution, $\langle , \rangle$ is a sesquilinear parahermitian form, and one has $\widetilde{\langle \underline{u}, \underline{v} \rangle} = \langle \underline{v}, \underline{u} \rangle$. Moreover, let $\boldsymbol{H} \in \mathbb{C}^{n\times n}[z, z^{-1}]$, then $\langle \boldsymbol{H}\underline{u}, \underline{v} \rangle = \langle \underline{u}, \widetilde{\boldsymbol{H}}\underline{v} \rangle$. If $\boldsymbol{H}$ is parahermitian, we then have $\langle \boldsymbol{H}\underline{u}, \underline{v} \rangle = \langle \underline{u}, \boldsymbol{H}\underline{v} \rangle$.

Let $\mathbb{C}^{n\times n}(z)$ be the ring of $n \times n$ matrices whose entries are rational functions. Let $\boldsymbol{H}$ be a proper rational matrix, the degree or the McMillan degree of $\boldsymbol{H}$ is defined as the sum of the degrees of the denominator polynomials in its Smith-McMillan form. It appears to be the minimal number of delays to implement a sytem with transfer matrix $\boldsymbol{H}$ [10, 11].

2 Invariant polynomial

When $\boldsymbol{M}$ is a polynomial matrix of $\mathbb{C}^{n\times n}[z]$, it is well known that one can always obtain a diagonal form called Smith form by pre-multiplying and post-multiplying $\boldsymbol{M}$ by unimodular matrices corresponding to several elementary row and column operations [10, 11]. $\boldsymbol{M}$ and $\boldsymbol{P} \in \mathbb{C}^{n\times n}[z]$ are Smith equivalent if they have the same Smith form, which is denoted $\boldsymbol{M} \overset{S}{\sim} \boldsymbol{P}$. As $\mathbb{C}[z, z^{-1}]$ is a PID, one can define a normal Smith form [8]. The involved L-unimodular matrices correspond to product of L-elementary row and column operations, defined as elementary operations replacing polynomial multiple by L-polynomial multiple.

Definition 1. *Let $\boldsymbol{M} \in \mathbb{C}^{n\times n}[z, z^{-1}]$ be of rank n, then there exist L-unimodular matrices $\boldsymbol{U}_1$, $\boldsymbol{U}_2 \in \mathbb{C}^{n\times n}[z, z^{-1}]$ such that $\boldsymbol{U}_1\boldsymbol{M}\boldsymbol{U}_2 = \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_i\}$ where λ_i divides λ_{i+1} and λ_i is unique up to a multiplication by a L-monomial.*

To ensure the uniqueness of the previous form, we will impose λ_i to be L-monic. $\boldsymbol{\Lambda}$ is called the L-Smith form of $\boldsymbol{M}$ and λ_i the L-invariant polynomials, they can be computed by $\lambda_i = \frac{\Delta_i(\boldsymbol{M})}{\Delta_{i-1}(\boldsymbol{M})}$, where $\Delta_i(\boldsymbol{M})$ is the L-monic gcd of all $i \times i$ minors of $\boldsymbol{M}$.

It is clear that the determinant of a full-rank L-polynomial matrix equals the product of its L- invariant polynomials up to L-monomial multiplication.

We say that $\boldsymbol{M}$,$\boldsymbol{P} \in \mathbb{C}^{n\times n}[z, z^{-1}]$ are L-Smith equivalent if they have the same L-Smith form, we note $\boldsymbol{M} \overset{LS}{\sim} \boldsymbol{P}$, and there exist $\boldsymbol{U}_1, \boldsymbol{U}_2$ L-unimodular such that $\boldsymbol{M} = \boldsymbol{U}_1\boldsymbol{P}\boldsymbol{U}_2$. We then have:

Property 2. *i. The L-invariant polynomials of a L-unimodular matrix are all equal to 1;*

ii. The L-invariant polynomials of a L-unitary matrix $\boldsymbol{U}$ are all equal to 1;

iii. The L-invariant polynomials of a parahermitian matrix $\boldsymbol{H}$ are self-inversive, that is: $\lambda_i(z) = e^{\jmath\theta_i} z^{m_i}\widetilde{\lambda}_i(z)$ with $\theta_i \in \mathbb{R}$ and $m_i = \deg(\lambda_i)$.

To prove i remark that $\det \boldsymbol{U}(z) = cz^{\alpha} = \det \boldsymbol{U}_1(z) \det \boldsymbol{\Lambda}(z) \det \boldsymbol{U}_2(z)$, so $\det \boldsymbol{\Lambda}$ is a L-monomial, and as λ_i are L-monic, the unique solutions are $\lambda_i = 1, \forall i$. Hence, $\boldsymbol{U} \overset{LS}{\sim} \boldsymbol{I}$.
To set ii, let $\det \boldsymbol{U}(z) = cz^{\alpha}\prod_{i=1}^{n}\lambda_i(z)$. Remark that as $\boldsymbol{U}\widetilde{\boldsymbol{U}} = \boldsymbol{I}$, one has $cz^{\alpha}\prod_{i=1}^{n}\lambda_i(z)c^*z^{-\alpha}\prod_{i=1}^{n}\widetilde{\lambda}_i(z) = 1$, so $\prod_{i=1}^{n}\lambda_i$ is invertible and as the λ_i are L-monic, the unique solutions are $\lambda_i = 1, \forall i$. Thus, $\boldsymbol{U} \overset{LS}{\sim} \boldsymbol{I}$ and paraunitary matrices are L-equivalent to L-unimodular matrices.

Finally, let $\boldsymbol{\Lambda}_H$ be the Smith form of $\boldsymbol{H}$: $\boldsymbol{H} = \boldsymbol{U}_1\boldsymbol{\Lambda}_H\boldsymbol{U}_2$. One has $\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{U}_2}\widetilde{\boldsymbol{\Lambda}_H}\widetilde{\boldsymbol{U}_1}$, with $\widetilde{\boldsymbol{U}_i}$ L-unimodular, so $\boldsymbol{\Lambda}_{\widetilde{H}} \overset{LS}{\sim} \widetilde{\boldsymbol{\Lambda}_H}$ with $\widetilde{\boldsymbol{\Lambda}_H} \in \mathbb{C}^{n\times n}[z^{-1}]$ and $\boldsymbol{\Lambda}_{\widetilde{H}} \in \mathbb{C}^{n\times n}[z]$ by definition. Let $\nu_i \in \mathbb{C}[z]$ be the monic reciprocal polynomial of $\widetilde{\lambda}_i$: $\widetilde{\lambda}_i(z) = \lambda_{i0}^* z^{-m_i}\nu_i(z)$ with $m_i = \deg(\lambda_i)$. One has $\widetilde{\boldsymbol{\Lambda}_H} \overset{LS}{\sim} \mathrm{diag}\{\nu_i\}$. As $\lambda_i \mid \lambda_{i+1}$, there exists $\beta_i \in \mathbb{C}[z]$ such that $\lambda_{i+1} = \lambda_i\beta_i$. Let α_i be the monic reciprocal polynomial of β_i, one has $\widetilde{\lambda}_{i+1}(z) = \lambda_{i0}^* z^{-m_i}\nu_i(z)\beta_{i0}^* z^{-b_i}\alpha_i(z)$ with $b_i = \deg(\beta_i)$ and $\nu_{i+1}(z) = \lambda_{(i+1)0}^{-1*}\lambda_{i0}^* z^{m_{i+1}-m_i-b_i}\alpha_i(z)\nu_i(z)$, so $\nu_i \mid \nu_{i+1}$. By unicity of the L-invariant polynom, one can then deduce that ν_i are the L-invariant polynomials of $\boldsymbol{\Lambda}_{\widetilde{H}}$ which are the same as those of $\boldsymbol{H}$ because $\boldsymbol{H}$ is parahermitian. Then, $\lambda_i(z) = \lambda_{i0}^{-1*} z^{m_i}\widetilde{\lambda}_i(z)$ and, for $z = 0$, $\lambda_{i0} = \lambda_{i0}^{-1*}$. □

3 Order and degree

As mentioned in introduction, order is defined for polynomial matrices, whether the indeterminate variable is z or z^{-1} and the degree of a matrix is only defined for proper rational (causal) matrices.

Definition 3. *Let $\boldsymbol{H} \in \mathbb{C}^{n\times n}[z, z^{-1}], \boldsymbol{H}(z) = \sum_{k=m}^{p}\boldsymbol{H}_k z^k$ with $\boldsymbol{H}_k \in \mathbb{C}^{n\times n}, m, p \in \mathbb{Z}, m \leq p, \boldsymbol{H}_m$ and $\boldsymbol{H}_p$ non equal to $\boldsymbol{0}$. Define*

$$\boldsymbol{\Xi}(z) = z^{-m}\boldsymbol{H}(z) \text{ and } \overline{\overline{\boldsymbol{H}}}(z) = z^{-p}\boldsymbol{H}(z)$$

with $\boldsymbol{\Xi} \in \mathbb{C}^{n\times n}[z]$ and $\overline{\overline{\boldsymbol{H}}} \in \mathbb{C}^{n\times n}[z^{-1}]$. The order of $\boldsymbol{H}$ is defined as the order of the associated polynomial matrix $\boldsymbol{\Xi}$, that is $p - m$, and the L-degree of $\boldsymbol{H}$ as the McMillan degree of the associated causal matrix $\overline{\overline{\boldsymbol{H}}}$.

Remark that, as for polynomials, when $\boldsymbol{M} \in \mathbb{C}^{n\times n}[z^{-1}]$, the above defined order differs from the classical one if $\boldsymbol{M}(z) = z^{-\alpha}\boldsymbol{Q}(z)$ with $\alpha \in \mathbb{N}^*$ and $\boldsymbol{Q} \in \mathbb{C}^{n\times n}[z^{-1}]$. Moreover, by definition, $\boldsymbol{\Xi}(z) = z^{p-m}\overline{\overline{\boldsymbol{H}}}(z)$, so the Smith-McMillan form of $\overline{\overline{\boldsymbol{H}}}(z)$ is $\frac{1}{z^{p-m}}\boldsymbol{S}(z)$ with $\boldsymbol{S}(z)$ is the Smith form of $\boldsymbol{\Xi}(z)$. In case of a parahermitian matrix, one can prove that $\boldsymbol{\Xi}(z)$ and $\overline{\overline{\boldsymbol{H}}}(z)$ are parconjugate of each other (but none is parahermitian except if constant).

Property 4. *Let $\boldsymbol{U} \in \mathbb{R}^{n\times n}[z, z^{-1}]$ be a paraunitary matrix, then the L-degree of $\boldsymbol{U}$ equals the order of $\boldsymbol{U}$.*

We shall prove this property for $n = 2$ with the help of the following lemma:

Lemma 5. *Let $\boldsymbol{U} \in \mathbb{R}^{2\times 2}[z^{-1}]$ be a FIR paraunitary matrix such that $\boldsymbol{U}_0 \neq \boldsymbol{0}$, then the L-degree of $\boldsymbol{U}$ equals the order of $\boldsymbol{U}$.*

First, let $\boldsymbol{H} \in \mathbb{R}^{2\times 2}[z^{-1}]$ be a FIR paraunitary matrix of degree N defined by

$$\boldsymbol{H}(z) = \boldsymbol{Z}(z)\ \boldsymbol{R}_1\ \boldsymbol{Z}(z)\ \boldsymbol{R}_2\ \ldots \boldsymbol{Z}(z)\ \ \boldsymbol{R}_{N-1}\boldsymbol{Z}(z) \quad (1)$$

with $\boldsymbol{Z}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \boldsymbol{R}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}$. It can be shown by induction that $\boldsymbol{H}(z) = \sum_{i=0}^{N} \boldsymbol{H}_i z^{-i}$ with $\boldsymbol{H}_0 = \operatorname{diag}\{\prod_{k=1}^{N-1}\cos\theta_k, 0\}$ and $\boldsymbol{H}_N = \operatorname{diag}\{0, \prod_{k=1}^{N-1}\cos\theta_k\}$.

Now, using factorization [11, p. 729] of a paraunitary matrix $\boldsymbol{U} \in \mathbb{R}^{2\times 2}[z^{-1}]$ of degree N: $\boldsymbol{U}(z) = \boldsymbol{R}_0\boldsymbol{H}(z)\boldsymbol{R}_N \operatorname{diag}\{1, \pm 1\}$ with $\boldsymbol{H}$ as in (1), then $\boldsymbol{U}_0 = \boldsymbol{R}_0\boldsymbol{H}_0\boldsymbol{R}_N \operatorname{diag}\{1, \pm 1\}$ and $\boldsymbol{U}_0 \neq \boldsymbol{0}$ whereas $\boldsymbol{H}_0 \neq \boldsymbol{0}$. As well, $\boldsymbol{U}_N = \boldsymbol{R}_0\boldsymbol{H}_N\boldsymbol{R}_N \operatorname{diag}\{1, \pm 1\} \neq \boldsymbol{0}$ whereas $\boldsymbol{H}_N \neq \boldsymbol{0}$, so the ordrer of $\boldsymbol{U}$ equals the degree N. □

Proof of Property 5. Finally, let $\boldsymbol{U} \in \mathbb{R}^{2\times 2}[z, z^{-1}]$ be a paraunitary matrix, let $\overline{\overline{\boldsymbol{U}}}$ be the associated FIR paraunitary matrix. By definition of $\overline{\overline{\boldsymbol{U}}}$, one has $\overline{\overline{\boldsymbol{U}}}_0 \neq \boldsymbol{0}$, so we can apply result of Lemma 5 and the degree of $\overline{\overline{\boldsymbol{U}}}$ equals its order, which is also the L-degree of $\boldsymbol{U}$. □

4 Polynomial Eigenvalue Decomposition

In polynomial matrices framework, eigenvalues are defined [9] as generalization of the determinantal roots of a order-1 polynomial matrix to an order-n polynomial matrix, that is roots of $\det(\lambda\boldsymbol{I} - \boldsymbol{M}) = 0$ to $\det(\sum_{k=0}^{m}\lambda^k\boldsymbol{M}_k) = 0$. Such eigenvalues are then elements of $\mathbb{C}$. The Polynomial Eigenvalue Decomposition (PEVD) [5] is another problem: one is looking for functions λ and vectors $\underline{v}$ such that, given $\boldsymbol{M} \in \mathbb{C}^{n\times n}[z, z^{-1}]$, one has $\boldsymbol{M}(z)\underline{v}(z) = \lambda(z)\underline{v}(z), \forall z$. As $\mathbb{C}^{n\times n}[z, z^{-1}]$ is not a vector field, we cannot set the existence of L-polynomial solutions.

In case of parahermitian matrices, it is shown in [5] that they can be almost diagonalized by means of paraunitary matrices. One can wonder if there exists an exact L-polynomial solution in this case. We will show in the following example that it is not always the case.

First, remark that if $\boldsymbol{H} \in \mathbb{C}^{n\times n}[z, z^{-1}]$ is parahermitian, and if there exist $\lambda \in \mathbb{C}[z, z^{-1}]$ and $\underline{u} \in \mathbb{C}^{n\times 1}[z, z^{-1}]$, $\underline{u} \neq \underline{0}$, such that $\boldsymbol{H}(z)\underline{u}(z) = \lambda(z)\underline{u}(z)$, then λ is a parahermitian L-polynomial. Indeed, $\langle \boldsymbol{H}\underline{u}, \underline{u}\rangle = \lambda\langle \underline{u}, \underline{u}\rangle$ and $\langle \boldsymbol{H}\underline{u}, \underline{u}\rangle = \langle \underline{u}, \boldsymbol{H}\underline{u}\rangle = \widetilde{\lambda}\langle \underline{u}, \underline{u}\rangle$, so, as $\underline{u} \neq \underline{0}$, $\lambda = \widetilde{\lambda}$.

Example 6. *Let* $\boldsymbol{H}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -2z^{-1} + 6 - 2z \end{bmatrix}$ *be a parahermitian matrix, its L-Smith form is* $\boldsymbol{S}(z) = \operatorname{diag}\{1, 1 - \frac{5}{2}z + z^2\}$. *Suppose there exists a paraunitary matrix* $\boldsymbol{U}$ *such that* $\widetilde{\boldsymbol{U}}\boldsymbol{H}\boldsymbol{U} = \boldsymbol{\Lambda}$. *According to Property 2,* $\boldsymbol{U}$ *is L-unimodular, so* $\boldsymbol{S} \overset{LS}{\sim} \boldsymbol{\Lambda}$, *and the normalized gcd's of their minors are equal, then* $gcd\{\lambda_1, \lambda_2\} = 1$ *and* $\lambda_1(z)\lambda_2(z) = c'z^{\alpha'}\det\boldsymbol{H}(z), c' \in \mathbb{C}, \alpha' \in \mathbb{Z}$. *But* λ_i, *if exist, are parahermitian, so one has (up to a permutation)* $\lambda_1(z) = c, c \in \mathbb{R}^*$ *and* $\lambda_2(z) = dz^{\beta}(1 - \frac{5}{2}z + z^2)$. *Parametrizing* $\boldsymbol{H}(z)\underline{v}(z) = c\underline{v}(z), c \in \mathbb{R}^*$ *leads to a system without any solution.*

A parahermitian L-polynomial that is positive definite on the unit-circle can always be factorized as [3]: $p = l\,\widetilde{l}$ with $l \in \mathbb{C}[z]$ minimum-phase. Otherwise, for a positive definite hermitian (constant) matrix $\boldsymbol{H}$ one has the Cholesky factorization $\boldsymbol{H} = \boldsymbol{L}\boldsymbol{L}^H$ where $\boldsymbol{L}$ is a lower triangular matrix. The Cholesky factorization cannot be extended to $\mathbb{C}^{n\times n}[z, z^{-1}]$ because $\mathbb{C}[z, z^{-1}]$ is not a field and the normalization step is not possible. Therefore, we prove the following.

Property 7. *Let* $\boldsymbol{H}(z) \in \mathbb{C}^{n\times n}[z, z^{-1}]$ *be a parahermitian matrix which is positive definite on* $\mathcal{C}$, *then there exists a rational lower triangular matrix* $\boldsymbol{L} \in \mathbb{C}^{n\times n}(z)$ *with all poles inside the unit disc such that* $\boldsymbol{H} = \boldsymbol{L}\widetilde{\boldsymbol{L}}$.

By hypothesis, on $\mathcal{C}$, one has $\boldsymbol{H} = \widetilde{\boldsymbol{H}} = \boldsymbol{H}^H$ and $\boldsymbol{H}$ is parahermitian. Hence, it admits an $\boldsymbol{L}\boldsymbol{D}\tilde{\boldsymbol{L}}$ factorization; the proof is constructive and follows the same lines as for constant hermitian matrices. Note that $\boldsymbol{L}$ and $\boldsymbol{D}$ have rational entries because divisions are required in the algorithm. Next, d_{ii} are parahermitian positive definite on $\mathcal{C}$ because $\boldsymbol{H}$ is. Hence they can be factorized into $d_{ii} = f_{ii}\widetilde{f_{ii}}$, where f_{ii} has all its poles inside the unit disk [3]. We pull the diagonal factor in the triangular one by defining $\boldsymbol{G} = \boldsymbol{L}\boldsymbol{F}$, and we have $\boldsymbol{H} = \boldsymbol{G}\widetilde{\boldsymbol{G}}$. However, some poles of $\boldsymbol{G}$ could be outside the unit disk, so that we are not finished. For $1 < k \leq n$, define $p_k(z)$ as the least common multiple of all unstable denominators present in the kth column of $\boldsymbol{G}$, and a new diagonal matrix $\boldsymbol{\Delta}$ with entries $\delta_{kk} = \frac{p_k}{\widetilde{p_k}}$. Obviously, $\boldsymbol{\Delta}\widetilde{\boldsymbol{\Delta}} = \boldsymbol{I}$. Now, matrix $\boldsymbol{G}\boldsymbol{\Delta}$ is triangular with all poles inside the unit disk, and $\boldsymbol{G}\boldsymbol{\Delta}\widetilde{\boldsymbol{\Delta}}\widetilde{\boldsymbol{G}} = \boldsymbol{H}$. □

Thanks to Property 7, every positive definite parahermitian matrix admits an $\boldsymbol{L}\widetilde{\boldsymbol{L}}$ factorization, where $\boldsymbol{L}$ is continuous on $\mathcal{C}$. Next, in [7] we have shown:

Property 8. *Let* $\boldsymbol{A}$ *be a continuous function from* $\mathcal{C}$ *to* $\mathbb{C}^{n\times n}$, *then there exist* $\boldsymbol{U}, \boldsymbol{V} \in \mathcal{C}(\mathbb{C}^{n\times n})$ *such that* $\boldsymbol{U}\boldsymbol{U}^H = \boldsymbol{I}$, $\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{I}$ *and*

$$\boldsymbol{A}(z) = \boldsymbol{V}(z)\boldsymbol{\Sigma}(z)\boldsymbol{U}^H(z), \forall z \in \mathcal{C}$$

with $\boldsymbol{\Sigma} = \operatorname{diag}\{\sigma_i\}$ *and* $\sigma_i \in \mathcal{C}(\mathbb{R}), i = 1$ *to* n.

This has been shown by maximizing the real part of $\underline{u}(z)^H\boldsymbol{A}(z)\underline{v}(z)$ on the unit circle with $\underline{u}$ and $\underline{v}$

continuous functions of unit norm for each $z \in \mathcal{C}$ ($\underline{u}(z)^H \underline{u}(z) = 1$), taking into account compactness of the involved sets, and proceeding by deflation [7]. We can then deduce:

Property 9. *Let $\boldsymbol{H} = \boldsymbol{A}\boldsymbol{A}^H$ with $\boldsymbol{A} \in \mathcal{C}(\mathbb{C}^{n\times n})$. Then, there exists $\boldsymbol{V} \in \mathcal{C}(\mathbb{C}^{n\times n})$ such that $\boldsymbol{V}\boldsymbol{V}^H = \boldsymbol{I}$ and*

$$\boldsymbol{H}(z) = \boldsymbol{V}(z)\boldsymbol{\Lambda}(z)\boldsymbol{V}(z)^H, \forall z \in \mathcal{C}$$

with $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_i\}, \lambda_i \in \mathcal{C}(\mathbb{R}), \lambda_i \geq 0, i$ =1 to n.

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be the paraunitary matrices on $\mathcal{C}$ of Property 8: $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^H$, so $\boldsymbol{H} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{U}^H\boldsymbol{U}\boldsymbol{\Sigma}^H\boldsymbol{V}^H = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^H\boldsymbol{V}^H$, and $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^H$ is a diagonal matrix which $\lambda_i = |\sigma_i|^2 \geq 0$. □
It is easy to show using Stone-Weierstrass theorem, that the set of L-polynomials defined on $\mathcal{C}$ is dense in $\mathcal{C}(\mathbb{C})$. Then, one can approximate each λ_i by a L-polynomial on $\mathcal{C}$. Each column of $\boldsymbol{V}$ can also be approximated by a L-polynomial vector and we obtain the final result:

Property 10. *Let $\boldsymbol{H} \in \mathbb{C}^{n\times n}[z, z^{-1}]$ be a parahermitian matrix, then there exist n L-polynomial vectors $\underline{v}_i \in \mathbb{C}^n[z, z^{-1}]$ and n L-polynomial λ_i positive on $\mathcal{C}$ such that*

$$\boldsymbol{H}(z)\underline{v}_i(z) \approx \lambda_i(z)\underline{v}_i(z)$$

with $\underline{v}_i(z)^H \underline{v}_j(z) \approx \delta_{ij}, \forall z \in \mathcal{C}$.

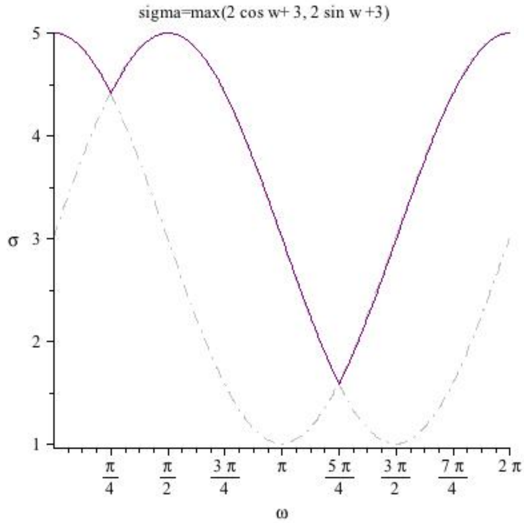
Because of lack of space, the proof is postponed to a full-length paper.

5 Concluding remarks

Example 6 pointed out that the L-polynomial PEVD does not always exist. On the other hand, the above property proved in [7] shows that an approximate PEVD exists, in the sense of the infinite norm of continuous functions on $\mathcal{C}$. This had been already guessed by the authors of [6], who devised numerical algorithms.

Appendix

The maximization of the real part of $\underline{u}^H\boldsymbol{A}\underline{v}$ on $\mathcal{C}$ in Property 8 should be carried out with care. Indeed, let $\boldsymbol{A}(z) = \mathrm{diag}\{z^{-1} + 3 + z, jz^{-1} + 3 - jz\}$ be a diagonal positive definite parahermitian matrix on $\mathcal{C}$. It can be shown that $\sigma_1(z) = \max\{|a_{11}(z)|, |a_{22}(z)|\}\, \forall z \in \mathcal{C}$. σ_1 is illustrated below, where ω stands for the argument of z. Clearly σ_1 is continuous but not differentiable, so its L-polynomial approximation will lead to a high degree polynomial. On the other hand, as $\boldsymbol{A}$ is diagonal, its eigenvalues are $z^{-1} + 3 + z$ and $jz^{-1} + 3 - jz$. The apparent swapping between σ_i's comes from the fact that ones imposed σ_i to be optimum for each z on the unit circle, which is not necessary. The permutation can indeed be fixed by only sorting σ_i at a given ω, *e.g.* at $\omega = 0$.



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