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# Existence and uniqueness of a blow-up solution for a parabolic problem with a *localized* *nonlinear* term via semigroup theory

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## Abstract

Here, we use the semigroup theory to establish the existence, uniqueness and blow-up for a classical solution of a semilinear parabolic problem with *localized nonlinear* term— a locally Lipschitz continuous function of the value of the solution at a point of a 1-dimensional domain. Our method, which uses Sobolev spaces and fractional power of operators, is in contrast with the classical ones (Green functions) which supply similar results in 1-dimensional settings.

## 1. Introduction

Let  $x_0$  be a fixed point in  $I = (0, 1)$  and denote its closure by  $\bar{I}$ . We study the semilinear parabolic initial-boundary value problem with a localized nonlinear term

$$\left. \begin{aligned} u_t(x, t) - \frac{1}{k(x)}(p(x)u_x(x, t))_x &= f(u(x_0, t)), \quad (x, t) \in I \times (0, \infty), \\ u(0, t) = 0 = u(1, t), \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \bar{I}, \end{aligned} \right\} \quad (1)$$

where  $k \in L^\infty(I)$ ,  $p \in L^\infty(I)$ ,  $u_0 \in H^2(I) \cap H_0^1(I)$  and  $f$  is locally Lipschitz continuous. A solution  $u$  of (1) is said to blow up at the point  $x = b$  in finite time  $t_b$  if there exists a sequence  $(x_n, t_n)$  such that  $(x_n, t_n) \rightarrow (b, t_b)$  and  $\lim_{n \rightarrow \infty} u(x_n, t_n) = \infty$ . The set consisting of all blow-up points of  $u$  is called

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the blow-up set of  $u$ . Our study is exclusively concerned with the question of existence and uniqueness of the blow-up solution of problem (1) and the blow-up point of such solution.

J. M. Chadam, A. Peirce and H. M. Yin [1] in 1992 studied the blow-up property of solutions to the problem

$$\begin{aligned} u_t - \Delta u &= f(u(x_0, t)), \quad (x, t) \in \Omega \times (0, T) \\ u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x), \quad x \in \bar{\Omega}, \end{aligned}$$

where  $T$  is a positive number,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$  while  $x_0$  is a fixed interior point of  $\Omega$ . They showed that under some conditions the solution blows up in finite time and the blow-up set is the whole region. In 2000, C. Y. Chan and J. Yang [2] studied the same question for the degenerate semilinear parabolic initial-boundary value problem

$$\begin{aligned} x^q u_t - u_{xx} &= f(u(x_0, t)), \quad (x, t) \in I \times (0, T), \\ u(0, t) &= 0 = u(1, t), \quad t \in (0, T), \\ u(x, 0) &= u_0(x), \quad x \in \bar{I}, \end{aligned}$$

where  $q$  is any nonnegative real number,  $f$  and  $u_0$  are given functions. By using Green function method, they proved that with suitable conditions,  $u$  blows up in finite time, and the blow-up set is the entire interval  $\bar{I}$ .

Our objective is to show existence, uniqueness and blow-up for a classical solution of problem (1) by using semigroup theory. Throughout this work, we assume the following:

- (H1)  $k \in L^\infty(I)$  and  $\exists k_m, k_M \in (0, +\infty)$  such that  $k_m < k(x) < k_M$  a.e.  $x \in I$ ,
- (H2)  $p \in L^\infty(I)$  and  $\exists p_m, p_M \in (0, +\infty)$  such that  $p_m < p(x) < p_M$  a.e.  $x \in I$ ,
- (H3)  $\forall M \in (0, +\infty) \exists L_M \in (0, +\infty)$  such that if for each  $s, s' \in \mathbb{R}^+$  with  $|s|, |s'| \leq M$ , then  $|f(s) - f(s')| \leq L_M |s - s'|$ .
- (H4)  $u_0 \in H^2(I) \cap H_0^1(I)$ .

In order to obtain existence and uniqueness of a solution of problem (1), we will consider its formally equivalent formulation in terms of a nonlinear evolution equation in the Hilbert space  $L^2(I)$  :

$$\left. \begin{aligned} \frac{du(t)}{dt} - Au(t) &= F(u) \quad \text{for } t > 0, \\ u(0) &= u_0, \end{aligned} \right\} \quad (2)$$

where  $A$  is the linear unbounded operator from  $D(A)$ , the domain of  $A$ , to  $L^2(I)$  with

$$D(A) = \left\{ v \in H_0^1(I) \mid \exists! w \in L^2(I) \text{ s.t.} \right. \\ \left. \int_I k(x)w(x)\varphi(x) dx = - \int_I p(x)D_x v(x)D_x \varphi(x) dx, \forall \varphi \in H_0^1(I) \right\},$$

and  $Av(x) = w(x)$  for all  $v \in D(A)$  and where  $F$  is defined by

$$u \in D(A) \longmapsto F(u) = f(u(x_0, t)) \in L^2(I).$$

It will be shown before showing proposition 3.1.6 that the definition of  $F$  is meaningful.

## 2. Main results

Our results comprise the following two theorems. The first one involves existence and uniqueness of a solution  $u$  of problem (2) (in the sense of semigroup theory) whereas the last one deals with the blow-up time of  $u$ .

**Theorem 2.1** *There exists a finite positive constant  $T$  such that the evolution problem (2) has a unique solution  $u \in C([0, T], D(A)) \cap C^1([0, T], L^2(I))$  defined by*

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)F(u(\tau))d\tau$$

where  $S(t)$  is an analytic semigroup generated by  $A$ .

**Theorem 2.2** *If  $[0, T_{\max})$  is the finite maximal time interval in which a continuous solution  $u$  of problem (2) exists, then  $|u(x_0, t)|$  is unbounded as  $t$  tends to  $T_{\max}$ .*

## 3. The proof of main results

Hereafter we use an inner product and a norm, equivalent to the usual one, on  $L^2(I)$  by

$$\langle v, w \rangle = \int_I k(x)v(x)w(x) dx, \text{ and } \|v\|_{L^2(I)} = \left( \int_I k(x)|v(x)|^2 dx \right)^{1/2}.$$

If  $D_x v$  denotes the distributional derivative with respect to  $x$  of the distribution  $v \in \mathcal{D}'(I)$ , we recall that

$$H^1(I) = \{ v \in L^2(I) \mid D_x v \in L^2(I) \}.$$

The Hilbert space  $H^1(I)$  here is equipped with the norm (equivalent to the usual one):

$$|v|_{H^1(I)}^2 = |v|_{L^2(I)}^2 + \int_I p(x) |D_x v(x)|^2 dx$$

whereas its closed subspace  $H_0^1(I) = \{v \in H^1(I) \mid v(0) = 0 = v(1)\}$  is equipped with

$$|v|_{H_0^1(I)}^2 = \int_I p(x) |D_x v(x)|^2 dx;$$

the norm induced by  $|\cdot|_{H^1(I)}$ .

### 3.1 The proof of Theorem 2.1

To get existence and uniqueness of a solution of problem (2), we need the following propositions referred to [4].

**Proposition 3.1.1** *If  $A$  is self-adjoint and generates a  $C_0$  uniformly bounded semigroup  $S(t)$  and  $g$  is Hölder continuous of exponent  $\alpha \in (0, 1]$ . Then the evolution equation:*

$$\frac{du(t)}{dt} = Au(t) + g(t) \text{ with } u(0) = u_0 \in D(A)$$

has a unique solution  $u$  such that

$$u \in C^1([0, \infty), L^2(I)) \cap C([0, \infty), D(A))$$

which can be expressed as

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)g(\tau)d\tau.$$

Observe that the operator  $A$  of problem (2) is given by

$$Av(x) = \frac{1}{k(x)} D_x(p(x)D_x v(x)).$$

To apply proposition 3.1.1 to such an operator, we show first that

**Proposition 3.1.2** *The operator  $A$  of problem (2) is  $m$ -dissipative and self-adjoint in  $L^2(I)$ .*

**Proof.** An  $m$ -dissipative property of  $A$  in  $L^2(I)$  is an immediate consequence of these two conditions:

1.  $\langle Av, v \rangle \leq 0$  for all  $v \in D(A)$ , and

2. for any  $\lambda > 0$ ,  $R(I - \lambda A) = L^2(I)$ , where  $R(I - \lambda A)$  and  $I$  denote the range of  $I - \lambda A$  and the identity operator on  $L^2(I)$  respectively.

Condition 1. follows directly from definition of  $A$ . To obtain condition 2., letting  $g \in L^2(I)$  and  $\lambda > 0$ , we need to give an existence of  $v \in H_0^1(I)$  with the property:

$$\frac{1}{\lambda} \int_I k(x)v(x)\varphi(x) dx + \int_I p(x)D_x v(x)D_x \varphi(x) dx = \frac{1}{\lambda} \int_I k(x)g(x)\varphi(x) dx$$

for each  $\varphi \in H_0^1(I)$ . Such an existence is guaranteed by Lax-Milgram theorem, and thus,  $A$  is  $m$ -dissipative.

In order to prove that  $A$  is a self-adjoint operator in  $L^2(I)$ , since  $A$  is  $m$ -dissipative in  $L^2(I)$ , it suffices to prove that  $A$  is symmetric, that is,  $\langle Av, \varphi \rangle = \langle v, A\varphi \rangle$  for all  $v$  and  $\varphi$  in  $D(A)$ . Indeed, definitions of  $D(A)$ ,  $Av$  and  $A\varphi$  yield

$$\langle Av, \varphi \rangle = - \int_I p(x)D_x v(x)D_x \varphi(x) dx = \langle v, A\varphi \rangle.$$

□

To solve problem (2), it is convenient to introduce the square root of  $-A$ ,  $(-A)^{\frac{1}{2}}$ . An elementary way to define  $(-A)^{\frac{1}{2}}$  is by considering the eigenvalues and eigenfunctions of  $-A$ . The operator  $(\lambda I - A)^{-1}$  is a bounded well-defined operator on  $L^2(I)$  with values in  $H_0^1(I)$  so that Rellich theorem (the embedding of  $H_0^1(I)$  into  $L^2(I)$  is compact) implies that  $(\lambda I - A)^{-1}$  is a compact operator on  $L^2(I)$ .

The following proposition is referred from [3].

**Proposition 3.1.3 (The spectral theory of self-adjoint compact operator)** *There exists a sequence  $(\lambda_n, \phi_n) \subset (0, +\infty) \times H_0^1(I)$  such that*

1.  $A\phi_n = -\lambda_n\phi_n$ .
2.  $\int_I k(x)\phi_n(x)\phi_m(x)dx = \delta_{nm}$ .
3.  $\int_I p(x)D_x\phi_n(x)D_x\phi_m(x)dx = \lambda_n\delta_{nm}$ .
4.  $v(x) = \sum_{n \in \mathbb{N}} \langle v, \phi_n \rangle \phi_n(x)$  for all  $v \in L^2(I)$ .
5.  $\|v\|_{L^2(I)}^2 = \sum_{n \in \mathbb{N}} \langle v, \phi_n \rangle^2$
6.  $D(A) = \left\{ v \in L^2(I) \mid \sum_{n \in \mathbb{N}} \lambda_n^2 \langle v, \phi_n \rangle^2 < +\infty \right\}$  and  $Av = - \sum_{n \in \mathbb{N}} \lambda_n \langle v, \phi_n \rangle \phi_n$  for each  $v \in D(A)$ .

7.  $S(t)v = \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \langle v, \phi_n \rangle \phi_n$  for all  $(v, t) \in L^2(I) \times [0, \infty)$ .

Now, we can define domain of  $(-A)^{\frac{1}{2}}$  by

$$D((-A)^{\frac{1}{2}}) = \left\{ v \in L^2(I) \mid \sum_{n \in \mathbb{N}} \lambda_n \langle v, \phi_n \rangle^2 < +\infty \right\}$$

and the unbounded self-adjoint operator  $(-A)^{\frac{1}{2}}$  in  $L^2(I)$  by

$$(-A)^{\frac{1}{2}}v = \sum_{n \in \mathbb{N}} \lambda_n^{\frac{1}{2}} \langle v, \phi_n \rangle \phi_n$$

for all  $v \in D((-A)^{\frac{1}{2}})$ . Moreover, we obtain the following propositions.

**Proposition 3.1.4** *For the operator  $A$  of problem (2),*

1.  $D((-A)^{\frac{1}{2}}) = H_0^1(I)$  and  $\left| (-A)^{\frac{1}{2}}v \right|_{L^2(I)} = |v|_{H_0^1(I)}$ .

2. If  $v \in D((-A)^{\frac{1}{2}})$ , then  $S(t)v \in D((-A)^{\frac{1}{2}})$  and

$$\left| (-A)^{\frac{1}{2}}S(t)v \right|_{L^2(I)} = \left| S(t)(-A)^{\frac{1}{2}}v \right|_{L^2(I)} \leq \left| (-A)^{\frac{1}{2}}v \right|_{L^2(I)}.$$

**Proof.** Let us prove result 1. first.

If  $v = \sum_{n \in \mathbb{N}} \langle v, \phi_n \rangle \phi_n$  for  $\phi_n \in H_0^1(I)$ , we have in the distributional sense:

$$D_x v = \sum_{n \in \mathbb{N}} \langle v, \phi_n \rangle D_x \phi_n$$

so that  $\sum_{n \in \mathbb{N}} \lambda_n \langle v, \phi_n \rangle^2 = \int_I p(x) |D_x v(x)|^2 dx = |v|_{H_0^1(I)}^2 < +\infty$ . Conversely, if  $v \in D((-A)^{\frac{1}{2}})$ , the sequence  $(V_N)$ , where

$$V_N = \sum_{n=1}^N \langle v, \phi_n \rangle \phi_n,$$

is Cauchy in  $H_0^1(I)$  because if  $N < M$ , then

$$\begin{aligned} |V_N - V_M|_{H_0^1(I)}^2 &= \int_I p(x) \left| \sum_{n=N+1}^M \langle v, \phi_n \rangle D_x \phi_n(x) \right|^2 dx \\ &= \sum_{n=N+1}^M \langle v, \phi_n \rangle^2 \int_I p(x) |D_x \phi_n(x)|^2 dx \\ &= \sum_{n=N+1}^M \lambda_n \langle v, \phi_n \rangle^2. \end{aligned}$$

Hence it converges to some  $V$  in  $H_0^1(I)$  ( $H_0^1(I)$  is a Hilbert space) and to  $v$  in  $L^2(I)$  so that  $v = V \in H_0^1(I)$ . The remaining equality has already been proven.

For result 2., because  $\sum_{n \in \mathbb{N}} \lambda_n e^{-2\lambda_n t} \langle v, \phi_n \rangle^2 \leq \sum_{n \in \mathbb{N}} \lambda_n \langle v, \phi_n \rangle^2$  for all  $t \geq 0$ , proposition 3.1.3 yields: if  $v \in D((-A)^{\frac{1}{2}})$ , then  $S(t)v \in D((-A)^{\frac{1}{2}})$  and  $(-A)^{\frac{1}{2}} S(t)v = S(t)(-A)^{\frac{1}{2}}v$  for  $t \geq 0$ .  $\square$

**Proposition 3.1.5** *There exists a  $C_0 > 0$  such that*

$$\left| (-A)^{\frac{1}{2}} S(t)v \right|_{L^2(I)} = |S(t)v|_{H_0^1(I)} \leq \frac{C_0}{t^{1/2}} |v|_{L^2(I)}$$

for all  $(v, t) \in L^2(I) \times (0, +\infty)$ .

**Proof.** It is not difficult to see that  $\left| (-A)^{\frac{1}{2}} S(t)v \right|_{L^2(I)} = |S(t)v|_{H_0^1(I)}$  for any  $v \in L^2(I)$ . Let  $v \in L^2(I)$ . Since the function  $s \in \mathbb{R}^+ \mapsto se^{-2s} \in \mathbb{R}^+$  is bounded, we have that there is a  $C_0 > 0$  such that

$$t \sum_{n \in \mathbb{N}} \lambda_n e^{-2\lambda_n t} \langle v, \phi_n \rangle^2 \leq C_0 \sum_{n \in \mathbb{N}} \langle v, \phi_n \rangle^2 = C_0 |v|_{L^2(I)}^2.$$

Therefore, the definition of  $(-A)^{\frac{1}{2}}$  yields that  $S(t)v \in D((-A)^{\frac{1}{2}})$  and that the estimate involved in proposition 3.1.5 is true.  $\square$

Note that the previous result implies that  $S(t)v \in D((-A)^{\frac{1}{2}})$  for all  $t > 0$  and all  $v \in L^2(I)$ , which, a priori, is not obvious for a standard semigroup  $T(t)$  on  $L^2(I)$ : usually  $T(t)v$  belongs to  $L^2(I)$  only but due to the self-adjointness of  $A$ , the semigroup  $S(t)$  is analytic (holomorphic) and consequently  $S(t)v \in D(A)$  for all  $t > 0$  and all  $v \in L^2(I)$ .

Presently, we are in a position to solve the evolution problem (2). Firstly, we define a mapping  $F$  by:

$$v \in H_0^1(I) \mapsto F(v) = f(v(x_0)) \in L^2(I). \quad (3)$$

Note that this definition is meaningful because  $v \in H_0^1(I)$  implies that  $v$  is continuous on  $\bar{I}$  so that  $v(x_0)$  has a meaning and  $F(v)$  is a constant on  $I$  and therefore belongs to  $L^2(I)$ .

**Proposition 3.1.6** *The mapping  $F$  defined by (3) is locally Lipschitz from  $D((-A)^{\frac{1}{2}})$  ( $= H_0^1(I)$ ) to  $L^2(I)$ .*

**Proof.** Let  $v, w \in H_0^1(I)$  ( $\hookrightarrow C(\bar{I})$ ) such that  $|v|_{C(\bar{I})}, |w|_{C(\bar{I})} \leq M$  with  $M$



being a positive constant. Then (H3) implies:

$$\begin{aligned}
|F(v) - F(w)|_{L^2(I)}^2 &\leq k_M |f(v(x_0)) - f(w(x_0))|^2 \\
&\leq k_M L_M^2 |v(x_0) - w(x_0)|^2 \\
&\leq k_M L_M^2 |v - w|_{C(\bar{I})}^2 \\
&\leq k_M L_M^2 C_s^2 |v - w|_{H_0^1(I)}^2,
\end{aligned}$$

where  $C_s$  is the constant involved in the Sobolev embedding  $H_0^1(I) \hookrightarrow C(\bar{I})$ .  $\square$

Next, due to proposition 3.1.4, we introduce a concept of mild solution for the evolution problem (2).

**Definition** *A function  $u$  is said to be a mild solution of problem (2) if there exists  $u \in C([0, \infty), H_0^1(I))$  such that*

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)F(u(\tau))d\tau, \quad \forall t \in [0, \infty),$$

$u_0$  being assumed to belong to  $H_0^1(I)$ .

We modify the proof of Theorem 2.5.1 of [5] to obtain the following result.

**Proposition 3.1.7** *There exists a  $T > 0$  such that problem (2) has a unique mild solution. Moreover, let  $u(t)$ ,  $\tilde{u}(t)$  be mild solutions corresponding to  $u_0$  and  $\tilde{u}_0$ , respectively. Then, for all  $t \in [0, T]$ , the following estimate holds:*

$$|u(t) - \tilde{u}(t)|_{H_0^1(I)} \leq |u_0 - \tilde{u}_0|_{H_0^1(I)} e^{2C_0 C_s k_M^{\frac{1}{2}} L_M T^{\frac{1}{2}}}.$$

**Proof.** Let  $M = |u_0|_{H_0^1(I)} + 1$  and  $L_M$  be the Lipschitz constant of  $f$ . Let  $T$  be a positive constant such that  $T < \frac{1}{4k_M C_0^2 C_s^2 L_M^2}$ . We define a mapping  $\Phi$  by:

$$v \in E \mapsto \Phi(v) = S(t)u_0 + \int_0^t S(t - \tau)F(v(\tau))d\tau$$

where

$$E = \left\{ v \in C([0, T], H_0^1(I)) \text{ such that } |v(t)|_{H_0^1(I)} \leq M \text{ for all } t \in [0, T] \right\},$$

equipped with the norm:

$$|v|_E = \sup_{t \in [0, T]} |v(t)|_{H_0^1(I)}.$$

We note that  $E$  is a closed convex subset of a Banach space  $C([0, T], H_0^1(I))$ . We would like to prove that  $\Phi$  is a contraction in  $E$ . Propositions 3.1.4, 3.1.5 and 3.1.6 imply:

$$\begin{aligned}
|\Phi(v)|_E &= \sup_{t \in [0, T]} \left| S(t)u_0 + \int_0^t S(t-\tau)F(v(\tau))d\tau \right|_{H_0^1(I)} \\
&\leq |u_0|_{H_0^1(I)} + \sup_{t \in [0, T]} \int_0^t |S(t-\tau)F(v(\tau))|_{H_0^1(I)} d\tau \\
&\leq |u_0|_{H_0^1(I)} + \sup_{t \in [0, T]} \int_0^t \frac{C_0}{(t-\tau)^{\frac{1}{2}}} \left( |f(0)|_{L^2(I)} + k_M^{\frac{1}{2}} L_M C_s |v|_{H_0^1(I)} \right) d\tau \\
&\leq |u_0|_{H_0^1(I)} + \left( C_0 |f(0)|_{L^2(I)} + C_0 k_M^{\frac{1}{2}} L_M C_s M \right) \sup_{t \in [0, T]} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \\
&\leq |u_0|_{H_0^1(I)} + 2C_0 \left( |f(0)|_{L^2(I)} + k_M^{\frac{1}{2}} L_M C_s M \right) T^{\frac{1}{2}}.
\end{aligned}$$

If  $T$  is chosen in such a way that

$$T < \min \left\{ \frac{1}{4k_M C_0^2 C_s^2 L_M^2}, \frac{1}{4C_0^2 \left( |f(0)|_{L^2(I)} + k_M^{\frac{1}{2}} L_M C_s M \right)^2} \right\},$$

then  $\Phi(v)$  is in  $E$  for any  $v \in E$ . Moreover, for any  $v_1, v_2 \in E$

$$\begin{aligned}
|\Phi(v_1) - \Phi(v_2)|_E &= \sup_{t \in [0, T]} \left| \int_0^t S(t-\tau) (F(v_1(\tau)) - F(v_2(\tau))) d\tau \right|_{H_0^1(I)} \\
&\leq C_0 \sup_{t \in [0, T]} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} |(F(v_1(\tau)) - F(v_2(\tau)))|_{L^2(I)} d\tau \\
&\leq C_0 k_M^{\frac{1}{2}} L_M C_s \sup_{t \in [0, T]} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau |v_1 - v_2|_E \\
&\leq 2C_0 k_M^{\frac{1}{2}} L_M C_s T^{\frac{1}{2}} |v_1 - v_2|_E.
\end{aligned}$$

That is,  $\Phi$  is a contraction in  $E$ . Thus,  $\Phi$  has a fixed point that is the mild solution to problem (2) in  $E$ . To show that the uniqueness also holds in  $C([0, T], H_0^1(I))$ , let  $u_1, u_2 \in C([0, T], H_0^1(I))$  be two solutions of problem (2) and let  $u = u_1 - u_2$ . Then

$$u(t) = \int_0^t S(t-\tau) (F(u_1(\tau)) - F(u_2(\tau))) d\tau.$$

Propositions 3.1.4, 3.1.5 and 3.1.6 imply:

$$\begin{aligned} |u(t)|_{H_0^1(I)} &= \left| \int_0^t S(t-\tau) (F(u_1(\tau)) - F(u_2(\tau))) d\tau \right|_{H_0^1(I)} \\ &\leq C_0 C_s k_M^{\frac{1}{2}} L_M \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} |u_1(\tau) - u_2(\tau)|_{H_0^1(I)} d\tau. \end{aligned}$$

By Gronwall inequality, we immediately conclude that  $|u(t)|_{H_0^1(I)} = 0$ , that is, the uniqueness in  $C([0, T], H_0^1(I))$  is proven. As before, we have

$$u(t) - \tilde{u}(t) = S(t)(u_0 - \tilde{u}_0) + \int_0^t S(t-\tau) (F(u(\tau)) - F(\tilde{u}(\tau))) d\tau.$$

Therefore,

$$\begin{aligned} &|u(t) - \tilde{u}(t)|_{H_0^1(I)} \\ &\leq |u_0 - \tilde{u}_0|_{H_0^1(I)} + C_0 C_s k_M^{\frac{1}{2}} L_M \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} |u(\tau) - \tilde{u}(\tau)|_{H_0^1(I)} d\tau. \end{aligned}$$

Gronwall inequality implies:

$$\begin{aligned} |u(t) - \tilde{u}(t)|_{H_0^1(I)} &\leq |u_0 - \tilde{u}_0|_{H_0^1(I)} e^{C_0 C_s k_M^{\frac{1}{2}} L_M \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} d\tau} \\ &\leq |u_0 - \tilde{u}_0|_{H_0^1(I)} e^{2C_0 C_s k_M^{\frac{1}{2}} L_M T^{\frac{1}{2}}}. \end{aligned}$$

Hence, this proposition is proven.  $\square$

By modifying the proof of Corollary 2.5.1 of [5] we establish the following result.

**Proposition 3.1.8** *The mild solution  $u$  is Hölder continuous of exponent  $\alpha$  ( $= \frac{1}{2}$ ) in  $t$  from  $[0, T]$  toward  $H_0^1(I)$  for any  $u_0 \in D(A) (= H^2(I) \cap H_0^1(I))$ .*

**Proof.** Let  $u_0 \in D(A)$ . For any  $h > 0$ . Let  $\tilde{u}(t) = u(t+h)$ . Then, we see that  $\tilde{u}$  is a mild solution of problem (2) with initial data  $u_0 = u(h)$ . Then,

$$\begin{aligned} |u(t+h) - u(t)|_{H_0^1(I)} &= |\tilde{u}(t) - u(t)|_{H_0^1(I)} \\ &\leq |u(h) - u_0|_{H_0^1(I)} e^{2C_0 C_s k_M^{\frac{1}{2}} L_M t^{\frac{1}{2}}}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
& |u(h) - u_0|_{H_0^1(I)} \\
& \leq |S(h)u_0 - u_0|_{H_0^1(I)} + \int_0^h |S(h-\tau)F(u(\tau))|_{H_0^1(I)} d\tau \\
& \leq \left| \int_0^h S(\tau)Au_0 d\tau \right|_{H_0^1(I)} + \int_0^h \frac{C_0}{(h-\tau)^{\frac{1}{2}}} |F(u(\tau))|_{H_0^1(I)} d\tau \\
& \leq \int_0^h |S(\tau)Au_0|_{H_0^1(I)} d\tau \\
& \quad + \int_0^h \frac{C_0}{(h-\tau)^{\frac{1}{2}}} \left( |F(u_0)|_{H_0^1(I)} + k_M^{\frac{1}{2}} L_M C_s |u(\tau) - u_0|_{H_0^1(I)} \right) d\tau \\
& \leq 2C_0 \left( |Au_0|_{L^2(I)} + |F(u_0)|_{H_0^1(I)} \right) h^{\frac{1}{2}} \\
& \quad + C_0 k_M^{\frac{1}{2}} L_M C_s \int_0^h \frac{|u(\tau) - u_0|_{H_0^1(I)}}{(h-\tau)^{\frac{1}{2}}} d\tau.
\end{aligned}$$

By Gronwall inequality, we have

$$|u(h) - u_0|_{H_0^1(I)} \leq 2C_0 \left( |Au_0|_{L^2(I)} + |F(u_0)|_{H_0^1(I)} \right) h^{\frac{1}{2}} e^{2C_0 k_M^{\frac{1}{2}} L_M C_s h^{\frac{1}{2}}}.$$

Thus, for any  $t_1, t_2 \in [0, T]$  such that  $t_1 + h = t_2$

$$\begin{aligned}
& |u(t_1) - u(t_2)|_{H_0^1(I)} \\
& \leq 2C_0 \left( |Au_0|_{L^2(I)} + |F(u_0)|_{H_0^1(I)} \right) e^{4C_0 C_s k_M^{\frac{1}{2}} L_M T^{\frac{1}{2}}} |t_1 - t_2|^{\frac{1}{2}}.
\end{aligned}$$

Hence  $u$  is Hölder continuous of exponent  $\alpha = \frac{1}{2}$  in  $t$ .  $\square$

Now we are in a position to prove theorem 2.1.

**Proof of Theorem 2.1.** Since  $F$  is locally Lipschitz and  $u$  is Hölder continuous of exponent  $\alpha = \frac{1}{2}$  in  $t$ ,  $F$  is also Hölder continuous of exponent  $\alpha = \frac{1}{2}$  in  $t$ . Hence, the result is a consequence of proposition 3.1.1.  $\square$

### 3.2 The proof of Theorem 2.2

Let us modify the proof of theorem 2.5.5 of [5] to obtain the following result.

**Proposition 3.2.1** *Let  $[0, T_{\max})$  be the maximal time interval in which the mild solution  $u$  of the evolution problem (2) exists. If  $T_{\max}$  is finite, then the solution  $u$  of problem (2) blows up in finite time, that is,*

$$\lim_{t \rightarrow T_{\max}} |u(t)|_{H_0^1(I)} = +\infty.$$

**Proof.** We will use the contraction argument to prove proposition 3.2.1. Suppose that there is a finite positive constant  $M$  and a sequence  $(t_n)$  such that

$$|u(t_n)|_{H_0^1(I)} \leq M \text{ as } t_n \rightarrow T_{\max}.$$

Consider the following problem:

$$\frac{dv(t)}{dt} = Av(t) + F(v) \text{ and } v(0) = u(t_n).$$

By proposition 3.1.7, the above problem has a unique local mild solution in  $[0, \delta]$  with  $\delta$  depending on  $M$ . We choose  $n$  large enough so that  $t_n + \delta > T_{\max}$ . Let

$$\tilde{u}(t) = \begin{cases} u(t), & \text{for } 0 \leq t \leq t_n, \\ v(t - t_n), & \text{for } t_n \leq t \leq t_n + \delta. \end{cases}$$

We next would like to show that  $\tilde{u}(t)$  is a mild solution of problem (2) in  $[0, t_n + \delta]$ , i.e.,  $\tilde{u}(t)$  satisfies the integral equation

$$\tilde{u}(t) = S(t)u_0 + \int_0^t S(t - \tau)F(\tilde{u}(\tau))d\tau \text{ for } 0 \leq t \leq t_n + \delta. \quad (4)$$

From

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)F(u(\tau))d\tau \text{ for } 0 \leq t \leq t_n,$$

and

$$v(t) = S(t)u(t_n) + \int_0^t S(t - \tau)F(v(\tau))d\tau \text{ for } 0 \leq t \leq \delta,$$

it is clear that for  $t \in [0, t_n]$ ,  $\tilde{u}(t)$  satisfies (4). For  $t \in [0, \delta]$ ,

$$\begin{aligned} \tilde{u}(t + t_n) &= v(t) \\ &= S(t + t_n)u_0 + \int_0^{t_n} S(t + t_n - \tau)F(u(\tau))d\tau + \int_0^t S(t - \tau)F(v(\tau))d\tau \\ &= S(t + t_n)u_0 + \int_0^{t_n} S(t + t_n - \tau)F(u(\tau))d\tau \\ &\quad + \int_{t_n}^{t+t_n} S(t + t_n - \tau)F(v(\tau - t_n))d\tau \\ &= S(t + t_n)u_0 + \int_0^{t_n} S(t + t_n - \tau)F(\tilde{u}(\tau))d\tau \\ &\quad + \int_{t_n}^{t+t_n} S(t + t_n - \tau)F(\tilde{u}(\tau))d\tau \\ &= S(t + t_n)u_0 + \int_0^{t+t_n} S(t + t_n - \tau)F(\tilde{u}(\tau))d\tau. \end{aligned}$$

Hence,  $\tilde{u}$  is a mild solution of problem (2) in  $[0, t_n + \delta]$  with  $t_n + \delta > T_{\max}$ . This contradicts the definition of  $T_{\max}$ . Therefore, the proof of proposition 3.2.1 is complete.  $\square$

We next prove Theorem 2.2

**Proof of Theorem 2.2.** Suppose that there is a positive constant  $M$  such that  $|u(x_0, t)| \leq M$  as  $t \rightarrow T_{\max}$ . Since

$$\begin{aligned} u(t) &= S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau \\ &= S(t)u_0 + \int_0^t f(u(x_0, \tau))S(t-\tau)\mathbf{1}d\tau \end{aligned}$$

where  $\mathbf{1}$  is a function in  $L^2(I)$  such that  $\mathbf{1}(x) = 1 \forall x \in I$ . Then, from proposition 3.1.4, we have

$$\begin{aligned} |u(t)|_{H_0^1(I)} &\leq |u_0|_{H_0^1(I)} + (|f(0)| + ML_M) \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} |\mathbf{1}|_{L^2(I)} d\tau \\ &= |u_0|_{H_0^1(I)} + 2(|f(0)| + ML_M) |\mathbf{1}|_{L^2(I)} t^{\frac{1}{2}}. \end{aligned}$$

Thus, as  $t \rightarrow T_{\max}$ ,  $|u(t)|_{H_0^1(I)}$  is bounded. This contradicts proposition 3.2.1. Hence, theorem 2.2 is proven.  $\square$

## Conclusion

In this paper, we prove existence, uniqueness of a blow-up solution of problem (1) via semigroup theory. It is in contrast with the Green's function method since we are dealing with functions in Sobolev space and fractional operator. The advantage of this method is that the same result can be extended in higher dimension. We also point out that the assumption (H3) on  $f$  guarantees a blow-up if  $T_{\max}$  is finite. Further assumptions of  $f$  is needed in order to show that  $T_{\max}$  is finite which we do not discuss here.

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