Remark on the Homogenization of a Micro-fibered Linearly Elastic Material

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Abstract

We consider the homogenization of a linear elliptic boundary value problem of elasticity. The isotropic elastic material is reinforced by a periodic distribution of very thin parallel fibers in which the Lamé coefficients are assumed to have high values. Bellieud and Gruais proved that the macroscopic behavior is the one of a generalized continuum medium involving an additional state variable accounting for the microstructure. Here we propose a proof of this result by studying the variational convergence of the energy functional.

Keywords: homogenization, variational convergence, capacitary problems.

1 Introduction

We intend to study the macroscopic behavior of a cylindrical micro-fibered structure made of a linearly isotropic elastic matrix surrounding a periodic distribution of very thin linearly isotropic elastic fibers of very high stiffness. As usual, we make no difference between the real physical space and $\mathbb{R}^3$ whose an orthonormal basis is denoted by $\{e^\alpha\}$ and, for all $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2). Let ω a bounded domain of $\mathbb{R}^2$, containing the origin, with a Lipschitz continuous boundary $\partial\omega$ and L a positive number so that $\Omega := \omega \times (0, L)$ is a reference configuration of the fibered structure which can be described as follows (see Figure 1).

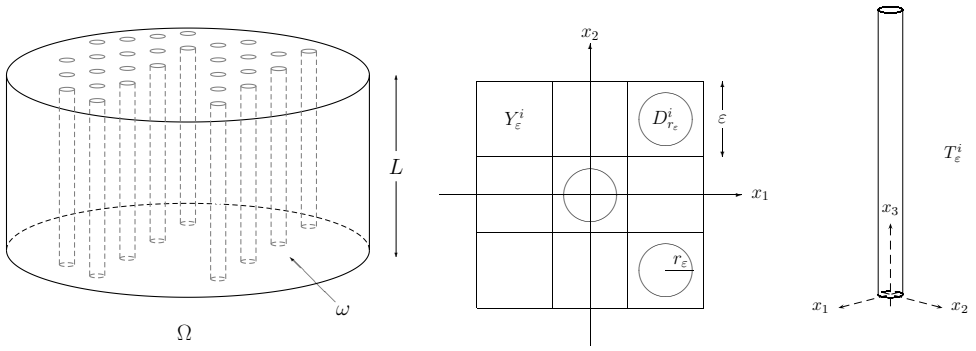

Figure 1: The fibered structure

For each $\varepsilon > 0$, $(Y_\varepsilon^i) \in I_\varepsilon$, where $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$ and $I_\varepsilon := \{i \in \mathbb{Z}^2 \mid Y_\varepsilon^i \subset \omega\}$, denotes a periodic distribution of cells. Let $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$ the family of disk of $\mathbb{R}^2$ centered at

$\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, and $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$. The set $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$ of thin parallel cylinders is the domain occupied by the fibers.

The Lamé coefficients λ_ε and μ_ε of the structure are such that

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_0 > 0, & \text{if } x \in \Omega \setminus T_\varepsilon \\ \lambda_{\varepsilon 1}, & \text{if } x \in T_\varepsilon \end{cases}, \quad \mu_\varepsilon(x) = \begin{cases} \mu_0 > 0, & \text{if } x \in \Omega \setminus T_\varepsilon \\ \mu_{\varepsilon 1}, & \text{if } x \in T_\varepsilon. \end{cases}$$

The structure is clamped on the part $\Gamma_0 := \omega \times \{0, L\}$ of the boundary $\partial\Omega$ of Ω, subjected to body forces of density f and to surface forces of density g on $\Gamma_1 := \partial\Omega \setminus \Gamma_0$.

The problem of finding the equilibrium configuration of the structure reads as

$$\begin{cases} -\operatorname{div} \sigma_\varepsilon = f & \text{in } \Omega, \\ \sigma_\varepsilon := \lambda_\varepsilon \operatorname{tr} e(u_\varepsilon) I + 2\mu_\varepsilon e(u_\varepsilon), & e(u_\varepsilon) = (\nabla u)_s := \frac{1}{2}(\nabla u_\varepsilon + \nabla u_\varepsilon^T), \\ u_\varepsilon = 0 & \text{on } \Gamma_0, \\ \sigma_\varepsilon n = g & \text{on } \Gamma_1. \end{cases} \tag{1}$$

where u_ε, σ_ε denote the displacement and stress fields and n is the unit outward normal. It is well-known that if $\lambda_{\varepsilon 1}$, $\mu_{\varepsilon 1} > 0$, $f \in L^2(\Omega, \mathbb{R}^3)$, $g \in L^2(\Gamma_1, \mathbb{R}^3)$, then the problem, which can also be written

$$(\mathcal{P}_\varepsilon) \quad \min \left\{ F_\varepsilon(w) - L(w) \mid w \in H^1_{\Gamma_0}(\Omega, \mathbb{R}^3) \right\}, \tag{2}$$

where

$$F_\varepsilon(w) := \int_\Omega W_\varepsilon(e(w))\, dx, \quad L(w) := \int_\Omega f \cdot w\, dx - \int_{\Gamma_1} g \cdot w\, ds,$$

and

$$W_\varepsilon(e) := \frac{1}{2}\lambda_\varepsilon \operatorname{tr}^2 e + \mu_\varepsilon |e|^2, \ \forall e \in \mathbb{S}^3 \text{ the space of symmetric } 3 \times 3 \text{ matrices},$$
$$H^1_{\Gamma_0}(\Omega, \mathbb{R}^3) = \{\, v \in H^1(\Omega, \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_0 \,\}$$

has a unique solution $\bar{u}_\varepsilon$.

To determine the macroscopic (or efficient) behavior of the micro-fibered structure, we aim to study the asymptotic behavior of $\bar{u}_\varepsilon$ when ε goes to zero. Let

$$k_\varepsilon := \mu_{\varepsilon 1} \frac{|T_\varepsilon|}{|\Omega|}, \quad l_\varepsilon := \frac{\lambda_{\varepsilon 1}}{\mu_{\varepsilon 1}},$$

and assume that, as $\varepsilon \to 0$,

$$\begin{gathered} r_\varepsilon \to 0, \quad \frac{r_\varepsilon}{\varepsilon} \to 0, \quad \lambda_{\varepsilon 1} \to +\infty, \quad \mu_{\varepsilon 1} \to +\infty, \\ k_\varepsilon \to k \in [0, +\infty], \quad r_\varepsilon^2 k_\varepsilon \to \kappa \in [0, +\infty], \quad l_\varepsilon \to l \in [0, +\infty), \\ (\varepsilon^2 |\ln r_\varepsilon|)^{-1} \to \gamma \in [0, +\infty). \end{gathered} \tag{3}$$

Let 1_{T_ε} the characteristic function of T_ε and $\mathcal{M}_b(\Omega, \mathbb{R}^3)$ the space of bounded $\mathbb{R}^3$-valued measures in Ω, it was proven in [4] that, as ε tends to zero, $\bar{u}_\varepsilon$ weakly converges in $H^1(\Omega, \mathbb{R}^3)$ toward $\bar{u}$ and $\bar{v}_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} \bar{u}_\varepsilon 1_{T_\varepsilon}$ weakly* converges in $\mathcal{M}_b(\Omega, \mathbb{R}^3)$ toward an element $\bar{v}$ of $L^2(\Omega, \mathbb{R}^3)$ solving

$$(\mathcal{P}^{\hom}) \quad \min \left\{ \Phi(u, v) - L(u) \mid (u, v) \in L^2(\Omega, \mathbb{R}^3)^2 \right\},$$

with

$$\Phi(u,v)=\begin{cases}\displaystyle\int_\Omega W_0(e(u))\,dx & \\ \displaystyle+\mu_0\pi\gamma\int_\Omega (v-u)^T\begin{pmatrix}\frac{\chi+1}{\chi} & 0 & 0\\ 0 & \frac{\chi+1}{\chi} & 0\\ 0 & 0 & 1\end{pmatrix}\cdot(v-u)\,dx & \\ \displaystyle+\frac{1}{2}\frac{3l+2}{2(l+1)}k\int_\Omega\left|\frac{\partial v_3}{\partial x_3}\right|^2 dx & \text{if } (u,v)\in\mathcal{D},\\ \displaystyle+\frac{1}{2}\frac{3l+2}{2(l+1)}\frac{\kappa}{4}\int_\Omega\left|\frac{\partial^2 v_1}{\partial x_3^2}\right|^2+\left|\frac{\partial^2 v_2}{\partial x_3^2}\right|^2 dx, & \\ +\infty, & \text{otherwise,}\end{cases}$$

and

$$W_0(e(u)):=\frac{1}{2}\lambda_0\,\mathrm{tr}^2\,e(u)+\mu_0|e(u)|^2,\quad \chi:=\frac{\lambda_0+3\mu_0}{\lambda_0+\mu_0},$$
$$\mathcal{D}:=H^1_{\Gamma_0}(\Omega,\mathbb{R}^3)\times\left\{\,v\in L^2(\omega,H^2_0(0,L;\mathbb{R}^3))\;\middle|\;\frac{\partial v_1}{\partial x_3}=\frac{\partial v_2}{\partial x_3}=0\text{ on }\Gamma_0\,\right\}.$$

Thus, the macroscopic behavior of the micro-fibered structure is the one of a so-called generalized elastic continuum medium involving an additional state variable and its first two derivatives. This additional state variable accounts for the microstructure in the extent where it describes the asymptotic behavior of a suitable scaling of the displacement field in the fibers. Our main concern is to understand this result more deeply and in a more general setting e.g., a different cross-section of the fibers, a more general behavior of the matrix or the fibers. Nevertheless, here, we confine to give another proof of the result of [4] by directly studying the variational convergence (as in the scalar case [5]) of F_ε and shall divide our proof into three steps:

1. a compactness property for each sequence (u_ε) such that $F_\varepsilon(u_\varepsilon)$ is bounded,
2. an upper bound equality for the sequence $(F_\varepsilon(u_\varepsilon))$,
3. a lower bound inequality for the sequence $(F_\varepsilon(u_\varepsilon))$.

2 A Different Approach

Actually, the result of [4] is a standard consequence [3] of the following three propositions:

Proposition 1. ***(Compactness property)*** *Let (u_ε) be a sequence such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon)$ is finite. Then (u_ε) is strongly relatively compact in $L^2(\Omega,\mathbb{R}^3)$ and (v_ε) is bounded in $L^1(\Omega,\mathbb{R}^3)$ and, up to a subsequence, (v_ε) weakly* converges in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$ to an element v of $L^2(\Omega,\mathbb{R}^3)$.*

Proposition 2. ***(Upper bound equality)*** *For all (u,v) in $L^2(\Omega,\mathbb{R}^3)^2$ with $\Phi(u,v)<+\infty$, there exists a sequence (u_ε) such that $u_\varepsilon\to u$ in $L^2(\Omega,\mathbb{R}^3)$, $v_\varepsilon\overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$ and*

$$\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)=\Phi(u,v).$$

Proposition 3. ***(Lower bound inequality)*** *For all u in $L^2(\Omega,\mathbb{R}^3)$ and for all sequence (u_ε) such that $u_\varepsilon\to u$ in $L^2(\Omega,\mathbb{R}^3)$, $v_\varepsilon\overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega,\mathbb{R}^3)$, one has:*

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\geq\Phi(u,v).$$

The convergence symbols $\to$, $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ are used for the strong convergence, the weak convergence and the weak* convergence, respectively. The proof of these propositions are presented in the following subsections, where, as a common practice, C denote various constants which may vary from line to line.

2.1 Proof of Proposition 1

A proof of this proposition can be found in [4].

2.2 Proof of Proposition 2

We split F_ε into three parts:

$$F_\varepsilon^1(w) := \int_{\Omega\setminus(B_\varepsilon\cup T_\varepsilon)} W_\varepsilon(e(w))\,dx,$$
$$F_\varepsilon^2(w) := \int_{B_\varepsilon} W_\varepsilon(e(w))\,dx, \quad F_\varepsilon^3(w) := \int_{T_\varepsilon} W_\varepsilon(e(w))\,dx,$$

where $B_\varepsilon := (D_{R_\varepsilon} \setminus \overline{D}_{r_\varepsilon}) \times (0, L)$, $D_{r_\varepsilon} := \cup_{i\in I_\varepsilon} D^i_{r_\varepsilon}$, $D_{R_\varepsilon} := \cup_{i\in I_\varepsilon} D^i_{R_\varepsilon}$, $D^i_{R_\varepsilon}$ is the disk of $\mathbb{R}^2$ centered at $\hat{x}^i_\varepsilon$ of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. We point out that our proof is in the same spirit as that of [5], where the main ingredient stems from [4] and essentially confine to the convergence of F^2_ε. We first assume u and v to be smooth on $\overline{\Omega}$ and construct an L^2-approximation u_ε of u by:

$$u_\varepsilon = \sum_{\alpha=1}^{3} \big(u_\alpha(e^\alpha - \theta^\alpha_\varepsilon) + \bar{w}_{\varepsilon\alpha}\theta^\alpha_\varepsilon\big). \tag{4}$$

Here, for each $\alpha \in \{1, 2, 3\}$, the vector field $\theta^\alpha_\varepsilon$ is first defined on the closure of $\omega_\varepsilon := \cup_{i\in I_\varepsilon} Y^i_\varepsilon$ as a $(-\varepsilon/2, \varepsilon/2)^2$-periodic element of $H^1(\Omega, \mathbb{R}^3)$ which does not depend on x_3 and satisfies $\theta^\alpha_\varepsilon = e^\alpha$ on D_{r_ε}, $\theta^\alpha_\varepsilon = 0$ on $\overline{\omega}_\varepsilon \setminus D_{R_\varepsilon}$. Next $\theta^\alpha_\varepsilon$ is assumed to vanish on $\overline{\omega}\setminus\omega_\varepsilon$ and

$$\bar{w}_\varepsilon(x) = w_\varepsilon(x) + V_\varepsilon(x) \tag{5}$$

where

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i\in I_\varepsilon} \left(\frac{1}{|D^i_{r_\varepsilon}|}\int_{D^i_{r_\varepsilon}} v(\hat{y}, x_3)\,d\hat{y}\right) 1_{Y^i_\varepsilon}(\hat{x}),$$

and V_ε stems from w_ε in such a way that $F^3_\varepsilon(u_\varepsilon)$ converges. The true expressions of V_ε can be found in [4] (formula (5.16) and (5.52) with ψ and φ in place of v and u).
As $R_\varepsilon \ll \varepsilon$ implies $\lim_{\varepsilon\to 0} |B_\varepsilon \cup T_\varepsilon| = 0$, we have

$$\lim_{\varepsilon\to 0} F^1_\varepsilon(u_\varepsilon) = \int_\Omega W_0(e(u))\,dx.$$

To find the limit of $F^2_\varepsilon(u_\varepsilon)$, we introduce

$$z_\varepsilon := \sum_{\alpha=1}^{3} (v-u)_\alpha\, e(\theta^\alpha_\varepsilon), \tag{6}$$

and compute the linearized strain of u_ε from (4):

$$e(u_\varepsilon) = z_\varepsilon + e(u) + \sum_{\alpha=1}^{3} \left[(\bar{w}_{\varepsilon\alpha} - v_\alpha)\, e(\theta^\alpha_\varepsilon) + (\theta^\alpha_\varepsilon \otimes \nabla(\bar{w}_\varepsilon - u)_\alpha)_s\right].$$

We claim that a good choice of $\theta^\alpha_\varepsilon$ yields

$$\lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon} W_0(e(u_\varepsilon))\,dx - \int_{B_\varepsilon} W_0(z_\varepsilon)\,dx\right) = 0.$$

Note that W_0, being convex and positively homogeneous of degree 2, satisfies (see [1]):

$$\forall\, \xi, \eta \in \mathbb{S}^3, \quad |W_0(\xi) - W_0(\eta)| \le C|\xi - \eta|(|\xi| + |\eta|) \tag{7}$$

so that Cauchy-Schwarz inequality implies

$$\Big|\int_{B_\varepsilon} W_0(e(u_\varepsilon)) - W_0(z_\varepsilon)\,dx\Big| \\ \leq C\left(\int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2\,dx\right)^{1/2}\left(\int_{B_\varepsilon} |e(u_\varepsilon)|^2\,dx + \int_{B_\varepsilon} |z_\varepsilon|^2\,dx\right)^{1/2}.$$

Because u and v are smooth, we have

$$|\nabla \bar{w}_\varepsilon| \leq C \text{ on } B_\varepsilon, \quad |\bar{w}_\varepsilon - v| \leq CR_\varepsilon \text{ on } B_\varepsilon,$$

consequently,

$$\int_{B_\varepsilon} |z_\varepsilon|^2\,dx \leq C\varepsilon^{-2}\sum_{\alpha=1}^{3}\int_{D(r_\varepsilon,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2\,d\hat{x}$$

with $D(r_\varepsilon, R_\varepsilon) = D(0,R_\varepsilon) \setminus \overline{D(0,r_\varepsilon)}$, where for all $R>0$, $D(0,R) := \{\hat{x}\in\mathbb{R}^2 \mid |\hat{x}|<R\}$ and

$$\int_{B_\varepsilon} |e(u_\varepsilon) - z_\varepsilon|^2\,dx \\ \leq CR_\varepsilon^2\varepsilon^{-2}\Big(1+\sum_{\alpha=1}^{3} R_\varepsilon^2\int_{D(0,R_\varepsilon)} |\theta_\varepsilon^\alpha|^2\,d\hat{x} + \int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2\,d\hat{x}\Big) \\ \leq CR_\varepsilon^2\varepsilon^{-2}\Big(1+\sum_{\alpha=1}^{3}\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha|^2\,d\hat{x}\Big)$$

by due account of the Korn inequality in $D(0,R_\varepsilon)$. Therefore,

$$\Big|\int_{B_\varepsilon} W_0(e(u_\varepsilon))\,dx - \int_{B_\varepsilon} W_0(z_\varepsilon)\,dx\Big| \\ \leq CR_\varepsilon\varepsilon^{-1}\Big(1+\sum_{\alpha=1}^{3}\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2\,d\hat{x}\Big)^{1/2}\cdot\varepsilon^{-1}\Big(\sum_{\alpha=1}^{3}\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2\,d\hat{x}\Big)^{1/2},$$

thus, assuming that $\theta_\varepsilon^\alpha$ satisfies

$$\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2\,d\hat{x} \leq \frac{C}{|\ln r_\varepsilon|}, \quad \forall\alpha=\{1,2,3\}, \tag{8}$$

it suffices to study the asymptotic behavior of $\int_{B_\varepsilon} W_0(z_\varepsilon)\,dx$. Let us denote the bilinear form associated with the quadratic form W_0 by w_0:

$$w_0(e,e') = \frac{1}{2}\lambda_0(\operatorname{tr} e)(\operatorname{tr} e') + \mu_0 e\cdot e', \quad \forall e,e'\in\mathbb{S}^3.$$

Note that

$$\int_{B_\varepsilon} W_0(z_\varepsilon)\,dx = \sum_{\alpha,\beta=1}^{3}\int_{B_\varepsilon} (v-u)_\alpha(v-u)_\beta w_0\big(e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big)\,dx \\ = \varepsilon^{-2}\sum_{\alpha,\beta=1}^{3}\Big(\int_{D(r_\varepsilon,R_\varepsilon)} w_0\big((e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big)\,d\hat{x} \\ \int_0^L \sum_{i\in I_\varepsilon} |Y_\varepsilon^i|(v-u)_\alpha(\hat{x}_\varepsilon^i, x_3)(v-u)_\beta(\hat{x}_\varepsilon^i, x_3)\,dx_3\Big) + O(\varepsilon).$$

Here, it clearly appears that in order to get the lowest upper bound for F_ε^2, $\theta_\varepsilon^\alpha$ has to be the unique solution of the capacitary problem

$$(\mathcal{P}_\varepsilon^{\text{cap},\,\alpha}) \quad \min\left\{\int_{D(r_\varepsilon,R_\varepsilon)} W_\varepsilon(e(\varphi))\,d\hat{x} \;\middle|\; \begin{array}{l}\varphi\in H^1((-\varepsilon,\varepsilon)^2,\mathbb{R}^3),\\ \varphi(\hat{x}) = e^\alpha \text{ on } D(0,r_\varepsilon) = \{\,|\hat{x}|<r_\varepsilon\,\},\\ \varphi(\hat{x}) = 0 \text{ on } (-\varepsilon,\varepsilon)^2\setminus D(0,R_\varepsilon).\end{array}\right\}$$

It is shown in [4] (see Appendix) that

$$\left.\begin{array}{l}\text{i) inequality (8) is true,}\\ \text{ii) } \exists w_0^{\rm cap}\in\mathbb{S}^3 \text{ such that}\\ \qquad \lim\limits_{\varepsilon\to 0}\varepsilon^{-2}\displaystyle\int_{D(r_\varepsilon,R_\varepsilon)} w_0\big((e(\theta_\varepsilon^\alpha),e(\theta_\varepsilon^\beta)\big)\,d\hat{x}=(w_0^{\rm cap})_{\alpha\beta}\quad\forall\alpha,\beta\in\{1,2,3\}.\end{array}\right\} \tag{9}$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(z_\varepsilon)\,dx=\int_\Omega w_0^{\rm cap}(v-u)\cdot(v-u)\,dx,$$

with (see Appendix)

$$w_0^{\rm cap}=\pi\gamma\mu_0\begin{pmatrix}\frac{\chi+1}{\chi} & 0 & 0\\ 0 & \frac{\chi+1}{\chi} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We complete the proof of the convergence of $F_\varepsilon^1(u_\varepsilon)$ and $F_\varepsilon^2(u_\varepsilon)$ for any (u,v) such that $\Phi(u,v)<\infty$ by approximation and diagonalization arguments. Eventually, as mentioned earlier, V_ε is chosen in such a way that a tedious computation shows that $F_\varepsilon^3(u_\varepsilon)$ has the expected limit.

2.3 Proof of Proposition 3

We assume here that $\liminf_\varepsilon F_\varepsilon(u_\varepsilon)<+\infty$. Compactness property yields that (u,v) belongs to $L^2(\Omega,\mathbb{R}^3)^2$.

We begin with the term $F_\varepsilon^2(u_\varepsilon)$. Let (u_η,v_η) be Lipschitz on $\overline{\Omega}$ with the property $\lim_{\eta\to 0}\|u_\eta-u\|_{L^2(\Omega,\mathbb{R}^3)}+\|v_\eta-v\|_{L^2(\Omega,\mathbb{R}^3)}=0$. Next we define an approximation $(v_\eta-u_\eta)_\varepsilon:=\sum_{i\in I_\varepsilon}(v_\eta-u_\eta)(\hat{x}_\varepsilon^i,x_3)1_{Y_\varepsilon^i}$ of $(v_\eta-u_\eta)$, and associate $z_{\eta\varepsilon}$ to (u_η,v_η) by (6). Let $\tilde{z}_{\eta\varepsilon}:=\sum_{\alpha=1}^3(v_\eta-u_\eta)_{\varepsilon\alpha}e(\theta_\varepsilon^\alpha)$. Because of local Lipschitz property (7) of W_0 and $(u,v)\in L^2(\Omega,\mathbb{R}^3)^2$, Cauchy-Schwarz inequality implies

$$\lim_{\varepsilon\to 0}\Big(\int_{B_\varepsilon} W_0(\tilde{z}_{\eta\varepsilon})\,dx-\int_{B_\varepsilon} W_0(z_{\eta\varepsilon})\,dx\Big)=0.$$

The proof of upper bound equality shows

$$\lim_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(z_{\eta\varepsilon})\,dx=\int_\Omega w_0^{\rm cap}(v_\eta-u_\eta)\cdot(v_\eta-u_\eta)\,dx.$$

Therefore, W_0, being convex and 2-positively homogeneous, the subdifferential inequality gives:

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(e(u_\varepsilon))\,dx\\ &\qquad\ge\liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0(\tilde{z}_{\eta\varepsilon})\,dx+\liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon})\cdot(e(u_\varepsilon)-\tilde{z}_{\eta\varepsilon})\,dx\\ &\qquad=-\int_\Omega w_0^{\rm cap}(v_\eta-u_\eta)\cdot(v_\eta-u_\eta)\,dx+\liminf_{\varepsilon\to 0}\int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon})\cdot e(u_\varepsilon)\,dx\end{aligned}\tag{10}$$

Letting $D^i(r_\varepsilon,R_\varepsilon):=D^i_{R_\varepsilon}\setminus\overline{D^i_{r_\varepsilon}}$, we have:

$$\int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon})\cdot e(u_\varepsilon)\,dx=\sum_{i\in I_\varepsilon}\sum_{\alpha=1}^3\int_0^L(v_\eta-u_\eta)_\alpha(\hat{x}_\varepsilon^i,x_3)\left(\int_{D^i(r_\varepsilon,R_\varepsilon)} W_0'(e(\theta_\varepsilon^\alpha))\cdot e(u_\varepsilon)\,d\hat{x}\right)dx_3.$$

If ν denotes the outer normal along to both $\partial D^i_{r_\varepsilon}$ and $\partial D^i_{R_\varepsilon}$. The very definition of $\theta_\varepsilon^\alpha$ as

a solution of $(\mathcal{P}_\varepsilon^{\mathrm{cap},\alpha})$ and Green formula imply:

$$\begin{aligned}
&\int_{D^i(r_\varepsilon,R_\varepsilon)} W_0'(e(\theta_\varepsilon^\alpha))\cdot e(u_\varepsilon)\,d\hat{x}\\
&\qquad = -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot u_\varepsilon\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot u_\varepsilon\,dl\\
&\qquad = -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot (u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot (u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl\\
&\qquad\quad + \int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot (\bar{\bar{u}}_\varepsilon-\bar{u}_\varepsilon)\,dl\\
&\qquad = -\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot (u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl + \int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot (u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl\\
&\qquad\quad + 2\sum_{\beta=1}^{3}(\bar{\bar{u}}_\varepsilon-\bar{u}_\varepsilon)^i_\beta \int_{D(r_\varepsilon,R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha),e(\theta_\varepsilon^\beta))\,d\hat{x},
\end{aligned}$$

where

$$\begin{aligned}
(\bar{\bar{u}}_\varepsilon)^i(x_3) &= \frac{1}{|\partial D^i_{r_\varepsilon}|}\int_{\partial D^i_{r_\varepsilon}} u_\varepsilon(\hat{x},x_3)\,dl, &\qquad \bar{\bar{u}}_\varepsilon(x) &= \sum_{i\in I_\varepsilon}(\bar{\bar{u}}_\varepsilon)^i(x_3)1_{Y^i_\varepsilon}(\hat{x}),\\
(\bar{u}_\varepsilon)^i(x_3) &= \frac{1}{|\partial D^i_{R_\varepsilon}|}\int_{\partial D^i_{R_\varepsilon}} u_\varepsilon(\hat{x},x_3)\,dl, &\qquad \bar{u}_\varepsilon(x) &= \sum_{i\in I_\varepsilon}(\bar{u}_\varepsilon)^i(x_3)1_{Y^i_\varepsilon}(\hat{x}),
\end{aligned}$$

$u_\varepsilon(\cdot,x_3)$ being, by Fubini's theorem, well defined in $H^1(\omega,\mathbb{R}^3)$ for a.e. $x_3\in(0,L)$. Actually, the standard estimates

$$\begin{aligned}
\int_0^L\int_{\partial D^i_{r_\varepsilon}} |u_\varepsilon-\bar{\bar{u}}^i_\varepsilon|\,dl\,dx_3 &\le r_\varepsilon\left\{\int_{T^i_\varepsilon}|\nabla u_\varepsilon|^2\,dx\right\}^{1/2},\\
\int_0^L\int_{\partial D^i_{R_\varepsilon}} |u_\varepsilon-\bar{u}^i_\varepsilon|\,dl\,dx_3 &\le R_\varepsilon\left\{\int_{D^i_{R_\varepsilon}\times(0,L)}|\nabla u_\varepsilon|^2\,dx\right\}^{1/2}
\end{aligned}$$

and the estimates (see Appendix)

$$|W_0'(e(\theta_\varepsilon^\alpha))\nu|_{L^\infty(\partial D(0,r_\varepsilon))}\le\frac{C}{r_\varepsilon|\ln r_\varepsilon|},\quad |W_0'(e(\theta_\varepsilon^\alpha))\nu|_{L^\infty(\partial D(0,R_\varepsilon))}\le\frac{C}{R_\varepsilon|\ln r_\varepsilon|}$$

for $\alpha=1,2,3$ imply that

$$\begin{aligned}
&\left|\sum_{i\in I_\varepsilon}\sum_{\alpha=1}^{3}\int_0^L (v_\eta-u_\eta)_\alpha(\hat{x}^i_\varepsilon,x_3)\left(\int_{\partial D^i_{r_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot(u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl\right)dx_3\right|\\
&\qquad\le\frac{C}{|\ln r_\varepsilon|}\sum_{i\in I_\varepsilon}\left\{\int_{T^i_\varepsilon}|\nabla u_\varepsilon|^2\,dx\right\}^{1/2}\le\frac{C\varepsilon}{\varepsilon^2|\ln r_\varepsilon|}\left\{\int_\Omega|\nabla u_\varepsilon|^2\,dx\right\}^{1/2}\le C\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{i\in I_\varepsilon}\sum_{\alpha=1}^{3}\int_0^L (v_\eta-u_\eta)_\alpha(\hat{x}^i_\varepsilon,x_3)\left(\int_{\partial D^i_{R_\varepsilon}} W_0'(e(\theta_\varepsilon^\alpha))\nu\cdot(u_\varepsilon-\bar{u}_\varepsilon)\,dl\right)dx_3\right|\\
&\qquad\le\frac{C}{|\ln r_\varepsilon|}\sum_{i\in I_\varepsilon}\left\{\int_{T^i_\varepsilon}|\nabla u_\varepsilon|^2\,dx\right\}^{1/2}\le\frac{C\varepsilon}{\varepsilon^2|\ln r_\varepsilon|}\left\{\int_\Omega|\nabla u_\varepsilon|^2\,dx\right\}^{1/2}\le C\varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{B_\varepsilon} W_0'(\tilde{z}_{\eta\varepsilon})\cdot e(u_\varepsilon)\,dx\\
&= 2\sum_{i\in I_\varepsilon}\sum_{\alpha,\beta=1}^{3}\int_0^L (v_\eta-u_\eta)_\alpha(\hat{x}_\varepsilon^i,x_3)\left(\int_{D(r_\varepsilon,R_\varepsilon)} w_0(e(\theta_\varepsilon^\alpha),e(\theta_\varepsilon^\beta))\,d\hat{x}\right)(\bar{\bar{u}}_\varepsilon-\bar{u}_\varepsilon)_\beta^i(x_3)\,dx_3\\
&= 2\int_\Omega w_0^{\mathrm{cap}}(v_\eta-u_\eta)\cdot(v-u)\,dx,
\end{aligned}$$

by due account of (9) and of the weak convergence in $L^2(\Omega,\mathbb{R}^3)$ of $(\bar{\bar{u}},\bar{u})$ toward (v,u) (see [4] P.68). Hence, letting η tends to zero, yields

$$\liminf_{\varepsilon\to 0} F_\varepsilon^2(u_\varepsilon)\geq\int_\Omega w_0^{\mathrm{cap}}(v-u)\cdot(v-u)\,dx.$$

Because $|B_\varepsilon\cup T_\varepsilon|$ tends to zero, a classical semi-continuity argument taking into account the convexity of W_0 yields

$$\liminf_{\varepsilon\to 0} F_\varepsilon^1(u_\varepsilon)\geq\int_\Omega W_0(e(u))\,dx.$$

For the third term $F_\varepsilon^3(u_\varepsilon)$, we may extend the strategy of [2] to all cases of relative behaviors of the parameters by due account of the function V_ε introduced by [4].

3 Concluding Remarks

Here was presented another proof of a result of [4] concerning the homogenization of a cylindrical fibered structure. Instead of passing to the limit on a formulation of the problem in terms of variational equality through appropriate sequence of oscillating test fields, we study the variational convergence of the energy functional. Hence, the ingredients in the construction of the appropriate oscillating test fields are clearly justified as providing the "best" upper bound. Thus, it seems possible to consider a more general cross section for the fibers (say $r_\varepsilon\Delta$ with $\partial\Delta$ smooth enough) and a more general quadratic bulk energy density W_M for the matrix in the extent where the solutions $\theta_\varepsilon^\alpha$ of the involved capacitary problems

$$\min\left\{\int_{(-\varepsilon,\varepsilon)^2} W_M(e(\varphi))\,d\hat{x}\;\middle|\;\begin{array}{l}\varphi\in H^1((-\varepsilon,\varepsilon)^2,\mathbb{R}^3),\\ \varphi(\hat{x})=e^\alpha \text{ on } r_\varepsilon\Delta,\\ \varphi(\hat{x})=0 \text{ on } (-\varepsilon,\varepsilon)^2\setminus D(0,R_\varepsilon).\end{array}\right\}$$

are such that

i) $\exists w_M^{\mathrm{cap}}\in\mathbb{S}^3$ such that $(w_M^{\mathrm{cap}})_{\alpha\beta}=\lim_{\varepsilon\to 0}\varepsilon^{-2}\int_{(-\varepsilon,\varepsilon)^2} w_M(e(\theta_\varepsilon^\alpha),e(\theta_\varepsilon^\beta))\,d\hat{x}$,

ii) $\lim_{\varepsilon\to 0}\int_{\varepsilon i+r_\varepsilon\partial\Delta} W_M'(e(\theta_\varepsilon^\alpha))\cdot(u_\varepsilon-\bar{\bar{u}}_\varepsilon)\,dl=\lim_{\varepsilon\to 0}\int_{\partial D_{R_\varepsilon}^i} W_M'(e(\theta_\varepsilon^\alpha))\cdot(u_\varepsilon-\bar{u}_\varepsilon)\,dl=0.$

A The Vector Capacitary Problem

Taking advantage of the cylindrical geometry, Bellieud and Gruais [4] showed that $\theta_\varepsilon^\alpha$ and $\sigma_\varepsilon^\alpha:=W_0'(e(\theta_\varepsilon^\alpha))$ are such that

$$\begin{aligned}
\theta_\varepsilon^1(x_1,x_2)=\theta_\varepsilon^2(x_2,x_1),\quad \theta_{\varepsilon 3}^1=\theta_{\varepsilon 3}^2=0,\\
\theta_\varepsilon^3=\frac{\ln(R_\varepsilon/r)}{\ln(R_\varepsilon/r_\varepsilon)}e^3 \text{ in } D(r_\varepsilon,R_\varepsilon),\quad r=|\hat{x}|,
\end{aligned}$$

$$\begin{aligned}
(\theta_{\varepsilon 1}^1+i\theta_{\varepsilon 2}^1)(\hat{x})=&\frac{A}{2\mu_0}\left(\chi(\ln z+\overline{\ln z})+\frac{z^2}{r_\varepsilon^2+R_\varepsilon^2}-\frac{z}{\bar{z}}-\frac{2z\bar{z}}{\chi(r_\varepsilon^2+R_\varepsilon^2)}\right.\\
&\left.+\frac{2\chi(r_\varepsilon^2\ln R_\varepsilon-R_\varepsilon^2\ln r_\varepsilon)}{R_\varepsilon^2-r_\varepsilon^2}+\frac{r_\varepsilon^2R_\varepsilon^2}{(r_\varepsilon^2+R_\varepsilon^2)\bar{z}^2}\right)+\frac{R_\varepsilon^2}{R_\varepsilon^2-r_\varepsilon^2},
\end{aligned}$$

with

$$\chi := \frac{\lambda_0 + 3\mu_0}{\lambda_0 + \mu_0}, \quad A := \frac{\mu_0}{\frac{R_\varepsilon^2 - r_\varepsilon^2}{\chi(r_\varepsilon^2 + R_\varepsilon^2)} - \chi \ln \frac{R_\varepsilon}{r_\varepsilon}},\ z \text{ the complex number } x_1 + ix_2,$$

and

$$\sigma_\varepsilon^1 \nu = \frac{\mu_0(\chi+1)(1+o(1))}{\chi r_\varepsilon |\ln r_\varepsilon|} e^1 \quad \text{on } \partial D(0, r_\varepsilon),$$

$$\sigma_\varepsilon^1 \nu = \frac{\mu_0(1+o(1))}{\chi R_\varepsilon |\ln r_\varepsilon|} \left[4(1 + \frac{1}{\chi}\cos^2\theta - (1 + \chi + \frac{2}{\chi})) \right] e^1 + 2(1 + \frac{1}{\chi}) \sin 2\theta e^2 \quad \text{on } \partial D(0, R_\varepsilon),$$

$$\sigma_\varepsilon^3 \nu = \frac{-\mu}{r \ln R_\varepsilon / r_\varepsilon} e^3 \quad \text{on the circle of radius } r.$$

Thus, for each α, $\beta = 1, 2, 3$,

i)

$$|\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0,r_\varepsilon))} \le \frac{C}{r_\varepsilon |\ln r_\varepsilon|}, \quad |\sigma_\varepsilon^\alpha \nu|_{L^\infty(\partial D(0,R_\varepsilon))} \le \frac{C}{R_\varepsilon |\ln r_\varepsilon|},$$

ii)

$$\int_{D(0,R_\varepsilon)} |e(\theta_\varepsilon^\alpha)|^2 \, d\hat{x} \le C \int_{D(0,R_\varepsilon)} W_0(e(\theta_\varepsilon^\alpha)) \, d\hat{x} \le \frac{C}{2} \int_{\partial D(0,R_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\alpha \, dl \le \frac{C}{|\ln r_\varepsilon|},$$

iii)

$$\begin{aligned}(w_0^{\text{cap}})_{\alpha\beta} &:= \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} w_0\big((e(\theta_\varepsilon^\alpha), e(\theta_\varepsilon^\beta)\big) \, d\hat{x} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon^{-2}}{2} \int_{\partial D(0,r_\varepsilon)} (\sigma_\varepsilon^\alpha \nu)_\beta \, dl,\end{aligned}$$

satisfies

$$w_0^{\text{cap}} = \pi\gamma\mu_0 \begin{pmatrix} \frac{\chi+1}{\chi} & 0 & 0 \\ 0 & \frac{\chi+1}{\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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