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► **To cite this version:**

Jean-Marc Azaïs, José R. Leon. CLT for Crossings of random trigonometric Polynomials. 2012.
hal-00747030v1

HAL Id: hal-00747030

<https://hal.science/hal-00747030v1>

Preprint submitted on 30 Oct 2012 (v1), last revised 29 May 2013 (v2)

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CLT for Crossings of random trigonometric Polynomials.

Jean-Marc Azaïs * José R. León †

April 3, 2012

1 Introduction

Let us consider the random trigonometric polynomial:

$$X_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^N (a_n \sin nt + b_n \cos nt),$$

where the coefficients a_n and b_n are independent standard Gaussian random variables and N is some integer.

The number of zeroes of such a process on the interval $[0, 2\pi)$ has been studied in the paper by Granville and Wigman [3] where a central limit theorem, as $N \rightarrow +\infty$ is proved.

The aim of this paper is twofold : firstly we extend their result to the number of crossings of every level and secondly we propose a simpler proof. The key point consist in proving that after a convenient scaling the process $X_N(t)$ converges in a certain sense to the stationary process $X(t)$ with covariance $\sin(t)/t$. The central limit theorem is then a consequence of the central limit theorem in large time for $X(t)$.

The organization of the paper is the following: in Section 2 we present basic calculations; Section 3 is devoted to the presentation of the Wiener chaos decomposition and to the study of the variance. Section 4 states the central limit theorem . Additional proofs are given in Section 5 and 6.

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2 Basic results and notation

$r_{X_N}(\tau)$ will be the covariance of the process $X_N(t)$ given by

$$r_{X_N}(\tau) := \mathbb{E}[X_N(0)X_N(\tau)] = \frac{1}{N} \sum_{n=1}^N \cos n\tau = \frac{1}{N} \cos\left(\frac{(N+1)\tau}{2}\right) \frac{\sin\left(\frac{N\tau}{2}\right)}{\sin\frac{\tau}{2}}. \quad (1) \quad \boxed{\text{f:rn}}$$

We define the process

$$Y_N(t) = X_N(t/N)$$

with covariance

$$r_{Y_N}(\tau) = r_{X_N}(\tau/N).$$

We have

$$r'_{Y_N}(\tau) = \frac{1}{2N \sin \frac{\tau}{2N}} \cos\left(\frac{2N+1}{2N}\tau\right) - \frac{\sin \tau}{4N^2 \sin^2 \frac{\tau}{2N}} \quad (2)$$

$$r''_{X_N}(\tau) = -\frac{\sin \frac{\tau}{2}}{2N \sin^2 \frac{\tau}{2N}} \left[\sin \frac{(N+1)\tau}{2N} \sin \frac{\tau}{2N} + \cos \frac{(N+1)\tau}{2N} \cos \frac{\tau}{2N} \right] \quad (3)$$

$$\begin{aligned} r''_{Y_N}(\tau) &= \frac{1}{N^2} r''_N\left(\frac{\tau}{N}\right) \\ &= \frac{\cos \frac{\tau}{2N} \cos \frac{(2N+1)}{2N}\tau - 2\frac{(2N+1)}{2} \sin \frac{\tau}{2N} \sin \frac{(2N+1)}{2N}\tau - \cos \tau}{4N^2 \sin^2 \frac{\tau}{2N}} \\ &\quad - \frac{(2N \sin \frac{\tau}{2N} \cos \frac{(2N+1)}{2N}\tau - \sin \tau) \cos \frac{\tau}{2N}}{4N^3 \sin^3 \frac{\tau}{2N}}. \end{aligned} \quad (4)$$

The convergence of Riemann sums to the integral implies simply that

$$r_{Y_N}(\tau) \rightarrow r(\tau) := \sin(\tau)/\tau \quad (5)$$

$$r'_{Y_N}(\tau) \rightarrow r'(\tau) = \cos(\tau)/\tau - \tau^{-2} \sin(\tau) \quad (6)$$

$$r''_{Y_N}(\tau) = \frac{1}{N^2} r''_N\left(\frac{\tau}{N}\right) \rightarrow r''(\tau) = -\frac{\sin(\tau)}{\tau} - 2\frac{\cos(\tau)}{\tau^2} + 2\frac{\sin(\tau)}{\tau^3} \quad (7)$$

And these convergence are uniform in every compact interval that does not contains zero.

We will need also the following upper-bounds that are easy

When $\tau \in [0, N\pi]$:

$$|r_{Y_N}(\tau)| \leq \pi/\tau \quad ; \quad |r'_{Y_N}(\tau)| \leq \frac{\pi}{2\tau} + \frac{\pi^2}{4\tau^2} \quad ; \quad |r''_{Y_N}(\tau)| \leq (\text{const})(\tau^{-1} + \tau^{-2} + \tau^{-3}) \quad (8) \quad \boxed{\text{f:ub}}$$

When $\tau \in [N\pi, 2N\pi]$:

$$\begin{aligned} |r_{Y_N}(\tau)| &\leq \pi/(2N\pi - \tau) \quad ; \quad |r'_{Y_N}(\tau)| \leq \frac{\pi}{2(2N\pi - \tau)} + \frac{\pi^2}{4(2N\pi - \tau)^2} \\ |r''_{Y_N}(\tau)| &\leq (\text{const})((2N\pi - \tau)^{-1} + (2N\pi - \tau)^{-2} + (2N\pi - \tau)^{-3}) \end{aligned} \quad (9) \quad \boxed{\text{f:ub2}}$$

We now compute the ingredients of the Rice formula

$$\mathbb{E}X_N^2(t) = 1, \quad \text{and} \quad \mathbb{E}(X'_N(t))^2 = \frac{1}{N} \sum_{n=1}^N n^2 = \frac{(N+1)(2N+1)}{6}.$$

Denoting by $N_{[0,2\pi]}^{X_N}(u)$ the numbers of crossings of the level u of X_N on the interval $[0, 2\pi)$, the Rice formula gives

$$\mathbb{E}[N_{[0,2\pi]}^{X_N}(u)] = 2\pi \cdot \sqrt{\mathbb{E}(X'_N(t))^2} \sqrt{2/\pi} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} = \frac{2}{\sqrt{3}} \sqrt{\frac{(N+1)(2N+1)}{2}} e^{-\frac{u^2}{2}}.$$

Hence

$$\lim_{N \rightarrow \infty} \frac{\mathbb{E}[N_{[0,2\pi]}^{X_N}(u)]}{N} = \frac{2}{\sqrt{3}} e^{-\frac{u^2}{2}}.$$

3 Spectral representation and Wiener Chaos

We can obtain a joint spectral representation of the processes X and Y_N . This chaining argument is one of our main tools. We have

$$X(t) = \int_0^1 \cos(t\lambda) dB_1(\lambda) + \int_0^1 \sin(t\lambda) dB_2(\lambda), \quad (10)$$

where B_1 and B_2 are two independent brownian motion. Using the same brownian motion we can write

$$Y_N(t) = \int_0^1 \sum_{n=1}^N \cos\left(\frac{nt}{N}\right) 1_{[\frac{n-1}{N}, \frac{n}{N})}(\lambda) dB_1(\lambda) + \int_0^1 \sum_{n=1}^N \sin\left(\frac{nt}{N}\right) 1_{[\frac{n-1}{N}, \frac{n}{N})}(\lambda) dB_2(\lambda).$$

It is easy to check, using isometry properties of stochastic integrals that $Y_N(t)$ has the desired covariance.

By defining the functions

$$\gamma_N^1(t, \lambda) = \sum_{n=1}^N \cos\left(\frac{nt}{N}\right) 1_{[\frac{n-1}{N}, \frac{n}{N})}(\lambda) \quad \text{and} \quad \gamma_N^2(t, \lambda) = \sum_{n=1}^N \sin\left(\frac{nt}{N}\right) 1_{[\frac{n-1}{N}, \frac{n}{N})}(\lambda).$$

we can write

$$Y_N(t) = \int_0^1 \gamma_N^1(t, \lambda) dB_1(\lambda) + \int_0^1 \gamma_N^2(t, \lambda) dB_2(\lambda). \quad (11)$$

In the sequel we are going to express the representation (10 and 11) in an Isonormal Process framework. Let define \mathfrak{H}^2 the Hilbert vectorial espace defined as

$$\{\mathbf{h} = (h_1, h_2) : \int_{\mathbb{R}} h_1^2(\lambda) d\lambda + \int_{\mathbb{R}} h_2^2(\lambda) d\lambda < \infty\},$$

with scalar product

$$\langle \mathbf{h}, \mathbf{g} \rangle = \int_{\mathbb{R}} h_1(\lambda) g_1(\lambda) d\lambda + \int_{\mathbb{R}} h_2(\lambda) g_2(\lambda) d\lambda.$$

The transformation

$$\mathbf{h} \rightarrow W(\mathbf{h}) := \int_{\mathbb{R}} h_1(\lambda) dB_1(\lambda) + \int_{\mathbb{R}} h_2(\lambda) dB_2(\lambda),$$

defines an isometry between \mathfrak{H}^2 and a Gaussian subspace of $L^2(\Omega, \mathcal{A}, P)$ where \mathcal{A} is the σ -field generated by $B_1(\lambda)$ and $B_2(\lambda)$.

Thus $W(\mathbf{h})_{\mathbf{h} \in \mathfrak{H}^2}$ is the Isonormal process associated to \mathfrak{H}^2 . By using the representations (10) and (11) readily we get

$$\begin{aligned} X(t) &= W(\mathbb{1}_{[0,1]}(\cdot, \cdot)(\cos t \cdot, \sin t \cdot)) \\ Y_N(t) &= W(\mathbb{1}_{[0,1]}(\cdot, \cdot)(\gamma_N^1(\cdot, t), \gamma_N^2(\cdot, t))) \\ \bar{X}'(t) &:= \frac{X'(t)}{\sqrt{1/3}} = W\left(\frac{\mathbb{1}_{[0,1]}(\cdot, \cdot)}{\sqrt{1/3}}(\cdot, \cdot)(-\sin t \cdot, \cos t \cdot)\right) \\ \bar{Y}'_N(t) &:= \frac{Y'_N(t)}{\sqrt{-r''_{Y_N}(0)}} = W\left(\frac{\mathbb{1}_{[0,1]}(\cdot, \cdot)}{\sqrt{-r''_{Y_N}(0)}}(\cdot, \cdot)((\gamma_N^1(\cdot, t))', (\gamma_N^2(\cdot, t))')\right). \end{aligned}$$

We are in disposition of introduce the Wiener's Chaos which is our second main tool. Let H_k be the Hermite's polynomial of degree k defined by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} (e^{-\frac{x^2}{2}}).$$

It is normalized such that for Y a standard Gaussian random variable we have $\mathbb{E}(H_k(Y)H_m(Y)) = \delta_{k,m}k!$. Consider $\{e_i\}_{i \in \mathbb{N}}$ an ortonormal basis for \mathfrak{H}^2 . Denoting as Λ the set the sequences $a = (a_1, a_2, \dots)$ $a_i \in \mathbb{N}$ such that all the terms except a finite number vanish. For $a \in \Lambda$ we set $a! = \prod_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multiindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

For each $n \geq 1$, we will denote by \mathcal{H}_n the closed subspace of $L^2(\Omega, \mathcal{A}, P)$ spanned by the random variables $\{\Phi_a, a \in \Lambda, |a| = n\}$. The space \mathcal{H}_n is the n th Wiener chaos associated with $B_1(\lambda)$ and $B_2(\lambda)$. If \mathcal{H}_0 denotes the space of constants we have the ortogonal decomposition

$$L^2(\Omega, \mathcal{A}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

For any Hermite's polynomial H_q , it holds

$$\begin{aligned} H_q(W(\mathbf{h})) = I_q(\mathbf{h}), & := \int_0^{+\infty} \dots \int_0^{+\infty} h^1(\lambda_1) \dots h^1(\lambda_q) dB_1(\lambda_1) \dots dB_1(\lambda_q) \\ & + \int_0^{+\infty} \dots \int_0^{+\infty} h^2(\lambda_1) \dots h^2(\lambda_q) dB_2(\lambda_1) \dots dB_2(\lambda_q), \end{aligned}$$

with $\mathbf{h} = (h^1, h^2)$. For instance as $Y_N(t) = W(\mathbb{1}_{[0,1]}(\cdot, \cdot)(\gamma_N^1(\cdot, t), \gamma_N^2(\cdot, t)))$, we obtain

$$\begin{aligned} H_2(Y_N(t)) = \int_0^1 \int_0^1 \gamma_N^1(\lambda_1, t) \gamma_{1,N}^1(\lambda_2, t) dB_1(\lambda_1) dB_1(\lambda_2) \\ + \int_0^1 \int_0^1 \gamma_N^2(\lambda_1, t) \gamma_N^2(\lambda_2, t) dB_2(\lambda_1) dB_2(\lambda_2). \end{aligned}$$

We now write the Wiener Chaos expansion for the number of crossings. As the absolute value function belongs to $\mathbb{L}^2(\mathbb{R}, \varphi(x)dx)$, where φ is the standard Gaussian density, we have $|x| = \sum_{k=0}^{\infty} a_{2k} H_{2k}(x)$ with

$$a_{2k} = 2 \frac{(-1)^{k+1}}{\sqrt{2\pi} 2^k k! (2k-1)}.$$

Because of the form of inequalities (8) and (9) it is shorter to study first $X_N(t)$ on $[0, \pi]$ (resp. $Y_N(t)$ and $X(t)$ on $[0, N\pi]$). The generalization to $[0, 2\pi]$ (resp. to $[0, 2N\pi]$) will be done in Section 4. By using the result of Kratz & León [4] or Th 10.10 in [2]

$$\begin{aligned} & \frac{1}{\sqrt{N\pi}} (N_{[0, \pi N]}^X(u) - \mathbb{E} N_{[0, \pi N]}^X(u)) \\ & = \sqrt{1/3} \varphi(u) \sum_{q=1}^{\infty} \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \frac{H_{q-2k}(u)}{(q-2k)!} \frac{a_{2k}}{\sqrt{N\pi}} \int_0^{\pi N} H_{q-2k}(X(s)) H_{2k}(\bar{X}'(s)) ds, \end{aligned} \tag{12} \quad \boxed{\text{f:kl}}$$

where $\lfloor x \rfloor$ is the integer part. We introduce the notation

$$f_q(u, x_1, x_2) = \varphi(u) \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \frac{H_{q-2k}(u)}{(q-2k)!} a_{2k} H_{q-2k}(x_1) H_{2k}(x_2).$$

$$I_q([0, \pi N]) = \frac{\sqrt{1/3}}{\sqrt{N\pi}} \int_0^{\pi N} f_q(u, X(t), \bar{X}'(t)) dt$$

so that

$$\frac{1}{\sqrt{N\pi}} (N_{[0, \pi N]}^X(u) - \mathbb{E} N_{[0, \pi N]}^X(u)) = \sum_{q=1}^{\infty} I_q([0, \pi N])$$

giving the decomposition in the Wiener chaos. The same type of expansion is also true for $N_{[0, \pi N]}^{Y_N}(u)$

$$\frac{1}{\sqrt{N}}(N_{[0, \pi N]}^{Y_N}(u) - \mathbb{E}N_{[0, \pi N]}^{Y_N}(u)) = \sum_{q=1}^{\infty} I_{q,N}([0, \pi N]) \quad (13) \quad \boxed{\text{f:k12}}$$

with obvious notation.

Our first goal is to compute the limit variance of (??). Our main tool will be the Arcones inequality. We define the norm

$$\|f_q\|^2 := \mathbb{E}f_q^2(u, X, Y),$$

Where (X, Y) is a bidimensional standard Gaussian vector. We have

$$\|f_q\|^2 = \varphi^2(u) \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \frac{H_{q-2k}^2(u)}{(q-2k)!} a_{2k}^2(2k)! \leq C \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} a_{2k}^2(2k)! \leq C,$$

where C is some constant that does not depend on q . Now we must introduce the Arcone's coefficient of dependence $\frac{\text{Ar} \cdot \text{Ar}}{[1]}$

$$\psi_N(\tau) = \sup \left(|r_{Y_N}(\tau)|, \left| \frac{r'_{Y_N}(\tau)}{\sqrt{-r''_{Y_N}(0)}} \right|, \left| \frac{r''_{Y_N}(\tau)}{r''_{Y_N}(0)} \right| \right).$$

The Arcones inequality says that if $\psi_N(s' - s) < 1$, it holds

$$|\mathbb{E}[f_q(u, Y_N(s), \bar{Y}'_N(s)) f_q(u, Y_N(s'), \bar{Y}'_N(s'))]| \leq \psi_N^q(s' - s) \|f_q\|^2.$$

We will use also the following Lemma the proof of which is given in Section 5

1:a **Lemma 1** For every $a > 0$, there exists a constant K_a such that

$$\sup_N \text{Var}(N_{[0, a]}^{Y_N}(u)) \leq K_a < \infty \quad (14) \quad \boxed{\text{f:lem}}$$

Chose some $\rho < 1$, using the inequality 8, we can choose a big enough such that for $\tau > a$ we have $\psi_N(\tau) < \frac{K}{\tau} \leq \rho$.

Then we partition $[0, N\pi]$ into $L = \lfloor \frac{N\pi}{a} \rfloor$ intervals J_1, \dots, J_L of length larger than a , and we set for short

$$N_\ell = N_{J_\ell}^{Y_N}(u)$$

Then we have

$$\text{Var}(N_{[0, N\pi]}^{Y_N}(u)) = \text{Var}(N_1 + \dots + N_L) = \sum_{\ell, \ell', |\ell - \ell'| \leq 1} \text{Cov}(N_\ell, N_{\ell'}) + \sum_{\ell, \ell', |\ell - \ell'| > 1} \text{Cov}(N_\ell, N_{\ell'}).$$

The first sum is easily shown to be $O(N)$ by applying Lemma 1 and the Cauchy-Schwarz inequality.

Let us look at a term of the second sum. Using the expansion (13) we set

$$\frac{N_\ell - \mathbb{E}(N_\ell)}{\sqrt{\pi N}} = \sum_{q=1}^{\infty} I_{q,N}(J_\ell),$$

Let us consider the terms corresponding to $q > 1$. The Arcones inequality implies that

$$\text{Cov}(I_{q,N}(J_\ell), I_{q,N}(J_{\ell'})) \leq \int_{J_\ell \times J_{\ell'}} \frac{1}{N\pi} r_{Y_N}''(0) (K/\tau)^q C ds dt \leq (\text{const}) \int_{J_\ell \times J_{\ell'}} \rho^{q-2} \tau^{-2} ds dt, \quad (15) \quad \boxed{\text{f:maj}}$$

where $\tau = s - t$. Summing over all pair on intervals and over $q \geq 2$ it is easy to check that this sum is bounded.

It remains to study the case $q = 1$. Since $H_1(X) = X$

$$I_{1,N}(J_\ell) = (N\pi)^{-1/2} \sqrt{-r_{Y_N}''(0)} u\phi(u) \int_{J_\ell} Y_N(s) ds.$$

So that

$$\sum_{\ell, \ell', |\ell - \ell'| > 1} \text{Cov}(I_{1,N}(J_\ell), I_{1,N}(J_{\ell'})) \leq (\text{const}) \frac{1}{N} \int_0^{\pi N} \int_0^{\pi N} r_{Y_N}(s - s') ds ds',$$

which is bounded because of the following result

$$\begin{aligned} \frac{1}{N} \int_0^{\pi N} \int_0^{\pi N} r_{Y_N}(s - s') ds ds' &= \frac{2}{N} \int_0^{\pi N} (\pi N - \tau) r_{Y_N}(\tau) d\tau \\ &= 2 \sum_{n=1}^N \int_0^{\pi N} \left(\pi - \frac{\tau}{N}\right) \frac{1}{N} \cos n \frac{\tau}{N} d\tau \\ &= 2 \sum_{n=1}^N \frac{1 - \cos n\pi}{n^2} \\ &= 4 \sum_{j=0}^N \frac{1}{(2j+1)^2} \\ &\rightarrow 4 \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = 4 \frac{\pi^2}{8} = \frac{\pi^2}{2}. \quad (16) \quad \boxed{\text{varq1}} \end{aligned}$$

■

p:1 **Proposition 1** For $q > 1$ we have

$$\text{Var}(I_{q,N}([0, \pi N])) \rightarrow \sigma_q^2 := \text{Var}(I_q([0, \pi N])) < +\infty \quad \text{as } N \rightarrow +\infty$$

For $q = 1$

$$\text{Var}(I_{1,N}([0, \pi N])) \rightarrow \frac{1}{3}u^2\phi^2(u)\pi.$$

In the case $u \neq 0$ this limit is different from

$$\text{Var}(I_1([0, \pi N])) = \frac{2}{3}u^2\phi^2(u)\pi.$$

This different behaviour, depending in which chaos we are, justifies the uses of Wiener Chaos decomposition.

Proof: Assume first that $q > 2$:

$$\begin{aligned} \mathbb{E}(I_{q,N}^2([0, N\pi])) &= -r''_{Y_N}(0)\varphi^2(u) \sum_{k_1=0}^{\lfloor \frac{q}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{q}{2} \rfloor} \frac{H_{q-2k_1}(u)}{(q-2k_1)!} a_{2k_1} \frac{H_{q-2k_2}(u)}{(q-2k_2)!} a_{2k_2} \\ &\frac{1}{N\pi} \int_0^{N\pi} \int_0^{N\pi} \mathbb{E}[H_{q-2k}(Y_N(s))H_{2k}\left(\frac{Y'_N(s')}{\sqrt{-r''_{Y_N}(0)}}\right)H_{q-2k}(Y_N(s'))H_{2k}\left(\frac{Y'_N(s)}{\sqrt{-r''_{Y_N}(0)}}\right)] ds' ds \\ &= -r''_{Y_N}(0)\varphi^2(u) \sum_{k_1=0}^{\lfloor \frac{q}{2} \rfloor} \sum_{k_2=0}^{\lfloor \frac{q}{2} \rfloor} \frac{H_{q-2k_1}(u)}{(q-2k_1)!} a_{2k_1} \frac{H_{q-2k_2}(u)}{(q-2k_2)!} a_{2k_2} \\ &2 \int_0^{\pi N} \left(1 - \frac{s}{N\pi}\right) \mathbb{E}[H_{q-2k}(Y_N(0))H_{2k}\left(\frac{Y'_N(0)}{\sqrt{-r''_{Y_N}(0)}}\right)H_{q-2k}(Y_N(s))H_{2k}\left(\frac{Y'_N(s)}{\sqrt{-r''_{Y_N}(0)}}\right)] ds. \end{aligned}$$

We now use the generalized Mehler formula (Lemma 10.7 page 270 of [Az:Wsc \[2\]](#)).

mehler

Lemma 2 Let (X_1, X_2, X_3, X_4) be a centered Gaussian vector with variance matrix

$$\Sigma = \begin{pmatrix} 1 & 0 & \rho_{13} & \rho_{14} \\ 0 & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & 0 \\ \rho_{14} & \rho_{24} & 0 & 1 \end{pmatrix}$$

Then, if $r_1 + r_2 = r_3 + r_4$,

$$\mathbb{E}(H_{r_1}(X_1)H_{r_2}(X_2)H_{r_3}(X_3)H_{r_4}(X_4)) = \sum_{(d_1, d_2, d_3, d_4) \in J} \frac{r_1!r_2!r_3!r_4!}{d_1!d_2!d_3!d_4!} \rho_{13}^{d_1} \rho_{14}^{d_2} \rho_{23}^{d_3} \rho_{24}^{d_4},$$

where J is the set of d_i 's satisfying : $d_i \geq 0$;

$$d_1 + d_2 = r_1 ; d_3 + d_4 = r_2 ; d_1 + d_3 = r_3 ; d_2 + d_4 = r_4. \quad (17) \quad \text{a:e:dd}$$

If $r_1 + r_2 \neq r_3 + r_4$ the expectation is equal to zero.

Using this lemma, there exist a finite set J_q and constants C_{q,k_1,k_2} such that

$$\begin{aligned} & \mathbb{E}[H_{q-2k_1}(Y_N(0))H_{2k_1}(\bar{Y}'_N(0))H_{q-2k_2}(Y_N(\tau))H_{2k_2}(\bar{Y}'_N(\tau))] \\ &= \sum_{J_q} C_{q,k_1,k_2} |r_{Y_N}(\tau)|^{2q-(2k_1+2k_2)-h_1} \left| \frac{r'_{Y_N}(\frac{\tau}{N})}{\sqrt{-r''_{Y_N}(0)}} \right|^{2h_1} \left| \frac{r''_{Y_N}(\tau)}{\sqrt{-r''_{Y_N}(0)}} \right|^{2k_1+2k_2-h_1}. \end{aligned} \quad (18) \quad \boxed{\text{diag}}$$

This clearly proves that

$$\begin{aligned} & \mathbb{E}[H_{q-2k_1}(Y_N(0))H_{2k_1}(\bar{Y}'_N(0))H_{q-2k_2}(Y_N(\tau))H_{2k_2}(\bar{Y}'_N(\tau))] \\ & \rightarrow \mathbb{E}[H_{q-2k_1}(X(0))H_{2k_1}(\bar{X}'(0))H_{q-2k_2}(X(\tau))H_{2k_2}(\bar{X}'N(\tau))] \end{aligned}$$

and the formula (15) gives a domination proving the convergence of the integral and the fact that σ_q^2 is finite.

Let us look to the case $q = 1$

$$\begin{aligned} \mathbb{E}(I_{1,N}^2([0, N\pi])) &= -r''_{Y_N}(0)\varphi^2(u)(ua_0)^2 \frac{1}{N\pi} \int_0^{N\pi} \int_0^{N\pi} \mathbb{E}(Y_N(s)Y_N(s')) ds' ds \\ &\rightarrow 1/3\varphi^2(u) \frac{2u^2}{\pi^2} \pi^2/2 = \frac{1}{3}u^2\phi^2(u). \end{aligned} \quad (19)$$

using (16).

On the other hand we have

$$\begin{aligned} \mathbb{E}(I_1^2([0, N\pi])) &= \frac{1}{3}\varphi^2(u)(ua_0)^2 \frac{1}{N\pi} \int_0^{N\pi} \int_0^{N\pi} \frac{\sin(s-s')}{s-s'} ds' ds \\ &= 1/3\varphi^2(u) \frac{2u^2}{\pi^2} 2 \int_0^{N\pi} (\pi - \tau/N) \frac{\sin(\tau)}{\tau} d\tau \rightarrow \frac{2}{3}u^2\phi^2(u). \end{aligned} \quad (20)$$

■

4 Central limit Theorem with a chaining argument

The covariance $r(t)$ of the limit process $X(t)$ is not a summable in the sense that

$$\int_0^{+\infty} |r(t)| dt = +\infty.$$

but it satisfies

$$\int_0^N r(t) dt \text{ converges as } N \rightarrow \infty,$$

for $q > 1$

$$\int_0^{+\infty} |r(t)|^q dt < +\infty$$

In fact a careful inspection of the proof of Theorem 1 in [\[7\]](#) or of the proof of Theorem 10.11 of [\[2\]](#) shows that it is enough to get the following central limit theorem

Theorem 1 As $t \rightarrow +\infty$,

$$\frac{1}{\sqrt{t}} (N_{[0,t]}^X(u) - \mathbb{E}(N_{[0,t]}^X(u))) \Rightarrow N(0, \frac{2}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty} \sigma_q^2(u)),$$

where \Rightarrow is the convergence in distribution.

The main idea is to use this result to extend it to the crossings of $Y_N(t)$. Our main result is the following

main **Theorem 2** As $N \rightarrow +\infty$,

$$\frac{1}{\sqrt{2N\pi}} (N_{[0,2N\pi]}^{Y_N}(u) - \mathbb{E}(N_{[0,2N\pi]}^{Y_N}(u))) \Rightarrow N(0, \frac{1}{3}u^2\phi^2(u) + \sum_{q=2}^{\infty} \sigma_q^2(u)),$$

Proof: Let us introduce the cross correlation:

$$\begin{aligned} \rho_N(s, t) &= \mathbb{E}(X(s)Y_N(t)) = \sum_{n=1}^N \int_{\frac{n-1}{N}}^{\frac{n}{N}} \cos(s\lambda - t\frac{n}{N}) d\lambda. \\ &= \sum_{n=1}^N \int_0^{\frac{1}{N}} \cos((s-t)\frac{n}{N} - sv) dv = \Re\left\{ \int_0^{\frac{1}{N}} e^{-isv} dv \sum_{n=1}^N e^{i(s-t)\frac{n}{N}} \right\} \\ &= \frac{\sin \frac{s}{N}}{\frac{s}{N}} \frac{1}{N} \sum_{n=1}^N \cos(s-t)\frac{n}{N} + \frac{1 - \cos \frac{s}{N}}{\frac{s^2}{2N^2}} \frac{s}{2N^2} \sum_{n=1}^N \sin(s-t)\frac{n}{N} \end{aligned}$$

So we can write

$$\rho_N(s, t) = \frac{\sin(s/N)}{s/N} r_{Y_N}(t-s) + \frac{1 - \cos(s/N)}{s^2/(2N^2)} \frac{s}{2N} \frac{1}{N} \sum_{n=1}^N \sin(s-t)\frac{n}{N}$$

The two functions $\frac{\sin(z)}{z}$ and $\frac{1 - \cos(z)}{z^2/2}$ are bounded, with bounded derivatives and $\frac{\sin(z)}{z}$ tend to 1 as z tends to 0. We have also

$$\left| \frac{1}{N} \sum_{n=1}^N \sin(s-t)\frac{n}{N} \right| = \left| \frac{2}{s-t} \frac{\sin \frac{(s-t)}{2}}{\frac{2N}{s-t}} \frac{\sin(\frac{N+1}{2N}(s-t))}{\sin \frac{(s-t)}{2N}} \right| \leq (const)|s-t|^{-1},$$

whenever $|s-t| < \pi N$.

We have already proved that $r_{Y_N}(s-t) = \frac{1}{N} \sum_{n=1}^N \cos((s-t)\frac{n}{N})$, converges to $r(s-t)$ uniformly on every compact that does not contains zero. The same result is true for the first two derivatives that converge respectively to the corresponding derivative of $r(s-t)$. In addition for larges values of $|s-t|$ these functions are bounded by $K|s-t|^{-1}$ and for each fixed s , $\frac{s}{2N^2} \sum_{n=1}^N \sin(s-t)\frac{n}{N} \rightarrow 0$. Using the derivation rules it is easy to see that this is enough to have

$$\begin{aligned}\rho_N(s, t) &\rightarrow r(s-t) \\ \frac{\partial \rho_N(s, t)}{\partial s} &= \mathbb{E}(X'(s)Y_N(t)) \rightarrow r'(s-t) \\ \frac{\partial \rho_N(s, t)}{\partial t} &= \mathbb{E}(X(s)Y'_N(t)) \rightarrow -r'(s-t) \\ \frac{\partial^2 \rho_N(s, t)}{\partial s \partial t} &= \mathbb{E}(X'(s)Y'_N(t)) \rightarrow -r''(s-t),\end{aligned}$$

again the convergence being uniform on every compact that does not contains zero. In additions these function are bounded by $(const)(s-t)^{-1}$.

We are now ready to prove the following lemma

chaine **Lemma 3** For $q \geq 2$

$$\lim_{N \rightarrow \infty} \mathbb{E}[I_{q,N}([0, N\pi]) - I_q([0, N\pi])]^2 = 0.$$

Proof:

$$\begin{aligned}\mathbb{E}[I_{q,N}([0, N\pi]) - I_q([0, N\pi])]^2 &= \mathbb{E}[I_{q,N}([0, N\pi])]^2 + \mathbb{E}[I_q([0, N\pi])]^2 \\ &\quad - 2\mathbb{E}[I_{q,N}([0, N\pi]).I_q([0, N\pi])].\end{aligned}$$

We have already shown that the two first terms tend to $\sigma_q^2(u)$. Only remains to prove that the third also does. But, since the cross correlation $\rho_N(s, t)$ shares all the properties of $r_{Y_N}(s-t)$, the same proof as in Section 3 proves that the limit is again $\sigma_q^2(u)$. ■

We now finish the proof of Theorem 2.

The case of $I_{1,N}([0, N\pi])$ is easy to handle since it is already a Gaussian variable and that its limit variance is easy to compute using 16 By Lemma 3 for $q > 2$, $I_{q,N}([0, N\pi])$ inherits of the asymptotic Gaussian behavior of $I_{q,N}([0, N\pi])$.

By Theorem xx of Peccatti and*** this is enough to obtain the normality of the sum.

It remains to extend the result to $[0, 2N\pi]$ ** ces deux dernier points pour Chichi**

5 Proof of Lemma 1

s:e

Let $U_{[0,a]}^{Y_N}(u)$ be the number of up-crossings of the level u by $Y_N(t)$ in the interval $[0, a]$ i.e. the number of instant t such that $Y_N(t) = u; Y_N'(t) > 0$. The Rolle theorem implies

$$N_{[0,a]}^{Y_N}(u) \leq 2U_{[0,a]}^{Y_N}(u) + 1.$$

So it suffices to prove the lemma for the number up-crossings. Writing U for $U_{[0,a]}^{Y_N}(u)$ for short, we have

$$\text{Var}(U) = \mathbb{E}(U(U - 1)) + \mathbb{E}(U) - (\mathbb{E}(U))^2.$$

We have already proved that the last two terms give a finite contribution after normalization. For studying the first one we define the function $\theta_N(t)$ by

$$r_{Y_N}(\tau) = 1 + \frac{r_{Y_N}''(0)}{2}\tau^2 + \theta_N(\tau).$$

and we use relation(4.14) of [\[2\]](#) ^{Az:Wsc} to get

$$\begin{aligned} \mathbb{E}(U(U - 1)) &= 2 \int_0^a (a - \tau) \mathbb{E}[|Y_N'(0)Y_N'(\tau)| | Y_N(0) = Y_N(\tau) = u] p_{Y_N(0), Y_N(\tau)}(u, u) d\tau \\ &\leq (\text{const})a \int_0^a \frac{\theta_N'(\tau)}{\tau^2} d\tau. \end{aligned}$$

By a Taylor-Lagange expansion we get

$$r_{Y_N}'(\tau) = r_{Y_N}'(0) + \frac{1}{6N^5} \sum_{n=1}^N n^4 \tau^3 \cos(\theta(n, N)),$$

with $\theta(n, N) \leq \tau/N$. We obtain that $|\theta_N'(\tau)| \leq (\text{const})\tau^3$, the constant being uniform in N . This gives the result. ■

6 Proof of Theorem 1

We will give a sketch of the proof for completeness. Let D_m be a diagonal matrix with diagonal terms $d_{ii} = \lim_{t \rightarrow \infty} \text{Var}(I_q([0, t]))$. Theorem 1 of Peccati & Tudor [\[7\]](#) ^{pe:tu} says that the random vector

$$I_{q_1}([0, t]), \dots, I_{q_m}([0, t]) \Rightarrow N(0, D_m),$$

if and only if each $I_{q_i}([0, t])$ converges in distribution towards $N(0, d_{ii})$. We will prove this last assertion.

Let us begin with the term corresponding to $q = 1$.

$$\begin{aligned} I_1([0, t]) &= \sqrt{1/3}\varphi(u)u a_0 \frac{1}{\sqrt{t}} \int_0^t X(s)ds = \sqrt{1/3} \frac{e^{-\frac{u^2}{2}} u}{\pi} \frac{1}{\sqrt{t}} \int_0^t X(s)ds \\ &\Rightarrow N\left(0, \frac{e^{-u^2} u^2}{3\pi^2} 2 \int_0^\infty \frac{\sin \tau}{\tau} d\tau\right) = N\left(0, e^{-u^2} \frac{u^2}{3}\right). \end{aligned} \quad (21)$$

pi*** For $q > 1$ we can adapt the proof of the cited references . The process X has $f(\lambda) = \frac{1}{2}1_{[-1,1]}(\lambda)$ as spectral density, if we symmetrize the spectrum. Let ϑ be an even function with $\int_{-\infty}^\infty |\lambda|^j |\vartheta(\lambda)| d\lambda < \infty$ $j = 1, 2$ and such that its Fourier Transform has support in $[-1, 1]$. By defining $\vartheta_\varepsilon = \frac{1}{\varepsilon}\vartheta(\frac{\cdot}{\varepsilon})$ and putting $f_\varepsilon(\lambda) = f * \vartheta_\varepsilon(\lambda)$ the following process

$$X_\varepsilon(t) = \int_0^\infty \cos(t\lambda) \sqrt{f_\varepsilon(\lambda)} dB_1(\lambda) + \int_0^\infty \sin(t\lambda) \sqrt{f_\varepsilon(\lambda)} dB_2(\lambda),$$

satisfies

$$r_\varepsilon(\tau) := \mathbb{E}X_\varepsilon(\tau)X_\varepsilon(0) = \int_0^\infty \cos(\tau\lambda) f_\varepsilon(\lambda) d\lambda = \frac{1}{2} \int_{-\infty}^\infty \cos t\lambda f_\varepsilon(\lambda) d\lambda = \frac{1}{2} r(\tau) \hat{\vartheta}(\varepsilon\tau).$$

Thus X_ε is a Gaussian $\frac{1}{\varepsilon}$ -dependent process, this process has variance one if $\hat{\vartheta}(0) = 2$, which is always possible. Moreover,

$$r'_\varepsilon(\tau) = \frac{1}{2}(r'(\tau)\hat{\vartheta}(\varepsilon\tau) + r(\tau)\varepsilon\hat{\vartheta}'(\varepsilon\tau))$$

and

$$r''_\varepsilon(\tau) = \frac{1}{2}(r''(\tau)\hat{\vartheta}(\varepsilon\tau) + 2r'(\tau)\varepsilon\hat{\vartheta}'(\varepsilon\tau) + r(\tau)\varepsilon^2\hat{\vartheta}''(\varepsilon\tau)).$$

These functions have bounded support and converge to r , r' and r'' respectively, functions that belong to $L^2(\mathbb{R})$. Recalling the set of indexes J_q of Lemma (2), we get by using Dominate Convergence Theorem

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \sum_{J_q} \int_0^{\frac{1}{\varepsilon}} |r_\varepsilon(\tau)|^{2q-(2k_1+2k_2)-h_1} \left| \frac{r'_\varepsilon(\tau)}{\sqrt{-r''_\varepsilon(0)}} \right|^{2h_1} \left| \frac{r''_\varepsilon(\tau)}{r''_\varepsilon(0)} \right|^{2k_1+2k_2-h_1} d\tau \\ &= \sum_{J_q} \int_0^\infty |r(\tau)|^{2q-(2k_1+2k_2)-h_1} \left| \frac{r'(\tau)}{\sqrt{-r''(0)}} \right|^{2h_1} \left| \frac{r''(\tau)}{r''(0)} \right|^{2k_1+2k_2-h_1} d\tau. \end{aligned}$$

The same result holds dropping the absolute value in the integrant. Let us define

$$I_{q,\varepsilon}([0, t]) = \sqrt{-r''_{X_\varepsilon}(0)} \frac{1}{\sqrt{t}} \int_0^t f_q(u, X_\varepsilon(s), \frac{X'_\varepsilon(s)}{\sqrt{-r''_{X_\varepsilon}(0)}}) ds.$$

The above result and Lemma 2, allow us to conclude that

$$\lim_{t \rightarrow \infty} \mathbb{E}[I_q^{t,\varepsilon}(f_q)]^2 = \frac{1}{\pi} \sigma_q^2(u)$$

We shall now to consider the convergence for the covariances.

$$\rho_\varepsilon(\tau) = \mathbb{E}[X_\varepsilon(\tau)X(0)] = \int_0^1 \cos \tau \lambda \sqrt{f_\varepsilon(\lambda)} d\lambda \rightarrow r(\tau),$$

$$\rho'_\varepsilon(\tau) = \mathbb{E}[X'_\varepsilon(\tau)X(0)] = - \int_0^1 \lambda \sin \tau \lambda \sqrt{f_\varepsilon(\lambda)} d\lambda \rightarrow -r'(\tau)$$

$$\rho''_\varepsilon(\tau) = \mathbb{E}[X'_\varepsilon(\tau)X'(0)] = - \int_0^1 \lambda^2 \cos \tau \lambda \sqrt{f_\varepsilon(\lambda)} d\lambda \rightarrow -r''(\tau).$$

Moreover,

$$\rho_\varepsilon(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \cos \tau \lambda \sqrt{f_\varepsilon(\lambda)} 1_{[-1,1]}(\lambda) d\lambda = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \cos \tau \lambda \sqrt{f_\varepsilon(\lambda)} \sqrt{\frac{1}{2} 1_{[-1,1]}(\lambda)} d\lambda$$

By using Fatou, Parseval and that $f_\varepsilon(\lambda) \rightarrow 2f(\lambda)$ in $L^2(\mathbb{R})$, we obtain easily

$$\begin{aligned} \int_0^\infty |r(\tau)|^2 d\tau &\leq \limsup_{\varepsilon \rightarrow 0} \int_0^\infty |\rho_\varepsilon(\tau)|^2 d\tau = \limsup_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{-\infty}^{\infty} |\rho_\varepsilon(\tau)|^2 d\tau \\ &= \limsup_{\varepsilon \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} |\sqrt{f_\varepsilon(\lambda)} \sqrt{\frac{1}{4} 1_{[-1,1]}(\lambda)}|^2 d\lambda = \int_0^\infty |r(\tau)|^2 d\tau. \end{aligned}$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty |\rho_\varepsilon(\tau)|^2 d\tau = \int_0^\infty |r(\tau)|^2 d\tau \quad (22)$$

In the same form we get

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty |\rho'_\varepsilon(\tau)|^2 d\tau = \int_0^\infty |r'(\tau)|^2 d\tau \text{ and } \lim_{\varepsilon \rightarrow 0} \int_0^\infty |\rho''_\varepsilon(\tau)|^2 d\tau = \int_0^\infty |r''(\tau)|^2 d\tau. \quad (23)$$

We will compute now

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{t \rightarrow \infty} \mathbb{E}[I_q^{t,\varepsilon}(f_q) - I_q^t(f_q)]^2 = \frac{2}{\pi} \sigma_q^2(u) - 2 \lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} \mathbb{E}[I_q^{t,\varepsilon}(f_q) I_q^t(f_q)].$$

This limit vanish if we can prove that the third term tends to $\frac{2}{\pi} \sigma_q^2(u)$ also. But this is a consequence again of Lemma 2 (22) and (23) cf. [5]. Let us sketch the proof that this is what really happens. Defining $d_{q,2k}(u) = \frac{H_{q-2k}(u)}{(q-2k)!} a_{2k}$, we have

$$\begin{aligned} \mathbb{E}[I_{q,\varepsilon}([0, t]), I_q([0, t])] &= -r''_X(0)\varphi^2(u) \sum_{k_1, k_2} d_{q, 2k_1}(u)d_{q, 2k_1}(u) \\ &\times \frac{1}{t} \int_0^t \int_0^t \mathbb{E}[H_{q-2k_1}(X^\varepsilon(s))H_{2k_1}\left(\frac{X'^\varepsilon(s)}{\sqrt{-r''_{X^\varepsilon}(0)}}\right)H_{q-2k_1}(X(s'))H_{2k_1}\left(\frac{X'(s')}{\sqrt{-r''_X(0)}}\right)dsds']. \end{aligned}$$

The integral is by Lemma 2 equal to

$$\begin{aligned} &\sum_{L_q} V_{q, 2k_1, 2k_2} \frac{1}{t} \int_0^t \int_0^t (1_{\{s > s'\}} \left[\frac{-\rho'_\varepsilon(s-s')}{\sqrt{-\rho''_\varepsilon(0)}} \right]^{2h_1} + 1_{\{s' > s\}} \left[\frac{-\rho'_\varepsilon(s'-s)}{\sqrt{-r''_X(0)}} \right]^{2h_1}) \\ &\times \rho_\varepsilon(|s-s'|)^{2q-(2k_1+2k_2)-h_1} \left[\frac{-\rho''_\varepsilon(|s-s'|)}{\sqrt{-\rho''_\varepsilon(0)}\sqrt{-r''_X(0)}} \right]^{2k_1+2k_2-h_1} dsds', \quad (24) \end{aligned} \quad \boxed{\text{diagrama}}$$

where L_q is a set of indexes and $V_{q, 2k_1, 2k_2}$ are fixed constant. Given that $q > 1$ and by using that the functions ρ_ε and its derivatives converge in L^2 towards their pointwise limit, it yields that this sum converges towards

$$2 \int_0^\infty \mathbb{E}[H_{q-2k_1}(X(0))H_{2k_1}\left(\frac{X'(0)}{\sqrt{-r''_X(0)}}\right)H_{q-2k_1}(X(\tau))H_{2k_1}\left(\frac{X'(\tau)}{\sqrt{-r''_X(0)}}\right)d\tau].$$

Obtaining the expected result.

The $\frac{1}{\varepsilon}$ -dependence entails that $I_{q,\varepsilon}([0, t])$ is asymptotically Gaussian and the proved contiguity in L^2 allows concluding the same for $I_{q,\varepsilon}([0, t])$, with asymptotic variance $\frac{1}{\pi} \sigma_q^2(u)$. The CLT for the crossings of X follows from the expansion

$$N_{[0,t]}^X = \sum_{q=1}^{\infty} I_q^t(f_q),$$

the asymptotic independence of the Gaussian limit in each chaos and the convergence of the variance. ■

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