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Fine properties of the subdifferential for a class of one-homogeneous functionals

A. Chambolle * M. Goldman † M. Novaga ‡

Abstract

We collect here some known results on the subdifferential of one-homogeneous functionals, which are anisotropic and nonhomogeneous variants of the total variation, and establish a new relationship between Lebesgue points of the calibrating field and regular points of the level lines of the corresponding calibrated function.

1 Introduction

In this note we recall some classical results on the structure of the subdifferential of first order one-homogeneous functionals, and we give new regularity results which extend and precise previous work by G. Anzellotti [5, 6, 7].

Given an open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, and a function $u \in C^1(\Omega) \cap BV(\Omega)$, we consider the functional

$$J(u) := \int_{\Omega} F(x, Du)$$

where $F: \Omega \times \mathbb{R}^d \to [0, +\infty)$ is continuous in x and $F(x, \cdot)$ is a smooth and uniformly convex norm on \mathbb{R}^d , for all $x \in \Omega$.

Since $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$, it is natural to consider J as a convex, l.s.c. function on the whole of $L^{d/(d-1)}(\Omega)$, with value $+\infty$ when $u \notin BV(\Omega)$ (see [2]). In this framework, for any $u \in L^{d/(d-1)}(\Omega)$ we can define the subgradient of a u in the duality $(L^{d/(d-1)}, L^d)$ as

$$\partial J(u) = \left\{ g \in L^d(\Omega) : J(v) \ge J(u) + \int_{\Omega} g(x)(v(x) - u(x)) \, dx \, \forall v \in L^{d/(d-1)}(\Omega) \right\}.$$

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The goal of this paper is to investigate the particular structure of the functions u and $g \in \partial J(u)$, when the subgradient is nonempty. Since J can be defined by duality as

$$J(u) = \sup \left\{ -\int_{\Omega} u(x) \operatorname{div} z(x) \, dx : z \in C_c^{\infty}(\Omega; \mathbb{R}^d), \ F^*(x, z(x)) = 0 \ \forall x \in \Omega \right\}$$

where $F^*(x,\cdot)$ is the Legendre-Fenchel transform of $F(x,\cdot)$ (it is equivalent to require that $F^{\circ}(x,z(x)) \leq 1$, $F^{\circ}(x,\cdot)$ being the convex polar of F defined in (4)), it is easy to see that such a g has necessarily the form $g = -\operatorname{div} z$, for some field $z \in L^{\infty}(\Omega; \mathbb{R}^d)$ with $F^*(x,z(x)) = 0$ a.e. in Ω .

Since by a formal integration by parts one gets $z \cdot Du = F(x, Du)$, |Du|-a.e., natural questions are: in what sense can this relation be true? can one assign a precise value to z on the support of the measure Du?

The first question has been answered by Anzelotti in the series of papers [5, 6, 7]. However, for the particular vector fields we are interested in, we can be more precise and obtain pointwise properties of z on the level sets of the function u. Indeed, we shall show that z has a pointwise meaning on all level sets of u, up to \mathcal{H}^{d-1} -negligible sets (which is much more than |Du|-a.e., as illustrated by the function $u = \sum_{n=1}^{+\infty} 2^{-n} \chi_{(0,x_n)}$, defined in the interval (0,1), with (x_n) a dense sequence in that interval).

We will therefore focus on the properties of the vector fields $z \in L^{\infty}(\Omega, \mathbb{R}^d)$ such that $F^*(x, z(x)) = 0$ a.e. in Ω and $g = -\operatorname{div} z \in L^d(\Omega)$, and such that there exists a function u such that for any $\phi \in C_c^{\infty}(\Omega)$,

$$-\int_{\Omega} \operatorname{div} z(x)u(x)\phi(x) dx = \int_{\Omega} u(x) z(x) \cdot \nabla \phi(x) dx + \int_{\Omega} \phi(x)F(x,Du).$$

In particular, one checks easily that u minimizes the functional

$$\int_{\Omega} F(x, Du) - \int_{\Omega} g(x)u(x) dx \tag{1}$$

among perturbations with compact support in Ω . Conversely, given $g \in L^d(\Omega)$ with $||g||_{L^d}$ sufficiently small, there exist functions u which minimize (1) under various types of boundary conditions, and corresponding fields z.

This kind of functionals appears in many contexts including image processing and plasticity [4, 17]. Notice also that, by the Coarea Formula [2], it holds

$$\int_{\Omega} F(x,Du) - \int_{\Omega} gu \, dx = \int_{\mathbb{R}} \left(\int_{\partial^* \{u>s\}} F(x,\nu) - \int_{\{u>s\}} g \, dx \right) \, ds \,,$$

where ν is the unit normal to $\{u > s\}$, and one can show (see for instance [10]) that the characteristic function of any level set of the form $\{u > s\}$ or $\{u \ge s\}$ is a minimizer of the geometric functional

$$\int_{\partial^* F} F(x, \nu) - \int_F g(x) \, dx \,. \tag{2}$$

The canonical example of such functionals is given by the total variation, corresponding to F(x, Du) = |Du|. In this case, (2) boils down to

$$P(E) - \int_{E} g(x) dx. \tag{3}$$

In [8], it is shown that every set with finite perimeter in Ω is a minimizer of (3) for some $g \in L^1(\Omega)$. However, if $g \in L^p(\Omega)$ with p > d, and E is a minimizer of (2), then ∂E is locally $C^{1,\alpha}$ for some $\alpha > 0$, out of a closed singular set of zero \mathscr{H}^{d-3} -measure [1]. When $g \in L^d(\Omega)$, the boundary ∂E is only of class C^{α} out of the singular set (see [3]). Since the Euler-Lagrange equation of (2) relates z to the normal to E, understanding the regularity of z is closely related to understanding the regularity of ∂E .

Our main result is that the Lebesgue points of z correspond to regular points of $\partial \{u > s\}$ or $\partial \{u \ge s\}$ (Theorem 3.7), and that the converse is true in dimension $d \le 3$ (Theorem 3.8).

2 Preliminaries

2.1 BV functions

We briefly recall the definition of function of bounded variation and set of finite perimeter. For a complete presentation we refer to [2].

Definition 2.1. Let Ω be an open set of \mathbb{R}^d , we say that a function $u \in L^1(\Omega)$ is a function of bounded variation if

$$\int_{\Omega} |Du| := \sup_{\substack{z \in \mathcal{C}_{c}^{1}(\Omega) \\ |z|_{c} \le 1}} \int_{\Omega} u \operatorname{div} z \ dx < +\infty.$$

We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω (when $\Omega = \mathbb{R}^d$ we simply write BV instead of $BV(\mathbb{R}^d)$).

We say that a set $E \subset \mathbb{R}^d$ is of finite perimeter if its characteristic function χ_E is of bounded variation and denote its perimeter in an open set Ω by $P(E,\Omega) := \int_{\Omega} |D\chi_E|$, and write simply P(E) when $\Omega = \mathbb{R}^d$.

Definition 2.2. Let E be a set of finite perimeter and let $t \in [0; 1]$. We define

$$E^{(t)} := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

We denote by $\partial E := (E^{(0)} \cup E^{(1)})^c$ the measure theoretical boundary of E. We define the reduced boundary of E by:

$$\partial^*E := \left\{ x \in \operatorname{Spt}(|D\chi_E|) \, : \, \nu^E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ exists and } |\nu^E(x)| = 1 \right\} \subset E^{\left(\frac{1}{2}\right)}.$$

The vector $\nu^{E}(x)$ is the measure theoretical inward normal to the set E.

Proposition 2.3. If E is a set of finite perimeter then $D\chi_E = \nu^E \ \mathcal{H}^{d-1} \sqcup \partial^* E$, $P(E) = \mathcal{H}^{d-1}(\partial^* E)$ and $\mathcal{H}^{d-1}(\partial E \setminus \partial^* E) = 0$.

Definition 2.4. We say that x is an approximate jump point of $u \in BV(\Omega)$ if there exist $\xi \in \mathbb{S}^{d-1}$ and distinct $a, b \in \mathbb{R}$ such that

$$\lim_{\rho \to 0} \frac{1}{|B_{\rho}^{+}(x,\xi)|} \int_{B_{\rho}^{+}(x,\xi)} |u(y) - a| \ dy = 0 \quad and \quad \lim_{\rho \to 0} \frac{1}{|B_{\rho}^{-}(x,\xi)|} \int_{B_{\rho}^{-}(x,\xi)} |u(y) - b| \ dy = 0,$$

where $B_{\rho}^{\pm}(x,\xi) := \{y \in B_{\rho}(x) : \pm (y-x) \cdot \xi > 0\}$. Up to a permutation of a and b and a change of sign of ξ , this characterize the triplet (a,b,ξ) which is then denoted by (u^+,u^-,ν_u) . The set of approximate jump points is denoted by J_u .

The following proposition can be found in [2, Proposition 3.92].

Proposition 2.5. Let $u \in BV(\Omega)$. Then, defining

$$\Theta_u := \{ x \in \Omega / \liminf_{\rho \to 0} \rho^{1-d} |Du|(B_\rho(x)) > 0 \},$$

there holds $J_u \subset \Theta_u$ and $\mathscr{H}^{d-1}(\Theta_u \backslash J_u) = 0$.

2.2 Anisotropies

Let $F(x,p): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a convex one-homogeneous function in the second variable such that there exists c_0 with

$$c_0|p| \le F(x,p) \le \frac{1}{c_0}|p| \qquad \forall (x,p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

We say that F is uniformly elliptic if for some $\delta > 0$, the function $p \mapsto F(p) - \delta |p|$ is still a convex function. We define the polar function of F by

$$F^{\circ}(x,z) := \sup_{\{F(x,p) \le 1\}} z \cdot p$$
 (4)

so that $(F^{\circ})^{\circ} = F$. It is easy to check that (* denoting the Legendre-Fenchel convex conjugate) $[F(x,\cdot)^2/2]^* = F^{\circ}(x,\cdot)^2/2$, in particular (if differentiable), $F(x,\cdot)\nabla_p F(x,\cdot)$ and $F^{\circ}(x,\cdot)\nabla_z F^{\circ}(x,\cdot)$ are inverse monotone operators. If we denote by F^* the convex conjugate of F with respect to the second variable, then $F^*(x,z) = 0$ if and only if $F^{\circ}(x,z) \leq 1$. If $F(x,\cdot)$ is differentiable then, for every $p \in \mathbb{R}^d$,

$$F(x,p) = p \cdot \nabla_p F(x,p)$$
 (Euler's identity)

and

$$z \in \{F^{\circ}(x,\cdot) \le 1\}$$
 with $p \cdot z = F(x,p) \iff z = \nabla_{p}F(x,p)$.

If F is elliptic and of class $C^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$, then F° is also elliptic and $C^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$. We will then say that F is a smooth elliptic anisotropy. Observe that, in this case, the function $F^2/2$ is also uniformly δ^2 -convex (this follows from the inequalities $D^2F(x,p) \geq \delta/|p|(I-p\otimes p/|p|^2)$ and $F(x,p) \geq \delta|p|$). In particular, for every $x,y,z \in \mathbb{R}^d$, there holds

$$F^{2}(x,y) - F^{2}(x,z) \ge 2\left(F(x,z)\nabla_{p}F(x,z)\right) \cdot (y-z) + \delta^{2}|y-z|^{2},\tag{5}$$

and a similar inequality holds for F° . We refer to [16] for general results on convex norms and convex bodies.

2.3 Pairings between measures and bounded functions

Following [5] we define a generalized trace [z, Du] for functions u with bounded variation and bounded vector fields z with divergence in L^d .

Definition 2.6. Let Ω be an open set with Lipschitz boundary, $u \in BV(\Omega)$ and $z \in L^{\infty}(\Omega, \mathbb{R}^d)$ with $\operatorname{div} z \in L^d(\Omega)$. We define the distribution [z, Du] by

$$\langle [z, Du], \psi \rangle = -\int_{\Omega} u \, \psi \, \mathrm{div} \, z - \int_{\Omega} u \, z \cdot \nabla \psi \qquad \forall \psi \in \mathcal{C}_{c}^{\infty}(\Omega).$$

Proposition 2.7. The distribution [z, Du] is a bounded Radon measure on Ω and if ν is the inward unit normal to Ω , there exists a function $[z, \nu] \in L^{\infty}(\partial\Omega)$ such that the generalized Green's formula holds,

$$\int_{\Omega} [z, Du] = -\int_{\Omega} u \operatorname{div} z - \int_{\partial \Omega} [z, \nu] u \, d\mathcal{H}^{d-1}.$$

The function $[z, \nu]$ is the generalized (inward) normal trace of z on $\partial\Omega$.

Given $z \in L^{\infty}(\Omega, \mathbb{R}^d)$, with div $z \in L^d(\Omega)$, we can also define the generalized trace of z on ∂E , where E is a set of locally finite perimeter. Indeed, for every bounded open set Ω with Lipschitz boundary, we can define as above the measure $[z, D\chi_E]$ on Ω . Since this measure is absolutely continuous with respect to $|D\chi_E| = \mathcal{H}^{d-1} \sqcup \partial^* E$ we have

$$[z, D\chi_E] = \psi_z(x) \mathcal{H}^{d-1} \sqcup \partial^* E$$

with $\psi_z \in L^{\infty}(\partial^* E)$ independent of Ω . We denote by $[z, \nu^E] := \psi_z$ the generalized (inward) normal trace of z on ∂E . If E is a bounded set of finite perimeter, by taking Ω strictly containing E, we have the generalized Gauss-Green Formula

$$\int_{E} \operatorname{div} z = -\int_{\partial^* E} [z, \nu^E] d\mathcal{H}^{d-1}.$$

Anzellotti proved the following alternative definition of $[z, \nu^E]$ [6, 7]

Proposition 2.8. Let $(x, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$. For any r > 0, $\rho > 0$ we let

$$C_{r,\rho}(x,\alpha) := \{ \xi \in \mathbb{R}^d : |(\xi - x) \cdot \alpha| < r, |(\xi - x) - [(\xi - x) \cdot \alpha]\alpha| < \rho \}.$$

There holds

$$[z,\alpha](x) = \lim_{\rho \to 0} \lim_{r \to 0} \frac{1}{2r\omega_{d-1}\rho^{d-1}} \int_{C_{r,\rho}(x,\alpha)} z \cdot \alpha$$

where ω_{d-1} is the volume of the unit ball in \mathbb{R}^{d-1} .

3 The subdifferential of anisotropic total variations

3.1 Characterization of the subdifferential

The following characterization of the subdifferential of J is classical and readily follows for example from the representation formula [9, (4.19)].

Proposition 3.1. Let F be a smooth elliptic anisotropy and $g \in L^d(\Omega)$ then u is a local minimizer of (1) if and only if there exists $z \in L^{\infty}(\Omega)$ with $\operatorname{div} z = g$, $F^*(x, z(x)) = 0$ a.e. and

$$[z, Du] = F(x, Du).$$

Moreover, for every $t \in \mathbb{R}$, for the set $E = \{u > t\}$ there holds $[z, \nu^E] = F(x, \nu^E)$ \mathscr{H}^{d-1} -a.e. on ∂E . We will say that such a vector field is a calibration of the set E for the minimum problem (2).

Remark 3.2. In [5], it is proven that if $z_{\rho}(x) := \frac{1}{|B_{\rho}(x)|} \int_{B_{\rho}(x)} z(y) \, dy$, then $z_{\rho} \cdot \nu^{E}$ weakly* converges to $[z, \nu^{E}]$ in $L^{\infty}_{loc}(\mathscr{H}^{d-1} \sqcup \partial^{*}E)$. Using (5) it is then possible to prove that if z calibrates E then z_{ρ} converges to $\nabla_{p}F(x,\nu^{E})$ in $L^{2}(\mathscr{H}^{d-1} \sqcup \partial^{*}E)$ yielding that up to a subsequence, $z_{\phi(\rho)}$ converges \mathscr{H}^{d-1} -a.e. to $\nabla_{p}F(x,\nu^{E})$. Unfortunately this is still a very weak statement since it is a priori impossible to recover from this the convergence of the full sequence z_{ρ} .

The main question we want to investigate now is whether we can give a classical meaning to $[z, \nu^E]$ (that is understand if $[z, \nu^E] = z \cdot \nu^E$). We observe that a priori the value of z is not well defined on ∂E which has zero Lebesgue measure (since z has Lebesgue points only a.e.). We let $S := \text{supp}(Du) \subset \Omega$ be the smallest closed set in Ω such that $|Du|(\Omega \setminus S) = 0$. The next result is classical.

Lemma 3.3 (Density estimate). There exists $\rho_0 > 0$ (depending on g) and a constant $\gamma > 0$ (which depends only on d), such that for any $B_{\rho}(x) \subset \Omega$ with $\rho \leq \rho_0$, and any level set E of u (that is, $E \in \{\{u > s\}, \{u \geq s\}, \{u \leq s\}, \{u \leq s\}, s \in \mathbb{R}\}$), if $|B_{\rho}(x) \cap E| < \gamma |B_{\rho}(x)|$ then $|B_{\rho/2}(x) \cap E| = 0$. As a consequence, E^0 and E^1 are open, ∂E is the topological boundary of E^1 , and (possibly changing slightly γ) if $x \in \partial E$, then $\mathscr{H}^{d-1}(\partial E \cap B_{\rho}(x)) \geq \gamma \rho^{d-1}$.

For a proof we refer to [13, 12]. This is not true anymore if $g \notin L^d(\Omega)$ [12]. If $\partial\Omega$ is Lipschitz, it is true up to the boundary.

Corollary 3.4. It follows that $u \in L^{\infty}_{loc}(\Omega)$ and $u \in C(\Omega \backslash \Theta_u)$.

Proof. For any ball $B_{\rho}(x) \subset \Omega$ and $\inf_{B_{\rho/2}(x)} u < a < b < \sup_{B_{\sigma/2}(x)} u$, one has

$$+\infty > |Du|(B_{\rho}(x)) \ge \int_{a}^{b} P(\{u > s\}, B_{\rho}(x)) ds \ge (b - a) \gamma \left(\frac{\rho}{2}\right)^{d-1},$$

so that $osc_{B_{\rho/2}(x)}(u)$ must be bounded and thus $u \in L^{\infty}_{loc}(\Omega)$. Moreover, if $x \in \Omega \backslash \Theta_u$ we find that $\lim_{\rho \to 0} osc_{B_{\rho}(x)}(u) = 0$ so that u is continuous at the point x.

It also follows from Lemma 3.3 that all points in the support of Du must be on the boundary of a level set of u:

Proposition 3.5. For any $x \in S$, there exists $s \in \mathbb{R}$ such that either $x \in \partial \{u > s\}$ or $x \in \partial \{u \geq s\}$.

Proof. First, if $x \notin S$ then $|Du|(B_{\rho}(x)) = 0$ for some $\rho > 0$ and clearly x cannot be on the boundary of a level set of u. On the other hand, if $x \in S$, then for any ball $B_{1/n}(x)$ (n large) there is a level s_n (uniformly bounded) with $\partial \{u > s_n\} \cap B_{1/n}(x) \neq \emptyset$ and by Hausdorff convergence, we deduce that either $x \in \partial \{u > s\}$ or $x \in \partial \{u \ge s\}$ where s is the limit of the sequence $(s_n)_n$ (which must actually converge).

The following stability property is classical (see e.g. [11]).

Proposition 3.6. Let E_n be local minimizers of (2), with a function $g = g_n \in L^d(\Omega)$, and converging in the L^1 -topology to a set E. Assume that the sets E_n are calibrated by z_n , that $z_n \stackrel{*}{\rightharpoonup} z$ weakly-* in L^{∞} and $g_n \to g = -\text{div } z \in L^d(\Omega)$, in $L^1(\Omega)$ as $n \to \infty$. Then z calibrates E, which is thus also a minimizer of (2).

In particular, one must notice that when $z_n \stackrel{*}{\rightharpoonup} z$ and $F^{\circ}(x, z_n) \leq 1$ a.e., then in the limit one still has $F^{\circ}(x, z) \leq 1$ a.e. (thanks to the convexity, and continuity w.r. the variable x).

3.2 The Lebesgue points of the calibration.

The next result shows that the regularity of the calibration z implies some regularity of the calibrated set.

Theorem 3.7. Let $\bar{x} \in \partial E$ be a Lebesgue point of z, with $E = \{u > t\}$ or $E = \{u \geq t\}$. Then, $\bar{x} \in \partial^* E$ and

$$z(\bar{x}) = \nabla_p F(\bar{x}, \nu^E(\bar{x})). \tag{6}$$

Proof. We follow [11, Th. 4.5] and let $z_{\rho}(y) := z(\bar{x} + \rho y)$. Since \bar{x} is a Lebesgue point of z, we have that $z_{\rho} \to \bar{z}$ in $L^{1}(B_{R})$, hence also weakly-* in $L^{\infty}(B_{R})$ for any R > 0, where $\bar{z} \in \mathbb{R}^{d}$ is a constant vector.

We let $E_{\rho} := (E - \bar{x})/\rho$ and $g_{\rho}(y) = g(\bar{x} + \rho y)$ (so that div $z_{\rho} = \rho g_{\rho}$). Observe that E_{ρ} minimizes

$$\int_{\partial^* E_\rho \cap B_R} F(\bar{x} + \rho y, \nu^{E_\rho}(y)) d\mathscr{H}^{d-1}(y) + \rho \int_{E_\rho \cap B_R} g_\rho(y) dy,$$

with respect to compactly supported perturbations of the set (in the fixed ball B_R). Also,

$$\|\rho g_{\rho}\|_{L^{d}(B_{R})} = \|g\|_{L^{d}(B_{\rho R})} \stackrel{\rho \to 0}{\longrightarrow} 0.$$

By Lemma 3.3, the sets E_{ρ} (and the boundaries ∂E_{ρ}) satisfy uniform density bounds, and hence are compact with respect to both local L^1 and Hausdorff convergence.

Hence, up to extracting a subsequence, we can assume that $E_{\rho} \to \bar{E}$, with $0 \in \partial \bar{E}$. Proposition 3.6 shows that \bar{z} is a calibration for the energy $\int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y)) d\mathcal{H}^{d-1}(y)$, and that \bar{E} is a minimizer calibrated by \bar{z} .

It follows that $[\bar{z}, \nu^{\bar{E}}] = F(\bar{x}, \nu^{\bar{E}}(y))$ for \mathscr{H}^{d-1} -a.e. y in $\partial \bar{E}$, but since \bar{z} is a constant, we deduce that $\bar{E} = \{y \cdot \bar{\nu} \geq 0\}$ with $\bar{\nu}/F(\bar{x}, \bar{\nu}) = \nabla_p F^{\circ}(\bar{x}, \bar{z})^1$. In particular the limit \bar{E} is unique, hence we obtain the global convergence of $E_{\rho} \to \bar{E}$, without passing to a subsequence.

We want to deduce that $\bar{x} \in \partial^* E$, with $\nu^E(\bar{x}) = F(\bar{x}, \nu^E(\bar{x})) \nabla_p F^{\circ}(\bar{x}, \bar{z})$, which is equivalent to (6). The last identity is obvious from the arguments above, so that we only need to show that

$$\lim_{\rho \to 0} \frac{D\chi_{E_{\rho}}(B_1)}{|D\chi_{E_{\rho}}|(B_1)} = \bar{\nu}. \tag{7}$$

Assume we can show that

$$\lim_{\rho \to 0} |D\chi_{E_{\rho}}|(B_R) = |D\chi_{\bar{E}}|(B_R) \ (= \omega_{d-1}R^{d-1})$$
 (8)

for any R > 0, then for any $\psi \in C_c^{\infty}(B_R; \mathbb{R}^d)$ we would get

$$\frac{1}{|D\chi_{E_{\rho}}|(B_R)} \int_{B_R} \psi \cdot D\chi_{E_{\rho}} = -\frac{1}{|D\chi_{E_{\rho}}|(B_R)} \int_{B_R \cap E_{\rho}} \operatorname{div} \psi(x) \, dx$$

$$\longrightarrow -\frac{1}{|D\chi_{\bar{E}}|(B_R)} \int_{B_R \cap \bar{E}} \operatorname{div} \psi(x) \, dx = \frac{1}{|D\chi_{\bar{E}}|(B_R)} \int_{B_R} \psi \cdot D\chi_{\bar{E}}$$

and deduce that the measure $D\chi_{E_{\rho}}/(|D\chi_{E_{\rho}}|(B_R))$ weakly-* converges to $D\chi_{\bar{E}}/(|D\chi_{\bar{E}}|(B_R))$. Using again (8)), we then obtain that

$$\lim_{\rho \to 0} \frac{D\chi_{E_{\rho}}(B_R)}{|D\chi_{E_{\rho}}|(B_R)} = \bar{\nu} \tag{9}$$

 $^{^1\}text{We}$ use here that $F(\bar{x},\cdot)\nabla F(\bar{x},\cdot)=[F^\circ(\bar{x},\cdot)\nabla F^\circ(\bar{x},\cdot)]^{-1},$ so that $\bar{z}=\nabla F(\bar{x},\nu^{\bar{E}}(y))$ implies both $F^\circ(\bar{x},\bar{z})=1$ and $\nu^{\bar{E}}(y)/F(\bar{x},\nu^{\bar{E}})(y)=\nabla F^\circ(\bar{x},\bar{z})$

for almost every R > 0. Since $D\chi_{E_{\rho}}(B_{\mu R})/(|D\chi_{E_{\rho}}|(B_{\mu R})) = D\chi_{E_{\rho/\mu}}(B_R)/(|D\chi_{E_{\rho/\mu}}|(B_R))$ for any $\mu > 0$, (9) holds in fact for any R > 0 and (7) follows, so that $\bar{x} \in \partial^* E$. It remains to show (8). First, we observe that, by minimality of E_{ρ} and \bar{E} plus the Hausdorff convergence of ∂E_{ρ} in balls, we can easily show the convergence of the energies

$$\lim_{\rho \to 0} \int_{\partial E_{\rho} \cap B_{R}} F(\bar{x} + \rho y, \nu^{E_{\rho}}(y)) d\mathcal{H}^{d-1}(y) + \rho \int_{E_{\rho} \cap B_{R}} g_{\rho}(y) dy$$

$$= \int_{\partial \bar{E} \cap B_{R}} F(\bar{x}, \nu^{\bar{E}}(y)) d\mathcal{H}^{d-1}(y)$$

and, by the continuity of F,

$$\lim_{\rho \to 0} \int_{\partial E_{\rho} \cap B_{R}} F(\bar{x}, \nu^{E_{\rho}}(y)) \, d\mathcal{H}^{d-1}(y) = \int_{\partial \bar{E} \cap B_{R}} F(\bar{x}, \nu^{\bar{E}}(y)) \, d\mathcal{H}^{d-1}(y). \tag{10}$$

Then, (7) follows from Reshetnyak's continuity theorem where, instead of using the Euclidean norm as reference norm, we use the uniformly convex norm $F(\bar{x}, \cdot)$ and the convergence of the measures $F(\bar{x}, D\chi_{E_{\rho}})$ to $F(\bar{x}, D\chi_{\bar{E}})$ (see [15, 11]).

In dimension 2 and 3 we can also show the reverse implication, proving that regular points of the boundary corresponds to Lebesgue points of the calibration. The idea is to show that the parameters r, ρ in Proposition 2.8 can be taken of the same order.

Theorem 3.8. Assume the dimension is d=2 or d=3. Let x,s be as in Proposition 3.5, E be a minimizer of (2) and assume $x \in \partial^* E$. Then x is a Lebesgue point of z and

$$z(x) = \nabla_p F(x, \nu^E)$$
.

Proof. We divide the proof into two steps.

Step 1. We first consider anisotropies F which are not depending on the x variable. Without loss of generality we assume x = 0. By assumption, there exists the limit

$$\overline{\nu} = \lim_{\rho \to 0} \frac{D\chi_E(B_\rho(0))}{|D\chi_E|(B_\rho(0))|} \tag{11}$$

and, without loss of generality, we assume that it coincides with the vector e_d corresponding to the last coordinate of $y \in \mathbb{R}^d$.

Also, if we let $E_{\rho} = E/\rho$, the sets E_{ρ} , E_{ρ}^{c} , ∂E_{ρ} converge in $B_{1}(0)$, in the Hausdorff sense (thanks to the uniform density estimates), respectively to $\{y_{d} \geq 0\}$, $\{y_{d} = 0\}$, $\{y_{d} \leq 0\}$. We also let $z_{\rho}(y) = z(\rho y)$ and $g_{\rho}(y) = g(\rho y)$, in particular $-\text{div }z_{\rho} = \rho g_{\rho}$. We let

$$\omega(\rho) = \sup_{x \in \Omega} \|g\|_{L^d(B_{\rho}(x) \cap \Omega)}$$
(12)

which is continuously increasing and goes to 0 as $\rho \to 0$, since $|g|^d$ is equi-integrable. We introduce the following notation: a point in \mathbb{R}^d is denoted by $y = (y', y_d)$, with $y' \in \mathbb{R}^{d-1}$.

We let $D_s := \{|y'| \le s\}$, $\bar{z} := \nabla F(\overline{\nu})$ and $D_s^t = \{D_s + \lambda \bar{z} : |\lambda| \le t\}$ and denote with ∂D_s the relative boundary of D_s in $\{y_d = 0\}$.

We choose $s \leq 1$, $0 < t \leq s$, (t is chosen small enough so that $D_s^t \subset B_1(0)$, that is $t < (1/|\bar{z}|)\sqrt{1-s^2}$). We integrate in D_s^t the divergence $\rho g_\rho = -\text{div } z_\rho = \text{div } (\bar{z} - z_\rho)$ against the function $(2\chi_E - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})}$, which vanishes for $y_d = \pm tF(\overline{\nu})$ if ρ is small enough (given t > 0), so that $\partial E_\rho \cap B_1(0) \subset \{|y_d| \leq tF(\overline{\nu})\}$. For y on the lateral boundary of the cylinder D_s^t , let $\xi(y)$ be the internal normal to $\partial D_s + (-t, t)\bar{z}$ at the point y. Using the fact that z_ρ is a calibration for E_ρ , we easily get that for almost all s,

$$\int_{D_s^t} \rho g_{\rho} \left((2\chi_E - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) dy$$

$$= \int_{\partial D_s + (-t,t)\bar{z}} \left((2\chi_E - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) \left[(\bar{z} - z_{\rho}), \xi(y) \right] d\mathcal{H}^{d-1}$$

$$- 2t \int_{\partial E_{\rho} \cap D_s^t} \left(\bar{z} \cdot \nu^{E_{\rho}} - F(\nu^{E_{\rho}}) \right) d\mathcal{H}^{d-1} + \int_{D_s^t} \left(1 - \frac{z_{\rho} \cdot \overline{\nu}}{F(\overline{\nu})} \right) dy . \quad (13)$$

Now since $F^{\circ}(\nabla F(\overline{\nu})) = 1$, there holds $\overline{z} \cdot \nu^{E_{\rho}} - F(\nu^{E_{\rho}}) \leq 0$ and using that $\overline{z} \cdot \xi(y) = 0$ on $\partial D_s + (-t, t)\overline{z}$, we get

$$\int_{D_{s}^{t}} \left(1 - \frac{z_{\rho} \cdot \overline{\nu}}{F(\overline{\nu})} \right) dy \leq \int_{D_{s}^{t}} \rho g_{\rho} \left((2\chi_{E} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) dy$$

$$\int_{\partial D_{s} + (-t,t)\overline{z}} \left((2\chi_{E} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) z_{\rho} \cdot \xi(y) d\mathcal{H}^{d-1} . \quad (14)$$

We claim that for $|\xi| \leq 1$ with $\xi \cdot \bar{z} = 0$, there holds

$$(\xi \cdot z_o)^2 \le C(F(\overline{\nu}) - \overline{\nu} \cdot z_o) \tag{15}$$

Since

$$(\xi \cdot z_{\rho})^2 \le |z_{\rho}|^2 - [z_{\rho} \cdot (\bar{z}/|\bar{z}|)]^2$$

it is enough to prove

$$|z_{\rho}|^2 - [z_{\rho} \cdot (\bar{z}/|\bar{z}|)]^2 \le C(F(\overline{\nu}) - \overline{\nu} \cdot z_{\rho}).$$

Using that $\overline{\nu}/F(\overline{\nu}) = \nabla F^{\circ}(\overline{z})$, from (5) applied to F° together with $F^{\circ}(\overline{z}) = 1 \geq F^{\circ}(z_{\rho})$, we find

$$(F(\overline{\nu}) - \overline{\nu} \cdot z_{\rho}) = F(\overline{\nu})(1 - z_{\rho} \cdot \nabla F^{\circ}(\overline{z})) \ge C|z_{\rho} - \overline{z}|^{2}.$$

which readily implies (15). We thus have

$$\int_{\partial D_{s}+(-t,t)\bar{z}} \left((2\chi_{E_{\rho}} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) (z_{\rho} \cdot \xi) d\mathcal{H}^{d-1}
\leq 2C\sqrt{F(\overline{\nu})}t \int_{\partial D_{s}+(-t,t)\bar{z}} \sqrt{1 - \frac{z_{\rho} \cdot \overline{\nu}}{F(\overline{\nu})}} d\mathcal{H}^{d-1}
\leq 2CF(\overline{\nu})t\sqrt{t} \left(\int_{\partial D_{s}+(-t,t)\bar{z}} \left(1 - \frac{z_{\rho} \cdot \overline{\nu}}{F(\overline{\nu})} \right) d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_{s})}. (16)$$

Now, we also have

$$\rho \int_{D_s^t} \left((2\chi_{E_\rho} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) g_\rho(y) \, dy \leq 2t\rho^{1-d} \int_{D_{\rho s}^{\rho t}} g(x) \, dx \\
\leq 2t\rho^{1-d} \|g\|_{L^d(B_{\rho s}(0))} |D_{\rho s}^{\rho t}|^{1-1/d} \leq ct^{2-1/d} s^{d-2+1/d} \omega(\rho s) \quad (17)$$

where here, $c = 2\mathcal{H}^{d-1}(D_1)^{1-1/d}$, and ω is defined in (12).

We choose a < 1, close to 1, and choose $t \in (0, (1/|\bar{z}|)\sqrt{1-a^2})$. If $\rho > 0$ is small enough (so that $\partial E_{\rho} \cap B_1$ is in $\{|y_d| \le tF(\overline{\nu})\}$), letting $f(s) := \int_{D_s^t} \left(1 - \frac{z_{\rho} \cdot \overline{\nu}}{F(\overline{\nu})}\right) dy$, we deduce from (14), (16) and (17) that for a.e. s with $t \le s \le a$, one has (possibly increasing the constant c)

$$f(s)^{2} \leq c \left(s^{d-2} t^{3} f'(s) + t^{4-2/d} s^{2d-4+2/d} \omega(\rho s)^{2} \right). \tag{18}$$

Unfortunately, this estimate does not give much information for d > 3. It seems it allows to conclude only whenever $d \in \{2,3\}$. Since the case d = 2 is simpler, we focus on d = 3. Estimate (18) becomes

$$f(s)^2 \le c \left(st^3 f'(s) + t^{10/3} s^{8/3} \omega(\rho s)^2 \right).$$
 (19)

Given M > 0, we fix a value t > 0 such that $\log(a/t) \ge cM$. If ρ is chosen small enough, then $\partial E_{\rho} \cap B_1(0) \subset \{|y_d| < tF(\overline{\nu})\}$, and (19) holds. It yields (assuming f(t) > 0, but if not, then the Proposition is proved)

$$-\frac{f'(s)}{f(s)^2} + \frac{1}{ct^3} \frac{1}{s} \le ct^{1/3} s^{5/3} \frac{\omega(\rho s)^2}{f(s)^2} \le ct^{1/3} s^{5/3} \frac{\omega(a\rho)^2}{f(t)^2}$$
 (20)

where we have used the fact that $t \leq s \leq a$ and f, ω are nondecreasing. Integrating (20) from t to a, after multiplication by t^3 we obtain

$$\frac{t^3}{f(a)} - \frac{t^3}{f(t)} + \frac{\log(a/t)}{c} \le \frac{3c}{8} t^{10/3} (a^{8/3} - t^{8/3}) \frac{\omega(a\rho)^2}{f(t)^2}.$$

Hence we get

$$\frac{t^3}{f(t)} + ca^{8/3}t^{-8/3}\omega(a\rho)^2 \frac{t^6}{f(t)^2} \ge M. \tag{21}$$

Eventually, we observe that

$$f(t) = \int_{D_t^t} \left(1 - \frac{z(\rho y) \cdot \overline{\nu}}{F(\overline{\nu})} \right) dy = \frac{1}{\rho^d} \int_{D_{at}^{\rho t}} \left(1 - \frac{z(x) \cdot \overline{\nu}}{F(\overline{\nu})} \right) dx \,,$$

so that (21) can be rewritten

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x) \cdot \overline{\nu}}{F(\overline{\nu})}\right) dx}{(\rho t)^3}\right)^{-1} \ge \frac{-1 + \sqrt{1 + 4Mca^{8/3}t^{-8/3}\omega(a\rho)^2}}{2ca^{8/3}t^{-8/3}\omega(a\rho)^2} \tag{22}$$

The value of t being fixed, we can choose the value of ρ small enough in order to have $4Mca^{8/3}t^{-8/3}\omega(a\rho)^2<1$, and (using $\sqrt{1+X}\geq 1+X/2-X^2/8$ if $X\in(0,1)$), (22) yields

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x) \cdot \overline{\nu}}{F(\overline{\nu})}\right) dx}{(\rho t)^3}\right)^{-1} \ge M - M^2 c a^{8/3} t^{-8/3} \omega (a\rho)^2 \ge \frac{3}{4} M.$$
(23)

It follows that

$$\limsup_{\varepsilon \to 0} \frac{\int_{D_{\varepsilon}^{\varepsilon}} \left(1 - \frac{z(x) \cdot \overline{\nu}}{F(\overline{\nu})}\right) dx}{\varepsilon^{3}} \le \frac{4}{3} M^{-1}$$
 (24)

and since M is arbitrary, 0 is indeed a Lebesgue point of z, with value $\bar{z} = \nabla F(\overline{\nu})$ (recall that $1 - \frac{z(x) \cdot \overline{\nu}}{F(\overline{\nu})} \ge (C/F(\overline{\nu}))|z(x) - \bar{z}|^2$).

Step 2. When F depends also on the x variable, the proof follows along the same lines as in Step 1, taking into account the errors terms in (14) and (16). Keeping the same notations as in Step 1 and setting $\bar{z} := \nabla_p F(0, \bar{\nu})$ we find that since $F^{\circ}(0, \bar{z}) \leq 1$, there holds $\bar{z} \cdot \nu^{E_{\rho}} \leq F(0, \nu^{E_{\rho}})$ and thus

$$\int_{\partial E_{\rho} \cap D_{z}^{t}} \bar{z} \cdot \nu^{E_{\rho}} - F(\rho x, \nu^{E_{\rho}}) d\mathscr{H}^{d-1} \leq \int_{\partial E_{\rho} \cap D_{z}^{t}} |F(0, \nu^{E_{\rho}}) - F(\rho x, \nu^{E_{\rho}})| d\mathscr{H}^{d-1} \leq C \rho s^{d-1}$$

where the last inequality follows from $t \leq s$ and the minimality of E_{ρ} inside D_s^t . Now since

$$(F^{\circ})^{2}(0,z_{\rho}) - (F^{\circ})^{2}(\rho x,z_{\rho}) \ge (F^{\circ})^{2}(0,z_{\rho}) - 1 \ge 2\frac{\bar{\nu}}{F(0,\bar{\nu})} \cdot (z_{\rho}-z) + \delta^{2}|z_{\rho}-z|^{2}$$

we find that (15) transforms into,

$$(\xi \cdot z_{\rho})^{2} \le C \left[(F(0, \bar{\nu}) - \bar{\nu} \cdot z_{\rho}) + ((F^{\circ})^{2} (0, z_{\rho}) - (F^{\circ})^{2} (\rho x, z_{\rho})) \right]$$

for every $|\xi| \leq 1$ and $\xi \cdot \bar{z} = 0$, from which we get

$$\begin{split} \int_{\partial D_s + (-t,t)\bar{z}} \left((2\chi_{E_\rho} - 1)t - \frac{\overline{\nu} \cdot y}{F(\overline{\nu})} \right) & (z_\rho \cdot \xi) \ d\mathcal{H}^{d-1} \\ & \leq 2CF(0,\overline{\nu})t\sqrt{t} \left(\int_{\partial D_s + (-t,t)\bar{z}} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0,\overline{\nu})} \right) \ d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_s)} \\ & + 2Ct \int_{\partial D_s + (-t,t)\bar{z}} \left| (F^\circ)^2 \left(0, z_\rho \right) - (F^\circ)^2 \left(\rho x, z_\rho \right) \right|^{1/2} \ d\mathcal{H}^{d-1} \\ & \leq CF(0,\overline{\nu})t\sqrt{t} \left(\int_{\partial D_s + (-t,t)\bar{z}} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0,\overline{\nu})} \right) \ d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_s)} + Ct\rho^{1/2} s^{d-1} t \,. \end{split}$$

Using these estimates, we finally get that, setting as before $f(s) := \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \overline{\nu}}{F(0, \overline{\nu})}\right) dy$, there holds

$$f(s)^2 \ \leq \ c \left(s^{d-2} t^3 f'(s) \ + \ t^{4-2/d} s^{2d-4+2/d} \omega(\rho s)^2 + \rho t s^{d-1} + \rho^{1/2} t^2 s^{d-1} \right) \, .$$

From this inequality, the proof can be concluded exactly as in *Step 1*.

Eventually, we can also give a locally uniform convergence result.

Proposition 3.9. For all $x \in \Omega$ we let

$$z_{\rho}(x) := \frac{1}{|B_{\rho}(0)|} \int_{B_{\rho}(x) \cap \Omega} z(y) \, dy \, .$$

Then, $F^{\circ}(x, z_{\rho}(x)) \to 1$ locally uniformly on S.

Proof. Given $K \subset \Omega$ a compact set, we can check that for any t > 0, there exists $\rho_0 > 0$ such that for any $x \in K \cap S$, if E^x is the level set of u through x, then for any $\rho \leq \rho_0$, the boundary of $(E^x - x)/\rho \cap B_1(0)$ lies in a strip of width 2t, that is, there is $\overline{\nu}^x \in \mathbb{S}^{d-1}$ with $\partial((E^x - x)/\rho) \cap B_1(0) \subset \{|y \cdot \overline{\nu}^x| \leq t\}$.

Indeed, if this is not the case, one can find t > 0, $\rho_k \to 0$, $x_k \in K \cap S$, such that $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ is not contained in any strip of width 2t. Up to a subsequence we may assume that $x_k \to x \in K \cap S$, and from the bound on the perimeter, that $(E^{x_k} - x_k)/\rho_k \cap B_1(0)$ converges to a local minimizer of $\int_{\partial E} F(0, \nu^E) d\mathcal{H}^{d-1}$ and is thus a halfspace.² Moreover, $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ converges in the Hausdorff sense (thanks to the density estimates) to a hyperplane. We easily obtain a contradiction.

The thesis follows when we observe that the proof of Proposition 3.8 can be reproduced by replacing the direction $\nu^{E^x}(x)$ (which exists only if x lies in the reduced boundary of E^x) with the direction $\overline{\nu}^x$ given above.

²If d = 2, this Bernstein result readily follows from the strict convexity of F, see [11, Prop 3.6] whereas for d = 3, see [18]. In the case of the area i.e when F(x, Du) = |Du| and $d \le 7$, see also [12, Rem 3.2].

3.3 A counterexample.

We provide an example where $g \in L^{d-\varepsilon}(\Omega)$, with $\varepsilon > 0$ arbitrarily small, and Theorem 3.8 does not hold.

Let $\Omega = B_1(0)$ be the unit ball of \mathbb{R}^d and let $E = \Omega \cap \{x_d \leq 0\}$. We shall construct a vector field $z : \Omega \to \mathbb{R}^d$ such that $z = \nu^E$ on $\partial E \cap \Omega$, $|z| \leq 1$ everywhere in Ω , div $z \in L^{d-\varepsilon}(\Omega)$, but 0 is not a Lebesgue point of z. Notice that E minimizes the functional (3) with g = div z. Letting $r_n \to 0$ be a decreasing sequence to be determined later, and let $B_n = B_{r_n}(x_n)$ with $x_n = 2r_n e_d$. Without loss of generality, we may assume $r_{n+1} < r_n/4$ so that the balls B_n are all disjoint. We define the vector field z as follows: $z(x) = e_d$ if $x \in \Omega \setminus \bigcup_n B_n$, and $z(x) = |x - x_n|e_d$ if $x \in B_n$. It follows that div z = 0 in $\Omega \setminus \bigcup_n B_n$ and $|\text{div } z| \leq 1/r_n$ in B_n , so that

$$\int_{\Omega} |\operatorname{div} z|^{d-\varepsilon} dx = \sum_{n} \int_{B_{n}} |\operatorname{div} z|^{d-\varepsilon} dx \le \omega_{d} \sum_{n} r_{n}^{\varepsilon} < +\infty$$

if we choose r_n converging to zero sufficiently fast, so that $g = -\text{div } z \in L^{d-\varepsilon}(\Omega)$. However, since $z \cdot e_d \leq 1/2$ in $B_{r_n/2}(x_n)$, we also have

$$\int_{B_{3r_n}(0)} z \cdot e_d \, dx \le |B_{3r_n}(0)| - \frac{1}{2} \left| B_{r_n/2}(x_n) \right|$$

so that

$$\frac{1}{|B_{3r_n}(0)|} \int_{B_{3r_n}(0)} z \cdot e_d \, dx \le 1 - \frac{1}{6^d} < 1 \,.$$

On the other hand, for $\delta \in (0, 1/6^d)$ we have

$$\frac{1}{|B_{r_n}(0)|} \int_{B_{r_n}(0)} z \cdot e_d \, dx \ge \frac{1}{|B_{r_n}(0)|} \left(|B_{r_n}(0)| - \sum_{i=n+1}^{\infty} |B_{r_i}(x_i)| \right) \ge 1 - \delta \,,$$

if we take the sequence r_n converging to 0 sufficiently fast. It follows that 0 is not a Lebesgue point of z.

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