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Fine properties of the subdifferential for a class of one-homogeneous functionals

A. Chambolle ^{*} M. Goldman [†] M. Novaga [‡]

Abstract

We collect here some known results on the subdifferential of one-homogeneous functionals, which are anisotropic and nonhomogeneous variants of the total variation, and establish a new relationship between Lebesgue points of the calibrating field and regular points of the level lines of the corresponding calibrated function.

1 Introduction

In this note we recall some classical results on the structure of the subdifferential of first order one-homogeneous functionals, and we give new regularity results which extend and precise previous work by G. Anzellotti [5, 6, 7].

Given an open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, and a function $u \in C^1(\Omega) \cap BV(\Omega)$, we consider the functional

$$J(u) := \int_{\Omega} F(x, Du)$$

where $F : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty)$ is continuous in x and $F(x, \cdot)$ is a smooth and uniformly convex norm on \mathbb{R}^d , for all $x \in \Omega$.

Since $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$, it is natural to consider J as a convex, l.s.c. function on the whole of $L^{d/(d-1)}(\Omega)$, with value $+\infty$ when $u \notin BV(\Omega)$ (see [2]). In this framework, for any $u \in L^{d/(d-1)}(\Omega)$ we can define the subgradient of a u in the duality $(L^{d/(d-1)}, L^d)$ as

$$\partial J(u) = \left\{ g \in L^d(\Omega) : J(v) \geq J(u) + \int_{\Omega} g(x)(v(x) - u(x)) dx \quad \forall v \in L^{d/(d-1)}(\Omega) \right\}.$$

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The goal of this paper is to investigate the particular structure of the functions u and $g \in \partial J(u)$, when the subgradient is nonempty. Since J can be defined by duality as

$$J(u) = \sup \left\{ - \int_{\Omega} u(x) \operatorname{div} z(x) dx : z \in C_c^{\infty}(\Omega; \mathbb{R}^d), F^*(x, z(x)) = 0 \ \forall x \in \Omega \right\}$$

where $F^*(x, \cdot)$ is the Legendre-Fenchel transform of $F(x, \cdot)$ (it is equivalent to require that $F^{\circ}(x, z(x)) \leq 1$, $F^{\circ}(x, \cdot)$ being the convex polar of F defined in (4)), it is easy to see that such a g has necessarily the form $g = -\operatorname{div} z$, for some field $z \in L^{\infty}(\Omega; \mathbb{R}^d)$ with $F^*(x, z(x)) = 0$ a.e. in Ω .

Since by a formal integration by parts one gets $z \cdot Du = F(x, Du)$, $|Du|$ -a.e., natural questions are: in what sense can this relation be true? can one assign a precise value to z on the support of the measure Du ?

The first question has been answered by Anzelotti in the series of papers [5, 6, 7]. However, for the particular vector fields we are interested in, we can be more precise and obtain pointwise properties of z on the level sets of the function u . Indeed, we shall show that z has a pointwise meaning on all level sets of u , up to \mathcal{H}^{d-1} -negligible sets (which is much more than $|Du|$ -a.e., as illustrated by the function $u = \sum_{n=1}^{+\infty} 2^{-n} \chi_{(0, x_n)}$, defined in the interval $(0, 1)$, with (x_n) a dense sequence in that interval).

We will therefore focus on the properties of the vector fields $z \in L^{\infty}(\Omega, \mathbb{R}^d)$ such that $F^*(x, z(x)) = 0$ a.e. in Ω and $g = -\operatorname{div} z \in L^d(\Omega)$, and such that there exists a function u such that for any $\phi \in C_c^{\infty}(\Omega)$,

$$- \int_{\Omega} \operatorname{div} z(x) u(x) \phi(x) dx = \int_{\Omega} u(x) z(x) \cdot \nabla \phi(x) dx + \int_{\Omega} \phi(x) F(x, Du).$$

In particular, one checks easily that u minimizes the functional

$$\int_{\Omega} F(x, Du) - \int_{\Omega} g(x) u(x) dx \tag{1}$$

among perturbations with compact support in Ω . Conversely, given $g \in L^d(\Omega)$ with $\|g\|_{L^d}$ sufficiently small, there exist functions u which minimize (1) under various types of boundary conditions, and corresponding fields z .

This kind of functionals appears in many contexts including image processing and plasticity [4, 17]. Notice also that, by the Coarea Formula [2], it holds

$$\int_{\Omega} F(x, Du) - \int_{\Omega} g u dx = \int_{\mathbb{R}} \left(\int_{\partial^* \{u > s\}} F(x, \nu) - \int_{\{u > s\}} g dx \right) ds,$$

where ν is the unit normal to $\{u > s\}$, and one can show (see for instance [10]) that the characteristic function of any level set of the form $\{u > s\}$ or $\{u \geq s\}$ is a minimizer of the geometric functional

$$\int_{\partial^* E} F(x, \nu) - \int_E g(x) dx. \tag{2}$$

The canonical example of such functionals is given by the total variation, corresponding to $F(x, Du) = |Du|$. In this case, (2) boils down to

$$P(E) - \int_E g(x) dx. \quad (3)$$

In [8], it is shown that every set with finite perimeter in Ω is a minimizer of (3) for some $g \in L^1(\Omega)$. However, if $g \in L^p(\Omega)$ with $p > d$, and E is a minimizer of (2), then ∂E is locally $C^{1,\alpha}$ for some $\alpha > 0$, out of a closed singular set of zero \mathcal{H}^{d-3} -measure [1]. When $g \in L^d(\Omega)$, the boundary ∂E is only of class C^α out of the singular set (see [3]). Since the Euler-Lagrange equation of (2) relates z to the normal to E , understanding the regularity of z is closely related to understanding the regularity of ∂E .

Our main result is that the Lebesgue points of z correspond to regular points of $\partial\{u > s\}$ or $\partial\{u \geq s\}$ (Theorem 3.7), and that the converse is true in dimension $d \leq 3$ (Theorem 3.8).

2 Preliminaries

2.1 BV functions

We briefly recall the definition of function of bounded variation and set of finite perimeter. For a complete presentation we refer to [2].

Definition 2.1. *Let Ω be an open set of \mathbb{R}^d , we say that a function $u \in L^1(\Omega)$ is a function of bounded variation if*

$$\int_{\Omega} |Du| := \sup_{\substack{z \in C_c^1(\Omega) \\ |z|_\infty \leq 1}} \int_{\Omega} u \operatorname{div} z \, dx < +\infty.$$

We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω (when $\Omega = \mathbb{R}^d$ we simply write BV instead of $BV(\mathbb{R}^d)$).

We say that a set $E \subset \mathbb{R}^d$ is of finite perimeter if its characteristic function χ_E is of bounded variation and denote its perimeter in an open set Ω by $P(E, \Omega) := \int_{\Omega} |D\chi_E|$, and write simply $P(E)$ when $\Omega = \mathbb{R}^d$.

Definition 2.2. *Let E be a set of finite perimeter and let $t \in [0; 1]$. We define*

$$E^{(t)} := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}.$$

We denote by $\partial E := (E^{(0)} \cup E^{(1)})^c$ the measure theoretical boundary of E . We define the reduced boundary of E by:

$$\partial^* E := \left\{ x \in \operatorname{Spt}(|D\chi_E|) : \nu^E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ exists and } |\nu^E(x)| = 1 \right\} \subset E^{(\frac{1}{2})}.$$

The vector $\nu^E(x)$ is the measure theoretical inward normal to the set E .

Proposition 2.3. *If E is a set of finite perimeter then $D\chi_E = \nu^E \llcorner \mathcal{H}^{d-1} \llcorner \partial^* E$, $P(E) = \mathcal{H}^{d-1}(\partial^* E)$ and $\mathcal{H}^{d-1}(\partial E \setminus \partial^* E) = 0$.*

Definition 2.4. *We say that x is an approximate jump point of $u \in BV(\Omega)$ if there exist $\xi \in \mathbb{S}^{d-1}$ and distinct $a, b \in \mathbb{R}$ such that*

$$\lim_{\rho \rightarrow 0} \frac{1}{|B_\rho^+(x, \xi)|} \int_{B_\rho^+(x, \xi)} |u(y) - a| dy = 0 \quad \text{and} \quad \lim_{\rho \rightarrow 0} \frac{1}{|B_\rho^-(x, \xi)|} \int_{B_\rho^-(x, \xi)} |u(y) - b| dy = 0,$$

where $B_\rho^\pm(x, \xi) := \{y \in B_\rho(x) : \pm(y - x) \cdot \xi > 0\}$. Up to a permutation of a and b and a change of sign of ξ , this characterizes the triplet (a, b, ξ) which is then denoted by (u^+, u^-, ν_u) . The set of approximate jump points is denoted by J_u .

The following proposition can be found in [2, Proposition 3.92].

Proposition 2.5. *Let $u \in BV(\Omega)$. Then, defining*

$$\Theta_u := \{x \in \Omega / \liminf_{\rho \rightarrow 0} \rho^{1-d} |Du|(B_\rho(x)) > 0\},$$

there holds $J_u \subset \Theta_u$ and $\mathcal{H}^{d-1}(\Theta_u \setminus J_u) = 0$.

2.2 Anisotropies

Let $F(x, p) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex one-homogeneous function in the second variable such that there exists c_0 with

$$c_0 |p| \leq F(x, p) \leq \frac{1}{c_0} |p| \quad \forall (x, p) \in \mathbb{R}^d \times \mathbb{R}^d.$$

We say that F is uniformly elliptic if for some $\delta > 0$, the function $p \mapsto F(p) - \delta |p|$ is still a convex function. We define the polar function of F by

$$F^\circ(x, z) := \sup_{\{F(x, p) \leq 1\}} z \cdot p \tag{4}$$

so that $(F^\circ)^\circ = F$. It is easy to check that (* denoting the Legendre-Fenchel convex conjugate) $[F(x, \cdot)^2/2]^* = F^\circ(x, \cdot)^2/2$, in particular (if differentiable), $F(x, \cdot) \nabla_p F(x, \cdot)$ and $F^\circ(x, \cdot) \nabla_z F^\circ(x, \cdot)$ are inverse monotone operators. If we denote by F^* the convex conjugate of F with respect to the second variable, then $F^*(x, z) = 0$ if and only if $F^\circ(x, z) \leq 1$.

If $F(x, \cdot)$ is differentiable then, for every $p \in \mathbb{R}^d$,

$$F(x, p) = p \cdot \nabla_p F(x, p) \quad (\text{Euler's identity})$$

and

$$z \in \{F^\circ(x, \cdot) \leq 1\} \text{ with } p \cdot z = F(x, p) \iff z = \nabla_p F(x, p).$$

If F is elliptic and of class $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$, then F° is also elliptic and $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\})$. We will then say that F is a smooth elliptic anisotropy. Observe that, in this case, the function $F^2/2$ is also uniformly δ^2 -convex (this follows from the inequalities $D^2F(x, p) \geq \delta/|p|(I - p \otimes p/|p|^2)$ and $F(x, p) \geq \delta|p|$). In particular, for every $x, y, z \in \mathbb{R}^d$, there holds

$$F^2(x, y) - F^2(x, z) \geq 2(F(x, z)\nabla_p F(x, z)) \cdot (y - z) + \delta^2|y - z|^2, \quad (5)$$

and a similar inequality holds for F° . We refer to [16] for general results on convex norms and convex bodies.

2.3 Pairings between measures and bounded functions

Following [5] we define a generalized trace $[z, Du]$ for functions u with bounded variation and bounded vector fields z with divergence in L^d .

Definition 2.6. *Let Ω be an open set with Lipschitz boundary, $u \in BV(\Omega)$ and $z \in L^\infty(\Omega, \mathbb{R}^d)$ with $\operatorname{div} z \in L^d(\Omega)$. We define the distribution $[z, Du]$ by*

$$\langle [z, Du], \psi \rangle = - \int_{\Omega} u \psi \operatorname{div} z - \int_{\Omega} u z \cdot \nabla \psi \quad \forall \psi \in \mathcal{C}_c^\infty(\Omega).$$

Proposition 2.7. *The distribution $[z, Du]$ is a bounded Radon measure on Ω and if ν is the inward unit normal to Ω , there exists a function $[z, \nu] \in L^\infty(\partial\Omega)$ such that the generalized Green's formula holds,*

$$\int_{\Omega} [z, Du] = - \int_{\Omega} u \operatorname{div} z - \int_{\partial\Omega} [z, \nu] u \, d\mathcal{H}^{d-1}.$$

The function $[z, \nu]$ is the generalized (inward) normal trace of z on $\partial\Omega$.

Given $z \in L^\infty(\Omega, \mathbb{R}^d)$, with $\operatorname{div} z \in L^d(\Omega)$, we can also define the generalized trace of z on ∂E , where E is a set of locally finite perimeter. Indeed, for every bounded open set Ω with Lipschitz boundary, we can define as above the measure $[z, D\chi_E]$ on Ω . Since this measure is absolutely continuous with respect to $|D\chi_E| = \mathcal{H}^{d-1} \llcorner \partial^* E$ we have

$$[z, D\chi_E] = \psi_z(x) \mathcal{H}^{d-1} \llcorner \partial^* E$$

with $\psi_z \in L^\infty(\partial^* E)$ independent of Ω . We denote by $[z, \nu^E] := \psi_z$ the generalized (inward) normal trace of z on ∂E . If E is a bounded set of finite perimeter, by taking Ω strictly containing E , we have the generalized Gauss-Green Formula

$$\int_E \operatorname{div} z = - \int_{\partial^* E} [z, \nu^E] \, d\mathcal{H}^{d-1}.$$

Anzellotti proved the following alternative definition of $[z, \nu^E]$ [6, 7]

Proposition 2.8. *Let $(x, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$. For any $r > 0$, $\rho > 0$ we let*

$$C_{r,\rho}(x, \alpha) := \{\xi \in \mathbb{R}^d : |(\xi - x) \cdot \alpha| < r, |(\xi - x) - [(\xi - x) \cdot \alpha]\alpha| < \rho\}.$$

There holds

$$[z, \alpha](x) = \lim_{\rho \rightarrow 0} \lim_{r \rightarrow 0} \frac{1}{2r\omega_{d-1}\rho^{d-1}} \int_{C_{r,\rho}(x, \alpha)} z \cdot \alpha$$

where ω_{d-1} is the volume of the unit ball in \mathbb{R}^{d-1} .

3 The subdifferential of anisotropic total variations

3.1 Characterization of the subdifferential

The following characterization of the subdifferential of J is classical and readily follows for example from the representation formula [9, (4.19)].

Proposition 3.1. *Let F be a smooth elliptic anisotropy and $g \in L^d(\Omega)$ then u is a local minimizer of (1) if and only if there exists $z \in L^\infty(\Omega)$ with $\operatorname{div} z = g$, $F^*(x, z(x)) = 0$ a.e. and*

$$[z, Du] = F(x, Du).$$

Moreover, for every $t \in \mathbb{R}$, for the set $E = \{u > t\}$ there holds $[z, \nu^E] = F(x, \nu^E)$ \mathcal{H}^{d-1} -a.e. on ∂E . We will say that such a vector field is a calibration of the set E for the minimum problem (2).

Remark 3.2. In [5], it is proven that if $z_\rho(x) := \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} z(y) dy$, then $z_\rho \cdot \nu^E$ weakly* converges to $[z, \nu^E]$ in $L^\infty_{loc}(\mathcal{H}^{d-1} \llcorner \partial^* E)$. Using (5) it is then possible to prove that if z calibrates E then z_ρ converges to $\nabla_p F(x, \nu^E)$ in $L^2(\mathcal{H}^{d-1} \llcorner \partial^* E)$ yielding that up to a subsequence, $z_{\phi(\rho)}$ converges \mathcal{H}^{d-1} -a.e. to $\nabla_p F(x, \nu^E)$. Unfortunately this is still a very weak statement since it is a priori impossible to recover from this the convergence of the full sequence z_ρ .

The main question we want to investigate now is whether we can give a classical meaning to $[z, \nu^E]$ (that is understand if $[z, \nu^E] = z \cdot \nu^E$). We observe that a priori the value of z is not well defined on ∂E which has zero Lebesgue measure (since z has Lebesgue points only a.e.). We let $S := \operatorname{supp}(Du) \subset \Omega$ be the smallest closed set in Ω such that $|Du|(\Omega \setminus S) = 0$. The next result is classical.

Lemma 3.3 (Density estimate). *There exists $\rho_0 > 0$ (depending on g) and a constant $\gamma > 0$ (which depends only on d), such that for any $B_\rho(x) \subset \Omega$ with $\rho \leq \rho_0$, and any level set E of u (that is, $E \in \{\{u > s\}, \{u \geq s\}, \{u < s\}, \{u \leq s\}, s \in \mathbb{R}\}$), if $|B_\rho(x) \cap E| < \gamma|B_\rho(x)|$ then $|B_{\rho/2}(x) \cap E| = 0$. As a consequence, E^0 and E^1 are open, ∂E is the topological boundary of E^1 , and (possibly changing slightly γ) if $x \in \partial E$, then $\mathcal{H}^{d-1}(\partial E \cap B_\rho(x)) \geq \gamma\rho^{d-1}$.*

For a proof we refer to [13, 12]. This is not true anymore if $g \notin L^d(\Omega)$ [12]. If $\partial\Omega$ is Lipschitz, it is true up to the boundary.

Corollary 3.4. *It follows that $u \in L_{loc}^\infty(\Omega)$ and $u \in C(\Omega \setminus \Theta_u)$.*

Proof. For any ball $B_\rho(x) \subset \Omega$ and $\inf_{B_{\rho/2}(x)} u < a < b < \sup_{B_{\rho/2}(x)} u$, one has

$$+\infty > |Du|(B_\rho(x)) \geq \int_a^b P(\{u > s\}, B_\rho(x)) ds \geq (b-a)\gamma \left(\frac{\rho}{2}\right)^{d-1},$$

so that $\text{osc}_{B_{\rho/2}(x)}(u)$ must be bounded and thus $u \in L_{loc}^\infty(\Omega)$. Moreover, if $x \in \Omega \setminus \Theta_u$ we find that $\lim_{\rho \rightarrow 0} \text{osc}_{B_\rho(x)}(u) = 0$ so that u is continuous at the point x . \square

It also follows from Lemma 3.3 that all points in the support of Du must be on the boundary of a level set of u :

Proposition 3.5. *For any $x \in S$, there exists $s \in \mathbb{R}$ such that either $x \in \partial\{u > s\}$ or $x \in \partial\{u \geq s\}$.*

Proof. First, if $x \notin S$ then $|Du|(B_\rho(x)) = 0$ for some $\rho > 0$ and clearly x cannot be on the boundary of a level set of u . On the other hand, if $x \in S$, then for any ball $B_{1/n}(x)$ (n large) there is a level s_n (uniformly bounded) with $\partial\{u > s_n\} \cap B_{1/n}(x) \neq \emptyset$ and by Hausdorff convergence, we deduce that either $x \in \partial\{u > s\}$ or $x \in \partial\{u \geq s\}$ where s is the limit of the sequence $(s_n)_n$ (which must actually converge). \square

The following stability property is classical (see e.g. [11]).

Proposition 3.6. *Let E_n be local minimizers of (2), with a function $g = g_n \in L^d(\Omega)$, and converging in the L^1 -topology to a set E . Assume that the sets E_n are calibrated by z_n , that $z_n \xrightarrow{*} z$ weakly-* in L^∞ and $g_n \rightarrow g = -\text{div } z \in L^d(\Omega)$, in $L^1(\Omega)$ as $n \rightarrow \infty$. Then z calibrates E , which is thus also a minimizer of (2).*

In particular, one must notice that when $z_n \xrightarrow{*} z$ and $F^\circ(x, z_n) \leq 1$ a.e., then in the limit one still has $F^\circ(x, z) \leq 1$ a.e. (thanks to the convexity, and continuity w.r. the variable x).

3.2 The Lebesgue points of the calibration.

The next result shows that the regularity of the calibration z implies some regularity of the calibrated set.

Theorem 3.7. *Let $\bar{x} \in \partial E$ be a Lebesgue point of z , with $E = \{u > t\}$ or $E = \{u \geq t\}$. Then, $\bar{x} \in \partial^* E$ and*

$$z(\bar{x}) = \nabla_p F(\bar{x}, \nu^E(\bar{x})). \quad (6)$$

Proof. We follow [11, Th. 4.5] and let $z_\rho(y) := z(\bar{x} + \rho y)$. Since \bar{x} is a Lebesgue point of z , we have that $z_\rho \rightarrow \bar{z}$ in $L^1(B_R)$, hence also weakly-* in $L^\infty(B_R)$ for any $R > 0$, where $\bar{z} \in \mathbb{R}^d$ is a constant vector.

We let $E_\rho := (E - \bar{x})/\rho$ and $g_\rho(y) = g(\bar{x} + \rho y)$ (so that $\operatorname{div} z_\rho = \rho g_\rho$). Observe that E_ρ minimizes

$$\int_{\partial^* E_\rho \cap B_R} F(\bar{x} + \rho y, \nu^{E_\rho}(y)) d\mathcal{H}^{d-1}(y) + \rho \int_{E_\rho \cap B_R} g_\rho(y) dy,$$

with respect to compactly supported perturbations of the set (in the fixed ball B_R). Also,

$$\|\rho g_\rho\|_{L^d(B_R)} = \|g\|_{L^d(B_{\rho R})} \xrightarrow{\rho \rightarrow 0} 0.$$

By Lemma 3.3, the sets E_ρ (and the boundaries ∂E_ρ) satisfy uniform density bounds, and hence are compact with respect to both local L^1 and Hausdorff convergence.

Hence, up to extracting a subsequence, we can assume that $E_\rho \rightarrow \bar{E}$, with $0 \in \partial \bar{E}$. Proposition 3.6 shows that \bar{z} is a calibration for the energy $\int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y)) d\mathcal{H}^{d-1}(y)$, and that \bar{E} is a minimizer calibrated by \bar{z} .

It follows that $[\bar{z}, \nu^{\bar{E}}] = F(\bar{x}, \nu^{\bar{E}}(y))$ for \mathcal{H}^{d-1} -a.e. y in $\partial \bar{E}$, but since \bar{z} is a constant, we deduce that $\bar{E} = \{y \cdot \bar{\nu} \geq 0\}$ with $\bar{\nu}/F(\bar{x}, \bar{\nu}) = \nabla_p F^\circ(\bar{x}, \bar{z})^1$. In particular the limit \bar{E} is unique, hence we obtain the global convergence of $E_\rho \rightarrow \bar{E}$, without passing to a subsequence.

We want to deduce that $\bar{x} \in \partial^* E$, with $\nu^E(\bar{x}) = F(\bar{x}, \nu^E(\bar{x})) \nabla_p F^\circ(\bar{x}, \bar{z})$, which is equivalent to (6). The last identity is obvious from the arguments above, so that we only need to show that

$$\lim_{\rho \rightarrow 0} \frac{D\chi_{E_\rho}(B_1)}{|D\chi_{E_\rho}|(B_1)} = \bar{\nu}. \quad (7)$$

Assume we can show that

$$\lim_{\rho \rightarrow 0} |D\chi_{E_\rho}|(B_R) = |D\chi_{\bar{E}}|(B_R) \quad (= \omega_{d-1} R^{d-1}) \quad (8)$$

for any $R > 0$, then for any $\psi \in C_c^\infty(B_R; \mathbb{R}^d)$ we would get

$$\begin{aligned} \frac{1}{|D\chi_{E_\rho}|(B_R)} \int_{B_R} \psi \cdot D\chi_{E_\rho} &= - \frac{1}{|D\chi_{E_\rho}|(B_R)} \int_{B_R \cap E_\rho} \operatorname{div} \psi(x) dx \\ &\longrightarrow - \frac{1}{|D\chi_{\bar{E}}|(B_R)} \int_{B_R \cap \bar{E}} \operatorname{div} \psi(x) dx = \frac{1}{|D\chi_{\bar{E}}|(B_R)} \int_{B_R} \psi \cdot D\chi_{\bar{E}} \end{aligned}$$

and deduce that the measure $D\chi_{E_\rho}/(|D\chi_{E_\rho}|(B_R))$ weakly-* converges to $D\chi_{\bar{E}}/(|D\chi_{\bar{E}}|(B_R))$. Using again (8), we then obtain that

$$\lim_{\rho \rightarrow 0} \frac{D\chi_{E_\rho}(B_R)}{|D\chi_{E_\rho}|(B_R)} = \bar{\nu} \quad (9)$$

¹We use here that $F(\bar{x}, \cdot) \nabla F(\bar{x}, \cdot) = [F^\circ(\bar{x}, \cdot) \nabla F^\circ(\bar{x}, \cdot)]^{-1}$, so that $\bar{z} = \nabla F(\bar{x}, \nu^{\bar{E}}(y))$ implies both $F^\circ(\bar{x}, \bar{z}) = 1$ and $\nu^{\bar{E}}(y)/F(\bar{x}, \nu^{\bar{E}}(y)) = \nabla F^\circ(\bar{x}, \bar{z})$

for almost every $R > 0$. Since $D\chi_{E_\rho}(B_{\mu R})/(|D\chi_{E_\rho}|(B_{\mu R})) = D\chi_{E_{\rho/\mu}}(B_R)/(|D\chi_{E_{\rho/\mu}}|(B_R))$ for any $\mu > 0$, (9) holds in fact for any $R > 0$ and (7) follows, so that $\bar{x} \in \partial^* E$.

It remains to show (8). First, we observe that, by minimality of E_ρ and \bar{E} plus the Hausdorff convergence of ∂E_ρ in balls, we can easily show the convergence of the energies

$$\begin{aligned} \lim_{\rho \rightarrow 0} \int_{\partial E_\rho \cap B_R} F(\bar{x} + \rho y, \nu^{E_\rho}(y)) d\mathcal{H}^{d-1}(y) + \rho \int_{E_\rho \cap B_R} g_\rho(y) dy \\ = \int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y)) d\mathcal{H}^{d-1}(y) \end{aligned}$$

and, by the continuity of F ,

$$\lim_{\rho \rightarrow 0} \int_{\partial E_\rho \cap B_R} F(\bar{x}, \nu^{E_\rho}(y)) d\mathcal{H}^{d-1}(y) = \int_{\partial \bar{E} \cap B_R} F(\bar{x}, \nu^{\bar{E}}(y)) d\mathcal{H}^{d-1}(y). \quad (10)$$

Then, (7) follows from Reshetnyak's continuity theorem where, instead of using the Euclidean norm as reference norm, we use the uniformly convex norm $F(\bar{x}, \cdot)$ and the convergence of the measures $F(\bar{x}, D\chi_{E_\rho})$ to $F(\bar{x}, D\chi_{\bar{E}})$ (see [15, 11]). \square

In dimension 2 and 3 we can also show the reverse implication, proving that regular points of the boundary corresponds to Lebesgue points of the calibration. The idea is to show that the parameters r, ρ in Proposition 2.8 can be taken of the same order.

Theorem 3.8. *Assume the dimension is $d = 2$ or $d = 3$. Let x, s be as in Proposition 3.5, E be a minimizer of (2) and assume $x \in \partial^* E$. Then x is a Lebesgue point of z and*

$$z(x) = \nabla_\rho F(x, \nu^E).$$

Proof. We divide the proof into two steps.

Step 1. We first consider anisotropies F which are not depending on the x variable. Without loss of generality we assume $x = 0$. By assumption, there exists the limit

$$\bar{\nu} = \lim_{\rho \rightarrow 0} \frac{D\chi_E(B_\rho(0))}{|D\chi_E|(B_\rho(0))} \quad (11)$$

and, without loss of generality, we assume that it coincides with the vector e_d corresponding to the last coordinate of $y \in \mathbb{R}^d$.

Also, if we let $E_\rho = E/\rho$, the sets $E_\rho, E_\rho^c, \partial E_\rho$ converge in $B_1(0)$, in the Hausdorff sense (thanks to the uniform density estimates), respectively to $\{y_d \geq 0\}, \{y_d = 0\}, \{y_d \leq 0\}$.

We also let $z_\rho(y) = z(\rho y)$ and $g_\rho(y) = g(\rho y)$, in particular $-\operatorname{div} z_\rho = \rho g_\rho$. We let

$$\omega(\rho) = \sup_{x \in \Omega} \|g\|_{L^d(B_\rho(x) \cap \Omega)} \quad (12)$$

which is continuously increasing and goes to 0 as $\rho \rightarrow 0$, since $|g|^d$ is equi-integrable. We introduce the following notation: a point in \mathbb{R}^d is denoted by $y = (y', y_d)$, with $y' \in \mathbb{R}^{d-1}$. We let $D_s := \{|y'| \leq s\}$, $\bar{z} := \nabla F(\bar{\nu})$ and $D_s^t = \{D_s + \lambda \bar{z} : |\lambda| \leq t\}$ and denote with ∂D_s the relative boundary of D_s in $\{y_d = 0\}$.

We choose $s \leq 1$, $0 < t \leq s$, (t is chosen small enough so that $D_s^t \subset B_1(0)$, that is $t < (1/|\bar{z}|)\sqrt{1-s^2}$). We integrate in D_s^t the divergence $\rho g_\rho = -\operatorname{div} z_\rho = \operatorname{div}(\bar{z} - z_\rho)$ against the function $(2\chi_E - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})}$, which vanishes for $y_d = \pm tF(\bar{\nu})$ if ρ is small enough (given $t > 0$), so that $\partial E_\rho \cap B_1(0) \subset \{|y_d| \leq tF(\bar{\nu})\}$. For y on the lateral boundary of the cylinder D_s^t , let $\xi(y)$ be the internal normal to $\partial D_s + (-t, t)\bar{z}$ at the point y . Using the fact that z_ρ is a calibration for E_ρ , we easily get that for almost all s ,

$$\begin{aligned} & \int_{D_s^t} \rho g_\rho \left((2\chi_E - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})} \right) dy \\ &= \int_{\partial D_s + (-t, t)\bar{z}} \left((2\chi_E - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})} \right) [(\bar{z} - z_\rho), \xi(y)] d\mathcal{H}^{d-1} \\ & \quad - 2t \int_{\partial E_\rho \cap D_s^t} (\bar{z} \cdot \nu^{E_\rho} - F(\nu^{E_\rho})) d\mathcal{H}^{d-1} + \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \bar{\nu}}{F(\bar{\nu})} \right) dy. \end{aligned} \quad (13)$$

Now since $F^\circ(\nabla F(\bar{\nu})) = 1$, there holds $\bar{z} \cdot \nu^{E_\rho} - F(\nu^{E_\rho}) \leq 0$ and using that $\bar{z} \cdot \xi(y) = 0$ on $\partial D_s + (-t, t)\bar{z}$, we get

$$\begin{aligned} \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \bar{\nu}}{F(\bar{\nu})} \right) dy &\leq \int_{D_s^t} \rho g_\rho \left((2\chi_E - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})} \right) dy \\ & \quad - \int_{\partial D_s + (-t, t)\bar{z}} \left((2\chi_E - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})} \right) z_\rho \cdot \xi(y) d\mathcal{H}^{d-1}. \end{aligned} \quad (14)$$

We claim that for $|\xi| \leq 1$ with $\xi \cdot \bar{z} = 0$, there holds

$$(\xi \cdot z_\rho)^2 \leq C(F(\bar{\nu}) - \bar{\nu} \cdot z_\rho) \quad (15)$$

Since

$$(\xi \cdot z_\rho)^2 \leq |z_\rho|^2 - [z_\rho \cdot (\bar{z}/|\bar{z}|)]^2$$

it is enough to prove

$$|z_\rho|^2 - [z_\rho \cdot (\bar{z}/|\bar{z}|)]^2 \leq C(F(\bar{\nu}) - \bar{\nu} \cdot z_\rho).$$

Using that $\bar{\nu}/F(\bar{\nu}) = \nabla F^\circ(\bar{z})$, from (5) applied to F° together with $F^\circ(\bar{z}) = 1 \geq F^\circ(z_\rho)$, we find

$$(F(\bar{\nu}) - \bar{\nu} \cdot z_\rho) = F(\bar{\nu})(1 - z_\rho \cdot \nabla F^\circ(\bar{z})) \geq C|z_\rho - \bar{z}|^2.$$

which readily implies (15). We thus have

$$\begin{aligned}
& \int_{\partial D_{s+(-t,t)\bar{z}}} \left((2\chi_{E_\rho} - 1)t - \frac{\bar{v} \cdot y}{F(\bar{v})} \right) (z_\rho \cdot \xi) d\mathcal{H}^{d-1} \\
& \leq 2C\sqrt{F(\bar{v})}t \int_{\partial D_{s+(-t,t)\bar{z}}} \sqrt{1 - \frac{z_\rho \cdot \bar{v}}{F(\bar{v})}} d\mathcal{H}^{d-1} \\
& \leq 2CF(\bar{v})t\sqrt{t} \left(\int_{\partial D_{s+(-t,t)\bar{z}}} \left(1 - \frac{z_\rho \cdot \bar{v}}{F(\bar{v})} \right) d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_s)}. \quad (16)
\end{aligned}$$

Now, we also have

$$\begin{aligned}
\rho \int_{D_s^t} \left((2\chi_{E_\rho} - 1)t - \frac{\bar{v} \cdot y}{F(\bar{v})} \right) g_\rho(y) dy & \leq 2t\rho^{1-d} \int_{D_{\rho s}^{\rho t}} g(x) dx \\
& \leq 2t\rho^{1-d} \|g\|_{L^d(B_{\rho s}(0))} |D_{\rho s}^{\rho t}|^{1-1/d} \leq ct^{2-1/d} s^{d-2+1/d} \omega(\rho s) \quad (17)
\end{aligned}$$

where here, $c = 2\mathcal{H}^{d-1}(D_1)^{1-1/d}$, and ω is defined in (12).

We choose $a < 1$, close to 1, and choose $t \in (0, (1/|\bar{z}|)\sqrt{1-a^2})$. If $\rho > 0$ is small enough (so that $\partial E_\rho \cap B_1$ is in $\{|y_d| \leq tF(\bar{v})\}$), letting $f(s) := \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \bar{v}}{F(\bar{v})} \right) dy$, we deduce from (14), (16) and (17) that for a.e. s with $t \leq s \leq a$, one has (possibly increasing the constant c)

$$f(s)^2 \leq c \left(s^{d-2} t^3 f'(s) + t^{4-2/d} s^{2d-4+2/d} \omega(\rho s)^2 \right). \quad (18)$$

Unfortunately, this estimate does not give much information for $d > 3$. It seems it allows to conclude only whenever $d \in \{2, 3\}$. Since the case $d = 2$ is simpler, we focus on $d = 3$. Estimate (18) becomes

$$f(s)^2 \leq c \left(st^3 f'(s) + t^{10/3} s^{8/3} \omega(\rho s)^2 \right). \quad (19)$$

Given $M > 0$, we fix a value $t > 0$ such that $\log(a/t) \geq cM$. If ρ is chosen small enough, then $\partial E_\rho \cap B_1(0) \subset \{|y_d| < tF(\bar{v})\}$, and (19) holds. It yields (assuming $f(t) > 0$, but if not, then the Proposition is proved)

$$-\frac{f'(s)}{f(s)^2} + \frac{1}{ct^3} \frac{1}{s} \leq ct^{1/3} s^{5/3} \frac{\omega(\rho s)^2}{f(s)^2} \leq ct^{1/3} s^{5/3} \frac{\omega(a\rho)^2}{f(t)^2} \quad (20)$$

where we have used the fact that $t \leq s \leq a$ and f, ω are nondecreasing. Integrating (20) from t to a , after multiplication by t^3 we obtain

$$\frac{t^3}{f(a)} - \frac{t^3}{f(t)} + \frac{\log(a/t)}{c} \leq \frac{3c}{8} t^{10/3} (a^{8/3} - t^{8/3}) \frac{\omega(a\rho)^2}{f(t)^2}.$$

Hence we get

$$\frac{t^3}{f(t)} + ca^{8/3} t^{-8/3} \omega(a\rho)^2 \frac{t^6}{f(t)^2} \geq M. \quad (21)$$

Eventually, we observe that

$$f(t) = \int_{D_t^t} \left(1 - \frac{z(\rho y) \cdot \bar{\nu}}{F(\bar{\nu})} \right) dy = \frac{1}{\rho^d} \int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x) \cdot \bar{\nu}}{F(\bar{\nu})} \right) dx,$$

so that (21) can be rewritten

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x) \cdot \bar{\nu}}{F(\bar{\nu})} \right) dx}{(\rho t)^3} \right)^{-1} \geq \frac{-1 + \sqrt{1 + 4Mca^{8/3}t^{-8/3}\omega(a\rho)^2}}{2ca^{8/3}t^{-8/3}\omega(a\rho)^2} \quad (22)$$

The value of t being fixed, we can choose the value of ρ small enough in order to have $4Mca^{8/3}t^{-8/3}\omega(a\rho)^2 < 1$, and (using $\sqrt{1+X} \geq 1 + X/2 - X^2/8$ if $X \in (0, 1)$), (22) yields

$$\left(\frac{\int_{D_{\rho t}^{\rho t}} \left(1 - \frac{z(x) \cdot \bar{\nu}}{F(\bar{\nu})} \right) dx}{(\rho t)^3} \right)^{-1} \geq M - M^2ca^{8/3}t^{-8/3}\omega(a\rho)^2 \geq \frac{3}{4}M. \quad (23)$$

It follows that

$$\limsup_{\varepsilon \rightarrow 0} \frac{\int_{D_\varepsilon^\varepsilon} \left(1 - \frac{z(x) \cdot \bar{\nu}}{F(\bar{\nu})} \right) dx}{\varepsilon^3} \leq \frac{4}{3}M^{-1} \quad (24)$$

and since M is arbitrary, 0 is indeed a Lebesgue point of z , with value $\bar{z} = \nabla F(\bar{\nu})$ (recall that $1 - \frac{z(x) \cdot \bar{\nu}}{F(\bar{\nu})} \geq (C/F(\bar{\nu}))|z(x) - \bar{z}|^2$).

Step 2. When F depends also on the x variable, the proof follows along the same lines as in *Step 1*, taking into account the errors terms in (14) and (16). Keeping the same notations as in *Step 1* and setting $\bar{z} := \nabla_p F(0, \bar{\nu})$ we find that since $F^\circ(0, \bar{z}) \leq 1$, there holds $\bar{z} \cdot \nu^{E_\rho} \leq F(0, \nu^{E_\rho})$ and thus

$$\int_{\partial E_\rho \cap D_s^t} \bar{z} \cdot \nu^{E_\rho} - F(\rho x, \nu^{E_\rho}) d\mathcal{H}^{d-1} \leq \int_{\partial E_\rho \cap D_s^t} |F(0, \nu^{E_\rho}) - F(\rho x, \nu^{E_\rho})| d\mathcal{H}^{d-1} \leq C\rho s^{d-1}$$

where the last inequality follows from $t \leq s$ and the minimality of E_ρ inside D_s^t . Now since

$$(F^\circ)^2(0, z_\rho) - (F^\circ)^2(\rho x, z_\rho) \geq (F^\circ)^2(0, z_\rho) - 1 \geq 2\frac{\bar{\nu}}{F(0, \bar{\nu})} \cdot (z_\rho - z) + \delta^2|z_\rho - z|^2$$

we find that (15) transforms into,

$$(\xi \cdot z_\rho)^2 \leq C \left[(F(0, \bar{\nu}) - \bar{\nu} \cdot z_\rho) + ((F^\circ)^2(0, z_\rho) - (F^\circ)^2(\rho x, z_\rho)) \right]$$

for every $|\xi| \leq 1$ and $\xi \cdot \bar{z} = 0$, from which we get

$$\begin{aligned}
& \int_{\partial D_{s+(-t,t)\bar{z}}} \left((2\chi_{E_\rho} - 1)t - \frac{\bar{\nu} \cdot y}{F(\bar{\nu})} \right) (z_\rho \cdot \xi) d\mathcal{H}^{d-1} \\
& \leq 2CF(0, \bar{\nu})t\sqrt{t} \left(\int_{\partial D_{s+(-t,t)\bar{z}}} \left(1 - \frac{z_\rho \cdot \bar{\nu}}{F(0, \bar{\nu})} \right) d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_s)} \\
& \quad + 2Ct \int_{\partial D_{s+(-t,t)\bar{z}}} \left| (F^\circ)^2(0, z_\rho) - (F^\circ)^2(\rho x, z_\rho) \right|^{1/2} d\mathcal{H}^{d-1} \\
& \leq CF(0, \bar{\nu})t\sqrt{t} \left(\int_{\partial D_{s+(-t,t)\bar{z}}} \left(1 - \frac{z_\rho \cdot \bar{\nu}}{F(0, \bar{\nu})} \right) d\mathcal{H}^{d-1} \right)^{\frac{1}{2}} \sqrt{\mathcal{H}^{d-2}(\partial D_s)} + Ct\rho^{1/2}s^{d-1}t.
\end{aligned}$$

Using these estimates, we finally get that, setting as before $f(s) := \int_{D_s^t} \left(1 - \frac{z_\rho \cdot \bar{\nu}}{F(0, \bar{\nu})} \right) dy$, there holds

$$f(s)^2 \leq c \left(s^{d-2}t^3 f'(s) + t^{4-2/d} s^{2d-4+2/d} \omega(\rho s)^2 + \rho t s^{d-1} + \rho^{1/2} t^2 s^{d-1} \right).$$

From this inequality, the proof can be concluded exactly as in *Step 1*. \square

Eventually, we can also give a locally uniform convergence result.

Proposition 3.9. *For all $x \in \Omega$ we let*

$$z_\rho(x) := \frac{1}{|B_\rho(0)|} \int_{B_\rho(x) \cap \Omega} z(y) dy.$$

Then, $F^\circ(x, z_\rho(x)) \rightarrow 1$ locally uniformly on S .

Proof. Given $K \subset \Omega$ a compact set, we can check that for any $t > 0$, there exists $\rho_0 > 0$ such that for any $x \in K \cap S$, if E^x is the level set of u through x , then for any $\rho \leq \rho_0$, the boundary of $(E^x - x)/\rho \cap B_1(0)$ lies in a strip of width $2t$, that is, there is $\bar{\nu}^x \in \mathbb{S}^{d-1}$ with $\partial((E^x - x)/\rho) \cap B_1(0) \subset \{y \cdot \bar{\nu}^x \leq t\}$.

Indeed, if this is not the case, one can find $t > 0$, $\rho_k \rightarrow 0$, $x_k \in K \cap S$, such that $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ is not contained in any strip of width $2t$. Up to a subsequence we may assume that $x_k \rightarrow x \in K \cap S$, and from the bound on the perimeter, that $(E^{x_k} - x_k)/\rho_k \cap B_1(0)$ converges to a local minimizer of $\int_{\partial E} F(0, \nu^E) d\mathcal{H}^{d-1}$ and is thus a halfspace.² Moreover, $\partial((E^{x_k} - x_k)/\rho_k) \cap B_1(0)$ converges in the Hausdorff sense (thanks to the density estimates) to a hyperplane. We easily obtain a contradiction.

The thesis follows when we observe that the proof of Proposition 3.8 can be reproduced by replacing the direction $\nu^{E^x}(x)$ (which exists only if x lies in the reduced boundary of E^x) with the direction $\bar{\nu}^x$ given above. \square

²If $d = 2$, this Bernstein result readily follows from the strict convexity of F , see [11, Prop 3.6] whereas for $d = 3$, see [18]. In the case of the area i.e when $F(x, Du) = |Du|$ and $d \leq 7$, see also [12, Rem 3.2].

3.3 A counterexample.

We provide an example where $g \in L^{d-\varepsilon}(\Omega)$, with $\varepsilon > 0$ arbitrarily small, and Theorem 3.8 does not hold.

Let $\Omega = B_1(0)$ be the unit ball of \mathbb{R}^d and let $E = \Omega \cap \{x_d \leq 0\}$. We shall construct a vector field $z : \Omega \rightarrow \mathbb{R}^d$ such that $z = \nu^E$ on $\partial E \cap \Omega$, $|z| \leq 1$ everywhere in Ω , $\operatorname{div} z \in L^{d-\varepsilon}(\Omega)$, but 0 is not a Lebesgue point of z . Notice that E minimizes the functional (3) with $g = \operatorname{div} z$.

Letting $r_n \rightarrow 0$ be a decreasing sequence to be determined later, and let $B_n = B_{r_n}(x_n)$ with $x_n = 2r_n e_d$. Without loss of generality, we may assume $r_{n+1} < r_n/4$ so that the balls B_n are all disjoint. We define the vector field z as follows: $z(x) = e_d$ if $x \in \Omega \setminus \cup_n B_n$, and $z(x) = |x - x_n|e_d$ if $x \in B_n$. It follows that $\operatorname{div} z = 0$ in $\Omega \setminus \cup_n B_n$ and $|\operatorname{div} z| \leq 1/r_n$ in B_n , so that

$$\int_{\Omega} |\operatorname{div} z|^{d-\varepsilon} dx = \sum_n \int_{B_n} |\operatorname{div} z|^{d-\varepsilon} dx \leq \omega_d \sum_n r_n^\varepsilon < +\infty$$

if we choose r_n converging to zero sufficiently fast, so that $g = -\operatorname{div} z \in L^{d-\varepsilon}(\Omega)$.

However, since $z \cdot e_d \leq 1/2$ in $B_{r_n/2}(x_n)$, we also have

$$\int_{B_{3r_n}(0)} z \cdot e_d dx \leq |B_{3r_n}(0)| - \frac{1}{2} |B_{r_n/2}(x_n)|$$

so that

$$\frac{1}{|B_{3r_n}(0)|} \int_{B_{3r_n}(0)} z \cdot e_d dx \leq 1 - \frac{1}{6^d} < 1.$$

On the other hand, for $\delta \in (0, 1/6^d)$ we have

$$\frac{1}{|B_{r_n}(0)|} \int_{B_{r_n}(0)} z \cdot e_d dx \geq \frac{1}{|B_{r_n}(0)|} \left(|B_{r_n}(0)| - \sum_{i=n+1}^{\infty} |B_{r_i}(x_i)| \right) \geq 1 - \delta,$$

if we take the sequence r_n converging to 0 sufficiently fast. It follows that 0 is not a Lebesgue point of z .

References

- [1] F.J. ALMGREN, R. SCHOEN, L. SIMON, *Regularity and singularity estimates on hypersurfaces minimizing elliptic variational integrals*, Acta Math., 139 (1977), 217-265.
- [2] L. AMBROSIO, N. FUSCO, D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Science Publications, 2000.
- [3] L. AMBROSIO, E. PAOLINI, *Partial regularity for quasi minimizers of perimeter*, Ricerche Mat. 48 (1999), suppl., 167-186.

- [4] F. ANDREU-VAILLO, V. CASELLES, J.M. MAZÒN, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*, collection “Progress in Mathematics” 223, Birkhäuser, 2004.
- [5] G. ANZELLOTTI, *Pairings between measures and bounded functions and compensated compactness*, Annali di Matematica Pura ed Applicata, 135 (1983), 1, 293-318.
- [6] G. ANZELLOTTI, *On the minima of functionals with linear growth*, Rend. Sem. Mat. Univ. Padova 75 (1986), 91–110.
- [7] G. ANZELLOTTI, *Traces of bounded vector fields and the divergence theorem*, unpublished.
- [8] E. BAROZZI, E. GONZALEZ, I. TAMANINI, *The mean curvature of a set of finite perimeter*, Proc. Amer. Math. Soc. 99 (1987), no. 2, 313–316.
- [9] G. BOUCHITTÉ, G. DAL MASO, *Integral representation and relaxation of convex local functionals on $BV(\Omega)$* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993), no. 4, 483–533.
- [10] A. CHAMBOLLE, *An algorithm for mean curvature motion*, Interfaces Free Bound. 6 (2006), no. 2, 195–218.
- [11] A. CHAMBOLLE, M. GOLDMAN, M. NOVAGA, *Plane-like minimizers and differentiability of the stable norm*, Preprint (2012).
- [12] E. GONZALEZ, U. MASSARI *Variational mean curvatures*, Rend. Sem. Mat. Univ. Politec. Torino 52 (1994), no. 1, 1–28.
- [13] U. MASSARI, *Frontiere orientate di curvatura media assegnata in L^p (Italian)*, Rend. Sem. Mat. Univ. Padova 53 (1975), 37–52.
- [14] E. PAOLINI, *Regularity for minimal boundaries in \mathbb{R}^n with mean curvature in L^n* , Manuscripta Math. 97 (1998), no. 1, 15–35.
- [15] Y.G. RESHETNYAK, *Weak Convergence of Completely Additive Vector Functions on a Set*, Siberian Math. J. 9 (1968), 1386–1394.
- [16] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, Cambridge university Press, 1993.
- [17] R. TEMAM, *Problèmes mathématiques en plasticité (French)*, Méthodes Mathématiques de l’Informatique, 12, Gauthier-Villars, Montrouge, 1983.

- [18] B. WHITE, *Existence of smooth embedded surfaces of prescribed topological type that minimize parametric even elliptic functionals on three-manifolds*, J. Diff. Geometry, 33 (1991), 413–443.