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Mathematical Modeling of Fiber Reinforced Structures by Homogenization

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Abstract : We present another proof of a study of Bellieud and Bouchitté that we expect to be more suitable to treat more general geometrical and physical cases. We consider the homogenization of the quasi-linear elliptic problem

$$\begin{aligned} -\operatorname{div} \sigma_\varepsilon &= f, \quad \sigma_\varepsilon = a_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon && \text{on } \Omega \\ u_\varepsilon &= u_0 && \text{on } \Gamma_0 \\ \sigma_\varepsilon \cdot n &= g && \text{on } \Gamma_1 \end{aligned}$$

where Ω is a bounded cylindrical open subset of \mathbb{R}^3 and $1 < p < +\infty$. The fibers occupy a set of thin parallel cylinders periodically distributed in Ω . The conductivity coefficient a_ε is ε -periodic and takes very high values in the fibers.

Keywords : Variational convergence, Homogenization

2000 Mathematics Subject Classification : ??? 2000 MSC ???

1 Introduction

Let $p \in (1, +\infty)$, we consider the homogenization of the elliptic problem

$$\begin{aligned} -\operatorname{div} \sigma_\varepsilon &= f, \quad \sigma_\varepsilon = a_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon && \text{on } \Omega \\ u_\varepsilon &= u_0 && \text{on } \Gamma_0 \\ \sigma_\varepsilon \cdot n &= g && \text{on } \Gamma_1 \end{aligned} \tag{1.1}$$

where $\Omega := \omega \times (0, L)$ with $L > 0$ and ω is a bounded domain of \mathbb{R}^2 with smooth boundary and containing the origin of coordinates. The homogenization study of (1.1) consists in examining the behavior of the sequence of the solution (u_ε) as ε tends to zero. The conductivity coefficient a_ε is ε -periodic and satisfies a uniform lower bound, Γ_0 is an open subset of $\partial\Omega$ with Hausdorff measure $H^2(\Gamma_0)$ strictly positive, $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, and n is the unit exterior normal on $\partial\Omega$. The boundary data u_0 is Lipschitz, while $(f, g) \in L^{p'}(\Omega) \times L^{p'}(\Gamma_1)$, $p' = p/(p-1)$.

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The problem (1.1) is related to the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min \left\{ F_\varepsilon(w) - L(w) : w \in W_{\Gamma_0}^{1,p}(\Omega) \right\},$$

where

$$\begin{aligned} W_{\Gamma_0}^{1,p}(\Omega) &:= \left\{ w \in W^{1,p}(\Omega) : w = u_0 \text{ on } \Gamma_0 \right\}, \\ F_\varepsilon(w) &:= \int_{\Omega} a_\varepsilon \phi_p(\nabla w) \, dx, \\ \phi_p(\xi) &:= \frac{1}{p} |\xi|^p, \quad \forall \xi \in \mathbb{R}^n, n = 1, 2, 3, \\ L(w) &:= \int_{\Omega} f w \, dx + \int_{\Gamma_1} g w \, dH^2. \end{aligned} \quad (1.2)$$

We are interested in the asymptotic behavior of $(\mathcal{P}_\varepsilon)$ as $\varepsilon \rightarrow 0$. We present another proof of a study of Bellieud and Bouchitté [2] that we expect to be more suitable to treat more general geometrical and physical cases.

The bases of the cylindrical domain Ω are denoted by $\omega_0 = \omega \times \{0\}$ and $\omega_L = \omega \times \{L\}$. For each ε , we consider a periodic distribution of cells $(Y_\varepsilon^i)_{i \in I_\varepsilon}$ such that $Y_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2) + (-\varepsilon/2, \varepsilon/2)^2$, and $I_\varepsilon := \{i \in \mathbb{Z}^2 : Y_\varepsilon^i \subset \omega\}$. Let $(D_{r_\varepsilon}^i)_{i \in I_\varepsilon}$ be the family of disks of \mathbb{R}^2 centered at $\hat{x}_\varepsilon^i := (\varepsilon i_1, \varepsilon i_2)$ of radius $r_\varepsilon \ll \varepsilon$, $T_\varepsilon^i := D_{r_\varepsilon}^i \times (0, L)$ and $T_\varepsilon := \cup_{i \in I_\varepsilon} T_\varepsilon^i$. The set of thin parallel cylinders T_ε represents the fibers (see Figure 1 and Figure 2). The conductivity coefficient a_ε is

$$a_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus T_\varepsilon, \\ \lambda_\varepsilon & \text{otherwise.} \end{cases}$$

We make the assumptions

$$r_\varepsilon \rightarrow 0, \quad \frac{r_\varepsilon}{\varepsilon} \rightarrow 0, \quad \lambda_\varepsilon \rightarrow +\infty, \quad k_\varepsilon := \lambda_\varepsilon \frac{r_\varepsilon^2}{\varepsilon^2} \rightarrow k, \quad k \geq 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (1.3)$$

In [2], it was shown that the asymptotic limit of $(\mathcal{P}_\varepsilon)$ is

$$\min \left\{ \Phi(u, v) - L(u) : (u, v) \in (L^p(\Omega))^2 \right\},$$

where

$$\Phi(u, v) = \begin{cases} \int_{\Omega} \phi_p(\nabla u) \, dx + \frac{k\pi}{p} \int_{\Omega} \left| \frac{\partial v}{\partial x_3} \right|^p \, dx + \frac{2\pi\gamma}{p} \int_{\Omega} |v - u|^p \, dx, \\ \text{if } \begin{cases} (u, v) \in W_{\Gamma_0}^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L)), \\ v = u_0 \text{ on } \Gamma_0 \cap (\omega_0 \cup \omega_L), \end{cases} \\ +\infty \quad \text{otherwise,} \end{cases} \quad (1.4)$$

and

$$[0, +\infty] \ni \gamma = \begin{cases} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} |\log r_\varepsilon|^{-1} & \text{if } p = 2, \\ \lim_{\varepsilon \rightarrow 0} \left| \frac{2-p}{p-1} \right|^{p-1} r_\varepsilon^{2-p} \varepsilon^{-2} & \text{if } p \neq 2. \end{cases} \quad (1.5)$$

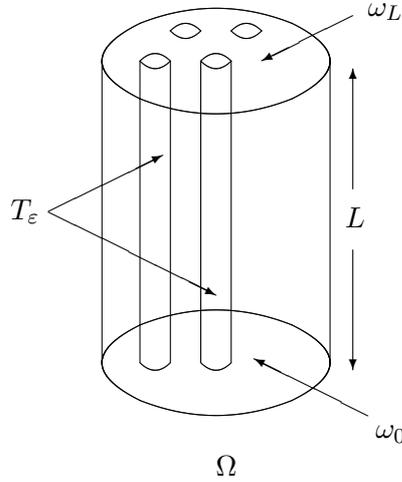


Figure 1: the domain $\Omega = \omega \times (0, L)$ occupied by a composite material

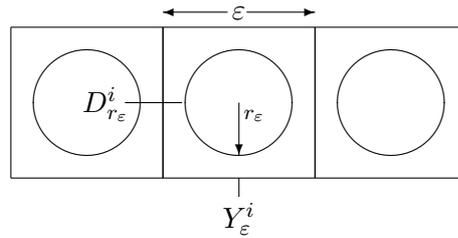


Figure 2: the circular cross section of the fiber, $Y_\epsilon^i \subset \omega$

Here, the boundary data u_0 is assumed to be Lipschitz in order to ensure that the infimum value of problem (\mathcal{P}_ϵ) remains finite as $\epsilon \rightarrow 0$. In case $k = +\infty$, we add further assumption

$$k_\epsilon r_\epsilon \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (1.6)$$

The conditions

$$k > 0 \quad \text{and} \quad \{ \gamma > 0 \quad \text{or} \quad \omega_0 \subset \Gamma_0 \quad \text{or} \quad \omega_L \subset \Gamma_0 \} \quad (1.7)$$

guarantee that the functional Φ is coercive in $W^{1,p}(\Omega) \times L^p(\omega, W^{1,p}(0, L))$.

We are concerned with the extension of this result to more general cross sections of the fibers and more general energy density than ϕ_p . The aim of this paper is therefore to provide another proof that we expect to be more suitable to treat such general cases. The steps of the proof in [2] are to successively establish:

- (i) a compactness property of the sequence (u_ϵ) such that $F_\epsilon(u_\epsilon) < C$,

- (ii) a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$,
- (iii) an upper bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$.

Here we replace the steps (ii) and (iii) by

- (ii') an upper equality of the sequence $(F_\varepsilon(u_\varepsilon))$,
- (iii') a lower bound inequality of the sequence $(F_\varepsilon(u_\varepsilon))$ which essentially uses a subdifferential inequality.

2 An Alternative Strategy

It consists, under (1.3), (1.5), (1.6) and (1.7), in proving the following three propositions. In the sequel, the symbols \rightarrow , \rightharpoonup and $\overset{*}{\rightharpoonup}$ stand for the strong convergence, the weak convergence and the weak star convergence, respectively. As usual, the letter C denotes various constants and for all $\xi = (\xi_1, \xi_2, \xi_3)$ in \mathbb{R}^3 , $\hat{\xi}$ stands for (ξ_1, ξ_2) .

Proposition 2.1 (compactness property) *Let (u_ε) be a sequence such that $\sup F_\varepsilon(u_\varepsilon)$ is finite. Then (u_ε) is strongly relatively compact in $L^p(\Omega)$ and (v_ε) , given by $v_\varepsilon := \frac{|\Omega|}{|T_\varepsilon|} 1_{T_\varepsilon} u_\varepsilon$, is bounded in $L^1(\Omega)$ and, up to a subsequence, (v_ε) weakly* converges in the space of bounded measures $\mathcal{M}_b(\Omega)$ to an element v of $L^p(\Omega)$.*

Proposition 2.2 (upper bound equality) *For all (u, v) in $(L^p(\Omega))^2$, such that $\Phi(u, v) < +\infty$, there exists a sequence (u_ε) such that $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$ and*

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \Phi(u, v).$$

Proposition 2.3 (lower bound inequality) *For all u in $L^p(\Omega)$ and for all sequences (u_ε) such that $u_\varepsilon \rightarrow u$ in $L^p(\Omega)$, $v_\varepsilon \overset{*}{\rightharpoonup} v$ in $\mathcal{M}_b(\Omega)$, one has:*

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \Phi(u, v).$$

The proofs of these propositions are presented in the following sections.

2.1 Proof of Proposition 2.1

Compactness property was already proved in [2].

2.2 Proof of Proposition 2.2

Our sole contribution is to prove that we can replace inequality by equality, for that we use the same approximation u'_ε of u as in [2]

$$u'_\varepsilon = (1 - \theta_\varepsilon)u + \theta_\varepsilon w_\varepsilon.$$

The function θ_ε is first defined on the closure of $\omega_\varepsilon := \cup_{i \in I_\varepsilon} Y_\varepsilon^i$ as a $(-\varepsilon/2, \varepsilon/2)^2$ -periodic continuous function which satisfies $0 \leq \theta_\varepsilon \leq 1$, $\theta_\varepsilon = 1$ on $D_\varepsilon := \cup_{i \in I_\varepsilon} D_{r_\varepsilon}^i$, $\theta_\varepsilon = 0$ on $\bar{\omega}_\varepsilon \setminus \cup_{i \in I_\varepsilon} D_{R_\varepsilon}^i$, where $D_{R_\varepsilon}^i$ is the disk of \mathbb{R}^2 centered at \hat{x}_ε^i of radius R_ε such that $r_\varepsilon \ll R_\varepsilon \ll \varepsilon$. Next θ_ε is assumed to vanish on $\bar{\omega} \setminus \omega_\varepsilon$ and

$$w_\varepsilon(\hat{x}, x_3) = \sum_{i \in I_\varepsilon} \left(\frac{1}{|D_{r_\varepsilon}^i|} \int_{D_{r_\varepsilon}^i} v(\hat{y}, x_3) d\hat{y} \right) 1_{Y_\varepsilon^i}(\hat{x}).$$

The approximation u'_ε does not satisfy the boundary condition on $\Gamma_0 \cap (\omega_0 \cup \omega_L)$ so that, as in [2], we introduce a sharper approximation

$$u_\varepsilon^\# := u\varphi_\varepsilon + u'_\varepsilon(1 - \varphi_\varepsilon).$$

Here φ_ε is a $C^\infty(\overline{\Omega})$ function which satisfies $\varphi_\varepsilon = 1$ on Γ_0 , $\varphi_\varepsilon = 0$ on $\overline{\Omega} \setminus \Sigma_\varepsilon$, $|\nabla\varphi_\varepsilon| \leq C/r_\varepsilon$ on $\overline{\Omega}$ where $\Sigma_\varepsilon := \{x \in \Omega : \text{dist}(x, \Gamma_0) < r_\varepsilon\}$. We assume that u and v are Lipschitz on $\overline{\Omega}$ and there exists $L > 0$ such that

$$\left| \frac{\partial v}{\partial x_3}(\hat{x}', x_3) - \frac{\partial v}{\partial x_3}(\hat{x}'', x_3) \right| < L|\hat{x}' - \hat{x}''| \quad \forall (\hat{x}', x_3), (\hat{x}'', x_3) \in \overline{\Omega}. \quad (2.1)$$

Letting Ψ be any continuous function on $\overline{\Omega}$ such that $0 \leq \Psi \leq 1$, we introduce F_ε^Ψ , Φ^Ψ defined by similar formulae as the ones of F_ε and Φ but with Ψdx in place of dx . We will prove the lemma:

Lemma 2.4

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^\Psi(u'_\varepsilon) = \Phi^\Psi(u, v).$$

If Lemma 2.4 is proved, then, by a classical approximation process, we can deduce

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u'_\varepsilon) = \Phi(u, v). \quad (2.2)$$

Finally, we complete the proof of (2.2) for any (u, v) such that $\Phi(u, v) < +\infty$ by approximation and diagonalization arguments.

Proof of Lemma 2.4. We split $F_\varepsilon^\Psi(u'_\varepsilon)$ in three parts

$$F_\varepsilon^\Psi(u'_\varepsilon) = F_\varepsilon^{\Psi 1}(u'_\varepsilon) + F_\varepsilon^{\Psi 2}(u'_\varepsilon) + F_\varepsilon^{\Psi 3}(u'_\varepsilon). \quad (2.3)$$

First, we consider

$$F_\varepsilon^{\Psi 1}(u'_\varepsilon) := \int_{\Omega \setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u'_\varepsilon) \Psi dx = \int_{\Omega \setminus B_\varepsilon \cup T_\varepsilon} \phi_p(\nabla u) \Psi dx,$$

where $B_\varepsilon := \cup_{i \in I_\varepsilon} D_{R_\varepsilon}^i \setminus \overline{D_{r_\varepsilon}^i} \times (0, L)$. Hence, the assumption $R_\varepsilon \ll \varepsilon$ yields $\lim_{\varepsilon \rightarrow 0} |B_\varepsilon \cup T_\varepsilon| = 0$ and, consequently,

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 1}(u'_\varepsilon) = \int_{\Omega} \phi_p(\nabla u) \Psi dx.$$

Next, we pay attention to

$$F_\varepsilon^{\Psi 2}(u'_\varepsilon) := \int_{B_\varepsilon} \phi_p(\nabla u'_\varepsilon) \Psi dx.$$

Writing

$$z_\varepsilon := (v - u) \widehat{\nabla} \theta_\varepsilon, \quad (2.4)$$

we obtain

$$\nabla u'_\varepsilon = z_\varepsilon + (w_\varepsilon - v)\nabla\theta_\varepsilon + (1 - \theta_\varepsilon)\nabla u + \theta_\varepsilon\nabla w_\varepsilon.$$

Let us show

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi \, dx = 0. \quad (2.5)$$

The function ϕ_p , being convex and positively homogeneous of degree p , satisfies

$$\forall \xi, \eta \in \mathbb{R}^n, \quad n = 1, 2, 3, \quad |\phi_p(\xi) - \phi_p(\eta)| \leq C|\xi - \eta|(|\xi|^{p-1} + |\eta|^{p-1}). \quad (2.6)$$

Therefore, Hölder inequality yields

$$\left| \int_{B_\varepsilon} (\phi_p(\nabla u'_\varepsilon) - \phi_p(z_\varepsilon))\Psi \, dx \right| \leq C \left(\int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p \, dx \right)^{\frac{1}{p}} \left(\int_{B_\varepsilon} |\nabla u'_\varepsilon|^p \, dx + \int_{B_\varepsilon} |z_\varepsilon|^p \, dx \right)^{\frac{1}{p'}}.$$

The smoothness of (u, v) implies

$$\left. \begin{aligned} u'_\varepsilon &= u \text{ on } \Omega \setminus (B_\varepsilon \cup T_\varepsilon), & |u'_\varepsilon| &\leq C \text{ on } \Omega, & |\nabla w_\varepsilon| &\leq C \text{ on } B_\varepsilon, \\ u'_\varepsilon &= w_\varepsilon \text{ on } T_\varepsilon, & |w_\varepsilon - v| &\leq CR_\varepsilon \text{ on } B_\varepsilon, \end{aligned} \right\} \quad (2.7)$$

so that

$$\begin{aligned} \int_{B_\varepsilon} |\nabla u'_\varepsilon|^p \, dx + \int_{B_\varepsilon} |z_\varepsilon|^p \, dx &\leq C\varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon) \, d\hat{x}, \\ \int_{B_\varepsilon} |\nabla u'_\varepsilon - z_\varepsilon|^p \, dx &\leq CR_\varepsilon^p \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon) \, d\hat{x}, \end{aligned}$$

where $D(r_\varepsilon, R_\varepsilon) = \{\hat{x} \in \mathbb{R}^2 : r_\varepsilon < |\hat{x}| < R_\varepsilon\}$. Hence, if we choose θ_ε such that

$$\exists M > 0; \quad \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon) \, d\hat{x} \leq M \quad \forall \varepsilon > 0, \quad (2.8)$$

then (2.5) is true. We finally have

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi^2}(u'_\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(z_\varepsilon)\Psi \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} |v - u|^p \phi_p(\widehat{\nabla}\theta_\varepsilon)\Psi \, dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon) \, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |v - u|^p(\hat{y}_\varepsilon^i, x_3)\Psi(\hat{y}_\varepsilon^i, x_3) \, dx_3 \quad (\text{with } \hat{y}_\varepsilon^i \in Y_\varepsilon^i) \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla}\theta_\varepsilon) \, d\hat{x} \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| |v - u|^p(\hat{y}_\varepsilon^i, x_3)\Psi(\hat{y}_\varepsilon^i, x_3) \, dx_3. \end{aligned}$$

Observe that $\lim_{\varepsilon \rightarrow 0} \int_0^L \sum_{i \in I_\varepsilon} |Y_\varepsilon^i| |v - u|^p (\hat{y}_\varepsilon^i, x_3) \Psi(\hat{y}_\varepsilon^i, x_3) dx_3 = \int_\Omega |v - u|^p \Psi dx$. To get the lowest upper bound in Proposition 2.2, it is clear that θ_ε has to be the solution of the capacity problem

$$(\mathcal{P}_\varepsilon^{\text{cap}}) \quad \min \left\{ \int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \varphi) d\hat{x} : \begin{array}{l} \varphi \in W^{1,p}(D(r_\varepsilon, R_\varepsilon)), \\ \varphi(\hat{x}) = 1 \text{ on } |\hat{x}| = r_\varepsilon, \\ \varphi(\hat{x}) = 0 \text{ on } |\hat{x}| = R_\varepsilon. \end{array} \right\}$$

As observed in [2], we have

$$\theta_\varepsilon = \begin{cases} \frac{\log R_\varepsilon - \log |\hat{x}|}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p = 2, \\ \frac{R_\varepsilon^s - |\hat{x}|^s}{R_\varepsilon^s - r_\varepsilon^s} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}) \end{cases}$$

and

$$\int_{D(r_\varepsilon, R_\varepsilon)} \phi_p(\widehat{\nabla} \theta_\varepsilon) d\hat{x} = \frac{2\pi}{p} \Gamma_p(r_\varepsilon, R_\varepsilon),$$

$$\text{where } \Gamma_p(r_\varepsilon, R_\varepsilon) := \begin{cases} \frac{1}{\log R_\varepsilon - \log r_\varepsilon} & \text{if } p = 2, \\ \left(\frac{s}{R_\varepsilon^s - r_\varepsilon^s} \right)^{p-1} & \text{if } p \neq 2 \quad (s = \frac{p-2}{p-1}). \end{cases}$$

Note that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \Gamma_p(r_\varepsilon, R_\varepsilon) = \gamma.$$

If $\gamma < +\infty$, then (2.8) is satisfied and

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = \frac{2\pi\gamma}{p} \int_\Omega |v - u|^p \Psi dx.$$

When $\gamma = +\infty$, it suffices to prove that $\lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 2}(u'_\varepsilon) = 0$. Due to (2.7), the result is a consequence of $F_\varepsilon^{\Psi 2}(u'_\varepsilon) \leq CR_\varepsilon^p \varepsilon^{-2} \Gamma_p(r_\varepsilon, R_\varepsilon)$, which tends to zero.

Now, we consider the remaining part

$$F_\varepsilon^{\Psi 3}(u'_\varepsilon) := \int_{T_\varepsilon} \lambda_\varepsilon \phi_p(\nabla w_\varepsilon) \Psi dx.$$

Recalling the assumption (2.1) on v and using the local Lipschitz property (2.6), we deduce

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon^{\Psi 3}(u'_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|T_\varepsilon|} \int_{T_\varepsilon} \lambda_\varepsilon \phi_p \left(\frac{\partial v}{\partial x_3} \right) \Psi dx \\ &= \frac{k\pi}{p} \int_\Omega \left| \frac{\partial v}{\partial x_3} \right|^p \Psi dx, \end{aligned}$$

as shown in [2]. ■

Now, we will prove the upper bound equality by using the sharper approximation $(u_\varepsilon^\#)$. We start with

$$F_\varepsilon(u_\varepsilon^\#) = \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) dx + \int_{\Omega \setminus \Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) dx. \quad (2.9)$$

Conditions (2.7) imply $|u'_\varepsilon - u| \leq C(r_\varepsilon 1_{T_\varepsilon} + R_\varepsilon 1_{B_\varepsilon})$. Hence

$$\int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) dx \leq C \left(|\Sigma_\varepsilon| + \int_{\Sigma_\varepsilon} a_\varepsilon(x) |\nabla u'_\varepsilon|^p dx + \lambda_\varepsilon |T_\varepsilon \cap \Sigma_\varepsilon| + \left(\frac{R_\varepsilon}{r_\varepsilon} \right)^p |\Sigma_\varepsilon| \right).$$

Lemma 2.4 implies that for every $\Psi \in C^0(\bar{\Omega}, [0, 1])$, such that $\Psi = 1$ on a small neighborhood of $\Gamma_0 \cap (\omega_0 \cup \omega_L)$,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) dx &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} a_\varepsilon \phi_p(\nabla u'_\varepsilon) \Psi dx \\ &= \int_{\Omega} \left(\phi_p(\nabla u) + \frac{k\pi}{p} \left| \frac{\partial v}{\partial x_3} \right|^p + \frac{2\pi\gamma}{p} |v - u|^p \right) \Psi dx. \end{aligned}$$

Thus, by letting Ψ tend to zero, we deduce

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u_\varepsilon^\#) dx = \lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) dx = 0,$$

and

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) dx &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} a_\varepsilon \phi_p(\nabla u'_\varepsilon) dx - \int_{\Sigma_\varepsilon} a_\varepsilon \phi_p(\nabla u'_\varepsilon) dx \right) \\ &= \int_{\Omega} \phi_p(\nabla u) + \frac{k\pi}{p} \left| \frac{\partial v}{\partial x_3} \right|^p + \frac{2\pi\gamma}{p} |v - u|^p dx, \end{aligned}$$

which proves the result for (u, v) smooth. We complete the proof by a standard approximation of (u, v) and a diagonalization argument [1].

2.3 Proof of Proposition 2.3

It is enough to consider $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Due to the compactness property, (u, v) is in $(L^p(\Omega))^2$. We first consider the term $F_\varepsilon^2(u_\varepsilon)$. Let (u_η, v_η) be Lipschitz on $\bar{\Omega}$ such that $\lim_{\eta \rightarrow 0} \|u_\eta - u\|_{L^p(\Omega)} + \|v_\eta - v\|_{L^p(\Omega)} = 0$. We define $(v_\eta - u_\eta)_\varepsilon := \sum_{i \in I_\varepsilon} (v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3) 1_{Y_\varepsilon^i}$ and $z_{\eta\varepsilon} := (v_\eta - u_\eta)_\varepsilon \widehat{\nabla} \theta_\varepsilon$. Because of the local Lipschitz property (2.6) of ϕ_p and $(u, v) \in (L^p(\Omega))^2$, Hölder inequality implies

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon}) - \phi_p(z_\varepsilon) dx = 0.$$

The proof of the upper bound equality allows us to write

$$\lim_{\varepsilon \rightarrow 0} \phi_p(z_{\eta\varepsilon}) = \frac{2\pi\gamma}{p} \int_{\Omega} |v_\eta - u_\eta|^p dx.$$

The convexity of ϕ_p and the fact that $\phi_p(\nabla u_\varepsilon) \geq \phi_p(\widehat{\nabla} u_\varepsilon)$ yield

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^2(u_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(\widehat{\nabla} u_\varepsilon) dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi_p(z_{\eta\varepsilon}) dx \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot (\widehat{\nabla} u_\varepsilon - z_{\eta\varepsilon}) dx. \end{aligned} \quad (2.10)$$

The very definition of ϕ_p implies

$$\begin{aligned} \phi'_p(\xi) &= |\xi|^{p-2} \xi && \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3, \\ \phi'_p(t\xi) &= \phi'_p(t) \phi'_p(\xi) && \forall (t, \xi) \in \mathbb{R} \times \mathbb{R}^n, \quad n = 1, 2, 3, \\ \phi'_p(\xi) \cdot \xi &= p\phi_p(\xi) && \forall \xi \in \mathbb{R}^n, \quad n = 1, 2, 3. \end{aligned}$$

Hence

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot z_{\eta\varepsilon} dx = 2\pi\gamma \int_{\Omega} |v_\eta - u_\eta|^p dx. \quad (2.11)$$

For the other term of (2.10), we have

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla} u_\varepsilon dx = \sum_{i \in I_\varepsilon} \int_0^L \phi'_p(v_\eta - u_\eta)(\hat{x}_\varepsilon^i, x_3) \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \widehat{\nabla} u_\varepsilon d\hat{x} dx_3,$$

where $D^i(r_\varepsilon, R_\varepsilon) = D_{R_\varepsilon}^i \setminus \overline{D_{r_\varepsilon}^i}$. Let ν be the outer normal on $\partial D^i(r_\varepsilon, R_\varepsilon)$, the very definition of θ_ε as a solution of $(\mathcal{P}_\varepsilon^{\text{cap}})$ yields

$$\begin{aligned} \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \widehat{\nabla} u_\varepsilon d\hat{x} &= \int_{\partial D^i(r_\varepsilon, R_\varepsilon)} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl \\ &= \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl + \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl \\ &= -\tilde{u}_\varepsilon^i \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) dl + \tilde{v}_\varepsilon^i \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl \end{aligned}$$

where $\tilde{u}_\varepsilon^i := \frac{\int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl}{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) dl} = \frac{1}{2\pi R_\varepsilon} \int_{\partial D_{R_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) u_\varepsilon dl$, $\tilde{u}_\varepsilon := \sum_{i \in I_\varepsilon} \tilde{u}_\varepsilon^i 1_{Y_\varepsilon^i}$,

$\tilde{v}_\varepsilon^i := \frac{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) v_\varepsilon dl}{\int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) dl} = \frac{1}{2\pi r_\varepsilon} \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) v_\varepsilon dl$, $\tilde{v}_\varepsilon := \sum_{i \in I_\varepsilon} \tilde{v}_\varepsilon^i 1_{Y_\varepsilon^i}$. Thus,

$$\begin{aligned} \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \widehat{\nabla} u_\varepsilon d\hat{x} &= (\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i) \int_{\partial D_{r_\varepsilon}^i} (\phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \nu) dl \\ &= (\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i) \int_{D^i(r_\varepsilon, R_\varepsilon)} \phi'_p(\widehat{\nabla} \theta_\varepsilon) \cdot \widehat{\nabla} \theta_\varepsilon d\hat{x} \\ &= 2\pi\Gamma_p(r_\varepsilon, R_\varepsilon) (\tilde{v}_\varepsilon^i - \tilde{u}_\varepsilon^i), \end{aligned}$$

and

$$\int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla} u_\varepsilon \, dx = 2\pi\Gamma_p(r_\varepsilon, R_\varepsilon) \int_{\Omega} \phi'_p((v_\eta - u_\eta)_\varepsilon)(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) \, dx.$$

It was shown in [2] that $(\tilde{v}_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup (v - u)$ in $L^p(\Omega)$. On the other hand, $(v_\eta - u_\eta)$ being smooth and ϕ'_p being continuous from $L^p(\Omega)$ to $L^{p'}(\Omega)$, we have $\phi'_p((v_\eta - u_\eta)_\varepsilon) \rightarrow \phi'_p(v_\eta - u_\eta)$ in $L^{p'}(\Omega)$. Hence,

$$\lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} \phi'_p(z_{\eta\varepsilon}) \cdot \widehat{\nabla} u_\varepsilon \, dx = 2\pi\gamma \int_{\Omega} \phi'_p(v_\eta - u_\eta)(v - u) \, dx. \quad (2.12)$$

Therefore, (2.10), (2.11) and (2.12) imply

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} F_\varepsilon^2(u_\varepsilon) &\geq \frac{2\pi\gamma}{p} \int_{\Omega} |v_\eta - u_\eta|^p \, dx \\ &\quad + 2\pi\gamma \left[\int_{\Omega} |v_\eta - u_\eta|^p \, dx - \int_{\Omega} \phi'_p(v_\eta - u_\eta)(v - u) \, dx \right]. \end{aligned}$$

The expected lower bound for $F_\varepsilon^2(u_\varepsilon)$ is obtained by letting η tend to zero.

To complete the proof it suffices to use the arguments of [2] concerning the lower bounds for $F_\varepsilon^1(u_\varepsilon)$, $F_\varepsilon^3(u_\varepsilon)$ and the fact that v belongs to $L^p(\omega, W^{1,p}(0, L))$.

2.4 The Final Result

The following theorem, a convergence result for the minimizer of $(\mathcal{P}_\varepsilon)$, is a standard consequence of the previous three propositions.

Theorem 2.5 *Let the assumptions (1.3) and (1.5) hold with $(k, \gamma) \in (0, +\infty)$. Then the unique solution \bar{u}_ε of $(\mathcal{P}_\varepsilon)$ converges weakly in $W^{1,p}(\Omega)$ to the unique solution \bar{u} of the problem*

$$\min \{ \min \{ \Phi(u, v) - L(u) : v \in L^p(\Omega) \} : u \in L^p(\Omega) \},$$

where Φ and L are defined by (1.4) and (1.2) respectively.

Proof. The proof of this theorem is the same as that in [2]. ■

3 Conclusions and Remarks

The previous analysis can be easily extended to the case when ϕ_p is replaced by any strictly convex function which satisfies

$$\exists M > 0, \exists r \in (1, p); \quad |W(\xi) - \phi_p(\xi)| \leq M|\xi|^r \quad \forall \xi \in \mathbb{R}^3, \quad (3.1)$$

the density function associated with $\Phi(u, v)$ becomes

$$W(\nabla u) + 2\pi\gamma|v - u|^p + W\left(\frac{\partial v}{\partial x_3}\right).$$

Indeed, (3.1) and Hölder inequality imply

$$\left| \int_{B_\varepsilon} W(\nabla u_\varepsilon) dx - \int_{B_\varepsilon} \phi_p(\nabla u_\varepsilon) dx \right| \leq M|B_\varepsilon|^{1-\frac{r}{p}} \int_{\Omega} |\nabla u_\varepsilon|^p dx,$$

while our arguments and those of [2] to derive the upper bound and lower bound respectively are valid when ϕ_p is replaced by any convex function satisfying a growth condition like

$$\exists \alpha, \beta > 0; \quad \alpha(|\xi|^p - 1) \leq W(\xi) \leq \beta(1 + |\xi|^p) \quad \forall \xi \in \mathbb{R}^3,$$

which is an obvious consequence of (3.1).

Eventually, the key arguments of our analysis are the identification of $\gamma, \theta_\varepsilon$ in terms of the solution of capacity problems and the use of the p -positive homogeneity and convexity of ϕ_p and of the fact that $\phi_p(\xi) \geq \phi_p(\hat{\xi})$. Thus, it is easy to guess what could be $\Phi(u, v)$, when ϕ_p is replaced by any strictly convex function and when the cross sections of the fibers are smooth star-shaped domains of \mathbb{R}^2 . We hope that our proposed strategy will be able to reduce and overcome the involved technical difficulties.

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