Spectral statistics for weakly correlated random potentials

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ABSTRACT. We study localization and derive stochastic estimates (in particular, Wegner and Minami estimates) for the eigenvalues of weakly correlated random discrete Schrödinger operators in the localized phase. We apply these results to obtain spectral statistics for general discrete alloy type models where the single site perturbation is neither of finite rank nor of fixed sign. In particular, for the models under study, the random potential exhibits correlations at any range.

RÉSUMÉ. On étudie la localisation et on obtient des estimées probabilistes (en particulier des estimées de Wegner et de Minami) pour les valeurs propres d'opérateurs de Schrödinger aléatoires discrets faiblement corrélés. Les résultats obtenus sont ensuite appliqués pour étudier les statistiques spectrales de modèles généraux de type alliage pour lesquels le potentiel de simple site n'est ni de signe constant ni de support compact. En particulier, pour les modèles étudiés, le potentiel aléatoire est corrélé à toutes les échelles.

1. Introduction: a model with long range correlations

Consider a single site potential $u : \mathbb{Z}^d \to \mathbb{R}$. Assume that

(S): $u \in \ell^1(\mathbb{Z}^d)$;

(H): the continuous function $\theta \mapsto \sum_{n\in\mathbb{Z}^d} u_n e^{in\theta}$ does not vanish on $\mathbb{R}^d$.

Let $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$ be real valued bounded i.i.d. random variables. Define the random ergodic Schrödinger operator H_ω as

$$H_\omega = -\Delta + \lambda \sum_{n\in\mathbb{Z}^d} \omega_n \tau_n u \tag{1.1}$$

where

- $\tau_n u$ is the potential u shifted to site n i.e. $(\tau_n u)_k = u(k-n)$
- the coupling constant λ is positive.

Define μ to be the common distribution of $(\omega_n)_{n\in\mathbb{Z}^d}$ and let S be the concentration function of μ, that is, $S(s) = \sup_{a\in\mathbb{R}} \mu([a, a+s])$ for $s \geq 0$.

Let us now assume that

(R): S is Lipschitz continuous at 0 i.e. there exists $C > 0$ such that $S(s) \leq Cs$ for $s \geq 0$.

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For $\Lambda \subset \mathbb{Z}^d$ a finite cube and $J \subset \mathbb{R}$ a measurable set, we define $P_\omega^{(\Lambda)}(J)$ to be the spectral projector of $H_\omega(\Lambda)$ that is H_ω restricted to Λ (with periodic boundary conditions) onto the energy interval J

$$P_\omega^{(\Lambda)}(J) = \mathbf{1}_J(H_\omega(\Lambda)).$$

Define N the integrated density of states of H_ω as

$$N(E) = \lim_{|\Lambda|\to+\infty} \frac{1}{|\Lambda|} \operatorname{tr} P_\omega^{(\Lambda)}((-\infty, E]) \tag{1.2}$$

where tr denotes the trace.
Almost surely, the limit (1.2) exists for all E real (see e.g. [12, 9]); it defines the distribution function of some probability measure, say, $dN(E)$, the support of which is the almost sure spectrum of H_ω (see e.g. [12, 9]). Let Σ denote the almost sure spectrum of H_ω.
We first prove

Theorem 1.1. *Under the assumptions (S), (H) and (R), for H_ω, one has*

1. *the integrated density of states N is uniformly Lipschitz continuous; N is almost everywhere differentiable*
2. *H_ω satisfies a Wegner and a Minami estimate, that is, there exists $C_\lambda > 0$ such that, for any bounded interval I and k, a positive integer, we have*

$$\mathbb{E}\left[\operatorname{tr}\left(P_\omega^{(\Lambda)}(I)\right) \cdot \left(\operatorname{tr}\left(P_\omega^{(\Lambda)}(I) - 1\right) \cdots \left(\operatorname{tr}\left(P_\omega^{(\Lambda)}(I) - (k-1)\right)\right]\right] \leq \left(C_\lambda |I| \, |\Lambda|\right)^k ;$$

3. *for λ sufficiently large, the whole spectrum of H_ω is localized i.e. there exists η such that, for any $L \geq 1$, if $\Lambda = \Lambda_L$ is the cube of center 0 and side-length L, one has, for any $L \geq 1$ and any $p > d$, there is $q = q_{p,d}$ so that, for any L large enough, the following holds with probability at least $1 - L^{-p}$: for any eigenvector $\varphi_{\omega,\Lambda,j}$ of $H_\omega(\Lambda)$, there exists a center of localization $x_{\omega,\Lambda,j}$ in Λ, so that for any $x \in \Lambda$, one has*

$$\|\varphi_{\omega,\Lambda,j}\|_x \leq L^q e^{-\eta|x - x_{\omega,\Lambda,j}|}.$$

As we shall see below, these conclusion holds for more general models of random Schrödinger operators with weakly dependent randomness (see Theorems 2.1, 2.2 and 2.3 below).

For $L \in \mathbb{N}$, let $\Lambda = \Lambda_L = [-L, L]^d \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ be a large box; recall that $H_\omega(\Lambda)$ is the operator H_ω restricted to Λ with periodic boundary conditions. Let $|\Lambda|$ be the volume of Λ i.e. $|\Lambda| = (2L+1)^d$.

$H_\omega(\Lambda)$ is an $|\Lambda| \times |\Lambda|$ real symmetric matrix. By point (2) of Theorem 1.1, we know that the eigenvalues of $H_\omega(\Lambda)$ are almost surely simple. Let us denote them ordered increasingly by $E_1(\omega, \Lambda) < E_2(\omega, \Lambda) < \cdots < E_{|\Lambda|}(\omega, \Lambda)$.

Let E_0 be an energy in Σ such that N is differentiable at E_0. Let $n(E_0)$ denote the derivative of N at the energy E_0. The local level statistics near E_0 is the point process defined by

$$\Xi(\xi, E_0, \omega, \Lambda) = \sum_{j=1}^{|\Lambda|} \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi) \tag{1.3}$$

where

$$\xi_j(E_0,\omega,\Lambda) = |\Lambda|\, n(E_0)\,(E_j(\omega,\Lambda) - E_0), \quad 1 \le j \le |\Lambda|. \tag{1.4}$$

We prove

Theorem 1.2. *In addition to (S), (H) and (R), assume that*

(D): *for some* $\eta > d - \frac{1}{2}$,

$$\limsup_{|n|\to+\infty} |n|^{\alpha}\,|u(n)| < +\infty.$$

Then, for λ *sufficiently large, for* E_0*, an energy in* Σ *such that* $n(E_0)$ *exist and be positive, when* $|\Lambda| \to +\infty$*, the point process* $\Xi(E_0,\omega,\Lambda)$ *converges weakly to a Poisson process on* $\mathbb{R}$ *with intensity* 1*. That is, for* $p > 0$ *arbitrary, for arbitrary non empty open two by two disjoint intervals* $I_1,\dots,I_p$ *and arbitrary integers* $k_1,\cdots,k_p$*, one has*

$$\lim_{|\Lambda|\to+\infty} \mathbb{P}\left(\left\{\omega;\ \begin{array}{c}\#\{j;\ \xi_j(E_0,\omega,\Lambda)\in I_1\} = k_1\\ \vdots \qquad\qquad \vdots \\ \#\{j;\ \xi_j(E_0,\omega,\Lambda)\in I_p\} = k_p\end{array}\right\}\right) = \frac{|I_1|^{k_1}}{k_1!}e^{-|I_1|}\dots\frac{|I_p|^{k_p}}{k_p!}e^{-|I_p|}. \tag{1.5}$$

When the single site potential u is supported in a single point, this result was first obtained [11]. This result is typical of the localized phase of random operators; a general analysis of the spectral statistics for the eigenvalues was developed in [6] (see also [7]). There, it was shown that, in the localized regime, under the additional "independence at a distance" (IAD) assumption, if the random model satisfies a Wegner and a Minami estimate (see below for more details), then the convergence to Poisson for the local statistics holds.

The (IAD) assumption requires that there exists some fixed positive distance, say, D such that, for any Λ and Λ' at least at a distance D apart from each other, the random operators $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent. For a general u chosen as above, this assumption is not fulfilled. There are long range correlations. We will show that they do not suffice to induce correlations between the eigenvalues asymptotically.

Remark 1.1. Note that if u is not too "lacunary" assumption (S) actually implies that $|u(n)| \lesssim |n|^{-d}$, that is, in particular that assumption (D) holds. From the analysis done in [6], it is quite clear that, for the model (1.1), if $|u(n)| = o(|n|^{-2d+1-\varepsilon})$ for some $\varepsilon > 0$, then, the correlation are much smaller than the typical spacing between the eigenvalues; thus, they should not play a role in the statistics.

As the proof of Theorem 1.2 shows (see, in particular, the proof of Lemma 3.4), condition (D) can be relaxed to

$$\limsup_{|n|\to+\infty} |n|^{d-1/2}\,\log|n|\,|u(n)| < \eta$$

for sufficiently small η.

Following the proof of Theorem 1.2 and that of [6, Theorem 1.13], one proves

Theorem 1.3. *Under the assumptions of Theorem 1.2, for $1 \leq j \leq |\Lambda|$, let $x_j(E_0, \omega, \Lambda)$ be a localization center associated to the eigenvalue $E_j(\omega, \Lambda)$ (see Theorem 1.1). Then, the joint (eigenvalue - localization center) local point process defined by*

$$\Xi^2_\Lambda(\xi, x; E_0, \Lambda) = \sum_{j=1}^{N} \delta_{\xi_j(E_0,\omega,\Lambda)}(\xi) \otimes \delta_{x_j(E_0\omega,\Lambda)/L}(x)$$

converges weakly to a Poisson point process on $\mathbb{R}\times[-1/2, 1/2]^d$ with intensity 1.

In [6], many other spectral statistics are studied; using the ideas developed there and in the present work, they can also be studied for the model (1.1).

To close this introduction, we note that, in [13], Theorem 1.2 is proved for H_ω under more restrictive assumptions: u is assumed to be of compact support and the common density of the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ is supposed to be sufficiently regular. In particular, under these assumptions, the operator H_ω satisfies the independence at a distance assumption.

2. Weakly correlated random potentials

On $\ell^2(\mathbb{Z}^d)$, consider the discrete Anderson model

$$H_{\tilde\omega} = -\Delta + \lambda V_{\tilde\omega} \text{ where } V_{\tilde\omega} = ((\tilde\omega_n \delta_{nm}))_{(n,m)\in\mathbb{Z}^d\times\mathbb{Z}^d} \tag{2.1}$$

and δ_{nm} is the Kronecker symbol and λ is positive coupling constant. Assume that the random variables $(\tilde\omega_n)_{n\in\mathbb{Z}^d}$ are non trivial, real valued and bounded. They are not assumed to be independent or identically distributed. Let $\Lambda \subset \mathbb{Z}^d$ be a finite cube and $J \subset \mathbb{R}$ a measurable set, we define $P^{(\Lambda)}_{\tilde\omega}(J)$ to be the spectral projector of $H_{\tilde\omega}(\Lambda)$, that is, $H_{\tilde\omega}$ restricted to Λ (with periodic boundary conditions) onto the energy interval J

$$P^{(\Lambda)}_{\tilde\omega}(J) = \mathbf{1}_J(H_{\tilde\omega}(\Lambda)).$$

For the operator $H_{\tilde\omega}$, we will now state a number of results on

- eigenvalue decorrelation estimates,
- localization,
- local representations of the eigenvalues of $H_{\tilde\omega}(\Lambda)$ in some energy interval.

In the case of independent random variables, these results are the basic building blocks of the analysis of the spectral statistics performed in [6]. We shall then show that these results can be applied to the model (1.1) and analyze the spectral statistics of this model to prove Theorem 1.2.

2.1. Basic estimates on eigenvalues - decorrelation estimates.

We first need some estimates on the occurrence of eigenvalues in given intervals as well as on their correlations. These give descriptions of the eigenvalues seen as random variables that have proved crucial in the study of spectral statistics. In these estimates, we actually do not need the random variables $(\tilde\omega_n)_{n\in\mathbb{Z}^d}$ to be identically distributed.

Define $\tilde\mu_m$ to be the distribution of $\tilde\omega_m$ conditioned on $(\tilde\omega_n)_{n\neq m}$ and let $\tilde S_m$

be the concentration function of $\tilde{\mu}_m$, that is, $\tilde{S}_m(s) = \sup_{a\in\mathbb{R}} \tilde{\mu}_m([a, a+s])$. For $\Lambda \subset \mathbb{Z}^d$, define

$$\tilde{S}_\Lambda := \sup_{m\in\Lambda} \operatorname*{ess\,sup}_{(\tilde{\omega}_n)_{n\neq m}} \tilde{S}_m$$

We prove

Theorem 2.1. *There exists $C > 0$ such that*

- *for any bounded interval I, we have*

$$\mathbb{E}\left[\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I)\right)\right] \leq C\tilde{S}_\Lambda(|I|)|\Lambda|; \tag{2.2}$$

- *for any two bounded intervals $I_1 \subset I_2$, we have*

$$\mathbb{E}\left[\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I_1)\right)\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I_2) - 1\right)\right)\right] \leq C^2\tilde{S}_\Lambda(|I_1|)\tilde{S}_\Lambda(|I_2|)|\Lambda|^2; \tag{2.3}$$

- *specializing to the case $I_1 = I_2 = I$, (2.3) reads*

$$\mathbb{E}\left[\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I)\right)\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I) - 1\right)\right)\right] \leq C^2\left(\tilde{S}_\Lambda(|I|)|\Lambda|\right)^2; \tag{2.4}$$

- *for any $k \geq 1$, and arbitrary intervals $I_1 \subset I_2 \subset \cdots \subset I_k$, one has*

$$\mathbb{E}\left[\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I_1)\right)\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I_2) - 1\right)\right)\cdots\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I_k) - (k-1)\right)\right)\right] \leq C^k|\Lambda|^k\prod_{j=1}^{k}\tilde{S}_\Lambda(|I_j|); \tag{2.5}$$

- *specializing to the case $I_1 = \cdots = I_k = I$, (2.5) reads*

$$\mathbb{E}\left[\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I)\right)\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I) - 1\right)\right)\cdots\left(\operatorname{tr}\left(P_{\tilde{\omega}}^{(\Lambda)}(I) - (k-1)\right)\right)\right] \leq C^k\left(\tilde{S}_\Lambda(|I|)|\Lambda|\right)^k. \tag{2.6}$$

Remark 2.1. In Theorem 2.1, as the proof shows, the results stay valid if the spectral projector $P_{\tilde{\omega}}^{(\Lambda)}(I)$ is replaced by the spectral projector on I for $H_{\tilde{\omega}}$ restricted to Λ with different boundary conditions.

Estimate (2.2) is the so called "Wegner estimate"; it was first obtained in [15] when the random variables $(\omega_n)_{n\in\mathbb{Z}^d}$ are independent and identically distributed. We refer the reader to [14] for more details on Wegner estimates for various models. Estimate (2.5) is the so called "Minami estimate"; it was first obtained in [11] when $(\tilde{\omega}_n)_n$ are independent and identically distributed (see also [8, 2]). Later, estimates (2.4) – (2.6) were proved for independent random variables in [3]. The method developed in [3] can be transposed verbatim to the present case as the proof below shows.

Proof of Theorem 2.1. The proof essentially consists in repeating verbatim the proof of [3, Theorem 2.1] replacing expectations by conditional expectations. We give some details for the reader's convenience.
We first recall the following well-known spectral averaging lemma (in a form specialized to our setting)

Lemma 2.1. *Let H_0 be a self-adjoint operator of $\ell^2(\mathbb{Z}^d)$ and define $H_t = H_0 + t\pi_0$ where $t \in \mathbb{R}$ and $\pi_0 = ((\delta_{n0}\delta_{0m}))_{(n,m)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ is the orthogonal projector on the unit vector $\delta_0 = (\delta_{n0})_{n\in\mathbb{Z}^d}$.*
Let $\mathbf{1}_I(H_t)$ be the spectral projector of H_t associated to the energy interval I.
Let μ be an arbitrary, non trivial, compactly supported probability measure on $\mathbb{R}$.
Then, for any bounded interval I, one has

$$\int_{\mathbb{R}} \langle \delta_0, \mathbf{1}_I(H_t)\delta_0\rangle d\mu(t) \leq 8\, S_\mu(|I|).$$

A proof of this result can be found e.g. in [4] or in the appendix of [3].
This immediately yields (2.2). Indeed, defining the vector $\delta_n = (\delta_{nm})_{m\in\mathbb{Z}^d}$ for $n \in \mathbb{Z}^d$, one writes

$$\mathbb{E}\left[\operatorname{tr}\left(P^{(\Lambda)}_{\tilde\omega}(I)\right)\right] = \sum_{n\in\Lambda} \mathbb{E}\left[\langle \delta_n, \mathbf{1}_I(H_{\tilde\omega}(\Lambda))\delta_n\rangle\right]. \tag{2.7}$$

Now, the operator $H_{\tilde\omega}(\Lambda)$ may be written as $H_{\tilde\omega}(\Lambda) = H^n_{\tilde\omega}(\Lambda) + \tilde\omega_n\pi_n$ where π_n is the orthogonal projector on δ_n. To evaluate the expectation in (2.7), in the nth term, we first compute the expectation with respect to $\tilde\omega_n$ conditioned on $(\tilde\omega_m)_{m\neq n}$ and, by Lemma 2.1, we get

$$\mathbb{E}_{\tilde\omega_n}\left[\langle \delta_n, \mathbf{1}_I(H_{\tilde\omega}(\Lambda))\delta_n\rangle \,|\, (\tilde\omega_m)_{m\neq n}\right] \leq 8\tilde S_n(|I|).$$

Plugging this into (2.7) and using the definition of $\tilde S_\Lambda$ immediately yields (2.2).
Let us now turn to the proof of (2.3); we won't give a detailed proof of inequality (2.5) as it is very similar to that of (2.3); we refer to [3] for details.
We recall [3, Lemma 4.1] specialized to our setting.

Lemma 2.2 ([3])**.** *Assume we are in the setting of Lemma 2.1. Assume moreover that $\operatorname{tr}(\mathbf{1}_I(H_0)) < +\infty$ for any $I \subset \mathbb{R}$.*
Then, for arbitrary $a < b$ real and $0 \leq s \leq t$, we have

$$\operatorname{tr}(\mathbf{1}_{(a,b]}(H_s)) \leq 1 + \operatorname{tr}(\mathbf{1}_{(a,b]}(H_t)).$$

To prove (2.3), we follow the proof of [3, Theorem 2.1]. Let $M = \sup_n \operatorname{ess\,sup} \tilde\omega_n$ and $m = \inf_n \operatorname{ess\,inf} \tilde\omega_n$. Pick $\tau_n \geq M$. Then, by Lemma 2.2, one computes

$$\begin{aligned}\operatorname{tr}(P^{(\Lambda)}_{\tilde\omega}(I_1))\left(\operatorname{tr}(P^{(\Lambda)}_{\tilde\omega}(I_2) - 1\right) &= \sum_{n\in\Lambda} \langle \delta_n, P^{(\Lambda)}_{\tilde\omega}(I_1)\delta_n\rangle \left(\operatorname{tr}(P^{(\Lambda)}_{\tilde\omega}(I_2) - 1\right)\\ &\leq \sum_{n\in\Lambda} \langle \delta_n, P^{(\Lambda)}_{\tilde\omega}(I_1)\delta_n\rangle \operatorname{tr}\left(P^{(\Lambda)}_{((\tilde\omega_m)_{m\neq n},\tau_n)}(I_2)\right).\end{aligned}$$

Thus, taking the expectation with respect to $\tilde\omega_n$ conditioned on $(\tilde\omega_m)_{m\neq n}$, using the spectral averaging lemma, Lemma 2.1, we obtain

$$\begin{aligned}\mathbb{E}\Big(\operatorname{tr}(P^{(\Lambda)}_{\tilde\omega}(I_1))\Big(\operatorname{tr}(P^{(\Lambda)}_{\tilde\omega}(I_2) - 1\Big) \,|(\tilde\omega_m)_{m\neq n}\Big)&\\ \leq 8\tilde S_\Lambda(|I_1|)\sum_{n\in\Lambda} \operatorname{tr}\left(P^{(\Lambda)}_{((\tilde\omega_m)_{m\neq n},\tau_n)}(I_2)\right).&\end{aligned} \tag{2.8}$$

for arbitrary $(\tau_n)_{n\in\Lambda}$ such that $\tau_n \geq M$. Thus, we can set $\tau_n = \tilde{\tau}_n + M - m$ where $\tilde{\tau}_n$ is the random variable $\tilde{\omega}_n$ conditioned on $(\tilde{\omega}_m)_{m\neq n}$.
We then take the expectation with respect to $\tilde{\omega}$ on both side in (2.8) to obtain

$$\begin{aligned}\mathbb{E}\left(\mathrm{tr}\,(P^{(\Lambda)}_{\tilde{\omega}}(I_1))\left(\mathrm{tr}\,(P^{(\Lambda)}_{\tilde{\omega}}(I_2)-1\right)\right) &\leq 8\tilde{S}_\Lambda(|I_1|)\\ &\quad\cdot\sum_{n\in\Lambda}\mathbb{E}\left(\mathrm{tr}\,\left(P^{(\Lambda)}_{((\tilde{\omega}_m)_{m\neq n},\tilde{\omega}_n+M-m)}(I_2)\right)\right)\\ &\leq C\tilde{S}_\Lambda(|I_1|)\sum_{n\in\Lambda}\tilde{S}_\Lambda(|I_2|)|\Lambda|\end{aligned}$$

where in the last step we have used estimate (2.2) (for a different set of random variables).
Thus, we obtain

$$\mathbb{E}\left(\mathrm{tr}\,(P^{(\Lambda)}_{\tilde{\omega}}(I_1))\left(\mathrm{tr}\,(P^{(\Lambda)}_{\tilde{\omega}}(I_2)-1\right)\right) \leq C\tilde{S}_\Lambda(|I_1|)\tilde{S}_\Lambda(|I_2|)|\Lambda|^2$$

that is, (2.3).
This completes the proof of Theorem 2.1. More details can be found in [3]. □

2.2. **Localization.** The second ingredient needed in our analysis is localization. Using the notations above, let us assume that

($\tilde{\mathbf{R}}$): There exists $C > 0$ such that, for $s \geq 0$, one has

$$\sup_{\Lambda\subset\mathbb{Z}^d}\tilde{S}_\Lambda(s) \leq Cs.$$

This in particular implies that, for any m, $\tilde{\mu}_m$, the distribution of $\tilde{\omega}_m$ conditioned on $(\tilde{\omega}_n)_{n\neq m}$ admits a density bounded by C.
Thus, we can apply the results of [1], in particular [1, criterion (1.12)] to obtain

Theorem 2.2. *Assume ($\tilde{R}$) holds. For λ sufficiently large, there exists $\eta = \eta_\lambda$ such that, for any $L \geq 1$, if $\Lambda = \Lambda_L$ is the cube of center 0 and side-length L, one has, for any $L \geq 1$*

$$\sup_{y\in\Lambda}\mathbb{E}\left\{\sum_{x\in\Lambda}e^{\eta|x-y|}\sup_{\substack{\mathrm{supp}\,f\subset I\\ |f|\leq 1}}\|\chi_x f(H_{\tilde{\omega},\Lambda})\chi_y\|_2\right\} < \infty. \tag{2.9}$$

Note that assumption ($\tilde{\mathrm{R}}$) guarantees that, for any $n \in \mathbb{Z}^d$, the distribution of $\tilde{\omega}_n$ conditioned on $(\tilde{\omega}_m)_{m\neq n}$ admits a density that is bounded by a constant that is independent of $(\tilde{\omega}_m)_{m\neq n}$ and n.
While in [1] the proofs are given in the case of independent random variables, in the beginning of [1, section 1.1], it is noted that, in the case of dependent random variables, regularity of the conditional distributions (in particular that implied by assumption ($\tilde{\mathrm{R}}$)) is sufficient to perform the same analysis.
As in [6, Theorem 6.1], we then obtain that, one has

Theorem 2.3. *Assume ($\tilde{R}$) holds. For λ sufficiently large, there exists $\eta = \eta_\lambda$ such that, for any $L \geq 1$, if $\Lambda = \Lambda_L$ is the cube of center 0 and side-length L, one has, for any $L \geq 1$, for all $p > d$, there is $q = q_{p,d}$ so that, for any L large enough, the following holds with probability at least $1 - L^{-p}$: for any eigenvector $\varphi_{\tilde{\omega},\Lambda,j}$ of $H_{\tilde{\omega}}(\Lambda)$, there exists a center of localization $x_{\tilde{\omega},\Lambda,j}$ in Λ such that, for any $x \in \Lambda$, one has*

$$\|\varphi_{\tilde{\omega},\Lambda,j}\|_x \leq L^q e^{-\eta|x - x_{\tilde{\omega},\Lambda,j}|}; \tag{2.10}$$

moreover, two localization centers are at most at a distance $C_p \log L$ away for each other (the positive constant C_p depends on p only).

Note that in the derivation of (2.10) from (2.9) in [6, the proof of Theorem 6.1] the assumption (IAD) was not used. Thus, this proof can be repeated verbatim here to derive Theorem 2.3.

2.3. **A representation theorem for the eigenvalues.** We are now in a position to prove a representation theorem for the eigenvalues of the random operator; it is the analogue of the representation theorem [6, Theorem 1.1]. To this effect, though it is not strictly necessary, it will be convenient to use the density of states of the model $H_{\tilde{\omega}}$. Therefore, it is convenient to assume now that the process $(\tilde{\omega}_n)_{n\in\mathbb{Z}^d}$ is $\mathbb{Z}^d$-ergodic (see e.g. [9]). For the model (1.1), this assumption is clearly satisfied.
Define N the integrated density of states of $H_{\tilde{\omega}}$ as

$$N(E) = \lim_{|\Lambda|\to+\infty} \frac{1}{|\Lambda|} \operatorname{tr} P_{\tilde{\omega}}^{(\Lambda)}((-\infty, E])$$

where tr denotes the trace. The existence of the above limit is a consequence of the ergodicity of the process $(\tilde{\omega}_n)_{n\in\mathbb{Z}^d}$ (see e.g. [9]).
Almost surely, the integrated density of states exists for all E real (see e.g. [12, 9]); it defines the distribution function of some probability measure, say, $dN(E)$, the support of which is the almost sure spectrum of $H_{\tilde{\omega}}$ (see e.g. [12, 9]). Let Σ denote the almost sure spectrum of $H_{\tilde{\omega}}$. To start, pick $\tilde{\rho}$ such that

$$0 \leq \tilde{\rho} < \frac{1}{1+d}. \tag{2.11}$$

Pick E_0 and I_Λ centered at E_0 such that $N(I_\Lambda) \asymp |\Lambda|^{-\alpha}$ for $\alpha \in (\alpha_{d,\tilde{\rho}}, 1)$ where $\alpha_{d,\tilde{\rho}}$ is defined as

$$\alpha_{d,\tilde{\rho}} := (1+\tilde{\rho})\frac{d+1}{d+2}. \tag{2.12}$$

Theorem 2.4. *Pick an energy E_0 such that N is differentiable at E_0 and such that $\dfrac{dN}{dE}(E_0) = n(E_0) > 0$. Pick I_Λ centered at E_0 such that $N(I_\Lambda) \asymp |\Lambda|^{-\alpha}$. There exists $\beta > 0$ and $\beta' \in (0, \beta)$ small so that $1 + \beta < \frac{2\alpha}{1+\tilde{\rho}}$ and, for $\ell \asymp L^\beta$ and $\ell' \asymp L^{\beta'}$, there exists a decomposition of Λ into disjoint cubes of the form $\Lambda_\ell(\gamma_j) := \gamma_j + [0,\ell]^d$ satisfying:*

- $\cup_j \Lambda_\ell(\gamma_j) \subset \Lambda$,
- $\operatorname{dist}(\Lambda_\ell(\gamma_j), \Lambda_\ell(\gamma_k)) \geq \ell'$ *if* $j \neq k$,
- $\operatorname{dist}(\Lambda_\ell(\gamma_j), \partial\Lambda) \geq \ell'$
- $|\Lambda \setminus \cup_j \Lambda_\ell(\gamma_j)| \lesssim |\Lambda|\ell'/\ell$,

and such that, for L sufficiently large, there exists a set of configurations $\mathcal{Z}_\Lambda$ s.t.:

- $\mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - |\Lambda|^{-(\alpha-\alpha_{d,\tilde{\rho}})}$,
- *for $\tilde{\omega} \in \mathcal{Z}_\Lambda$, each centers of localization associated to $H_{\tilde{\omega}}(\Lambda)$ belong to some $\Lambda_\ell(\gamma_j)$ and each box $\Lambda_\ell(\gamma_j)$ satisfies:*
  1. *the Hamiltonian $H_{\tilde{\omega},\Lambda_\ell(\gamma_j)}$ has at most one eigenvalue in I_Λ, say, $E_j(\tilde{\omega}, \Lambda_\ell(\gamma_j))$;*
  2. *$\Lambda_\ell(\gamma_j)$ contains at most one center of localization, say $x_{k_j}(\tilde{\omega}, \Lambda)$, of an eigenvalue of $H_{\tilde{\omega}}(\Lambda)$ in I_Λ, say $E_{k_j}(\tilde{\omega}, \Lambda)$;*
  3. *$\Lambda_\ell(\gamma_j)$ contains a center $x_{k_j}(\tilde{\omega}, \Lambda)$ if and only if $\sigma(H_{\tilde{\omega},\Lambda_\ell(\gamma_j)}) \cap I_\Lambda \neq \emptyset$; in which case, one has*

$$|E_{k_j}(\tilde{\omega}, \Lambda) - E_j(\tilde{\omega}, \Lambda_\ell(\gamma_j))| \leq e^{-\eta \ell'/2} \quad \text{and} \quad \operatorname{dist}(x_{k_j}(\tilde{\omega}, \Lambda), \Lambda \setminus \Lambda_\ell(\gamma_j)) \geq \ell'. \tag{2.13}$$

where η is given by Theorem 2.3.

In particular, if $\tilde{\omega} \in \mathcal{Z}_\Lambda$, all the eigenvalues of $H_{\tilde{\omega}}(\Lambda)$ are described by (2.13).

The proof of this result is a verbatim repetition the proof of [6, Theorem 1.1] verbatim. The independence at a distance is not used; only the localization and the Wegner and Minami estimates are used.
Note that now the "local" Hamiltonians $(H_{\tilde{\omega},\Lambda_\ell(\gamma_j)})_j$ are not necessarily stochastically independent.
In our application of Theorem 2.4 to the model (1.1), the choice we will make for the parameters α, β', β and $\tilde{\rho}$ will be quite different from the one made in [6]. While in [6] the authors wanted to maximize the admissible size f $|I_\Lambda$ (i.e. minimize α), here, our primary concern will be to control the non independence of the "local" Hamiltonians $(H_{\tilde{\omega},\Lambda_\ell(\gamma_j)})_j$. To control their correlations, we will pick β' close to 1. Thus, α and β will also have to be close to 1 and $\tilde{\rho}$ will be close to 0.

2.4. **The local distribution of the eigenvalues.** The final ingredient needed for the analysis of the spectral statistics is a precise description of the distribution of the spectrum of the "local" Hamiltonians $(H_{\tilde{\omega},\Lambda_\ell(\gamma_j)})_j$ constructed in Theorem 2.4 within I_Λ.
Pick ℓ large and $1 \ll \ell' \ll \ell$. Consider a cube Λ of side-length ℓ i.e. $\Lambda = \Lambda_\ell$ and an interval $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$ (i.e. I_Λ is contained in the localization region). Consider the following random variables:

- $X = X(\Lambda, I_\Lambda) = X(\Lambda, I_\Lambda, \ell')$ is the Bernoulli random variable

$$X = \mathbf{1}_{H_{\tilde{\omega}}(\Lambda) \text{ has exactly one eigenvalue in } I_\Lambda \text{ with localization center in } \Lambda_{\ell-\ell'}};$$

- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$ is the eigenvalue of $H_{\tilde{\omega}}(\Lambda)$ in I_Λ conditioned on $X = 1$;
- $\tilde{\xi} = \tilde{\xi}(\Lambda, I_\Lambda) = \dfrac{\tilde{E}(\Lambda, I_\Lambda) - a_\Lambda}{b_\Lambda - a_\Lambda}$.

Clearly $\tilde{\xi}$ is valued in $[0, 1]$; let $\tilde{\Xi}$ be its distribution function.
We now describe the distribution of these random variables as $|\Lambda| \to +\infty$ and $|I_\Lambda| \to 0$. One proves

Lemma 2.3 ([6]). *Assume λ is sufficiently large and pick η as in Theorem 2.3. One has*

$$|\mathbb{P}(X=1) - N(I_\Lambda)|\Lambda|| \lesssim (|\Lambda||I_\Lambda|)^{1+\rho} + N(I_\Lambda)|\Lambda|\ell'\ell^{-1} + |\Lambda|e^{-\eta\ell'/2} \tag{2.14}$$

where $N(E)$ denotes the integrated density of states of $H_{\tilde\omega}$.
One has, for all $x, y \in [0,1]$,

$$\left|(\tilde\Xi(x) - \tilde\Xi(y))\,\mathbb{P}(X=1)\right| \lesssim |x-y||I_\Lambda||\Lambda|. \tag{2.15}$$

Moreover, setting $N(x,y,\Lambda) := [N(a_\Lambda + x|I_\Lambda|) - N(a_\Lambda + y|I_\Lambda|)]|\Lambda|$, one has

$$\begin{aligned}\left|(\tilde\Xi(x) - \tilde\Xi(y))\,\mathbb{P}(X=1) - N(x,y,\Lambda)\right| \\ \lesssim (|\Lambda||I_\Lambda|)^{1+\rho} + |N(x,y,\Lambda)|\,\ell'\ell^{-1} + |\Lambda|e^{-\eta\ell'/2}.\end{aligned} \tag{2.16}$$

The proof of this result in [6, Lemma 2.2] relies solely on the localization estimates obtained in Theorem 2.3, the Wegner estimate (2.2) and the Minami estimate (2.4). Thus, it can be repeated verbatim in the present setting for the model $H_{\tilde\omega}$.

3. The long range correlation model (1.1) seen as weakly dependent model of the type (2.1)

We shall now show that the long range correlation model (1.1) satisfies the assumptions of Theorems 2.1, 2.2, 2.3 and 2.4 and Lemma 2.3. The only non trivial assumption to verify is (R̃).

3.1. **A linear change of random variables.**

Consider the Banach space $\ell^\infty(\mathbb{Z}^d)$ (endowed with its natural norm). Consider a bounded linear mapping $M : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d)$ that admits a bounded inverse. Consider now $\omega = (\omega_n)_{n\in\mathbb{Z}^d}$ independent random variables that are all bounded the same constant. Define the random variables

$$\tilde\omega = (\tilde\omega_n)_{n\in\mathbb{Z}^d} = M(\omega_n)_{n\in\mathbb{Z}^d} = M\omega. \tag{3.1}$$

Fix n_0 and m_0 in $\mathbb{Z}^d$ arbitrary and write

$$\begin{pmatrix}\tilde\omega_{m_0}\\ \tilde\omega^{m_0}\end{pmatrix} = \begin{pmatrix}a & B\\ A & C\end{pmatrix}\begin{pmatrix}\omega_{n_0}\\ \omega^{n_0}\end{pmatrix} \tag{3.2}$$

where $\tilde\omega^{m_0} = (\tilde\omega_m)_{m\neq m_0}$ and $\omega^{m_0} = (\omega_m)_{m\neq m_0}$. So, we consider M as a map from $\ell^2(\mathbb{Z}^d) = \mathbb{C}\delta_{n_0} \oplus \ell^2(\mathbb{Z}^d \setminus \{n_0\})$ to $\ell^2(\mathbb{Z}^d) = \mathbb{C}\delta_{m_0} \oplus \ell^2(\mathbb{Z}^d \setminus \{m_0\})$. Assume that

(I): $a \neq 0$ and C is one-to-one.

Let S_m be the concentration function of the random variable ω_m and $\tilde S_m$ be the concentration function of the random variable $\tilde\omega_m$ conditioned on $(\tilde\omega_n)_{n\neq m}$. Then, one has

Lemma 3.1. *Under assumption (I), there exists $\kappa > 0$ such that, for $x \geq 0$ and for any bounded sequence $(\tilde\omega_n)_{n\neq m}$, one has $\tilde S_{m_0}(x) = S_{n_0}(\kappa\, x)$.*

Proof. As M is invertible with bounded inverse, one can write

$$M^{-1} = \begin{pmatrix} \tilde{a} & \tilde{B} \\ \tilde{A} & \tilde{C} \end{pmatrix}$$

considering M^{-1} as a map from $\ell^2(\mathbb{Z}^d) = \mathbb{C}\delta_{m_0} \oplus \ell^2(\mathbb{Z}^d \setminus \{m_0\})$ to $\ell^2(\mathbb{Z}^d) = \mathbb{C}\delta_{n_0} \oplus \ell^2(\mathbb{Z}^d \setminus \{n_0\})$. We, thus, obtain that

$$\begin{pmatrix} \tilde{a} & \tilde{B} \\ \tilde{A} & \tilde{C} \end{pmatrix} \begin{pmatrix} a & B \\ A & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} a & B \\ A & C \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{B} \\ \tilde{A} & \tilde{C} \end{pmatrix}. \tag{3.3}$$

Solving (3.2) in $\tilde{\omega}^{m_0}$ and ω_{n_0}, we obtain $\tilde{\omega}_{m_0} = a\omega_{n_0} + B\omega^{n_0}$ and $\tilde{\omega}^{m_0} - \omega_{n_0}A = C\omega^{n_0}$. Thus, by the first equality in (3.3), we have

$$\tilde{C}\tilde{\omega}^{m_0} - \omega_{n_0}\tilde{C}A = \omega^{n_0} - (B\omega^{n_0})\tilde{A}. \tag{3.4}$$

Applying B to this, we get

$$(1 - B\tilde{A})B\omega^{n_0} = B\tilde{C}\tilde{\omega}^{m_0} - \omega_{n_0}B\tilde{C}A \tag{3.5}$$

We claim that, as C is one-to-one, one has $B\tilde{A} \neq 1$. Indeed, if $B\tilde{A} = 1$, then $\tilde{A} \neq 0$ and, by the first equality in (3.3), $\ker C = \operatorname{span} \tilde{A}$. This contradicts our assumption on C.
Hence, from (3.4), (3.5) and (3.3), we obtain

$$\tilde{\omega}_{m_0} = \frac{a}{1 - B\tilde{A}}\,\omega_{n_0} - D\,\tilde{\omega}^{m_0}$$

where D is a bounded linear form on $\ell^\infty(\mathbb{Z}^d \setminus \{m_0\})$.
This immediately yields Lemma 3.1 if one sets $\kappa = \left|\dfrac{1 - B\tilde{A}}{a}\right|$. □

Keeping the notations of the proof of Lemma 3.1, we easily checks that the assumption (I) is equivalent to the assumption

(I'): $a \cdot \tilde{a} \neq 0$.

One can apply this to convolutions i.e. assume that M is a convolution, that is, it is given by a matrix $M = ((\hat{M}_{m-n}))_{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d}$ (where $\hat{M}_n \in \mathbb{R}$). Assume that

(H1) $\displaystyle\sum_{n\in\mathbb{Z}^d} |\hat{M}_n| < +\infty$,

(H2) the function $\theta \mapsto M(\theta) = \displaystyle\sum_{n\in\mathbb{Z}^d} \hat{M}_n e^{in\theta}$ does not vanish on $\mathbb{R}^d$.

Consider now $(\omega_n)_{n\in\mathbb{Z}^d}$ independent random variables that are all bounded by the same constant and define the random variables $(\tilde{\omega}_n)_{n\in\mathbb{Z}^d}$ by (3.1). Then, as a consequence of Lemma 3.1, we prove

Theorem 3.1. *Assume (H1) and (H2). There exists $\kappa > 0$ and $k \in \mathbb{Z}^d$ such that, for $n \in \mathbb{Z}^d$ and $x \geq 0$, for any $(\tilde{\omega}_m)_{m\neq}$, one has $\tilde{S}_n(x) = S_{n+k}(\kappa\, x)$.*

Proof. The fact that, under assumptions (H1) and (H2), M is invertible on $\ell^\infty(\mathbb{Z}^d)$ with bounded inverse is an immediate consequence of Wiener's $1/f$-theorem (see e.g. [5, Chapter 2.4]).

Let $(\hat{M}_n^{-1})_{n\in\mathbb{Z}^d}$ denote the Fourier coefficients of the function $\theta \mapsto M^{-1}(\theta)$. As $M \cdot M^{-1} = 1$, one has $\sum_{n\in\mathbb{Z}^d} \hat{M}_n \hat{M}_{-n}^{-1} = 1$. Thus, we pick k such that

$$\hat{M}_{-k}^{-1} \hat{M}_k \neq 0. \tag{3.6}$$

Fix $n_0 \in \mathbb{Z}^d$ and write the decomposition (3.2) for $m_0 = n_0 + k$ and n_0. Then, the coefficient a, the vector A, the linear form B and the operator C in (3.2) are given by

$$a = \hat{M}_k,\ A = (\hat{M}_{k-n})_{n\neq 0},\ B = (\hat{M}_{k+m})_{m\neq 0} \text{ and } C = ((\hat{M}_{k+m-n}))_{\substack{m\neq 0\\ n\neq 0}}.$$

Note that a, A, B and C do not depend on n_0.
In the same way, using the notations of the proof of Lemma 3.1, one computes

$$\tilde{a} = \hat{M}_{-k}^{-1},\ \tilde{A} = (\hat{M}_{-k-n}^{-1})_{n\neq 0},\ \tilde{B} = (\hat{M}_{-k+m}^{-1})_{m\neq 0} \text{ and } \tilde{C} = ((\hat{M}_{-k+m-n}^{-1}))_{\substack{m\neq 0\\ n\neq 0}}.$$

Note that $\tilde{a}$, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ do not depend on n_0.
By construction (see (3.6)), we have $a \cdot \tilde{a} \neq 0$; hence, assumption (I'), thus, assumption (I) is satisfied and the statement of Theorem 3.1 for $\tilde{S}_{n_0}$ and S_{n_0+k} follows immediately from Lemma 3.1. Recalling that a, A, B, C, $\tilde{a}$, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ do not depend on n_0, we obtain the full statement of Theorem 3.1. □

To show that the long range correlation model (1.1) satisfies the assumption ($\tilde{\text{R}}$) (thus, that the conclusions of Theorems 2.1, 2.2, 2.3 and 2.4 and Lemma 2.3 hold), we first note that H_ω defined in (1.1) can be rewritten as $H_{\tilde{\omega}}$ defined in (2.1) where $\tilde{\omega} = M\omega$ and M is the convolution associated to the multiplier $\theta \mapsto M(\theta) = \sum_{n\in\mathbb{Z}^d} u_n e^{in\theta}$. Thus, the summability of u and the assumption (H) guarantee that assumptions (H1) and (H2) are satisfied. That for the model (1.1) assumption ($\tilde{\text{R}}$) is satisfied is then an immediate consequence of Theorem 3.1 and assumption (R).

3.2. **The proof of Theorem 1.2.** We are now in a position to prove Theorem 1.2. To simplify notations, we assume that the random variables $(\omega_n)_n$ are centered; this comes up to shifting the operator H_ω by a constant.

Recall that ξ_j is defined in (1.4). For $p > 0$ arbitrary, for arbitrary, non empty, open, two by two disjoint intervals $I_1, \dots, I_p$ and arbitrary integers $k_1, \cdots, k_p$, consider the event

$$\Omega^\Lambda_{I_1,k_1;I_2,k_2;\cdots;I_p,k_p} := \bigcap_{l=1}^{p} \{\omega;\ \#\{j;\ \xi_j(\tilde{\omega},\Lambda) \in I_l\} = k_l\}\,.$$

In view of Theorem 2.4, as $\mathbb{P}(\mathcal{Z}_\Lambda) \underset{|\Lambda|\to+\infty}{\to} 1$, Theorem 1.2 will be proved if we prove that

$$\lim_{|\Lambda|\to+\infty} \mathbb{P}\left(\Omega^\Lambda_{I_1,k_1;I_2,k_2;\cdots;I_p,k_p} \cap \mathcal{Z}_\Lambda\right) = \frac{|I_1|^{k_1}}{k_1!} e^{-|I_1|} \dots \frac{|I_p|^{k_p}}{k_p!} e^{-|I_p|}. \tag{3.7}$$

Pick ε small, in particular, smaller than

- the length of the smallest of the intervals $I_1, \dots, I_p$,
- the smallest distance between two distinct intervals among $I_1, \dots, I_p$.

Define

$$I_j^{+,\varepsilon} = I_j \cup [-\varepsilon/2, \varepsilon/2] \quad \text{and} \quad I_j^{-,\varepsilon} = I_j \cap ({}^cI_j + [-\varepsilon/2, \varepsilon/2]).$$

Clearly, $I_j^{-,\varepsilon} \subset I_j \subset I_j^{+,\varepsilon}$.
For a cube Λ and an interval I, define the Bernoulli random variable $X_{\Lambda_\ell,\ell',I}$ by

$$X_{\Lambda_\ell,\ell',I} = \mathbf{1}_{H_\omega(\Lambda_\ell) \text{ has an e.v. in } E_0+|\Lambda| n(E_0)]^{-1} I \text{ with localization center in } \Lambda_{\ell-\ell'}}.$$

Here, the length scales ℓ and ℓ' are taken as in Theorem 2.4 that is $\ell \asymp L^\beta$ and $\ell' \asymp L^{\beta'}$. Notice that, using the notations of section 2.4, one has $X_{\Lambda_\ell,\ell',I} = X(\Lambda_\ell, N^{-1}[N(E_0) + |\Lambda|^{-1}J], \ell')$ where, as it is assumed that N is differentiable at E_0 and that $n(E_0) = N'(E_0) > 0$, one has $|J\Delta I| = o(1)$ as $\Lambda \to +\infty$; here, $J\Delta I$ denotes the symmetric difference between I and J i.e. $J\Delta I = (I \setminus J) \cup (J \setminus I)$.
Consider now the decomposition in cubes $(\Lambda_\ell(\gamma_j))_{1\le j\le J}$ given by Theorem 2.4; then, $J = L^{d(1-\beta)}(1 + o(1))$. The choice of the parameters $\tilde\rho$, α, β and β' is the following. The parameter β' will be the determining parameter and will be chosen close to 1 (see Lemma 3.4 below); thus, the parameters α and β will also be close to 1 and $\tilde\rho$ will be chosen close to 0; indeed, in Theorem 2.4, one requires

$$0 < \beta' < \beta, \quad 0 < \alpha < 1, \quad 0 < \tilde\rho \quad \text{and} \quad 1 + \beta < \frac{2\alpha}{1+\tilde\rho}.$$

By Theorem 2.4, in particular (2.13), and the Wegner estimate (2.2), we know that, for L sufficiently large, one has

$$\bigcap_{l=1}^p \left\{ \omega;\ \sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),\ell',I_l^{-,\varepsilon}} = k_l \right\} \bigcap \mathcal{Z}_\Lambda \subset \Omega^\Lambda_{I_1,k_1;I_2,k_2;\cdots;I_p,k_p} \cap \mathcal{Z}_\Lambda,$$

$$\Omega^\Lambda_{I_1,k_1;I_2,k_2;\cdots;I_p,k_p} \cap \mathcal{Z}_\Lambda \subset \bigcap_{l=1}^p \left\{ \omega;\ \sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),\ell',I_l^{+,\varepsilon}} = k_l \right\} \bigcap \mathcal{Z}_\Lambda.$$

Hence, as $\mathbb{P}(\mathcal{Z}_\Lambda) \underset{|\Lambda|\to+\infty}{\to} 1$, it suffices to prove that, for any $\delta > 0$, there exists $\varepsilon > 0$ small such that

$$\begin{aligned} \liminf_{|\Lambda|\to+\infty} \mathbb{P}\left(\bigcap_{l=1}^p \left\{ \sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),\ell',I_l^{-,\varepsilon}} = k_l \right\} \right) &\ge \prod_{j=1}^p \frac{|I_j|^{k_j}}{k_j!} e^{-|I_j|} - \delta, \\ \limsup_{|\Lambda|\to+\infty} \mathbb{P}\left(\bigcap_{l=1}^p \left\{ \sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),\ell',I_l^{+,\varepsilon}} = k_l \right\} \right) &\le \prod_{j=1}^p \frac{|I_j|^{k_j}}{k_j!} e^{-|I_j|} + \delta. \end{aligned} \tag{3.8}$$

The main loss compared to the case when the (IAD) assumption holds is that the random variables $(X_{\Lambda_\ell(\gamma_j),\ell',I_l^{\pm,\varepsilon}})_j$ are not independent anymore. We will show that this dependence can be controlled using assumption (D) and the decorrelation estimates obtained in Theorem 2.1.

Now, for $\ell'' \leq \ell' \leq \ell$ and $\gamma \in \mathbb{Z}^d$, define the auxiliary operator $\tilde{H}_\omega(\Lambda_\ell(\gamma), \ell'')$ to be the operator

$$\tilde{H}_{\omega,\ell''} := -\Delta + \lambda \sum_{n\in\mathbb{Z}^d} \tilde{\omega}_n \tau_n(u_{\ell''}) \text{ where } u_{\ell''}(m) = \begin{cases} u(m) & \text{if } |m| \leq \ell'', \\ 0 & \text{if not} \end{cases}$$

restricted to $\Lambda_\ell(\gamma)$ (with periodic boundary conditions).
We prove

Lemma 3.2. *There exists ℓ_0 such that if $\ell'' \geq \ell_0$, then, for λ sufficiently large, the whole spectrum of $\tilde{H}_{\omega,\ell''}$ is localized, in particular, the conclusions of Theorems 2.2 and 2.3 hold for this operator for a value η independent of ℓ''. Moreover, the spectral estimates given in Theorem 2.1 also hold for $\tilde{H}_{\omega,\ell''}$ with constants independent of ℓ''.*

Proof of Lemma 3.2. Clearly, to prove Lemma 3.2, it suffices to

(1) prove that if u satisfies (S) and (H), there exists ℓ_0 such that, for $\ell'' \geq \ell_0$, the potential $u_{\ell''}$ satisfies (S) and (H) uniformly in ℓ'',
(2) reapply the arguments explained above for H_ω.

Assume that

$$m := \min_{\theta\in\mathbb{R}^d} \left| \sum_{n\in\mathbb{Z}^d} u_n e^{in\theta} \right| > 0.$$

Thus, by (S), we can pick ℓ_0 such that $\displaystyle\sum_{|n|\geq\ell_0} |u(n)| \leq m/2$; then, for $\ell'' \geq \ell_0$, we have that

$$\min_{\theta\in\mathbb{R}^d} \left| \sum_{n\in\mathbb{Z}^d} u^{\ell''}_n e^{in\theta} \right| \geq \min_{\theta\in\mathbb{R}^d} \left| \sum_{n\in\mathbb{Z}^d} u_n e^{in\theta} \right| - m/2 \geq m/2.$$

This completes the proof of Lemma 3.2. □

We also define the Bernoulli random variable $X_{\Lambda_\ell,\ell',\ell'',I}$ by

$$X_{\Lambda_\ell,\ell',\ell'',I} = \mathbf{1}_{\tilde{H}_{\omega,\ell''}(\Lambda_\ell) \text{ has an e.v. in } E_0+|\Lambda|n(E_0)]^{-1} I \text{ with localization center in } \Lambda_{\ell-\ell'}}.$$

Then, clearly, by Theorem 2.4, one has that

Lemma 3.3. *Assume $\ell'' \leq \ell/3$. For I a real interval, the random variables $(X_{\Lambda_\ell(\gamma_j),\ell',\ell'',I})_{1\leq j\leq J}$ are two by two independent.*

We also prove

Lemma 3.4. *Assume (D) holds. Recall that $\eta > d - 1/2$ and assume that $\beta' \in \left(\dfrac{d}{2\eta - d + 1}, 1\right)$. Set $\delta := -d + (2\eta - d + 1)\beta' > 0$.*
Then, for any $a < b$, any $\varepsilon \in (0, (b-a)/2)$ and any $p > 0$, for L sufficiently large, with probability at least $1 - L^{-p}$, for all $1 \leq j \leq J$, one has

$$X_{\Lambda_\ell(\gamma_j),2\ell'/3,\ell'/3,(a+\varepsilon,b-\varepsilon)} \leq X_{\Lambda_\ell(\gamma_j),\ell',(a,b)} \leq X_{\Lambda_\ell(\gamma_j),4\ell'/3,\ell'/3,(a-\varepsilon,\beta+\varepsilon)}. \tag{3.9}$$

Proof of Lemma 3.4. By Hoeffding's inequality (see e.g. [10]), we know that, for some $C > 0$ (depending only on the essential supremum of the random variables $(|\omega_n|)_n$), for $\ell' = L^{\beta'}/3$, for L sufficiently large, one has

$$\mathbb{P}\left(\left|\sum_{|m|\geq \ell'} \omega_{n-m} u(m)\right| \geq \varepsilon L^{-d}\right) \leq \exp\left(-\frac{\varepsilon L^{-d}}{C\sum_{|m|\geq \ell'} |u(m)|^2}\right) \leq e^{-\varepsilon L^{\delta}} \tag{3.10}$$

where we recall that $\delta = -d + (2\eta - d + 1)\beta' > 0$; here, we have used assumption (D), the fact that $\eta > d - 1/2$ and picked $\dfrac{d}{2\eta - d + 1} < \beta' < 1$. Hence, with a probability at least $1 - L^d e^{-\varepsilon L^{\delta}}$, we have that

$$\sup_{1\leq j\leq J} \sup_{m\in\Lambda_\ell(\gamma_j)} \left|\left(\sum_{n\in\mathbb{Z}^d} \omega_n \tau_n u\right)(m) - \tilde{\omega}_m\right| \leq \varepsilon L^{-d}.$$

that is

$$\sup_{1\leq j\leq J} \left\|\tilde{H}_{\omega,,\ell''}(\Lambda_\ell(\gamma_j)) - H_\omega(\Lambda_\ell(\gamma_j))\right\| \leq \varepsilon L^{-d}. \tag{3.11}$$

Next, we prove

Lemma 3.5. *Assume that the conclusions of Theorem 2.3 hold for the random operator H_ω for all L sufficiently large. Fix $p > 0$ and $r > 0$. Let q and η be given by Theorem 2.3.*
Then, for L sufficiently large, with probability at least $1 - L^{-p}$, for $L' \leq L$ and $\gamma \in \Lambda_L$ such that $\Lambda_{L'+\eta^{-1}(q+r+d/2)\log L}(\gamma) \subset \Lambda_L$ (q being defined in Theorem 2.3), if there exists $\varphi \in \ell^2(\Lambda_L)$ such that

- *supp$\varphi \subset \Lambda_{L'}(\gamma)$,*
- *$\|\varphi\| = 1$ and $\|(H_\omega(\Lambda_L) - E)\varphi\| \leq \varepsilon L^{-d}$,*

then, $H_\omega(\Lambda_L)$ has an eigenvalue in $[E - 2\varepsilon L^{-d}, E + 2\varepsilon L^{-d}]$ with localization center in $\Lambda_{L'+\eta^{-1}(q+r+d/2)\log L}(\gamma)$.

Proof of Lemma 3.5. Pick φ as in Lemma 3.5. As $H_\omega(\Lambda_L)$ is self-adjoint and $\|(H_\omega(\Lambda_L) - E)\varphi\| \leq \varepsilon$, $H_\omega(\Lambda_L)$ has an eigenvalue in $[E - \varepsilon, E + \varepsilon]$. Expand φ in the basis of eigenfunctions of $H_\omega(\Lambda_L)$: $\varphi = \sum_i \langle\varphi, \varphi_i\rangle\varphi_i$. If the localization centers of φ_i are outside of $\Lambda_{L'+\eta^{-1}(q+r+d/2)\log L}(\gamma)$, as supp$\varphi \subset \Lambda_{L'}(\gamma)$, one has $|\langle\varphi, \varphi_i\rangle| \lesssim L^{-r-d/2}$. On the other hand,

$$\varepsilon^2 L^{-2d} \geq \|(H - E)\varphi\|^2 = \sum_i |\langle\varphi, \varphi_i\rangle|^2 |E - E_i|^2. \tag{3.12}$$

Thus, if the conclusion of Lemma 3.5 does not hold, then, as the total number of eigenvalues is bounded by L^d, we have

$$1 - \sum_{|E-E_i|>2\varepsilon L^{-d}} |\langle\varphi, \varphi_i\rangle|^2 = \sum_{|E-E_i|\leq 2\varepsilon L^{-d}} |\langle\varphi, \varphi_i\rangle|^2 \lesssim L^{-2r},$$

thus, by (3.12),

$$\varepsilon^2 L^{-2d} \geq \|(H-E)\varphi\|^2 = \sum_i |\langle \varphi, \varphi_i\rangle|^2 |E-E_i|^2 \geq 4\varepsilon^2 L^{-2d}\left(1-O(L^{-2r})\right).$$

which is absurd for L sufficiently large.
Lemma 3.5 is proved. □

Now, we can use (3.11) and apply Lemma 3.5 in turn to $H_\omega(\Lambda_\ell(\gamma_j), \ell')$ and to $H_\omega(\Lambda_\ell(\gamma_j))$ to obtain that, for L sufficiently large, with probability at least $1-L^{-p}$ ($p>0$ fixed arbitrary),

- if $H_\omega(\Lambda_\ell(\gamma_j))$ has an eigenvalue in (a,b) with loc. center in $\Lambda_{\ell-\ell'}(\gamma_j)$ then $H_{\omega,,\ell''}(\Lambda_\ell(\gamma_j))$ has an eigenvalue in $(a-\varepsilon L^{-d}, b+\varepsilon L^{-d})$ with loc. center in $\Lambda_{\ell-\ell'+\ell''}(\gamma_j)$;
- if $H_{\omega,,\ell''}(\Lambda_\ell(\gamma_j))$ has an eigenvalue in (a,b) with loc. center in $\Lambda_{\ell-\ell'}(\gamma_j)$ then $H_\omega(\Lambda_\ell(\gamma_j))$ has an eigenvalue in $(a-\varepsilon L^{-d}, b+\varepsilon L^{-d})$ with loc. center in $\Lambda_{\ell-\ell'+\ell''}(\gamma_j)$.

This, then, implies (3.9) and completes the proof of Lemma 3.4. □

Equipped with Lemma 3.4, in view of (3.8), to prove Theorem 1.2, it suffices to prove that, for any $a>0$ and $b>0$, for any $\delta \in (0,1/3]$, there exists $\varepsilon>0$ small such that

(3.13)
$$\liminf_{|\Lambda|\to+\infty} \mathbb{P}\left(\bigcap_{l=1}^p \left\{\sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{-,\varepsilon}} = k_l\right\}\right) \geq \prod_{j=1}^p \frac{|I_j|^{k_j}}{k_j!} e^{-|I_j|} - \delta,$$
$$\limsup_{|\Lambda|\to+\infty} \mathbb{P}\left(\bigcap_{l=1}^p \left\{\sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = k_l\right\}\right) \leq \prod_{j=1}^p \frac{|I_j|^{k_j}}{k_j!} e^{-|I_j|} + \delta.$$

We will only prove the second inequality, the first one being proved in the same way.
First, by (2.14) of Lemma 2.3, by Lemma 3.4 and by the definition of $(I_l^{\pm,\varepsilon})_l$, for L sufficiently large and our choice of ε, one has

(3.14)
$$\begin{aligned}\mathbb{P}(X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{\pm,\varepsilon}} = 1) &= (|I_l| \pm \varepsilon) L^{d(\beta-1)}(1+o(1)) \\ &= (|I_l| \pm \varepsilon) J^{-1}(1+o(1)).\end{aligned}$$

Then, we compute

(3.15)
$$\begin{aligned}&\mathbb{P}\left(\bigcap_{l=1}^p \left\{\sum_{j=1}^J X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = k_l\right\}\right) \\ &= \sum_{\substack{K_l\subset\{1,\cdots,\tilde N\}\\ \#K_l=k_l,\ 1\leq l\leq p}} \mathbb{P}\left(\bigcap_{l=1}^p \left\{\omega;\ \begin{array}{l}\forall j\in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=1\\ \forall j\notin K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=0\end{array}\right\}\right).\end{aligned}$$

For $1 \le l \le p$, pick $K_l \subset \{1, \cdots, J\}$ such that $(\#K_l)_{1\le l\le p} = (k_l)_{1\le l\le p}$. For $1 \le j \le J$, define $\kappa_j = \#\{1 \le l \le p;\ \gamma_j \in K_l\} = \sum_{l=1}^{p} \mathbf{1}_{\gamma_j \in K_l}$. Then, one has

$$\sum_{j=1}^{J} \kappa_j = \sum_{l=1}^{p} \sum_{j=1}^{J} \mathbf{1}_{\gamma_j \in K_l} = k_1 + \cdots + k_p. \tag{3.16}$$

As $\cup_l I_l^{+,\varepsilon} \subset I_C := [-C, C]$ (for some $C > 0$), thanks to the decorrelation estimates (2.6) for $H_{\omega,,\ell''}(\Lambda_\ell(\gamma_j))$ (see Lemma 3.2) and using their stochastic independence, one has the following a priori bound

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{l=1}^{p} \left\{\omega;\ \begin{matrix} \forall j \in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 1 \\ \forall j \notin K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 0 \end{matrix}\right\}\right) \\ &\le \prod_{j=1}^{J} \mathbb{P}\left\{\omega; H_{\omega,,\ell''}(\Lambda_\ell(\gamma_j)) \text{ has at least } \kappa_j \text{ e.v. in } I_C\right\} \\ &\le \prod_{j=1}^{J} (CL^{-d}|\Lambda_\ell(\gamma_j)|)^{\kappa_j} \le CJ^{-k_1-k_2-\cdots-k_p} \end{aligned} \tag{3.17}$$

as $J = L^d|\Lambda_\ell(\gamma_j)|^{-1}(1+o(1)) = L^{d(1-\beta)}(1+o(1))$; here, we have used (3.16). By (3.15), we have

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{l=1}^{p} \left\{\sum_{j=1}^{J} X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = k_l\right\}\right) \\ &= \sum_{\substack{K_l\subset\{1,\cdots,\tilde N\} \\ \#K_l=k_l,\ 1\le l\le p \\ \forall l\neq l',\ K_l\cap K_{l'}=\emptyset}} \mathbb{P}\left(\bigcap_{l=1}^{p} \left\{\omega;\ \begin{matrix} \forall j \in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 1 \\ \forall j \notin K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 0 \end{matrix}\right\}\right) \\ &\quad + \sum_{\substack{K_l\subset\{1,\cdots,\tilde N\} \\ \#K_l=k_l,\ 1\le l\le p \\ \exists l\neq l',\ K_l\cap K_{l'}\neq\emptyset}} \mathbb{P}\left(\bigcap_{l=1}^{p} \left\{\omega;\ \begin{matrix} \forall j \in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 1 \\ \forall j \notin K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 0 \end{matrix}\right\}\right). \end{aligned} \tag{3.18}$$

One sets and, by Lemmas 2.3 and 3.4, one computes

$$p_l^+ := \mathbb{P}\left(X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}} = 1\right) = |I_l^{+,\varepsilon}|J^{-1}(1+o(1)).$$

We also will use [6, Lemma 4.1]

Lemma 3.6. *With the choice of $(I_l)_{1\le l\le p}$ made above, under the assumptions of Theorem 1.2, with our choice of ℓ and ℓ', for L sufficiently large, one has*

$$\mathbb{P}\left(\sum_{l=1}^{p} X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{\pm,\varepsilon}} = 0\right) = 1 - (1-o(1))J^{-1}\sum_{l=1}^{p} |I_l^{\pm,\varepsilon}|.$$

For $1 \le l \le p$, pick $K_l \subset \{1, \cdots, J\}$ such that $(\#K_l)_{1\le l\le p} = (k_l)_{1\le l\le p}$ and $K_l \cap K_{l'} = \emptyset$ if $l \neq l'$. For such $(K_l)_{1\le l\le p}$, using the stochastic independence

of the operators $(H_{\omega,,\ell''}(\Lambda_\ell(\gamma_j)))_{1\leq j\leq J}$, one computes

$$
\begin{aligned}
&\mathbb{P}\left(\bigcap_{l=1}^{p}\left\{\omega;\ \begin{array}{r}\forall j\in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=1\\ \forall j\notin K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=0\end{array}\right\}\right)\\
&=\mathbb{P}\left(\bigcap_{l=1}^{p}\left\{\begin{array}{r}\forall j\in K_l,\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=1\\ \omega;\ \forall j\notin \bigcup_{l'\neq l}K_{l'},\ X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=0\end{array}\right\}\right)\\
&=\prod_{j\notin\cup_{l=1}^{p}K_l}\mathbb{P}\left(\sum_{l=1}^{p}X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=0\right)\\
&\qquad\qquad\cdot\prod_{l=1}^{p}\prod_{j\in K_l}\mathbb{P}\left(X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=1\right)\\
&=\left(1-\sum_{l=1}^{p}p_l^+\right)^{J-(k_1+\cdots+k_p)}\prod_{l=1}^{p}\left[p_l^+\right]^{k_l}(1+o(1))\\
&=\left(\prod_{l=1}^{p}e^{-|I_l^{+,\varepsilon}|}|I_l^{+,\varepsilon}|^{k_l}\right)J^{-k_1-\cdots-k_p}(1+o(1)).
\end{aligned}
\tag{3.19}
$$

On the other hand, as $k_1+\cdots+k_p$ is bounded, when $J\to+\infty$, that is, when $|\Lambda|\to+\infty$, one has

$$
\sum_{\substack{K_l\subset\{1,\cdots,J\}\\ \#K_l=k_l,\ 1\leq l\leq p}}1=\prod_{l=1}^{p}\binom{J}{k_l}\quad\text{and}\quad\sum_{\substack{K_l\subset\{1,\cdots,J\}\\ \#K_l=k_l,\ 1\leq l\leq p\\ \exists l\neq l',\ K_l\cap K_{l'}\neq\emptyset}}1=o\left(\prod_{l=1}^{p}\binom{J}{k_l}\right).
\tag{3.20}
$$

Thus, for L sufficiently large, plugging the a-priori bound (3.17), the bounds (3.20) and (3.19) into (3.18), we compute

$$
\begin{aligned}
&\mathbb{P}\left(\bigcap_{l=1}^{p}\left\{\sum_{j=1}^{J}X_{\Lambda_\ell(\gamma_j),a\ell',b\ell',I_l^{+,\varepsilon}}=k_l\right\}\right)\\
&=\left(\prod_{l=1}^{p}e^{-|I_l^{+,\varepsilon}|}|I_l^{+,\varepsilon}|^{k_l}\right)\prod_{l=1}^{p}\binom{J}{k_l}J^{-k_1-\cdots-k_p}+o\left(J^{-k_1-k_2-\cdots-k_p}\prod_{l=1}^{p}\binom{J}{k_l}\right)\\
&=\prod_{l=1}^{p}\frac{e^{-|I_l^{+,\varepsilon}|}|I_l^{+,\varepsilon}|^{k_l}}{k_l!}+o(1)=\prod_{l=1}^{p}\frac{e^{-|I_l|}|I_l|^{k_l}}{k_l!}+O(\varepsilon).
\end{aligned}
$$

This proves the second inequality in (3.13), thus, in (3.8). The first one is proved in the same way. The proof of Theorem 1.2 is complete. □

References

[1] Michael Aizenman, Jeffrey H. Schenker, Roland M. Friedrich, and Dirk Hundertmark. Finite-volume fractional-moment criteria for Anderson localization. *Comm. Math. Phys.*, 224(1):219–253, 2001. Dedicated to Joel L. Lebowitz.

[2] Jean V. Bellissard, Peter D. Hislop, and Günter Stolz. Correlation estimates in the Anderson model. *J. Stat. Phys.*, 129(4):649–662, 2007.

[3] Jean-Michel Combes, François Germinet, and Abel Klein. Generalized eigenvalue-counting estimates for the Anderson model. *J. Stat. Phys.*, 135(2):201–216, 2009.
[4] Jean-Michel Combes, Peter D. Hislop, and Frédéric Klopp. An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.*, 140(3):469–498, 2007.
[5] E. Brian Davies. *Linear operators and their spectra*, volume 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.
[6] François Germinet and Frédéric Klopp. Spectral statistics for random Schrödinger operators in the localized regime. To appear in JEMS. ArXiv http://arxiv.org/abs/1011.1832, 2010.
[7] François Germinet and Frédéric Klopp. Spectral statistics for the discrete Anderson model in the localized regime. In *Spectra of random operators and related topics*, RIMS Kôkyûroku Bessatsu, B27, pages 11–24. Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.
[8] Gian Michele Graf and Alessio Vaghi. A remark on the estimate of a determinant by Minami. *Lett. Math. Phys.*, 79(1):17–22, 2007.
[9] Werner Kirsch. An invitation to random Schrödinger operators. In *Random Schrödinger operators*, volume 25 of *Panor. Synthèses*, pages 1–119. Soc. Math. France, Paris, 2008. With an appendix by Frédéric Klopp.
[10] Pascal Massart. *Concentration inequalities and model selection*, volume 1896 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 33rd Summer School on Probability Theory held in Saint-Flour, July 6–23, 2003, With a foreword by Jean Picard.
[11] Nariyuki Minami. Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.*, 177(3):709–725, 1996.
[12] Leonid Pastur and Alexander Figotin. *Spectra of random and almost-periodic operators*, volume 297 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
[13] Martin Tautenhahn and Ivan Veselić. Minami's estimate: beyond rank one perturbation and monotonicity. ArXiv http://arxiv.org/abs/1210.3542, 2012.
[14] Ivan Veselić. *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, volume 1917 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.
[15] Franz Wegner. Bounds on the density of states in disordered systems. *Z. Phys. B*, 44(1-2):9–15, 1981.

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