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► **To cite this version:**

Christophe Dutang, Claude Lefèvre, Stéphane Loisel. On an asymptotic rule  $A+B/u$  for ultimate ruin probabilities under dependence by mixing. *Insurance: Mathematics and Economics*, 2013, 53 (3), pp.774-785. hal-00746251v2

**HAL Id: hal-00746251**

**<https://hal.science/hal-00746251v2>**

Submitted on 30 Aug 2013

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# On an asymptotic rule $A + B/u$ for ultimate ruin probabilities under dependence by mixing

C. Dutang<sup>\*†</sup>, C. Lefèvre<sup>‡</sup>, S. Loisel<sup>\*</sup>

June 12, 2013

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The purpose of this paper is to point out that an asymptotic rule  $A + B/u$  for the ultimate ruin probability applies to a wide class of dependent risk processes, in continuous or discrete time. That dependence is incorporated through a mixing model in the individual claim amount distributions. Several special mixing distributions are examined in detail and some close-form formulas are derived. Claim tail distributions and the dependence structure are also investigated.

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Keywords: Ultimate ruin probability; Discrete and continuous time; Mixing model; Asymptotics; Tail distribution; Archimedean copulas.

## 1 Introduction

Traditionally, the surplus  $(U_t)_{t \geq 0}$  of an insurance company at time  $t$  is represented by  $U_t = u + ct - \sum_{i=1}^{N_t} X_i$ , where  $u$  is the initial surplus,  $c$  is the premium rate,  $(X_i)_{i \geq 1}$  are the successive claim amounts and  $(N_t)_{t \geq 0}$  is the claim arrival process (the claim waiting times are denoted by  $(T_i)_{i \geq 1}$ ). In the Cramér-Lundberg model,  $(N_t)_{t \geq 0}$  is modeled by a Poisson process,  $(X_i)_{i \geq 1}$  are independent and identically distributed (i.i.d.) random variables and claim severities  $(X_i)_{i \geq 1}$  are independent of the claim waiting times  $(T_i)_{i \geq 1}$ . Andersen (1957) generalized this model by proposing a renewal process for the claim arrival process  $(N_t)_{t \geq 0}$ .

Since then, extensions have been proposed in many directions. Asmussen & Rolski (1991) studied ruin models with phase-type distributions for both claim severities  $X_i$  and claim waiting times  $T_i$ . Gerber & Shiu (1998) unified the analysis of ruin measures in the Cramér-Lundberg model, including the deficit at ruin, the claim causing the ruin or the ruin probability, by introducing a so-called discounted penalty function. Gerber & Shiu (2005), Song et al. (2010) and many others extended the Gerber-Shiu approach to a wider class of risk models. Many variants of the Sparre Andersen model have also been proposed such as for non-homogeneous claim arrivals (e.g. Lu & Garrido (2005), Albrecher & Asmussen (2006)), reinsurance treaties (e.g. Centeno (2002), Dimitrova & Kaishev (2010)), multivariate risks (e.g. Picard et al. (2003), Cai & Li (2005), Collamore (1996)) and dependent risks (e.g. Albrecher & Boxma (2004), Boudreault et al. (2006), Albrecher & Teugels (2006), Lefèvre & Loisel (2009), Lefèvre & Picard (2011), Ignatov & Kaishev (2012)).

The ultimate ruin probability  $\psi(u) = P(\exists t > 0 : U_t < 0 | U_0 = u)$  is a major risk measure and has received considerable attention in the literature. For the Sparre Andersen model with light-tailed claim

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amounts,  $\psi(u) \sim Ce^{-\gamma u}$  as  $u \rightarrow \infty$ , where  $\gamma$  is the positive root of a simple equation involving the moment generating function of  $X_i$  (e.g. Asmussen & Albrecher (2010)). With heavy-tailed claim amounts, the ruin probability is in the class of Hall of order 1 since  $\psi(u) \sim C/u^\alpha$  as  $u \rightarrow \infty$  (e.g. Embrechts & Veraverbeke (1982), Klüppelberg & Stadtmüller (1998), Reiss & Thomas (2007)). For models with dependence, Albrecher & Teugels (2006) investigated the case where the claim amounts and claim waiting times,  $(X_i, T_i)_{i \geq 1}$ , are correlated. They obtained an exponential decrease for  $\psi(u)$  in the case of light-tailed claim sizes. In a recent paper, Albrecher et al. (2011) discussed the ruin probability when there is dependence by mixing in the claim amounts  $(X_i)_{i \geq 1}$  or the claim waiting times  $(T_i)_{i \geq 1}$ ; see also Constantinescu et al. (2011). They derived an asymptotic formula  $\psi(u) \sim A + B/u$  as  $u \rightarrow +\infty$  for Pareto correlated claims or inter-arrival times.

The present paper considers the same insurance context where the claim amounts  $(X_i)_{i \geq 1}$  or the claim waiting times  $(T_i)_{i \geq 1}$  are dependent through a mixing model. Such a form of dependence can translate some uncertainty on the model parameters due to incomplete available informations. For instance, due to reporting or claim settlement delays, it might take a long time for the insurer to realize that claims are more adverse than expected. While an instantaneously informed insurer could manage parameter uncertainty with credibility techniques (e.g. Trufin & Loisel (2013)), an insurer suffering from information delays would in practice undergo high losses due to unfavorable parameter values before being able to react. In some markets, a solution could be to stop the business. Thus,  $\psi(u)$  must then be interpreted as the probability of being ruined or of stopping the business if no premium adjustment is possible. Now, another possible situation of mixing is when the model parameters are not univoquely fixed but depend on exogeneous socio-economic factors. As suggested by a referee, another problem of interest would consist in accounting for the influence of an exterior environment that modifies the parameters in the course of time.

More precisely, we are concerned with a continuous time model in which the claim amounts have a mixed exponential distribution, and a discrete time model in which the claim amounts have a mixed zero-modified geometric distribution. In Section 2, we derive exact formulas for the ultimate ruin probability under several particular mixing distributions. In Section 3, we establish the approximating rule  $A + B/u$  as  $u \rightarrow \infty$ , as well as some asymptotic refinements to this rule. Section 4 is devoted to the analysis of the asymptotic claim tail distribution  $P(X > x)$  as  $x \rightarrow \infty$ . To close, Section 5 briefly discusses the dependence structure involved. Except otherwise stated, all numerical illustrations are done with the R statistical software (R Core Team (2012)).

## 2 A class of mixing risk models

This section presents the class of dependent risk processes under study, in a continuous or discrete time scale. We first derive the ultimate ruin probability when claim amounts have a mixed exponential distribution or a mixed zero-modified geometric distribution. We then deduce explicit formulas for several specific mixing distributions.

### 2.1 Continuous time framework

Consider the context of the classical Cramér-Lundberg model. The free surplus of an insurance company at time  $t$  is modeled by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where  $u$  is the initial surplus,  $c$  is the premium rate,  $(X_i)_{i \geq 1}$  are the claim amounts and  $(N_t)_{t \geq 0}$  is the Poisson claim arrival process with intensity  $\lambda$ . The  $X_i$ 's are independent of the claim arrival process. This time, however, we assume that the  $X_i$ 's are i.i.d. random variables conditionally on a latent random variable  $\Theta$  (given  $\Theta = \theta$ , they are distributed as  $(X|\Theta = \theta)$ ). The variable  $\Theta$  can be used to translate, for example, the uncertainty in the claim amounts or their heterogeneity. Its distribution is assumed to be continuous.

Ruin occurs as soon as the surplus process becomes negative. Given  $\Theta = \theta$ , the ruin probability is thus defined as

$$\psi(u, \theta) = P(\exists t > 0 : U_t < 0 | U_0 = u, \Theta = \theta). \quad (1)$$

To determine this probability, a standard method consists in looking at the state of the surplus after the first claim arrival. This leads to an integro-differential equation that can be solved by using Laplace-Stieltjes transforms (e.g. Asmussen & Albrecher (2010)). For exponentially distributed claims, i.e. when  $(X|\Theta = \theta) \sim \mathcal{E}(\theta)$ , one gets from (1) the well-known formula

$$\psi(u, \theta) = \min \left( \frac{\lambda}{\theta c} e^{-u(\theta - \lambda/c)}, 1 \right),$$

where  $\min(\dots) < 1$  under the net profit condition  $\theta > \lambda/c \equiv \theta_0$ . Integrating over the parameter  $\theta$  yields for the (global) ruin probability

$$\psi(u) = F_{\Theta}(\theta_0) + I(u, \theta_0), \quad (2)$$

where  $F_{\Theta}$  is the distribution function of  $\Theta$  and

$$I(u, \theta_0) = \int_{\theta_0}^{\infty} \frac{\theta_0}{\theta} e^{-u(\theta - \theta_0)} dF_{\Theta}(\theta). \quad (3)$$

Note that formula (2) is given as (5) in Albrecher et al. (2011). This expression for  $\psi(u)$  can also be interpreted from a regulatory point of view. Suppose that a regulator supervises a set of insurers who face exponentially distributed claims, each insurer having its proper parameter. Then, the term independent of  $u$ ,  $F_{\Theta}(\theta_0)$ , may be viewed as the proportion of insurers that will bankrupt irrespectively of their initial reserves, while the  $u$ -dependent term  $I(u, \theta_0)$  corresponds to the proportion of insurers that will bankrupt with  $u$  as initial reserves, the distribution of  $\Theta$  translating the heterogeneity among insurers.

We now examine two particular distributions for the latent variable  $\Theta$  that are discussed in Albrecher et al. (2011). Firstly, we consider for  $\Theta$  a gamma distribution  $\mathcal{G}a(\alpha, \lambda)$  with density

$$f_{\Theta}(\theta) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta}, \quad \text{thus } F_{\Theta}(\theta) = \frac{\gamma(\alpha, \lambda\theta)}{\Gamma(\alpha)}, \quad \theta > 0, \quad (4)$$

where  $\gamma(\dots)$  (resp.  $\Gamma(\dots)$ ) denotes the incomplete lower gamma function (resp. the gamma function); e.g. Olver et al. (2010). The resulting claim amount  $X$  has a Pareto type II distribution with parameters  $\alpha$  and  $\lambda$ , whose survival function is

$$P(X > x) = \frac{1}{(1 + x/\lambda)^{\alpha}}, \quad x \geq 0.$$

Making the change of variable  $y = \theta(\lambda + u)$ , the integral  $I(u, \theta_0)$  in (3) can be expressed in terms of the incomplete upper gamma function  $\Gamma(\dots)$  (see Olver et al. (2010)). From (2) and (3), we then obtain the following (known) result.

**Proposition 2.1.** *In the continuous time model with  $(X|\Theta = \theta) =_d \mathcal{E}(\theta)$  and  $\Theta =_d \mathcal{G}a(\alpha, \lambda)$ ,*

$$\psi(u) = \frac{\gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{\lambda^{\alpha}\theta_0}{\Gamma(\alpha)} \frac{\Gamma(\alpha - 1, \theta_0(\lambda + u))}{(\lambda + u)^{\alpha-1}} e^{\theta_0 u}, \quad (5)$$

*provided  $\alpha > 1$  (i.e. if the density of  $(X|\Theta = \theta)$  is log-concave).*

Secondly, consider for  $\Theta$  a stable 1/2 distribution (also called a Lévy distribution  $\mathcal{L}e(\alpha)$ ) with density

$$f_{\Theta}(\theta) = \frac{\alpha}{2\sqrt{\pi}\theta^3} e^{-\alpha^2/(4\theta)}, \quad \text{thus } F_{\Theta}(\theta) = \text{erfc} \left( \frac{\alpha}{2\sqrt{\theta}} \right), \quad \theta > 0, \quad (6)$$

where  $\text{erfc}(\dots)$  denotes the complementary error function; e.g. Olver et al. (2010). We choose the same parametrization of the Lévy distribution as in Albrecher et al. (2011). Note that there exists another parametrization which involves a location parameter  $\delta$  and a shape parameter  $\gamma$  (e.g. Nolan (2012)). These

two parametrizations are equivalent by setting  $\gamma = \alpha^2/2$  and  $\delta = 0$ . The resulting claim distribution is a Weibull distribution with parameters  $1/2$  and  $1/\alpha^2$  whose tail distribution is

$$P(X > x) = e^{-\alpha\sqrt{x}}, \quad x \geq 0.$$

The computation of  $I(u, \theta)$  in the Lévy case is more elaborate. Applying the change of variable  $x = u\theta$  yields

$$I(u, \theta_0) = \frac{\theta_0 \alpha \sqrt{u^3}}{2\sqrt{\pi}} e^{u\theta_0} \int_{u\theta_0}^{\infty} \frac{1}{\sqrt{x^5}} e^{-x - \alpha^2 u / (4x)} dx. \quad (7)$$

The latter integral is related to the generalized error function, a particular case of the generalized incomplete upper gamma function, which is defined as

$$\Gamma(a, x, b) = \int_x^{\infty} t^{a-1} e^{-t-b/t} dt;$$

see e.g. Chaudry & Zubair (2002). So, in formula (7), we deal with  $\Gamma(-3/2, \theta_0 u, \alpha^2 u/4)$ . As for the usual incomplete gamma function, the function  $\Gamma(., ., .)$  satisfies a recurrence equation on the parameter  $a$  given by

$$\Gamma(a+1, x, b) = a\Gamma(a, x, b) + b\Gamma(a-1, x, b) + x^a e^{-x-b/x};$$

see Theorem 2.2 in Chaudry & Zubair (2002). Using this equation, we are able to compute  $\Gamma(-3/2, x, b)$  in terms of  $\Gamma(-1/2, x, b)$  and  $\Gamma(1/2, x, b)$ , which can be both expressed in terms of the standard error function; see A for details. This leads us to the following (known) expression for the ruin probability:

**Proposition 2.2.** *In the continuous time model with  $(X|\Theta = \theta) =_d \mathcal{E}(\theta)$  and  $\Theta =_d \mathcal{L}e(\alpha)$ ,*

$$\begin{aligned} \psi(u) = \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{\theta_0 \sqrt{u}}{\alpha} e^{u\theta_0} & \left[ \left(1 - \frac{1}{\alpha\sqrt{u}}\right) e^{\alpha\sqrt{u}} \operatorname{erfc}(d_+) \right. \\ & \left. + \left(1 + \frac{1}{\alpha\sqrt{u}}\right) e^{-\alpha\sqrt{u}} \operatorname{erfc}(d_-) - \frac{2}{\sqrt{\pi}u\theta_0} e^{-u\theta_0 - \alpha^2/(4\theta_0)} \right], \end{aligned} \quad (8)$$

where  $d_+ = \sqrt{u\theta_0} + \alpha/(2\sqrt{\theta_0})$  and  $d_- = \sqrt{u\theta_0} - \alpha/(2\sqrt{\theta_0})$ .

## 2.2 Discrete time framework

The compound binomial risk model, introduced by Gerber (1988), is the discrete-time analog of the Cramér-Lundberg model. Here too, we are going to construct an extended version of this model by using a similar mixing approach. The insurance portfolio is now examined at times  $t \in \mathbb{N}_0 = \{0, 1, \dots\}$ . The claim amounts during the successive periods form a sequence of  $\mathbb{N}$ -valued random variables  $(X_i)_{i \geq 1}$  that are i.i.d. conditionally on a latent random variable  $\Theta$ . In discrete time, the premium per period is generally set to 1 and corresponds to a prepayment to cover the risk during that period. The surplus of the insurance company at time  $t \in \mathbb{N}$  is then given by

$$U_t = u + t - \sum_{i=1}^t X_i,$$

where  $u \in \mathbb{N}$  is the initial surplus. When the claims are independent, this model is named compound binomial because the number of strictly positive claims has a binomial distribution  $\mathcal{B}(t, 1-q)$  where  $q = P(X=0)$ . The net profit condition which avoids the a.s. ruin is given here by  $E(X) < 1$ .

Since the surplus may become null (with a positive probability), it is necessary to specify if ruin occurs when the surplus becomes non-positive or strictly negative. For example, in Gerber (1988), ruin occurs at the first time when the surplus reaches or goes below 0, so that

$$\psi_G(u) = P(\exists t \in \mathbb{N} : U_t \leq 0 | U_0 = u),$$

while in Shiu (1989), ruin occurs at the first time when the surplus becomes negative, so that

$$\psi_S(u) = P(\exists t \in \mathbb{N} : U_t < 0 | U_0 = u).$$

Graphically,  $\psi_G(u)$  (resp.  $\psi_S(u)$ ) is the probability that the surplus crosses the level 0 (resp.  $-1$  as the claim amounts are  $\mathbb{N}_0$ -valued). Note that  $\psi_G(u) = \psi_S(u - 1)$ . In the sequel, we work with the ruin probability  $\psi_S(u) \equiv \psi(u)$ .

Formulas for  $\psi(u)$  are available in the general case and for special claim amount distributions (see e.g. Willmot (1993), Picard & Lefevre (2003), Lefevre & Loisel (2008), Li et al. (2009)). The case where the claims are geometrically distributed is discussed in detail by Sundt & dos Reis (2007). In particular, assuming a geometric decreasing tail for the ruin probability, they proved (Theorem 1) that the claim amount distribution is necessarily of geometric type. It is also shown that *for claim amounts with a zero-modified geometric distribution  $\mathcal{G}e(q, \rho)$*  (see B), then

$$\psi(u) = \min \left( \frac{1-q}{\rho} \left( \frac{1-\rho}{q} \right)^{u+1}, 1 \right), \quad (9)$$

where  $\min(\dots) < 1$  under the net profit condition  $\rho > 1 - q$ .

The formula (9) can be easily extended by using again a mixing approach. Specifically, let us now suppose that  $(X|\Theta = \theta) \sim \mathcal{G}e(q, e^{-\theta})$ . Let  $\psi(u, \theta)$  be the corresponding ruin probability. Then, the (global) ruin probability is given by

$$\psi(u) = \bar{F}_\Theta(\theta_0) + I(u, \theta_0), \quad (10)$$

where  $\theta_0 = -\log(1 - q)$ ,  $\bar{F}_\Theta \equiv 1 - F_\Theta$  is the tail distribution of  $\Theta$  and

$$I(u, \theta_0) = \int_0^{\theta_0} \frac{1-q}{e^{-\theta}} \left( \frac{1-e^{-\theta}}{q} \right)^{u+1} dF_\Theta(\theta). \quad (11)$$

Compared to the continuous setting, the integral in (11) is taken over the interval  $[0, \theta_0]$  rather than  $[\theta_0, \infty[$  in (3). This is a consequence of the parametrizations.

**Remark 2.3.** *The usual geometric distribution  $\mathcal{G}e(\rho)$  is not convenient because the net profit condition is  $\rho > 1/2$ , which is restrictive on the parameter  $\rho$ . In that case, however, one could consider, for example, a geometric distribution  $(X|\Theta = \theta) \sim \mathcal{G}e(1/(1 + \theta))$ , which leads to*

$$\psi(u) = \bar{F}_\Theta(1) + \int_0^1 \theta^{u+2} dF_\Theta(\theta).$$

*Choosing for  $\Theta$  a uniform distribution  $U(0, p)$  with  $p \leq 1$  yields the surprisingly simple result  $\psi(u) = p^{u+2}/(u + 3)$ . This formula is also interesting by showing that the decrease of  $\psi(u)$  is geometric when  $p < 1$  and polynomial when  $p = 1$ .*

Let us now derive close-form ruin probabilities for three particular distributions of  $\Theta$ . Firstly, we consider an exponential distribution  $\Theta \sim \mathcal{E}(\lambda)$  of density given in (4) with  $\alpha = 1$ . We will use the following integral:

$$I_1(a, b, x) = \int_0^x (e^{-\theta})^a (1 - e^{-\theta})^b d\theta = \int_{e^{-x}}^1 p^{a-1} (1-p)^b dp = \bar{\beta}(a, b+1, e^{-x}), \quad x > 0,$$

for  $a > 0$ ,  $b > -1$ , where  $\bar{\beta}(a, b+1, e^{-x}) = \beta(a, b+1, 1 - e^{-x})$  is a complementary incomplete beta function (e.g. Chaudry & Zubair (2002)). Note that  $\bar{\beta}(a, b+1, e^{-x}) \rightarrow \beta(a, b+1)$  as  $x \rightarrow \infty$ . Writing  $I_1(\lambda + 1, k, \infty)$ , we see that the probability mass function of  $X$  is given by

$$P(X = k) = q\delta_{k0} + (1 - \delta_{k0})\lambda(1 - q)\beta(\lambda + 1, k), \quad k \in \mathbb{N}_0, \quad (12)$$

where  $\delta_{ij}$  denotes the Kronecker delta. With the presence of a beta function one recognizes the zero-modified Yule-Simon distribution (e.g. Simon (1955)) which is relevant in the study of word frequencies. Moreover, we see that  $I(u, \theta_0) = (1 - q)/q^{u+1} \times I_1(\lambda - 1, u + 1, \theta_0)$ . Thus, (10) and (11) yield the following result.

**Proposition 2.4.** *In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{E}(\lambda)$ ,*

$$\psi(u) = (1-q)^\lambda + \frac{\lambda(1-q)}{q^{u+1}} \bar{\beta}(\lambda-1, u+2, 1-q), \quad (13)$$

provided  $\lambda > 1$ .

Secondly, consider for  $\Theta$  a gamma distribution  $\mathcal{G}a(\alpha, \lambda)$  of density given in (4). We here use the integral

$$I_2(a, n, b, x) = \int_0^x (e^{-\theta})^a (1-e^{-\theta})^n \theta^b d\theta = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \int_0^x e^{-\theta(a+n-j)} \theta^b d\theta, \quad x > 0,$$

for  $a > 0, b > -1, n \in \mathbb{N}_0$ , which can be rewritten as

$$I_2(a, n, b, x) = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \int_0^{(a+n-j)x} \frac{y^b}{(a+n-j)^{b+1}} e^{-y} dy,$$

and substituting  $n-j$  for  $j$ ,

$$I_2(a, n, b, x) = \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{\gamma(b+1, x(a+j))}{(a+j)^{b+1}},$$

where  $\gamma(\cdot, \cdot)$  is the incomplete lower gamma function (see (4)). As  $x \rightarrow \infty$ , only the term  $\gamma(b+1, \tilde{x})$  changes and tends to  $\Gamma(b+1)$ . Writing  $I_2(\lambda, k-1, \alpha-1, \infty)$ , we see that the probability mass function of  $X$  is

$$P(X = k) = q\delta_{k0} + (1-\delta_{k0})(1-q) \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j \lambda^\alpha}{(\lambda+j)^\alpha}, \quad k \in \mathbb{N}_0. \quad (14)$$

From (10), (11) and using  $I_2(\lambda-1, u+1, \alpha-1, \theta_0)$ , we deduce the formula below.

**Proposition 2.5.** *In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{G}a(\alpha, \lambda)$ ,*

$$\psi(u) = \frac{\Gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{1-q}{q^{u+1}} \sum_{j=0}^{u+1} \binom{u+1}{j} (-1)^j \frac{\gamma(\alpha, \theta_0(\lambda+j-1))}{\Gamma(\alpha)} \left( \frac{\lambda}{\lambda+j-1} \right)^\alpha, \quad (15)$$

where  $\theta_0 = -\log(1-q)$ , provided  $\lambda > 1$ .

Finally, consider for  $\Theta$  a Lévy distributed  $\mathcal{L}e(\alpha)$  of density given in (6). We use the integral

$$I_3(a, n, b, x) = \int_0^x (e^{-\theta})^a (1-e^{-\theta})^n \theta^{-3/2} e^{-b/\theta} d\theta = 2 \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \int_{1/\sqrt{x}}^\infty e^{-(a+n-j)/y^2} e^{-by^2} dy, \quad x > 0,$$

for  $a > 0, b > 0, n \in \mathbb{N}_0$ , after doing a change of variable  $1/\theta = y^2$ . This integral is linked to the generalized incomplete upper gamma function (e.g. Chaudry & Zubair (2002)). From C, we get

$$I_3(a, n, b, x) = \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{\sqrt{\pi}}{2\sqrt{b}} \left[ e^{2\sqrt{b(a+j)}} \operatorname{erfc} \left( \frac{\sqrt{b}}{\sqrt{x}} + \sqrt{a+j}\sqrt{x} \right) + e^{-2\sqrt{b(a+j)}} \operatorname{erfc} \left( \frac{\sqrt{b}}{\sqrt{x}} - \sqrt{a+j}\sqrt{x} \right) \right].$$

As  $x \rightarrow \infty$ ,

$$I_3(a, n, b, \infty) = \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{\sqrt{\pi}}{\sqrt{b}} e^{-2\sqrt{b}\sqrt{a+j}}.$$

Writing  $I_3(0, k-1, \alpha^2/4, \infty)$ , we obtain for the probability mass function of  $X$

$$P(X = k) = (1 - q) \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-\alpha\sqrt{j}}, \quad k \in \mathbb{N}_0. \quad (16)$$

The ruin probability is computed using  $I_3(-1, u+1, \alpha^2/4, \theta_0)$ .

**Proposition 2.6.** *In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{L}e(\alpha)$ ,*

$$\begin{aligned} \psi(u) = 1 - \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{1-q}{4q^{u+1}} \sum_{j=0}^{u+1} \binom{u+1}{j} (-1)^j \left[ e^{\alpha\sqrt{j-1}} \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}} + \sqrt{j-1}\sqrt{\theta_0}\right) \right. \\ \left. + e^{-\alpha\sqrt{j-1}} \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}} - \sqrt{j-1}\sqrt{\theta_0}\right) \right], \end{aligned} \quad (17)$$

where  $\sqrt{-1} = i$  and  $\theta_0 = -\log(1-q)$ .

One can check that in (17), the term for  $j = 0$  is a real number. Note that the results in the discrete time model are more complex than in the continuous time model.

### 3 Asymptotics: the ultimate ruin probabilities

This Section is the core of the paper. Our purpose is to establish an asymptotic rule  $A + B/u$ , as  $u \rightarrow \infty$ , for the ultimate ruin probability in the mixing risk models of Section 2. Furthermore, we will obtain an expansion of this probability as a power series of  $1/u$ .

For the asymptotic analysis, we use the standard Landau notation  $\mathcal{O}(\cdot)$ ,  $o(\cdot)$  and  $\sim$  (e.g. Jones (1997)). Integration by part is a standard tool in this field (see e.g. Olver et al. (2010), Gordon (1994)); it is extensively applied below. A comprehensive and updated list of known asymptotics for special functions (such as the Stirling formula for the gamma function) is also provided by Olver et al. (2010).

#### 3.1 Continuous time framework

We begin by presenting the main result for the continuous time risk model of Section 2.

**Theorem 3.1.** *Consider the continuous time model with  $(X|\Theta = \theta) \sim \mathcal{E}(\theta)$ . Let  $\theta_0 = \lambda/c$ .*

(i) *If the maximum of the function  $g(\theta) \equiv F_\Theta(\theta)(1/\theta^2 + u/\theta)$ ,  $\theta \geq \theta_0$ , is attained at  $\theta = \theta_0$ , then an upper bound for  $\psi(u)$  with  $u > 0$  is*

$$\psi(u) \leq F_\Theta(\theta_0) + \frac{1}{u} \frac{F_\Theta(\theta_0)}{\theta_0}. \quad (18)$$

(ii) *If  $\Theta$  has a continuous distribution of density  $f_\Theta$  such that for some integer  $k \geq 1$ ,  $f_\Theta$  is  $C^{k-1}$  a.e. on  $[\theta_0, \infty)$  and  $f_\Theta^{(k)}$  is Lebesgue integrable on  $[\theta_0, \infty)$ , then*

$$\psi(u) = F_\Theta(\theta_0) + \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{u^{i+1}} + o\left(\frac{1}{u^k}\right), \quad (19)$$

where  $h(x) \equiv \theta_0 f_\Theta(x + \theta_0)/(x + \theta_0)$  so that

$$h^{(i)}(0) = \sum_{j=0}^i (-1)^j \frac{i!}{(i-j)! \theta_0^j} f_\Theta^{(i-j)}(\theta_0).$$



*Proof.* (i) By (2) and (3),

$$\psi(u) = F_{\Theta}(\theta_0) + \int_{\theta_0}^{\infty} \psi_u(\theta) dF_{\Theta}(\theta), \quad \text{with } \psi_u(\theta) = \frac{\theta_0}{\theta} e^{-u(\theta-\theta_0)},$$

where  $\theta_0 = \lambda/c$ . Both  $\psi_u$  and  $F_{\Theta}$  are bounded functions on  $[\theta_0, \infty)$ . They are also of bounded variation since monotone. In addition,  $\psi_u$  is continuous. So,  $F_{\Theta}$  is Stieltjes integrable with respect to  $\psi_u$  (see e.g. Theorem 12.1 of Hildebrandt (1971)). We may then apply integration by part (see e.g. Theorem 12.14 of Gordon (1994)), which gives

$$\psi(u) = F_{\Theta}(\theta_0) + \lim_{b \rightarrow \infty} \psi_u(b)F_{\Theta}(b) - \psi_u(\theta_0)F_{\Theta}(\theta_0) - \int_{\theta_0}^{\infty} F_{\Theta}(\theta) d\psi_u(\theta).$$

As  $\psi_u$  is continuously differentiable, the Stieltjes integral above reduces to a Riemann integral. One has

$$\psi'_u(\theta) = -\theta_0(1/\theta^2 + u/\theta)e^{-u(\theta-\theta_0)},$$

$\psi_u(\theta_0) = 1$  and  $\psi_u(b)F_{\Theta}(b) \rightarrow 0$  as  $b \rightarrow \infty$ . Therefore, we obtain

$$\psi(u) = \theta_0 \int_{\theta_0}^{+\infty} F_{\Theta}(\theta)(1/\theta^2 + u/\theta)e^{-u(\theta-\theta_0)} d\theta \leq \theta_0 \max_{\theta \geq \theta_0} \{F_{\Theta}(\theta)(1/\theta^2 + u/\theta)\} \int_{\theta_0}^{\infty} e^{-u(\theta-\theta_0)} d\theta,$$

yielding the upper bound (18).

(ii) Let  $I(u, \theta_0) = \int_{\theta_0}^{\infty} \psi_u(\theta) dF_{\Theta}(\theta)$ . Suppose that  $\Theta$  has a continuous distribution with density  $f_{\Theta}$ . Making the change of variable  $t = \theta - \theta_0 \geq 0$  gives

$$I(u, \theta_0) = \int_0^{\infty} h(t)e^{-ut} dt,$$

where

$$h(t) = \theta_0 f_{\Theta}(t + \theta_0)/(t + \theta_0),$$

i.e.  $I(u, \theta_0)$  is the Laplace transform of  $h$ . The minimum condition for integration by part is the absolute continuity of  $h$  (see e.g. Theorem 12.5 of Gordon (1994)). By the fundamental theorem of calculus, absolute continuity of  $h$  on an interval  $[0, b]$  is equivalent to  $h$  is a.e. differentiable on  $[0, b]$  with  $h'$  Lebesgue integrable. Since  $t \mapsto \theta_0/(t + \theta_0)$  is  $C^{\infty}$  on  $[0, b]$ ,  $h$  is absolutely continuous on  $[0, b]$  if and only if so is  $f_{\Theta}$ . Now, consider the assumption made on  $f_{\Theta}$  in the statement above. For  $k = 1$ ,  $f_{\Theta}(t + \theta_0)$  is a.e. differentiable on  $\mathbb{R}_+$  with  $f'_{\Theta}$  Lebesgue integrable, hence  $h$  is absolutely continuous. Applying integration by part, we then have

$$\int_0^b h(t)e^{-ut} dt = \frac{h(0)}{u} - \frac{h(b)e^{-bu}}{u} + \frac{1}{u} \int_0^b h'(t)e^{-ut} dt,$$

so that letting  $b \rightarrow \infty$ ,

$$I(u, \theta_0) = \frac{h(0)}{u} + \frac{1}{u} \int_0^{+\infty} h'(t)e^{-ut} dt.$$

For a larger  $k \geq 1$ , the assumption that  $f_{\Theta}$  is  $C^{k-1}$  a.e. on  $[\theta_0, \infty)$  with  $f_{\Theta}^{(k)}$  Lebesgue integrable implies that  $h^{(i)}$  is absolutely continuous for all  $1 \leq i \leq k$ . Applying  $k$  times the integration by part theorem, we get

$$I(u, \theta_0) = \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{u^{i+1}} + \frac{1}{u^k} \int_0^{\infty} h^{(k)}(t)e^{-ut} dt.$$

Moreover, by a property of the Laplace transform (see e.g. Chapter 19 of Jeffrey & Dai (2008)),

$$\int_0^{\infty} h^{(k)}(t)e^{-ut} dt \rightarrow 0 \quad \text{as } u \rightarrow \infty,$$

and using the Leibnitz formula,

$$h^{(i)}(t) = \sum_{j=0}^i \binom{i}{j} \frac{(-1)^j j! \theta_0}{(\theta_0 + t)^{j+1}} f_{\Theta}^{(i-j)}(t + \theta_0).$$

Therefore, we deduce for  $\psi(u)$  the approximating expansion (19). □

**Remark 3.2.** For  $k = 1$ , formula (19) gives the asymptotic rule  $A + B/u$ :

$$\psi(u) = F_{\Theta}(\theta_0) + \frac{f_{\Theta}(\theta_0)}{u} + o\left(\frac{1}{u}\right),$$

when  $f_{\Theta}$  is  $C^0$  a.e. on  $[\theta_0, \infty)$  with  $f'_{\Theta}$  Lebesgue integrable. For  $k = \infty$ ,

$$\psi(u) \sim F_{\Theta}(\theta_0) + \sum_{i=0}^{\infty} \frac{h^{(i)}(0)}{u^{i+1}} \quad \text{as } u \rightarrow \infty,$$

when  $f_{\Theta}$  is  $C^{\infty}$  a.e. on  $[\theta_0, \infty)$ . Note that this is related to an asymptotic result (2.3.2) of Olver et al. (2010) since  $I(u, \theta_0)$  is a Laplace transform.

**Remark 3.3.** A sufficient condition for  $f_{\Theta}$  to be a.e. differentiable is local Lipschitz continuity. This is a consequence of the Rademacher theorem (see e.g. Clarke & Bessis (1999)).

**Remark 3.4.** A similar approach is applicable when mixing the waiting times  $(T_1, T_2, \dots)$  instead of the claim amounts. Following section 3 of Albrecher et al. (2011), we get now

$$\psi(u) = \bar{F}_{\Lambda}(\lambda_0) + \int_0^{\lambda_0} \psi_u(\lambda) dF_{\Lambda}(\lambda), \quad \text{with } \psi_u(\lambda) = \frac{\lambda}{\lambda_0} e^{-u/\theta(1-\lambda/\lambda_0)},$$

where  $\lambda_0 = \theta c$ . A result similar to Theorem 3.1 can be derived here too. In particular, a first order asymptotic expansion gives

$$\psi(u) = \bar{F}_{\Lambda}(\lambda_0) + \frac{f_{\Lambda}(\lambda_0)}{cu} + o\left(\frac{1}{u}\right).$$

Let us go back to the two special cases studied in Subsection 2.1. From formula (19) with  $k = 2$  (for instance), we deduce the following approximations for  $\psi(u)$ .

**Proposition 3.5.** In the continuous time model with  $(X|\Theta = \theta) \sim \mathcal{E}(\theta)$  and  $\Theta \sim \mathcal{G}a(\alpha, \lambda)$ ,

$$\psi(u) = \frac{\gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{\lambda^{\alpha}\theta_0^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\theta_0} \left( \frac{1}{u} + \frac{1}{u^2} \left( \frac{\alpha-1}{\theta_0} - \lambda \right) + o\left(\frac{1}{u^2}\right) \right). \quad (20)$$

**Remark 3.6.** By using known asymptotics to approximate the gamma function in (5) (see e.g. Olver et al. (2010)), one gets

$$\psi(u) = \frac{\gamma(\alpha, \theta_0\lambda)}{\Gamma(\alpha)} + \frac{\lambda^{\alpha}\theta_0^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\theta_0} \left( \frac{1}{\lambda+u} + \frac{\alpha-1}{(\lambda+u)^2\theta_0} + o\left(\frac{1}{u^2}\right) \right),$$

which is similar to formula (20), the main difference being  $1/(\lambda+u)$  substituted for  $1/u$  (both are of the same order for large  $u$ ).

**Proposition 3.7.** In the continuous time model with  $(X|\Theta = \theta) \sim \mathcal{E}(\theta)$  and  $\Theta \sim \mathcal{L}e(\alpha)$ ,

$$\psi(u) = \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{\alpha}{2\sqrt{\pi}\theta_0^3} e^{-\alpha^2/4\theta_0} \left( \frac{1}{u} + \frac{1}{u^2} \left( \frac{\alpha^2}{4\sqrt{\theta_0}} - \frac{3}{2\theta_0} \right) + o\left(\frac{1}{u^2}\right) \right). \quad (21)$$

**Remark 3.8.** Using in (8) known asymptotics for the complementary error function (Olver et al. (2010)) leads to an explosive term  $e^{\alpha\sqrt{u}}$ . So, a term-by-term asymptotic is not appropriate and a uniform expansion of the function  $\Gamma(-3/2, x, b)$  is needed when both  $x$  and  $b$  are large.

In Figures 1a and 1b, we compare the two approximations (20) and (21) with their close-form counterparts (5) and (8) as well as their empirical estimates. We observe that all close-form formulas diverge for  $u \geq 500$  because of roundoff errors. Their second-order approximations, however, remain numerically stable. Furthermore, empirical estimates for a 10,000 sample size oscillate around the asymptotics even with  $u$  moderately large, for both cases. This illustrates the accuracy of the asymptotics derived in Theorem 3.1.

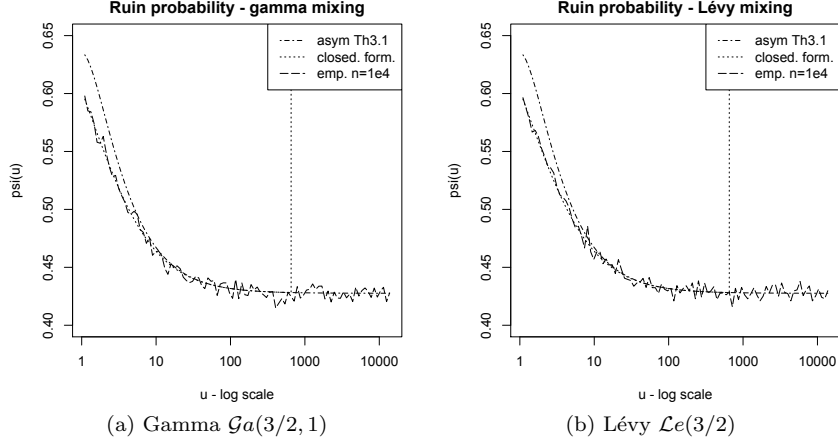


Figure 1: Exact and approximated ruin probabilities, with empirical estimates.

### 3.2 Discrete time framework

We now discuss the discrete time risk model of Section 2.

**Theorem 3.9.** Consider the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, \theta)$ . Let  $\theta_0 = -\log(1 - q)$ .

(i) A lower bound for  $\psi(u)$  with  $u \geq 0$  is

$$\psi(u) \geq \bar{F}_\Theta(\theta_0)(1 - q) \left(1 + \frac{q}{u + 2}\right). \quad (22)$$

(ii) If  $\Theta$  has a continuous distribution with density  $f_\Theta$  such that for some integer  $k \geq 1$ ,  $f_\Theta$  is  $C^{k-1}$  a.e. on  $[0, \theta_0]$  and  $f_\Theta^{(k)}$  is bounded on  $[0, \theta_0]$ , then

$$\psi(u) = \bar{F}_\Theta(\theta_0) + q(1 - q) \sum_{i=0}^{k-1} \frac{(-1)^i \tilde{h}^{(i)}(1)}{(u + 2) \dots (u + 2 + i)} + o\left(\frac{1}{(u + 2) \dots (u + 2 + k - 1)}\right), \quad (23)$$

where  $\tilde{h}(x) \equiv f_\Theta(-\log(1 - xq))/(1 - xq)^2$ .

*Proof.* (i) By (10) and (11),

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \int_0^{\theta_0} \psi_u(\theta) dF_\Theta(\theta), \quad \text{with } \psi_u(\theta) = \frac{1 - q}{e^{-\theta}} \left(\frac{1 - e^{-\theta}}{q}\right)^{u+1},$$

where  $\theta_0 = -\log(1 - q)$ . Let us rewrite the Stieltjes integral by substituting  $-d\bar{F}_\Theta(\theta)$  for  $dF_\Theta(\theta)$ . Both  $\psi_u$  and  $\bar{F}_\Theta$  are of bounded variation on  $[0, \theta_0]$ , and  $\psi_u$  is continuous. So,  $\bar{F}_\Theta$  is Stieltjes integrable with respect to  $\psi_u$ . An integration by part (see e.g. Theorem 12.14 of Gordon (1994)) then yields

$$\psi(u) = \bar{F}_\Theta(\theta_0) - \psi_u(\theta_0)\bar{F}_\Theta(\theta_0) + \psi_u(0)\bar{F}_\Theta(0) + \int_0^{\theta_0} \bar{F}_\Theta(\theta) d\psi_u(\theta) = \int_0^{\theta_0} \bar{F}_\Theta(\theta) d\psi_u(\theta),$$

using  $\psi_u(\theta_0) = 1$  and  $\psi_u(0) = 0$ . As  $\psi_u$  is continuously differentiable, the Stieltjes integral above reduces to a Riemann integral. By inserting the expression derived for  $\psi_u'(\theta)$ , we get

$$\psi(u) = \int_0^{\theta_0} \bar{F}_\Theta(\theta)(1 - q)e^\theta \left(\frac{1 - e^{-\theta}}{q}\right)^{u+1} d\theta + \int_0^{\theta_0} \bar{F}_\Theta(\theta) \frac{1 - q}{q}(u + 1) \left(\frac{1 - e^{-\theta}}{q}\right)^u d\theta.$$

Let  $J(u) = \int_0^{\theta_0} [(1 - e^{-\theta})/q]^u d\theta$ . We then see that

$$\psi(u) \geq \bar{F}_\Theta(\theta_0)(1 - q)J(u + 1) + \bar{F}_\Theta(\theta_0)(u + 1)\frac{1 - q}{q}J(u).$$

Now, making the change of variable  $qx = 1 - e^{-t}$  in  $J(u)$  shows that

$$J(u) = q \int_0^1 \frac{1}{1 - xq} x^u dx \geq \frac{q}{u + 1}.$$

Therefore, we deduce for  $\psi(u)$  the lower bound stated in (22).

(ii) Let  $I(u, \theta_0) = \int_0^{\theta_0} \psi_u(\theta) dF_\Theta(\theta)$ . The mixing variable  $\Theta$  has a continuous distribution with density  $f_\Theta$ . By the change of variable  $x = (1 - e^{-\theta})/q$ ,

$$I(u, \theta_0) = q(1 - q) \int_0^1 \frac{h(x)}{(1 - xq)^2} x^{u+1} dx,$$

where

$$h(x) = f_\Theta(g(x)) \quad \text{with} \quad g(x) = -\log(1 - xq).$$

The minimum condition for an integration by part is the absolute continuity of  $h(x)/(1 - xq)^2$ . As  $x \mapsto 1/(1 - xq)^2$  is  $C^\infty$ , one must show that  $h = f_\Theta \circ g$  is absolutely continuous. Note that this is not necessarily true if both  $f_\Theta$  and  $g$  are absolutely continuous. In fact, if  $g$  is absolutely continuous, then  $h$  is absolutely continuous if and only if  $f_\Theta$  is locally Lipschitz (see e.g. Merentes (1991)). By the Rademacher theorem,  $f_\Theta$  is locally Lipschitz, so  $h$  is absolutely continuous. For  $k = 1$ , integration by part gives

$$I(u, \theta_0) = q(1 - q) \left\{ \tilde{h}(1) - \int_0^1 \tilde{h}'(x) \frac{x^{u+2}}{u+2} dx \right\},$$

where  $\tilde{h}(x)$  is defined as in (23). For  $k \geq 1$ , the assumptions made on  $f_\Theta$  in the statement (ii) imply that  $h^{(i)}$  is absolutely continuous for all  $1 \leq i \leq k$ . Applying  $k$  times integration by part, we get

$$I(u, \theta_0) = q(1 - q) \left\{ \sum_{i=0}^{k-1} \frac{(-1)^i \tilde{h}^{(i)}(1)}{(u+2) \dots (u+2+i)} + (-1)^k \int_0^1 \tilde{h}^{(k)}(x) \frac{x^{u+1+k}}{(u+2) \dots (u+2+k-1)} dx \right\}.$$

Note that computing  $h^{(i)}$  is not very easy (see e.g. Huang et al. (2006)). As  $f_\Theta^{(i)}$ ,  $1 \leq i \leq k$ , are bounded on  $[0, \theta_0]$ , one can check that  $\tilde{h}^{(k)}$  is bounded on  $[0, 1]$ , hence the announced result (23).  $\square$

**Remark 3.10.** For  $k = 1$ , formula (23) gives the asymptotic rule  $A + B/u$ :

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \frac{1}{u+2} \frac{qf_\Theta(\theta_0)}{1-q} + o\left(\frac{1}{u+2}\right),$$

when  $f_\Theta$  is  $C^0$  a.e. on  $[0, \theta_0]$  with  $f'_\Theta$  bounded. For  $k = \infty$ ,

$$\psi(u) \sim \bar{F}_\Theta(\theta_0) + q(1 - q) \sum_{i=0}^{+\infty} \frac{(-1)^i \tilde{h}^{(i)}(1)}{(u+2) \dots (u+2+i)} \quad \text{as } u \rightarrow \infty,$$

when  $f_\Theta$  is  $C^\infty$  a.e. on  $[0, \theta_0]$ .

Let us examine below the three special cases presented in Subsection 2.2. From (23) with  $k = 1$  and using the corresponding expressions of  $\tilde{h}(x)$ , we deduce for  $\psi(u)$  the approximations given below. Numerical illustrations as in Figure 1 can be obtained here too.

**Proposition 3.11.** In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{E}(\lambda)$ ,

$$\psi(u) = (1 - q)^\lambda + \frac{1}{u+2} \lambda q(1 - q)^{\lambda-1} + o\left(\frac{1}{u+2}\right). \quad (24)$$

**Proposition 3.12.** *In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{G}a(\alpha, \lambda)$ ,*

$$\psi(u) = \frac{\Gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{1}{u+2} \frac{\lambda^\alpha}{\Gamma(\alpha)} q(1-q)^{\lambda-1} \theta_0^{\alpha-1} + o\left(\frac{1}{u+2}\right). \quad (25)$$

**Proposition 3.13.** *In the discrete time model with  $(X|\Theta = \theta) =_d \mathcal{G}e(q, e^{-\theta})$  and  $\Theta =_d \mathcal{L}e(\alpha)$ ,*

$$\psi(u) = 1 - \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{1}{u+2} \frac{q\alpha\theta_0^{-3/2}}{2(1-q)\sqrt{\pi}} e^{-\alpha^2/4\theta_0} + o\left(\frac{1}{u+2}\right). \quad (26)$$

**Remark 3.14.** *One could try to use known asymptotics of the incomplete beta function (see Olver et al. (2010)) to derive such approximations. In the exponential case, inserting in (13) an asymptotic for  $\tilde{\beta}(\lambda - 1, u + 2, 1 - q)$  provides the same formula (24). In the gamma and Lévy cases, proceeding similarly in (15) and (17) does not seem to lead easily to the asymptotics (25) and (26).*

## 4 Asymptotics: the claim tail distributions

In this Section, we investigate the asymptotic behavior of the claim tail distribution, i.e.  $P(X > x)$  as  $x \rightarrow \infty$ , for the mixing models of Section 2. The analysis of the discrete time model is rather complex. For this study, we will follow a method based on complex analysis and contour integrals developed e.g. in Flajolet & Sedgewick (1995).

### 4.1 Continuous case

For an exponential conditional claim distribution,

$$P(X > x) = \int_0^\infty e^{-\theta x} dF_\Theta(\theta),$$

i.e. the Laplace transform of the random variable  $\Theta$ . Applying the integration by part directly on this Stieltjes integral does not lead to interesting results. Let us examine the case where  $\Theta$  has a continuous distribution.

**Proposition 4.1.** *Consider the continuous time model where  $(X|\Theta = \theta) =_d \mathcal{E}(\theta)$  and  $\Theta$  has a density  $f_\Theta$ .*

(i) *If  $f_\Theta$  is a.e. differentiable on  $\mathbb{R}_+$  and  $f'_\Theta$  is Lebesgue-integrable, then*

$$P(X > x) = \frac{f_\Theta(0)}{x} + o\left(\frac{1}{x}\right). \quad (27)$$

(ii) *If  $f_\Theta$  is  $C^\infty$  in the neighborhood of the origin, then*

$$P(X > x) \underset{x \rightarrow \infty}{\sim} \sum_{k=0}^{\infty} \frac{f_\Theta^{(k)}(0)}{x^{k+1}}. \quad (28)$$

(iii) *If  $f_\Theta$  can be expanded in the neighborhood of the origin as*

$$f_\Theta(\theta) \underset{\theta \rightarrow 0}{\sim} \sum_{k=0}^{\infty} f_k \theta^{(k+\eta)/\mu-1},$$

*for some  $\eta, \mu > 0$ , then*

$$P(X > x) \underset{x \rightarrow \infty}{\sim} \sum_{k=0}^{\infty} \Gamma\left(\frac{k+\eta}{\mu}\right) \frac{f_k}{x^{(k+\eta)/\mu}}. \quad (29)$$

*Proof.* For (i),  $f_{\Theta}$  satisfies the minimum condition for an integration by parts, which gives

$$P(X > x) = \left[ f_{\Theta}(\theta) \frac{e^{-\theta x}}{-x} \right]_0^{\infty} + \frac{1}{x} \int_0^{\infty} e^{-\theta x} f'_{\Theta}(\theta) d\theta,$$

i.e. (27). Formulas (28) and (29) directly follow from Propertiy 2.3(i) and the Watson lemma 2.3(ii) in Olver et al. (2010), respectively.  $\square$

**Remark 4.2.** *Parts (i) and (ii) of this proposition may be not applicable when  $f_{\Theta}$  is not defined or is zero at the origin. This justifies the part (iii). The important role played by the behavior of  $f_{\Theta}$  at the origin can be explained through the Laplace method (see e.g. Olver et al. (2010)).*

Let us consider the two special cases discussed previously. If  $\Theta =_d \mathcal{G}a(\alpha, \lambda)$ , then  $f_{\Theta}$  is given by (4) and can be expanded as

$$f_{\Theta}(\theta) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \theta^{\alpha+k-1},$$

so that by (29),

$$P(X > x) \underset{x \rightarrow \infty}{\sim} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k + \alpha)}{\Gamma(\alpha)k!} \left( \frac{\lambda}{x} \right)^{k+\alpha}.$$

This (asymptotic) decrease of the survival function is consistent with the fact that  $X$  is known to be Pareto type II distributed (see before, just below (4)).

When  $\Theta =_d \mathcal{L}e(\alpha)$ ,  $f_{\Theta}$  is given by (6). Although this function is not defined at zero, the density converges to zero since

$$f_{\Theta}(1/t) = \frac{\alpha t^{3/2}}{2\sqrt{\pi}} e^{-\alpha^2 t/4} \xrightarrow[t \rightarrow \infty]{} 0.$$

However, it is not possible to get an expansion of  $f_{\Theta}(1/t)$  using again an expansion of the exponential function. Therefore, Theorem 4.1 is not useful in this case. Remember, however, that the survival function is known (see before, just below (6)).

## 4.2 Discrete case

For a zero-modified geometric claim distribution,

$$P(X > x) = (1 - q) \int_0^{\infty} (1 - e^{-\theta})^x dF_{\Theta}(\theta).$$

Here too, we meet difficulties to work with this integral. Suppose that  $\Theta$  has a density  $f_{\Theta}$ . A change of variable  $y = -\log(1 - e^{-\theta})$  is not relevant because that logarithmic function is unbounded near 0. Let us try  $y = 1 - e^{-\theta}$ , which gives

$$P(X > x) = (1 - q) \int_0^1 \frac{f_{\Theta}(-\log(1 - y))}{1 - y} y^x dy.$$

An integration by part requires the existence of the limit

$$\lim_{y \rightarrow 1} \frac{f_{\Theta}(-\log(1 - y))}{1 - y} = \lim_{t \rightarrow \infty} f_{\Theta}(t) e^t.$$

This condition is strong and will be satisfied only for light-tailed distributions with certain parameter values. For example, when  $\Theta =_d \mathcal{E}(\lambda)$ , it imposes that  $\lambda > 1$ .

Another way to tackle the asymptotic analysis is to write that, for  $x \in \mathbb{N}_0$ ,

$$P(X > x) = \sum_{k=0}^x \binom{x}{k} (1-q)(-1)^k L_{\Theta}(k), \quad (30)$$

where  $L_{\Theta}$  is the Laplace transform of  $\Theta$  at point  $k$ . This binomial sum having alternating signs, it is not easy to estimate its behavior for large values of  $x$ . In fact, (30) falls within the wider framework of binomial alternating sums defined by

$$S_n(\phi) = \sum_{k=n_0}^n \binom{n}{k} (-1)^k \phi(k), \quad (31)$$

where  $0 \leq n_0 \leq n$ ,  $\phi$  is a real function and  $n$  is large.

#### 4.2.1 Through complex analysis

To deal with such sums, a standard method consists in using complex analysis and contour integrals. Flajolet & Sedgewick (1995) provide a comprehensive treatment of the topic. A brief presentation of their analysis is given below. Consider the complex extension of the function  $\phi$  introduced in (31). Their Lemma 1 gives the Rice integral representation of  $S_n(\phi)$  for an analytic function  $\phi$ :

$$S_n(\phi) \equiv \sum_{k=n_0}^n \binom{n}{k} (-1)^k \phi(k) = \frac{(-1)^n}{2\pi i} \oint_{\gamma} \phi(s) \frac{n!}{s(s-1)\dots(s-n)} ds, \quad (32)$$

where  $\phi$  is analytic in a domain that contains  $[n_0, \infty)$  and  $\gamma$  is a closed curve in the domain that encircles  $[n_0, n]$  but does not include any of the integers  $0, \dots, n_0 - 1$ . Let  $f$  denote the integrand in the r.h.s. of (32). By the residue theorem (see e.g. Chapter 10 of Bak & Newman (2010)), if  $f$  is analytic except at a countable number of isolated singularities inside  $\gamma$ , then the contour integral equals the sum of the residues of  $f$  at the singularities.

The goal is to derive some asymptotics as  $n \rightarrow \infty$ . So, the domain  $\gamma$  is extended to the positive real half-plane  $\{z : \text{Re}(z) > 0\}$ . Moreover, it is not known whether  $|f(z)|$  explodes or not as  $\text{Re}(z) \rightarrow \infty$ . Their Theorem 1 deals with the case where  $\phi$  is a rational function.

**Theorem** (Flajolet and Sedgewick). *Let  $\phi$  be a rational function analytic on  $[n_0, \infty)$ . Then, except for a finite number of values of  $n$ ,*

$$\sum_{k=n_0}^n \binom{n}{k} (-1)^k \phi(k) = -(-1)^n \sum_s \text{Res} \left( \phi(s) \frac{n!}{s(s-1)\dots(s-n)} \right),$$

where the sum is extended to all poles  $s$  of  $f$  not on  $[n_0, \infty)$ .

This is proved by choosing for  $\gamma$  a large circle centered at 0. One can see that the integral converges to 0 for  $n > r$  where  $r$  is the degree of  $\phi$ , thus its value is exactly 0. Applying the residue theorem, one gets that the integral also equals  $S_n(\phi)$  plus the sum of the residues of (32) at the other poles of  $f$ , hence the result.

Their Theorem 2 is concerned with the case where  $\phi$  is a meromorphic function (i.e. analytic except on a set of isolated points) that is sufficiently well conditioned on large contours or in half-planes. An analogous result is derived by adapting slightly the previous approach.

Finally, the authors examine the case where  $\phi$  has algebraic or logarithmic singularities. The estimation here is made by means of contours of Hankel type. Attention is mainly paid to the study of  $S_n(\phi)$  when  $\phi = s^{-\lambda}$  for nonintegral  $\lambda$ . In particular, it is shown in their Theorem 3 that

$$S_n(\sqrt{s}) \sim -\frac{1}{\sqrt{\pi} \log n} + \frac{\gamma_e}{2\sqrt{\pi}(\log n)^3}, \quad (33)$$

where  $\gamma_e$  is the Euler-Mascheroni constant. Table 1 then provides the correspondences between the nature of singularities and the asymptotic behavior of  $S_n$ .

Singularities at $s_0 \notin \mathbb{N}$	Asymptotics for $S_n$
simple/multiple pole: $(s - s_0)^{-r}$	$\Gamma(-s_0)n^{s_0} \frac{(\log n)^{r-1}}{(r-1)!}$
algebraic singularity: $(s - s_0)^\lambda$	$\Gamma(-s_0)n^{s_0} \frac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}$
logarithmic singularity: $(s - s_0)^\lambda (\log(s - s_0))^r$	$\Gamma(-s_0)n^{s_0} \frac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)} (\log \log n)^r$

Table 1: (Flajolet and Sedgewick). Correspondences between singularities and asymptotics.

#### 4.2.2 Application to claim tails

To begin with, suppose that  $\Theta =_d \mathcal{E}(\lambda)$ . This case is simple because  $P(X > x) = (1 - q)\beta(\lambda, x + 1)$  for all  $x$ . So, using asymptotics for the beta function as  $x$  is large (see e.g. Olver et al. (2010)), we get

$$P(X > x) \underset{x \rightarrow \infty}{\sim} (1 - q) \frac{\Gamma(\lambda + 1)}{(x + 1)^\lambda}, \quad (34)$$

which decreases like a discrete Pareto (Zipf) distribution. This tail behavior of a Yule-Simon distribution is reported in Simon (1955).

Consider now the case where  $\Theta =_d \mathcal{G}a(\alpha, \lambda)$ . We here use the binomial alternating representation (31) in which  $\phi(k) = (1 - q)L_\Theta(k) = (1 - q)[\lambda/(k + \lambda)]^\alpha$ . From Table 1, we then obtain

$$P(X > x) \underset{x \rightarrow \infty}{\sim} (1 - q) \frac{\lambda^\alpha \Gamma(\lambda)}{\Gamma(\alpha)} \frac{(\log x)^{\alpha-1}}{x^\lambda}, \quad (35)$$

which decreases slightly slower than (34) due to the logarithm in the numerator.

When  $\Theta =_d \mathcal{L}e(\alpha)$ , we go back to (31) in which  $\phi(k) = (1 - q)L_\Theta(k) = (1 - q)e^{-\alpha\sqrt{k}}$ . One may make the approximaton  $\phi(k) \sim 1 - \alpha\sqrt{k} + o(k)$ . Moreover, the constant 1 has no influence in the computation of the integral contour. Using (33), we thus get

$$P(X > x) \underset{x \rightarrow \infty}{\sim} \frac{(1 - q)\alpha}{\sqrt{\pi} \log x} - \frac{(1 - q)\alpha\gamma_e}{2\sqrt{\pi}(\log x)^3}, \quad (36)$$

which decreases extremely slowly. Such a tail behaviour is heavier than for a Pareto distribution. In the continuous case, a similar behaviour is obtained with the log-Cauchy distribution, for example.

In Figure 2, we compare the three approximations (34), (35) and (36) with the exact tail distributions obtained from (12), (14) and (16). Formula (12) is easily tractable since the incomplete beta function is available in most softwares, e.g. via the `pbeta` function in R. For (14) and (16), we have to evaluate two alternating binomial sums. These computations are particularly unstable because the central combinatorial factors tend to infinity for large  $n$ .

In modern computers, a real number is stored in eight bytes (i.e. 64 bits), for which only 53 bits are reserved for the precision (see e.g. [http://en.wikipedia.org/wiki/Double-precision\\_floating-point\\_format](http://en.wikipedia.org/wiki/Double-precision_floating-point_format)). In our numerical experiments, the sums  $S_n(\phi)$  become unstable for  $n \geq 48$  with the standard double precision. To compute the binomial sums for large  $n$ , we have no other way than to use high precision floating-point arithmetic libraries such as the GMP library of Grandlund Torbjorn & the GMP Devel. Team (2011), or the MPFR library of Fousse et al. (2011) (which allows us to work with a high number of bits, say 500 or 1000). Using the Rmpfr package of Maechler (2012), we are then able to assess the claim survival function.



Figures 2a, 2b and 2c correspond to the cases where  $\Theta$  is exponential, gamma and Lévy distributed, respectively. In the first two figures, the tail distribution shows a Pareto-type behavior, as we observe a straight line for the log-scale graphs. The third figure exhibits a heavier tail behavior. Note that the asymptotic is not as reliable as for other cases. The heavier the tail distribution of  $\Theta$ , the larger the error. These numerical experiments also illustrate the importance of high precision arithmetic libraries for numerical issues in actuarial problems.

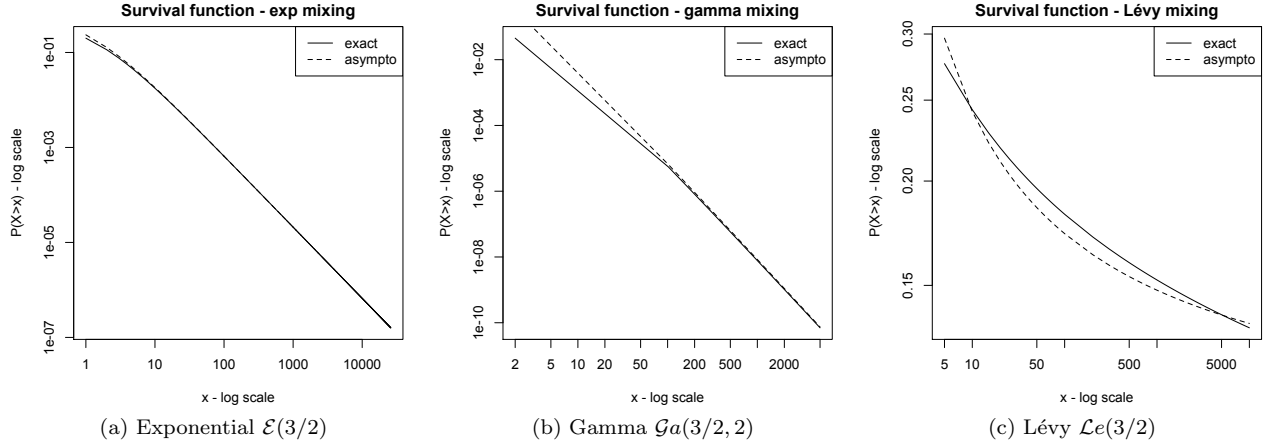


Figure 2: Exact and asymptotic claim survival functions.

## 5 On the dependence structure involved

To close, we briefly discuss the dependence structure in the mixing risk models of Section 2. For that, we will use the well-known concept of copula (see e.g. Joe (1997), Nelsen (2006)).

For the continuous model, we recall the following property derived by Albrecher et al. (2011).

**Theorem** (Albrecher et al.). *In the continuous time model with  $(X|\Theta = \theta) =_d \mathcal{E}(\theta)$ , the dependence structure corresponds to an Archimedean survival copula with generator  $\phi = L_{\Theta}^{-1}$ , the inverse Laplace transform of  $\Theta$ .*

Let us now consider the discrete model, in the bivariate case for clarity. By the Sklar theorem, a copula  $C$  of a random vector is not unique outside the support of the vector. This unidentifiability is a major cause of difficulty in the discrete case: if  $(X_1, X_2)$  are  $\mathbb{N}_0^2$ -valued,  $C$  is unique on  $\mathbb{N}_0^2$  but not on  $\mathbb{R}^2 \setminus \mathbb{N}_0^2$ .

A comprehensive review about copulas for discrete distributions is provided by Genest & Nešlehová (2007). Let  $(X_1, X_2)$  be a random vector with joint distribution function  $F(x_1, x_2)$  and marginal d.f.  $F_1(x_1)$ ,  $F_2(x_2)$ . Their starting point (Proposition 1) is that, denoting by

- (1)  $\mathcal{A}$  the class of functions  $A : [0, 1]^2 \rightarrow [0, 1]$  such that  $F(x_1, x_2) = A(F_1(x_1), F_2(x_2))$  for all  $x_1, x_2 \in \mathbb{R}$ ,
- (2)  $\mathcal{C}_F \subset \mathcal{A}$  the set of copulas  $C$  such that  $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$  for all  $x_1, x_2 \in \mathbb{R}$ ,
- (3)  $B : [0, 1]^2 \rightarrow [0, 1]$  the function defined by  $B(u, v) = F(F_1^{-1}(u), F_2^{-1}(v))$  for all  $u, v \in [0, 1]$ ,
- (4)  $D$  the distribution function of the vector  $(F_1(X_1), F_2(X_2))$ ,

then both  $B$  and  $D$  belong to the class  $\mathcal{A}$ , but  $B$  is not a distribution function and  $D$  is a distribution function but not a copula.

In Section 3, the authors analyze the importance of this unidentifiability issue by measuring the size of the set of copulas  $\mathcal{C}_F$ . In Section 4, they study the impact on copula models by pointing out that the copula

does no longer characterize the dependence and each  $C \in \mathcal{C}_F$  leads to different values of Kendall's  $\tau$  and Spearman's  $\rho$ .

Nevertheless, there exists a copula  $C_F^{\boxtimes} \in \mathcal{C}_F$  such that  $X_1 \perp X_2$  iff  $C_F^{\boxtimes}$  is the independent copula,  $\tau(X_1, X_2) = \tau(C_F^{\boxtimes})$  and  $\rho(X_1, X_2) = \rho(C_F^{\boxtimes})$ . This copula is obtained by linear interpolation of the d.f.  $D$  of  $(F_1(X_1), F_2(X_2))$  for all  $(u, v) \in \text{Ran}(F_1) \times \text{Ran}(F_2)$ . It is also the copula associated with the vector  $(X_1 + U_1, X_2 + U_2)$  where  $U_1, U_2$  are two independent  $(0, 1)$ -uniform random variables. The application to the present context is easy.

**Proposition 5.1.** *Consider the discrete model with  $(X|\Theta = \theta) =_d \mathcal{G}(q, e^{-\theta})$ . Then, the distribution function  $D$  of  $(F_1(X_1), F_2(X_2))$  is given by*

$$D(u, v) = \begin{cases} 0 & \text{if } u < q \text{ or } v < q, \\ 1 - (1 - q)(H_{\Theta}(\lfloor l_u \rfloor) + H_{\Theta}(\lfloor l_v \rfloor)) + (1 - q)^2 H_{\Theta}(\lfloor l_u \rfloor + \lfloor l_v \rfloor) & \text{otherwise,} \end{cases} \quad (37)$$

for  $u, v \in [0, 1]$ , where  $H_{\Theta}(x) = E[(1 - e^{-\Theta})^x]$ ,  $l_z = H_{\Theta}^{-1}((1 - z)/(1 - q))$  for  $z = u, v \geq q$  and  $\lfloor x \rfloor$  is the floor function.

The interpolated copula  $C_F^{\boxtimes}$  for  $(X_1, X_2)$  is then constructed as usual, i.e.

$$C_F^{\boxtimes}(u, v) = \bar{\lambda}_u \bar{\lambda}_v D(u_i, v_j) + \lambda_u \bar{\lambda}_v D(u_{i+1}, v_j) + \bar{\lambda}_u \lambda_v D(u_i, v_{j+1}) + \lambda_u \lambda_v D(u_{i+1}, v_{j+1}),$$

for  $(u_i, v_j) \in \text{Ran}(F_1) \times \text{Ran}(F_2)$  such that  $(u, v) \in [u_i, u_{i+1}] \times [v_j, v_{j+1}]$ , where  $\lambda_z = 1 - \bar{\lambda}_z = (z - z_i)/(z_{i+1} - z_i)$  ( $z = u, v$ ).

*Proof.* By assumption,

$$P(X \leq x|\Theta = \theta) = q + (1 - q)[1 - (1 - e^{-\theta})^{\lfloor x \rfloor}] = 1 - (1 - q)(1 - e^{-\theta})^{\lfloor x \rfloor}, \quad x \in \mathbb{R}_+.$$

Thus, using the function  $H_{\Theta}$  defined above,

$$P(X \leq x) = 1 - (1 - q)H_{\Theta}(\lfloor x \rfloor), \quad x \in \mathbb{R}_+,$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2) = 1 - (1 - q)(H_{\Theta}(\lfloor x_1 \rfloor) + H_{\Theta}(\lfloor x_2 \rfloor)) + (1 - q)^2 H_{\Theta}(\lfloor x_1 \rfloor + \lfloor x_2 \rfloor), \quad x_1, x_2 \in \mathbb{R}_+.$$

Now, defining  $l_z$  as indicated before,

$$P(F_1(X_1) \leq u, F_2(X_2) \leq v) = \begin{cases} 0 & \text{if } u < q \text{ or } v < q, \\ P(X_1 \leq l_u, X_2 \leq l_v) & \text{otherwise,} \end{cases}$$

for  $u, v \in [0, 1]$ , hence (37). The expression of  $C_F^{\boxtimes}$  follows directly by linear interpolation of  $D$ .  $\square$

**Remark 5.2.** *Using floor functions on marginals of a continuous copula was considered in several previous papers (e.g. Joe (1997) and Frees & Wang (2006)).*

## Acknowledgements

This work is partially funded by the Research Chair *Actuariat Durable* sponsored by Milliman, the research chair *Management de la Modélisation* sponsored by BNP Paribas Cardif and the AXA research fund. C. Lefèvre received support from a grant *Crédit aux Chercheurs* of the Belgian FNRS and the *ARC project IAPAS* of the Fédération Wallonie-Bruxelles. C. Dutang gratefully acknowledges financial support from the *Swiss National Science Foundation*, project 200021-124635/1. The authors thank Dominik Kortschak, the Editor and two Referees for many useful remarks and suggestions.

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## A Computation of $\Gamma(-3/2, x, b)$

The function  $\Gamma(a, x, b)$  satisfies a recurrence on the parameter  $a$ , namely

$$\Gamma(a + 1, x, b) = a\Gamma(a, x, b) + b\Gamma(a - 1, x, b) + x^a e^{-x-b/x},$$

see e.g. Chaudry & Zubair (2002). Thus,

$$\Gamma(-3/2, x, b) = \frac{1}{b} \left( \Gamma(1/2, x, b) + 1/2\Gamma(-1/2, x, b) - x^{-1/2} e^{-x-b/x} \right).$$

Now,  $\Gamma(a, x, b)$  has a simple expression in terms of the error function when  $a = 1/2$  and  $-1/2$  (Chaudry & Zubair (2002)):

$$\begin{aligned} \Gamma(1/2, x, b) &= \frac{\sqrt{\pi}}{2} \left[ e^{2\sqrt{b}} \operatorname{erfc} \left( \sqrt{x} + \frac{\sqrt{b}}{\sqrt{x}} \right) + e^{-2\sqrt{b}} \operatorname{erfc} \left( \sqrt{x} - \frac{\sqrt{b}}{\sqrt{x}} \right) \right], \\ \Gamma(-1/2, x, b) &= \frac{\sqrt{\pi}}{2\sqrt{b}} \left[ -e^{2\sqrt{b}} \operatorname{erfc} \left( \sqrt{x} + \frac{\sqrt{b}}{\sqrt{x}} \right) + e^{-2\sqrt{b}} \operatorname{erfc} \left( \sqrt{x} - \frac{\sqrt{b}}{\sqrt{x}} \right) \right]. \end{aligned}$$

Therefore, we get

$$\Gamma(-3/2, x, b) = \frac{\sqrt{\pi}}{2b} \left[ \left( 1 - \frac{1}{2\sqrt{b}} \right) e^{2\sqrt{b}} \operatorname{erfc}(d_+) + \left( 1 + \frac{1}{2\sqrt{b}} \right) e^{-2\sqrt{b}} \operatorname{erfc}(d_-) - \frac{2}{\sqrt{\pi x}} e^{-x-b/x} \right],$$

with  $d_+ = \sqrt{x} + \sqrt{b}/\sqrt{x}$  and  $d_- = \sqrt{x} - \sqrt{b}/\sqrt{x}$ . For formula (7), we need  $\Gamma(-3/2, \theta_0 u, \alpha^2 u/4)$  which directly follows. We then deduce that

$$\begin{aligned} I(u, \theta_0) &= \frac{\theta_0 \sqrt{u}}{\alpha} e^{u\theta_0} \left[ \left( 1 - \frac{1}{\alpha\sqrt{u}} \right) e^{\alpha\sqrt{u}} \operatorname{erfc}(d_+) + \left( 1 + \frac{1}{\alpha\sqrt{u}} \right) e^{-\alpha\sqrt{u}} \operatorname{erfc}(d_-) \right. \\ &\quad \left. - \frac{2}{\sqrt{\pi u \theta_0}} e^{-u\theta_0 - \alpha^2/(4\theta_0)} \right], \end{aligned}$$

which gives, after reordering the terms, a formula for  $\psi(u)$  of Example 2.4 in Albrecher et al. (2011):

$$I(u, \theta_0) = \frac{\theta_0}{\alpha^2} e^{-\alpha/(4\theta_0)} \left[ e^{(c\alpha+2\lambda\sqrt{u})^2/(4\lambda c)} (-1 + \alpha\sqrt{u}) \operatorname{erfc}(d_+) + e^{(c\alpha-2\lambda\sqrt{u})^2/(4\lambda c)} (1 + \alpha\sqrt{u}) \operatorname{erfc}(d_-) - \frac{2\alpha}{\sqrt{\pi\theta_0}} \right].$$

## B Zero-modified geometric distribution

A random variable  $X$  of geometric distribution  $\mathcal{G}(p)$  has the probability mass function  $P(X = k) = p(1-p)^k$ ,  $k \in \mathbb{N}_0$ , where  $0 \leq p \leq 1$ ; so,  $P(X > k) = (1-p)^{k+1}$ . This distribution depends on one parameter only, which can be restrictive. A generalization with two parameters is the zero-modified geometric distribution  $\mathcal{G}(q, \rho)$  whose probability mass function is

$$P(X = k) = \begin{cases} q & \text{for } k = 0, \\ (1-q)\rho(1-\rho)^{k-1} & \text{for } k \geq 1, \end{cases}$$

yielding  $P(X > k) = (1-q)(1-\rho)^k$ .

## C Integral of error function type

Consider the following integral

$$J(a, b, x) = \int_x^\infty e^{-ay^2 - b/y^2} dy,$$

where  $a, b, x > 0$ . For its computation, the SAGE mathematical software (Stein et al. (2011)) suggests to do a change of variable  $t$  in order to get the integral of  $e^{-t^2}$ . As the equation  $t^2 = ay^2 + b/y^2$  does not have a unique solution, the trick is to split the integral  $J(a, b, x)$ . Since we have

$$2\sqrt{a}dy = \sqrt{a}dy + \frac{\sqrt{b}}{-y^2}dy + \sqrt{a}dy - \frac{\sqrt{b}}{-y^2}dy,$$

we rewrite  $J(a, b, x)$  as

$$2\sqrt{a}J(a, b, x) = \int_x^\infty e^{-ay^2 - b/y^2} d\left(\sqrt{ay} + \frac{\sqrt{b}}{y}\right) + \int_x^\infty e^{-ay^2 - b/y^2} d\left(\sqrt{ay} - \frac{\sqrt{b}}{y}\right),$$

Taking  $t = \sqrt{ay} + \sqrt{b}/y$  in the former integral and  $t = \sqrt{ay} - \sqrt{b}/y$  in the latter then leads to

$$2\sqrt{a}J(a, b, x) = e^{2\sqrt{ab}} \int_{\tilde{x}_1}^\infty e^{-t^2} dt + e^{-2\sqrt{ab}} \int_{\tilde{x}_2}^\infty e^{-t^2} dt,$$

with  $\tilde{x}_1 = \sqrt{ax} + \sqrt{b}/x$  and  $\tilde{x}_2 = \sqrt{ax} - \sqrt{b}/x$ . Hence,

$$J(a, b, x) = \frac{\sqrt{\pi}}{4\sqrt{a}} \left[ e^{2\sqrt{ab}} \operatorname{erfc}\left(\sqrt{ax} + \frac{\sqrt{b}}{x}\right) + e^{-2\sqrt{ab}} \operatorname{erfc}\left(\sqrt{ax} - \frac{\sqrt{b}}{x}\right) \right].$$

This result is in line with Theorem 3.1 of Chaudry & Zubair (2002) and is closely related to the generalized error function

$$e^{-2\sqrt{b}} \operatorname{erfc}(x; b) = \frac{1}{2} \left[ e^{2\sqrt{b}} \operatorname{erfc}\left(x + \frac{\sqrt{b}}{x}\right) + e^{-2\sqrt{b}} \operatorname{erfc}\left(x - \frac{\sqrt{b}}{x}\right) \right].$$

For  $x = 0$ , we get

$$J(a, b, 0) = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-2\sqrt{ab}}.$$

Now, take  $b = -1$  inside  $J(a, b, x)$  (although  $b > 0$  a priori). Putting  $\sqrt{-1} = i$  (the imaginary number) yields

$$J(a, -1, x) = \frac{\sqrt{\pi}}{4\sqrt{a}} \left[ e^{2i\sqrt{a}} \operatorname{erfc} \left( \sqrt{a}x + \frac{i}{x} \right) + e^{-2i\sqrt{a}} \operatorname{erfc} \left( \sqrt{a}x - \frac{i}{x} \right) \right].$$

This number is of type  $z_1 z_2 + \bar{z}_1 \bar{z}_2$  where  $z_1 = e^{2i\sqrt{a}}$  and  $z_2 = \operatorname{erfc}(\sqrt{a}x + i/x)$  ( $\bar{z}$  is the conjugate of  $z$ ). One can check that  $z_1 z_2 + \bar{z}_1 \bar{z}_2 = 2|z_1 z_2| \cos(\arg(z_1) + \arg(z_2)) \in \mathbb{R}$ . Thus,  $J(a, -1, x)$  may be defined by the above expression.