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To cite this version:

Christophe Dutang, Claude Lefèvre, Stéphane Loisel. The "A+B/u" rule for discrete and continuous time risk models with dependence. 2012. hal-00746251v1

HAL Id: hal-00746251

https://hal.science/hal-00746251v1

Preprint submitted on 28 Oct 2012 (v1), last revised 30 Aug 2013 (v2)

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The $A + B/u$ rule for discrete and continuous time risk models with dependence

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Abstract

The purpose of this paper is to point out that an asymptotic rule $A + B/u$ for the ultimate ruin probability applies to a wide class of dependent risk models, in discrete and continuous time. Dependence is incorporated through a mixing approach among claim amounts or claim inter-arrival times, leading to a systemic risk behavior. Ruin corresponds here either to classical ruin, or to stopping the activity after realizing that it is not profitable at all, when one has little possibility to increase premium income rate. Several special cases for which closed formulas are derived, are also investigated in some detail.

Keywords: Ruin probability; Frailty models; Mixing; Archimedean copulas; Asymptotics

1. Introduction

Traditionally, the surplus $(U_t)_{t\geq 0}$ of an insurance company at time t is represented by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where u is the initial surplus, c is the premium rate, $(X_i)_{i\geq 1}$ are the successive claim amounts and $(N_t)_{t\geq 0}$ is the claim arrival process (the claim waiting times are denoted by $(T_i)_{i\geq 1}$). In the Cramér-Lundberg model, $(N_t)_{t\geq 0}$ is modeled by a Poisson process, $(X_i)_{i\geq 1}$ are independent and identically distributed (i.i.d.) random variables and claim severities $(X_i)_{i\geq 1}$ are independent of the claim waiting times $(T_i)_{i\geq 1}$. Andersen (1957) generalized the Cramér-Lundberg model by proposing a general renewal process for the claim arrival process $(N_t)_{t\geq 0}$.

Since then, extensions have been proposed in many directions. Asmussen and Rolski (1991) studied ruin models with phase-type distributions for both claim severities X_i and claim waiting times T_i. Gerber and Shiu (1998) unified the analysis of ruin measures in the Cramér-Lundberg model, including the deficit at ruin, the claim causing the ruin or the ruin probability, by introducing a so-called discounted penalty function. Gerber and Shiu (2005), Song et al. (2010) and many others extended the Gerber-Shiu approach

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to a wider class of risk models. Various generalizations of the Sparre Andersen model have been proposed, such as for non-homogeneous claim arrivals (e.g. Lu and Garrido (2005), Albrecher and Asmussen (2006)), reinsurance treaties (e.g. Centeno (2002*a*,*b*)), multivariate risks (e.g. Cai and Li (2005), Collamore (1996)) and dependent risks (e.g. Albrecher and Boxma (2004), Boudreault et al. (2006), Albrecher and Teugels (2006)).

The ultimate ruin probability, i.e. $\psi(u) = P(\exists t > 0 : U_t < 0 | U_0 = u)$, is a major ruin measure and has received a considerable attention in the literature. For the Sparre Andersen model, with light-tailed claim amounts, $\psi(u) \sim Ce^{-\gamma u}$ as $u \to \infty$, where γ is the positive root of a simple equation involving the moment generating function of X_i (see e.g. Asmussen and Albrecher (2010)). For heavy-tailed claim amounts, the ruin probability is in the class of Hall of order 1 since $\psi(u) \sim C/u^\alpha$ as $u \to \infty$ (e.g., Embrechts and Veraverbeke (1982), Klüppelberg and Stadtmüller (1998), Reiss and Thomas (2007)). Concerning models with dependence, Albrecher and Teugels (2006), e.g., studied the ruin probability when claim size and claim waiting times, $(X_i, T_i)_i$, are correlated; they obtained again an exponential decrease for $\psi(u)$ in the case of light-tailed claim sizes. In a recent paper, Albrecher et al. (2011) investigated study the ruin probability when there is dependence by mixing among the claim sizes $(X_i)_i$ or the claim waiting times $(T_i)_i$, see also Constantinescu et al. (2011). They derived here an asymptotic formula $\psi(u) - A \sim B/u$ (as $u \to +\infty$) for Pareto correlated claims or inter-occurrence times.

The main purpose of the present work is to show that this asymptotic rule (that we call $A + B/u$ rule) applies to a wide class of dependent risk models in discrete and continuous time. Dependence is incorporated through a mixing approach among claim amounts $(X_i)_{i\geq 1}$ or claim inter-arrival times $(T_i)_{i\geq 1}$. This corresponds to some uncertainty on the model parameters, due to incomplete information. In some cases (for example when one launches a new line of business in a remote country), it takes time to collect claim data and it might be complicated to react and adjust premium income rate. Ruin corresponds here to either classical ruin or stopping the activity before ruin but after losing a large amount of money, realizing that the activity is not profitable. Sufficient conditions are also given under which the ruin probability can be expanded as a series of terms a_k/u^k, $k \geq 0$, and the coefficients a_k, $k \geq 0$, are given.

Risk models are also often formulated in discrete time. In fact, such models are often more appropriate in insurance because the surplus of the company is usually examined after regular time periods. Li et al. (2009) provided a review of standard risk models in discrete time. Our starting point is when claim amounts have a geometric distribution, which implies an exponential decrease for $\psi(u)$. Adopting a mixing approach, we will focus on three particular cases of special interest. We also obtain asymptotics for the tail of the resulting claim distributions and then discuss the dependence structure involved.

The paper is organized as follows. Section 2 describes the mixing approach for both continuous and discrete time models. Section 3 establishes the asymptotic rule $A+B/u$ and some variants. Section 4 focuses on special features of the discrete time model. Except mentioned otherwise, all numerical illustrations are done with the R statistical software (R Core Team (2012)).

2. Model formulation

This section is devoted to the presentation of dependent risk models, first in the continuous time framework and then in the discrete time framework. In addition to a general formula of the ruin probability under the mixing approach, we present examples of explicit ruin probability formulas with specific mixing distributions for both time scales.

2.1. Continuous time framework

In this subsection, we present the continuous time framework based on the classic Cramér-Lundberg model. The free surplus of an insurance company at time t is modeled by

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i,$$

where u is the initial surplus, c is the premium rate, $(X_i)_i$ are the claim amounts and $(N_t)_{t\geq 0}$ is the Poisson claim arrival process with intensity λ. We assume that the $(X_i)_i$ are i.i.d. conditionally on a latent variable Θ (distributed as $X|\Theta = \theta$); they are independent of the claim arrival process. Θ can be interpreted as the heterogeneity in the claim process. In such setting, the claim sizes $(X_1, \ldots, X_n)$ are dependent random variables.

Ruin occurs as soon as the surplus process becomes negative. Conditionally on $\Theta = \theta$, the ruin probability is thus defined as

$$\psi(u, \theta) = P(\exists t > 0 : U_t < 0 | U_0 = u, \Theta = \theta).$$

To determine such a probability, a standard method consists in looking at the state of the surplus after the first claim arrival. This leads to an integro-differential equation that can be solved by using Laplace-Stieltjes transforms, see e.g. Asmussen and Albrecher (2010). In the case of exponentially distributed claims $(X_i)_i \sim \mathcal{E}(\theta)$, we have the well-known following formula

$$\psi(u, \theta) = \min\left(\frac{\lambda}{\theta c} e^{-u(\theta - \frac{\lambda}{c})}, 1\right),$$

where the min is equivalent to the net profit condition $\theta > \lambda/c$. Integrating over the parameter θ yields the ruin probability,

$$\psi(u) = F_\Theta(\theta_0) + I(u, \theta_0), \tag{1}$$

where $\theta_0 = \lambda/c$, F_Θ denotes the distribution function of Θ and

$$I(u, \theta_0) = \int_{\theta_0}^{+\infty} \frac{\theta_0}{\theta} e^{-u(\theta - \theta_0)} dF_\Theta(\theta). \tag{2}$$

(1) is nothing else than Equation (5) of Albrecher et al. (2011).

Now, we briefly present the results for two particular distributions of the latent variable Θ, reported in Albrecher et al. (2011). Firstly, we consider for Θ a gamma distribution $\mathcal{G}a(\alpha, \lambda)$ with density

$$f_\Theta(\theta) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta}, \text{thus } F_\Theta(\theta) = \frac{\gamma(\alpha, \lambda\theta)}{\Gamma(\alpha)}, \theta > 0,$$

where $\gamma(.,.)$ (resp. $\Gamma(.)$) denotes the incomplete lower gamma function (the gamma function), see Olver et al. (2010). The resulting claim generic variable X has a Pareto type II distribution with parameter $\mathcal{P}a(\alpha, \lambda)$, whose survival function is

$$P(X > x) = \frac{1}{\left(1 + \frac{x}{\lambda}\right)^\alpha}, x \geq 0.$$

Using the change of variable $y = \theta(\lambda + u)$, the integral $I(u, \theta_0)$ can be expressed in terms of the incomplete upper gamma function $\Gamma(.,.)$, see Olver et al. (2010). We get

$$\psi(u) = \frac{\gamma(\alpha, \theta_0\lambda)}{\Gamma(\alpha)} + \frac{\lambda^\alpha \theta_0}{\Gamma(\alpha)} e^{\theta_0 u} \frac{\Gamma(\alpha - 1, \theta_0(\lambda + u))}{(\lambda + u)^{\alpha-1}}. \tag{3}$$

Note that the formula is only valid when the shape parameter verifies $\alpha > 1$, i.e. the density of $X/\Theta = \theta$ is log-concave.

Secondly, consider for Θ a stable 1/2 distribution (also called a Lévy distribution $\mathcal{L}e(\alpha)$) with density

$$f_\Theta(\theta) = \frac{\alpha}{2\sqrt{\pi\theta^3}} e^{-\alpha^2/4\theta}, \text{thus } F_\Theta(\theta) = \text{erfc}\left(\frac{\alpha}{2\sqrt{\theta}}\right), \theta > 0,$$

where erfc(.) denotes the complementary error function, see Olver et al. (2010). We choose the same parametrization of the Lévy distribution as in Albrecher et al. (2011), but there exists another parametrization, see Nolan (2012), which involves a location parameter δ and a shape parameter γ. These two parametrizations are equivalent by setting $\gamma = \alpha^2/2$ and $\delta = 0$. The resulting claim distribution is a Weibull distribution $\mathcal{W}e(1/2, 1/\alpha^2)$ for which the distribution tail is

$$P(X > x) = e^{-\alpha\sqrt{x}}, x \geq 0.$$

Unlike the previous case, the computation of $I(u, \theta)$ in the Lévy case is more complicated. Using this time the change of variable $x = u\theta$, we get

$$I(u, \theta_0) = \frac{\theta_0 \alpha \sqrt{u^3}}{2\sqrt{\pi}} e^{u\theta_0} \int_{u\theta_0}^{+\infty} \frac{1}{\sqrt{x^5}} e^{-x-\alpha^2 u/(4x)} dx. \tag{4}$$

The latter integral is related to the generalized error function, a particular case of the generalized incomplete upper gamma function, which is defined as

$$\Gamma(a, x; b) = \int_x^{+\infty} t^{a-1} e^{-t-b/t} dt,$$

see e.g. Chaudry and Zubair (1994, 2002). In Equation (4), we use $\Gamma(-3/2, \theta_0 u; \alpha^2 u/4)$. As for the classic incomplete gamma function, the function $\Gamma(., ., .)$ satisfies a recurrence equation on the parameter a,

$$\Gamma(a+1, x; b) = a\Gamma(a, x; b) + b\Gamma(a-1, x; b) + x^a e^{-x-b/x},$$

see Theorem 2.2 of Chaudry and Zubair (2002). Using this equation, we are able to compute $\Gamma(-3/2, x; b)$ in terms of $\Gamma(-1/2, x; b)$ and $\Gamma(1/2, x; b)$, which can be both expressed in terms of the (classic) error function, see Appendix A for details. We get

$$\begin{aligned}\psi(u) = \text{erfc}(\alpha/2\sqrt{\theta_0}) + \frac{\theta_0\sqrt{u}}{\alpha} e^{u\theta_0} \Bigg[&\left(1 - \frac{1}{\alpha\sqrt{u}}\right) e^{\alpha\sqrt{u}} \text{erfc}\,(d_+) \\ &+ \left(1 + \frac{1}{\alpha\sqrt{u}}\right) e^{-\alpha\sqrt{u}} \text{erfc}\,(d_-) - \frac{2}{\sqrt{\pi u \theta_0}} e^{-u\theta_0 - \alpha^2/(4\theta_0)} \Bigg],\end{aligned} \tag{5}$$

where $d_+ = \sqrt{u\theta_0} + \alpha/(2\sqrt{\theta_0})$ and $d_- = \sqrt{u\theta_0} - \alpha/(2\sqrt{\theta_0})$.

2.2. Discrete time framework

The compound binomial risk model, introduced by Gerber (1988), is the discrete time analog of the Cramér-Lundberg model. Here too, we construct an extended version of this model by using a mixing approach. We are going to derive the ruin probability, for this risk process, as well as explicit formulas for three special cases. The insurance portfolio is now examined at times $t \in \mathbb{N}$. The successive claim amounts form a sequence of $\mathbb{N}$-valued random variables that are conditionally independent w.r.t. Θ (distributed as $X|\Theta = \theta$). The premium per period is constant and is equal to 1, and corresponds to a prepayment to cover the risk during that period. The surplus of the insurance company at time t is then given by

$$U_t = u + t - \sum_{i=1}^{t} X_i,$$

where $u \in \mathbb{N}$ is the initial surplus. When the claims are independent, this model is named compound binomial, because the number of strictly positive claims has a binomial distribution $\mathcal{B}(t, 1-q)$ where $q = P(X = 0)$. The net profit condition is $E(X) < 1$ in order to avoid the a.s. ruin.

The definition of ruin probability has to be made precise since there is a non-zero probability for the surplus to be zero. In other words, we must specify if the ruin of the insurance company occurs when $U_t < 0$ or $U_t \leq 0$. Gerber (1988) considers the ruin as the first time the process U reaches 0, i.e.

$$\psi_G(u) = P(\exists t \in \mathbb{N}^+ : U_t \leq 0 | U_0 = u).$$

Shiu (1989) considers the ruin as the first time the process U becomes strictly negative:

$$\psi_S(u) = P(\exists t \in \mathbb{N}^+ : U_t < 0 | U_0 = u).$$

Graphically, ψ_G is the probability that the surplus process crosses the level 0 while ψ_S is the probability that the surplus crosses the level -1, since U_t is integer-valued. We can switch from one formula to the other using the relation $\psi_G(u) = \psi_S(u-1)$. For the rest of the paper, we consider the ruin probability ψ_S.

Closed formulas for the ruin probability ψ_S are available (see e.g. Willmot (1993), Sundt and dos Reis (2007), Lefèvre and Loisel (2008)). Sundt and dos Reis (2007) derived the ruin probability when X is geometrically distributed. More precisely, assuming a geometric decreasing tail for the ruin probability, they proved that the claim amount distribution is of geometric type (see proof of Theorem 1 of Sundt and dos Reis (2007)). In this framework, when the claim amount follows a 0-modified geometric distribution $\mathcal{G}e(q, \rho)$, see Appendix B, then the ultimate ruin probability is given by

$$\psi_S(u) = \min\left(\frac{1-q}{\rho}\left(\frac{1-\rho}{q}\right)^{u+1}, 1\right). \tag{6}$$

where the minimum is equivalent to the net profit condition $\rho > 1 - q$, ensuring the term in power of u does not explode.

At our disposal, we have two closed formulas for the infinite time ruin probability. Now, let us extend the formula (6) by using again a mixing approach. We choose this formula rather than the previous one because of its tractability. Specifically, we suppose that $X_i/\Theta = \theta \sim \mathcal{G}e(q, e^{-\theta})$, then the overall ruin probability is

$$\psi(u) = \bar{F}_\Theta(\theta_0) + I(u, \theta_0), \tag{7}$$

where $\theta_0 = -\log(1-q)$ and

$$I(u, \theta_0) = \int_0^{\theta_0} \frac{1-q}{e^{-\theta}}\left(\frac{1-e^{-\theta}}{q}\right)^{u+1} dF_\Theta(\theta). \tag{8}$$

Compared to the continuous setting, (1) and (2), the integral in (8) is done over the interval $[0, \theta_0]$ for $I(u, \theta)$ rather than the interval $[\theta_0, +\infty[$. This is due to the fact that $\psi_S(u, \theta)$ is decreasing function of θ in the considered parametrization.

We do not choose the classic geometric distribution $\mathcal{G}e(\rho)$, because the net profit condition ($\rho > 1/2$) is restrictive on the type of parametrization for the parameter ρ. However, in that case, one could consider, for example, a geometric distribution $X/\Theta = \theta \sim \mathcal{G}e(1/(1+\theta))$. This leads to a ruin probability

$$\psi(u) = \int_0^1 \theta^{u+2} dF_\Theta(1) + \bar{F}_\Theta(1).$$

Choosing a uniform distribution $\Theta \sim \mathcal{U}(0, p)$ with $p \leq 1$ yields the surprisingly simple formula $\psi(u) = p^{u+2}/(u+3)$. This simple ruin probability is particularly interesting, because whether $p < 1$ or $p = 1$, the decrease of the ruin probability switches a geometric speed to a polynomial speed. In this special setting, the ruin probability is also explicit when Θ is beta distributed.

We present here results for three different distributions of Θ. Firstly, we consider an exponential distribution $\Theta \sim \mathcal{E}(\lambda)$. We use the following definite integral

$$I_1(a, b, x) = \int_0^x \left(e^{-\theta}\right)^a \left(1 - e^{-\theta}\right)^b d\theta = \int_{e^{-x}}^1 p^a (1-p)^b \frac{dp}{p} = \bar{\beta}(a, b+1, e^{-x}),$$

for $x > 0$. $I_1(a, b, x)$ reduces to the beta function $\beta(a, b+1)$ when x tends to infinity. Using $I_1(\lambda+1, k, +\infty)$, the mass probability function of the claim distribution is given

$$P(X = k) = q\delta_{k0} + (1 - \delta_{k0})\lambda(1 - q)\beta(\lambda + 1, k),$$

where δ_{ij} denotes the Kronecker product. With the presence of a beta function in this mass probability function, one can recognize the zero-modified Yule-Simon distribution, see e.g. Simon (1955). This distribution appears in the study of word frequency. The survival function is given by

$$P(X > k) = \lambda(1 - q)\beta(\lambda, k + 1).$$

Using $I(u, \theta_0) = I_1(\lambda - 1, u + 1, \theta_0)$, the ruin probability can be derived.

Proposition 2.1. *Let us consider the discrete time framework of Subsection 2.2 with a latent variable Θ exponentially distributed $\mathcal{E}(\lambda)$.*

$$\psi(u) = (1 - q)^\lambda + \frac{\lambda(1 - q)}{q^{u+1}}\bar{\beta}(\lambda, u + 1, 1 - q), \forall u \geq 0.$$

Secondly, consider Θ follows a gamma distribution $\mathcal{G}a(\alpha, \lambda)$. We use the following integral

$$I_2(a, n, b, x) = \int_0^x \left(e^{-\theta}\right)^a \left(1 - e^{-\theta}\right)^n \theta^b d\theta = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \int_0^x e^{-\theta(a+n-j)}\theta^b d\theta,$$

yielding

$$I_2(a, n, b, x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \int_0^{\tilde{x}} \frac{y^b}{(a + n - j)^{b+1}} e^{-y} dy.$$

with $\tilde{x} = x(a + j)$. Substituting $n - j$ to j gives

$$I_2(a, n, b, x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{\gamma(b + 1, \tilde{x})}{(a + j)^{b+1}},$$

where $\gamma(., .)$ denotes the incomplete lower gamma function. When x tends to infinity, only the term $\gamma(b+1, \tilde{x})$ changes and tends to $\Gamma(b + 1)$. With the integral $I_2(\lambda, k - 1, \alpha - 1, +\infty)$ (resp. $I_2(\lambda, k, \alpha - 1, +\infty)$), the resulting claim distribution has a mass probability function (resp. a survival function)

$$P(X = k) = q\delta_{k0} + (1 - \delta_{k0})(1 - q) \sum_{j=0}^{k-1} \binom{k - 1}{j} \frac{(-1)^j \lambda^\alpha}{(\lambda + j)^\alpha} \quad \text{and} \quad P(X > k) = (1 - q) \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j \lambda^\alpha}{(\lambda + j)^\alpha}.$$

Using $I_2(\lambda - 1, u + 1, \alpha - 1, \theta_0)$, the ruin probability can be deduced.

Proposition 2.2. *Let us consider the discrete time framework of Subsection 2.2 with a latent variable Θ gamma distributed $\mathcal{G}a(\alpha, \lambda)$.*

$$\psi(u) = \frac{\Gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{1 - q}{q^{u+1}} \sum_{j=0}^{u+1} \binom{u + 1}{j} (-1)^j \frac{\gamma(\alpha, \theta_0(\lambda + j - 1))}{\Gamma(\alpha)} \left(\frac{\lambda}{\lambda + j - 1}\right)^\alpha,$$

with $\lambda > 1$, $\theta_0 = -\log(1 - q)$ and for $u \geq 0$.

Finally, consider Θ is Lévy distributed $\mathcal{L}e(\alpha)$. We use the integral

$$I_3(a, n, b, x) = \int_0^x \left(e^{-\theta}\right)^a \left(1 - e^{-\theta}\right)^n \theta^{-3/2} e^{-\frac{b}{\theta}} d\theta = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} 2 \int_{\tilde{x}}^{\infty} e^{-\frac{a+n-j}{y^2}} e^{-by^2} dy.$$

with a change of variable and $\tilde{x} = x^{-1/2}$. This integral is linked to the generalized incomplete upper gamma function. Using Appendix C, we get

$$I_3(a,n,b,x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{\sqrt{\pi}}{2\sqrt{b}} \left[e^{2\sqrt{b(a+j)}} \mathrm{erfc}\left(\frac{\sqrt{b}}{\sqrt{x}} + \sqrt{a+j}\sqrt{x} \right) \right.$$
$$\left. + e^{-2\sqrt{b(a+j)}} \mathrm{erfc}\left(\frac{\sqrt{b}}{\sqrt{x}} - \sqrt{a+j}\sqrt{x} \right) \right].$$

When x tends to infinity, we have

$$I_3(a,n,b) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{\sqrt{\pi}}{\sqrt{b}} e^{-2\sqrt{b}\sqrt{a+j}}.$$

Using $I_3(0,k-1,\alpha^2/4)$ and $I_3(0,k,\alpha^2/4)$, the mass probability and survival functions are given by

$$P(X=k) = (1-q) \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-\alpha\sqrt{j}} \quad \text{and} \quad P(X>k) = (1-q) \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-\alpha\sqrt{j}}.$$

The expressions derived when Θ is Lévy distributed, are much more complex than in the continuous time framework. In Subsection 3.2, we study asymptotics for the survival function. The ruin probability can be computed using $I_3(-1,u+1,\alpha^2/4,\theta_0)$.

Proposition 2.3. *Let us consider the discrete time framework of Subsection 2.2 with a latent variable Θ Lévy distributed $\mathcal{L}e(\alpha)$.*

$$\psi(u) = erfc\left(\frac{\alpha}{2\sqrt{\theta_0}} \right) + \frac{1-q}{4q^{u+1}} \sum_{j=0}^{u+1} \binom{u+1}{j} (-1)^j \left[e^{\alpha\sqrt{j-1}} erfc\left(\frac{\alpha}{2\sqrt{\theta_0}} + \sqrt{j-1}\sqrt{\theta_0} \right) \right.$$
$$\left. + e^{-\alpha\sqrt{j-1}} erfc\left(\frac{\alpha}{2\sqrt{\theta_0}} - \sqrt{j-1}\sqrt{\theta_0} \right) \right],$$

with the convention $\sqrt{-1} = i$, $\theta_0 = -\log(1-q)$ and for $u \geq 0$.

3. Asymptotics – the $A + B/u$ rule

This section is the core of the paper, where we establish the $A + B/u$ asymptotic rule for the ultimate ruin probability for both continuous and discrete time models. We also obtain an expansion of the ruin probability as a power series of $1/u$. Finally, we investigate the asymptotic behavior of the resulting claim distribution, which requires a special treatment with complex analysis.

We use the standard Landau notation of the asymptotic analysis $\mathcal{O}()$, $o()$ and $\sim$; see e.g. Jones (1997), Olver et al. (2010). Integration by part is a standard tool to derive integral asymptotics as pointed in Olver et al. (2010). Olver et al. (2010) also give a comprehensive and updated list of known asymptotics of the so-called special functions (such as the Stirling formula for the gamma function). Integration by part will be extensively used in the next two subsections; see e.g. Gordon (1994) and the references therein.

3.1. Continuous time framework

In this subsection, we present and show the $A + B/u$ rule for the continuous time model.

Theorem 3.1. *Let us consider the continuous time framework of Subsection 2.1 with a positive latent variable Θ and $\theta_0 = \lambda/c$.*

(i) If the maximum (in t) of $F_\Theta(t)\left(\frac{1}{t^2}+\frac{u}{t}\right)$ is attained at θ_0, then for all $u>0$, the ruin probability is bounded

$$\psi(u) \leq F_\Theta(\theta_0) + \frac{1}{u} \times \frac{F_\Theta(\theta_0)}{\theta_0}.$$

(ii) If Θ has a continuous distribution with density f_Θ such that f_Θ is almost everywhere differentiable on $[\theta_0, +\infty[$ and f'_Θ being a Lebesgue-integrable, then

$$\psi(u) = F_\Theta(\theta_0) + \frac{f_\Theta(\theta_0)}{u} + o\left(\frac{1}{u}\right).$$

(iii) If in addition f_Θ is $C^{k\text{-}1}$ almost everywhere on $[\theta_0, +\infty[$ and $f_\Theta^{(k)}$ is Lebesgue integrable and bounded on $[\theta_0, +\infty[$, then

$$\psi(u) = F_\Theta(\theta_0) + \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{u^{i+1}} + o\left(\frac{1}{u^k}\right)$$

where $h(x) = \theta_0 f_\Theta(x+\theta_0)/(x+\theta_0)$, so that

$$h^{(i)}(0) = \sum_{j=0}^{i} (-1)^j \frac{i!}{(i-j)!\theta_0^j} f_\Theta^{(i-j)}(\theta_0).$$

(iv) If f_Θ is C^∞ on $[\theta_0, +\infty[$, then

$$\psi(u) \underset{u\to+\infty}{\sim} F_\Theta(\theta_0) + \sum_{i=0}^{+\infty} \frac{h^{(i)}(0)}{u^{i+1}}.$$

Proof. (i) From (1) and (2), the ruin probability is given by

$$\psi(u) = F_\Theta(\theta_0) + \int_{\theta_0}^{+\infty} \psi_u(\theta) dF_\Theta(\theta), \quad \text{with} \quad \psi_u(\theta) = \frac{\theta_0}{\theta} e^{-u(\theta-\theta_0)},$$

where $\theta_0 = \lambda/c$. Both ψ_u and F_Θ are bounded functions on $[\theta_0, +\infty[$. They also have bounded variations since they are monotone. In addition, ψ_u is continuous. So by Corollary 7.1.23 of Silvia (1999) or Theorem 12.1 of (Hildebrandt, 1971, Chap. 2), F_Θ is Stieltjes integrable with respect to the function ψ_u. Then, we apply the integration by part theorem on $\int \psi_u dF_\Theta$ reported in Theorem 12.14 of Gordon (1994). We get

$$\psi(u) = F_\Theta(\theta_0) + \lim_{b\to+\infty} \psi_u(b)F_\Theta(b) - \psi_u(\theta_0)F_\Theta(\theta_0) - \int_{\theta_0}^{+\infty} F_\Theta(t) d\psi_u(t).$$

Since ψ_u is continuously differentiable, the Stieltjes integral $\int F_\Theta d\psi_u$ reduces to a Riemann integral. We have

$$\psi'_u(\theta) = \frac{-1}{\theta^2}\theta_0 e^{-u(\theta-\theta_0)} + \frac{\theta_0}{\theta}(-u)e^{-u(\theta-\theta_0)} = -\theta_0\left(\frac{1}{\theta^2}+\frac{u}{\theta}\right)e^{-u(\theta-\theta_0)}.$$

Furthermore, $\psi_u(\theta_0) = 1$ and

$$\lim_{b\to+\infty} \psi_u(b)F_\Theta(b) = 0.$$

Therefore, we obtain

$$\psi(u) = \theta_0 \int_{\theta_0}^{+\infty} F_\Theta(t)\left(\frac{1}{t^2}+\frac{u}{t}\right)e^{-u(t-\theta_0)}dt \leq \theta_0 \max_{t\in[\theta_0,+\infty[} F_\Theta(t)\left(\frac{1}{t^2}+\frac{u}{t}\right)\int_{\theta_0}^{+\infty} e^{-u(t-\theta_0)}dt.$$

Since the maximum is attained at θ_0, we get

$$\psi(u) \leq F_\Theta(\theta_0) + \frac{1}{u} \times \frac{F_\Theta(\theta_0)}{\theta_0}.$$

(ii) Let $I(u,\theta_0) = \int_{\theta_0}^{+\infty} \psi_u(\theta) dF_\Theta(\theta)$. We assume a continuous distribution for the mixing variable Θ and make the change of variable $t = \theta - \theta_0$, we get

$$I(u,\theta_0) = \int_0^{+\infty} \frac{\theta_0}{\theta_0 + t} f_\Theta(t+\theta_0) e^{-ut} dt.$$

We easily recognize a Laplace transform of the function h defined as

$$h(t) = \frac{\theta_0}{\theta_0 + t} f_\Theta(t+\theta_0).$$

The minimum condition to apply an integration by part theorem is to require h to be absolutely continuous, see e.g. Theorem 12.5 of Gordon (1994). Using the fundamental theorem of calculus, absolute continuity of h on $[a,b]$ is equivalent to h is almost everywhere differentiable $[a,b]$ with h' being Lebesgue integrable on $[a,b]$. Since $t \mapsto \theta_0/(\theta_0+t)$ is C^∞ on $[0,b]$ for $b>0$, h is absolutely continuous on $[0,b]$ if and only if f_Θ is. By assumption, f_Θ is almost everywhere differentiable on $\mathbb{R}_+$ with f'_Θ being Lebesgue integrable, hence h is absolutely continuous. Thus we have

$$\int_0^b h(t) e^{-ut} dt = \left[h(t) \frac{e^{-ut}}{-u} \right]_0^b + \frac{1}{u} \int_0^b h'(t) e^{-ut} dt = \frac{h(0)}{u} - \frac{h(b) e^{-bu}}{u} + \frac{1}{u} \int_0^b h'(t) e^{-ut} dt.$$

As b tends to infinity, we get

$$I(u,\theta_0) = \frac{h(0)}{u} + \frac{1}{u} \int_0^{+\infty} h'(t) e^{-ut} dt.$$

Using a property of the Laplace transform, see e.g. Chapter 19 of Jeffrey and Dai (2008), we have

$$\int_0^{+\infty} h'(t) e^{-ut} dt \underset{u \to +\infty}{\longrightarrow} 0.$$

Finally, we conclude

$$\psi(u) = F_\Theta(\theta_0) + \frac{f_\Theta(0)}{u} + o\left(\frac{1}{u}\right).$$

(iii) As f_Θ is C^{k-1} almost everywhere on $[\theta_0, +\infty[$ and $f_\Theta^{(k)}$ is Lebesgue integrable, then $h^{(i)}$ is absolute continous for all $i \leq k$. Applying k times the integration by part theorem, we get

$$I(u,\theta_0) = \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{u^{i+1}} + \frac{1}{u^k} \int_0^{+\infty} h^{(k)}(t) e^{-ut} dt.$$

Similarly if $h^{(k)}(t)$ is bounded on $[\theta_0, +\infty[$, then the latter term is controlled by $o\big(1/u^k\big)$. Let g be the function $t \mapsto \frac{\theta_0}{\theta_0+t}$. The ith-order derivative of h, if it exists, can be derived by the Leibniz formula

$$h^{(i)}(t) = \sum_{j=0}^{i} \binom{i}{j} g^{(j)}(t) f_\Theta^{(i-j)}(t+\theta_0) \quad \text{with} \quad g^{(j)}(t) = \frac{(-1)^j j! \theta_0}{(\theta_0+t)^{j+1}}.$$

Thus, we have

$$\psi(u) = F_\Theta(\theta_0) + \sum_{i=0}^{k-1} \frac{h^{(i)}(0)}{u^{i+1}} + o\left(\frac{1}{u^k}\right) \quad \text{with} \quad h^{(i)}(0) = \sum_{j=0}^{i} (-1)^j \frac{i!}{(i-j)! \theta_0^j} f_\Theta^{(i-j)}(\theta_0).$$

(iv) if f_Θ is C^∞, we have

$$\psi(u) \underset{u \to +\infty}{\sim} F_\Theta(\theta_0) + \sum_{i=0}^{+\infty} \frac{h^{(i)}(0)}{u^{i+1}}.$$

Unsurprisingly, we get back to asymptotic result (2.3.2) of (Olver et al., 2010, Chapter 2), since $I(u,\theta)$ is a Laplace transform.

□

Remark 1. *A sufficient condition for f_Θ to be almost everywhere differentiable is local Lipchitzness. This is a consequence of the Rademacher theorem, see e.g. Clarke and Bessis (1999).*

Remark 2. *A similar approach can be done when mixing the waiting times $(T_1, T_2, \dots)$. Using Albrecher et al. (2011)'s Section 3, we have*

$$\psi(u) = \bar{F}_\Lambda(\lambda_0) + \int_0^{\lambda_0} \psi_u(\lambda) dF_\Lambda(\lambda), \quad \textit{with} \;\; \psi_u(\lambda) = \frac{\lambda}{\lambda_0} e^{-u/\theta(1-\lambda/\lambda_0)}, \lambda_0 = \theta c.$$

We give here only the first terms of the series expansion assuming Λ has a continuous distribution

$$\psi(u) = \bar{F}_\Lambda(\lambda_0) + \frac{1}{cu} f_\Lambda(\lambda_0) + o\left(\frac{1}{u}\right).$$

Below, we present asymptotics for the two special cases analyzed in Subsection 2.1, based on known asymptotics listed in Olver et al. (2010). When Θ is gamma distributed, we have

$$\psi(u) = \frac{\gamma(\alpha, \theta_0 \lambda)}{\Gamma(\alpha)} + \frac{\lambda^\alpha \theta_0^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\theta_0} \left(\frac{1}{\lambda + u} + \frac{\alpha - 1}{(\lambda + u)^2 \theta_0} + o\left(\frac{1}{u^2}\right) \right).$$

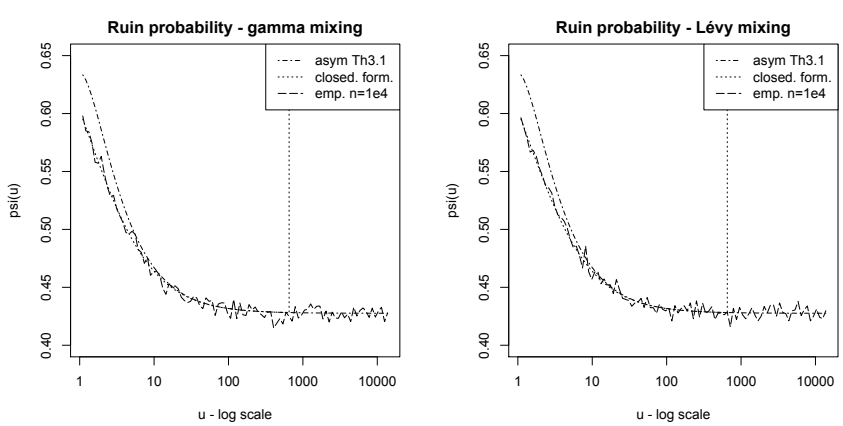
If we use Theorem 3.1, we get

$$\psi(u) = \frac{\gamma(\alpha, \lambda\theta_0)}{\Gamma(\alpha)} + \frac{\lambda^\alpha \theta_0^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\theta_0} \left(\frac{1}{u} + \frac{1}{u^2} \left(\frac{\alpha - 1}{\theta_0} - \lambda \right) + o\left(\frac{1}{u^2}\right) \right). \tag{9}$$

These two expressions are similar with only different denominators $1/u$ against $1/(\lambda + u)$, but this does not matter for large values of u.

When Θ is Lévy distributed, the term $I(u, \theta_0)$ contains two terms linked with the error complementarity function. There exists expansion formula for the error function, cf. Olver et al. (2010), but unfortunately the asymptotic of $I(u, \theta_0)$ leads to an explosive term $e^{\alpha\sqrt{u}}$. We conclude that a term-by-term asymptotic is not appropriate, a uniform expansion of the original function $\Gamma(3/2, x, b)$ is needed, when both x and b are large. But, we can still use Theorem 3.1 to get

$$\psi(u) = \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{\alpha}{2\sqrt{\pi\theta_0^3}} e^{-\alpha^2/4\theta_0} \left(\frac{1}{u} + \frac{1}{u^2} \left(\frac{\alpha^2}{4\sqrt{\theta_0}} - \frac{3}{2\theta_0} \right) + o\left(\frac{1}{u^2}\right) \right). \tag{10}$$



(a) Gamma $\mathcal{G}a(3/2, 1)$ (b) Lévy $\mathcal{L}e(3/2)$

Figure 1: Ruin probabilities

On Figures 1a and 1b, we compare these two asymptotics (Equations (9) and (10)) with their closed-formula counterparts (Equations (3) and (5) respectively) and their empirical estimates. All closed formulas diverge for u around 500 because of computation errors, however their second-order approximations remain numerically stable. Furthermore, empirical estimates of the ruin probability for a 10,000 sample size oscillate around the asymptotics for both mixing distributions. This illustrates the accuracy of the asymptotics derived from Theorem 3.1.

3.2. Discrete time framework

Now, let us turn our attention to the discrete-time framework, where the approach of this subsection shares strong similarities with the previous subsection.

Theorem 3.2. *Let us consider the discrete time framework of Subsection 2.2 with a positive latent variable Θ and $\theta_0 = -\log(1-q)$.*

(i) For all $u \geq 0$, the ruin probability is lower bounded

$$\bar{F}_\Theta(\theta_0) + F_\Theta(\theta_0)\frac{q}{u+2} \leq \psi(u).$$

(ii) If Θ has a continuous distribution with density f_Θ such that f_Θ is almost everywhere differentiable on $[0,\theta_0]$ with f_Θ, f'_Θ being bounded, then

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \frac{1}{u+2} \times \frac{q f_\Theta(\theta_0)}{1-q} + o\left(\frac{1}{u+2}\right).$$

(iii) If in addition f_Θ is $C^{k\text{-}1}$ almost everywhere on $[0,\theta_0]$ and successive derivatives of f_Θ are bounded on $[0,\theta_0]$, then

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \sum_{i=0}^{k-1} \frac{\tilde{h}^{(i)}(0)}{(u+2)\dots(u+2+i)} + o\left(\frac{1}{(u+2)\dots(u+2+k)}\right),$$

with $\tilde{h}(x) = f_\Theta(-\log(1-xq))/(1-xq)^2$.

(iv) If f_Θ is C^∞ on $[0,\theta_0]$, then

$$\psi(u) \underset{u\to+\infty}{\sim} \bar{F}_\Theta(\theta_0) + \sum_{i=0}^{+\infty} \frac{\tilde{h}^{(i)}(0)}{(u+2)\dots(u+2+i)}.$$

Proof. (i) From (7) and (8), the ruin probability is given by

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \int_0^{\theta_0} \psi_u(\theta) dF_\Theta(\theta), \quad \text{with} \quad \psi_u(\theta) = \frac{1-q}{e^{-\theta}}\left(\frac{1-e^{-\theta}}{q}\right)^{u+1},$$

where $\theta_0 = -\log(1-q)$. Firstly, we change the right-hand side Stieltjes integral by using the survival function $\bar{F}_\Theta$ rather than the cumulative distribution function. We get

$$\psi(u) = \bar{F}_\Theta(\theta_0) - \int_0^{\theta_0} \psi_u(\theta) d\bar{F}_\Theta(\theta).$$

Secondly, it is easy to see that both ψ_u and $\bar{F}_\Theta$ are also of bounded variation on $[0,\theta_0]$. They also have bounded variations since they are monotone. In addition, ψ_u is continuous. So by Corollary 7.1.23 of Silvia

(1999), $\bar{F}_\Theta$ is Stieltjes integrable with respect to the function ψ_u. Then we apply the integration by part theorem on $\int \psi_u d\bar{F}_\Theta$ reported in Theorem 12.14 of Gordon (1994). We get

$$\psi(u) = \bar{F}_\Theta(\theta_0) - \psi_u(\theta_0)\bar{F}_\Theta(\theta_0) + \psi_u(0)\bar{F}_\Theta(0) + \int_0^{\theta_0} \bar{F}_\Theta(t) d\psi_u(t) = \int_0^{\theta_0} \bar{F}_\Theta(t) d\psi_u(t),$$

using $\psi_u(\theta_0) = 1$ and $\psi_u(0) = 0$. Since ψ_u is continuously differentiable, the Stieltjes integral $\int \bar{F}_\Theta d\psi_u$ reduces to a Riemann integral. We have

$$\psi_u'(\theta) = (1-q)e^\theta \left(\frac{1-e^{-\theta}}{q}\right)^{u+1} + \frac{1-q}{q}(u+1)\left(\frac{1-e^{-\theta}}{q}\right)^u.$$

Therefore, we obtain

$$\psi(u) = \int_0^{\theta_0} (1-q)e^t \bar{F}_\Theta(t) \left(\frac{1-e^{-t}}{q}\right)^{u+1} dt + \int_0^{\theta_0} \frac{1-q}{q}(u+1)\bar{F}_\Theta(t)\left(\frac{1-e^{-t}}{q}\right)^u dt.$$

Let $J(u) = \int_0^{\theta_0} ((1-e^{-t})/q)^u d\theta$. Making the change of variable $qx = 1 - e^{-t}$, we have

$$J(u) = q\int_0^1 \frac{1}{1-xq} x^u dx \quad \text{and} \quad q \times \frac{1}{u+1} \le J(u) \le \frac{q}{1-q} \times \frac{1}{u+1}.$$

Furthermore, we have

$$\max_{\theta\in[0,\theta_0]} \bar{F}_\Theta(\theta)e^\theta \ge \frac{1}{1-q}, \min_{\theta\in[0,\theta_0]} \bar{F}_\Theta(\theta)e^\theta \le \bar{F}_\Theta(\theta_0).$$

Therefore, the ruin probability is bounded as

$$(1-q)\bar{F}_\Theta(\theta_0)J(u+1) + (u+1)\frac{1-q}{q}\bar{F}_\Theta(\theta_0)J(u) \le \psi(u) \le J(u+1) + (u+1)\frac{1-q}{q}J(u).$$

This yields

$$F_\Theta(\theta_0)\frac{q}{u+2} + F_\Theta(\theta_0) \le \psi(u) \le \frac{q}{1-q} \times \frac{1}{u+2} + 1.$$

(ii) Let $I(u,\theta_0) = \int_0^{\theta_0} \psi_u(\theta) dF_\Theta(\theta)$. We assume a continuous distribution for the mixing variable Θ and make the change of variable $x = (1-e^{-\theta})/q$, for which $qdx = e^{-\theta}d\theta$, we get

$$I(u,\theta_0) = q(1-q)\int_0^1 \frac{f_\Theta(-\log(1-xq))}{(1-xq)^2} x^{u+1} dx.$$

Let h be $f_\Theta \circ g$ with $g(x) = -\log(1-xq)$. The minimum condition to apply an integration by part theorem is to require the integrand function $(h(x)/(1-xq)^2)$ to be absolutely continuous. As $x \mapsto 1/(1-xq)^2$ are C^∞, we must show h is absolutely continuous. But $h = f_\Theta \circ g$ is not necessarily continuous if both f_Θ and h are absolutely continuous. According to Merentes (1991), if g is absolutely continous, then $f_\Theta \circ g$ is absolutely continuous if and only if f_Θ is locally Lipschitzian. Using the Rademacher theorem, we deduce that f_Θ is locally Lipschitizan, so h is absolutely continuous. We obtain

$$I(u,\theta_0) = q(1-q)\left[\frac{h(x)}{(1-xq)^2}\frac{x^{u+2}}{u+2}\right]_0^1 - q(1-q)\underbrace{\int_0^1 \left(\frac{h'(x)}{(1-xq)^2} + \frac{2qh(x)}{(1-xq)^3}\right)\frac{x^{u+2}}{u+2}dx}_{J(u)}.$$

The first term equals to

$$\frac{qf_\Theta(\theta_0)}{(1-q)(u+2)}$$

while the integral term is controlled as

$$|J(u)| \le \sup_{x\in[0,1]} \left|\frac{qf_\Theta'(g(x))}{(1-xq)^3} + \frac{2qf_\Theta(g(x))}{(1-xq)^3}\right| \int_0^1 \frac{x^{u+2}}{u+2}dx = C\frac{1}{(u+2)(u+3)} = o\left(\frac{1}{u+2}\right),$$

since f_Θ and f'_Θ are bounded on $[0,\theta_0]$. Combining the two preceding results, we get to

$$\psi(u) = \bar{F}_\Theta(\theta_0) + \frac{q f_\Theta(\theta_0)}{(1-q)(u+2)} + o\left(\frac{1}{u+2}\right).$$

(iii) As f_Θ is C^{k-1} almost everywhere on $[0,\theta_0]$ and $f_\Theta^{(k)}$ is Lebesgue integrable, then $h^{(i)}$ is absolute continous for all $i \leq k$. Applying k times the integration by part theorem, we get

$$I(u,\theta_0) = \sum_{i=0}^{k-1} \frac{\tilde{h}^{(i)}(0)}{(u+2)\dots(u+2+i)} + \int_0^{\theta_0} \tilde{h}^{(k)}(t) \frac{x^{u+2+k}}{(u+2)\dots(u+2+k)} dx,$$

where $\tilde{h}(x) = h(x)/(1-xq)^2$. Since successive derivatives $f_\Theta^{(i)}$ are bounded on $[0,\theta_0]$, the integral term is controlled by $o\big(1/u^k\big)$. The expression of the ith order derivative for a composition $f \circ g$ is complex, see Huang et al. (2006).

(iv) If f_Θ is C^∞ on $[0,\theta_0]$, we have

$$\psi(u) \underset{u\to+\infty}{\sim} \bar{F}_\Theta(\theta_0) + \sum_{i=0}^{+\infty} \frac{\tilde{h}^{(i)}(0)}{(u+2)\dots(u+2+i)}.$$

□

We examine below the three special cases studied in Subsection 2.2. Only one asymptotic is available via known asymptotics of the incomplete beta function asymptotic, see Olver et al. (2010). If Θ follows an exponential distribution, this gives

$$\psi(u) = (1-q)^\lambda + \lambda(1-q)^\lambda \frac{1}{u+2} + o\left(\frac{1}{u+2}\right).$$

Using Theorem 3.2, with $\tilde{h}(x) = \lambda(1-xq)^\lambda/(1-xq)^2$, leads to the same expansion.

For the two other distributions, gamma and Lévy, we have to use Theorem 3.2, as no asymptotic is available. When Θ is gamma distribution, the function $\tilde{h}$ is

$$\tilde{h}(x) = \frac{1}{(1-xq)^2}(1-xq)^\lambda \left(\log\left(\frac{1}{1-xq}\right)\right)^{\alpha-1} \frac{\lambda^\alpha}{\Gamma(\alpha)}.$$

Thus,

$$\psi(u) \underset{u\to+\infty}{\sim} \frac{\Gamma(\alpha,\lambda\theta_0)}{\Gamma(\alpha)} + \frac{\lambda^\alpha}{\Gamma(\alpha)}(1-q)^{\lambda-1}\theta_0^{\alpha-1}\frac{1}{u+2} + o\left(\frac{1}{u+2}\right).$$

with $\lambda > 1$ and $\theta_0 = -\log(1-q)$.

When Θ is Lévy distributed, the function $\tilde{h}$ is

$$\tilde{h}(x) = \frac{\alpha}{2\sqrt{\pi}} \log\left(\frac{1}{1-xq}\right)^{-3/2} e^{-\frac{\alpha^2}{4\log\left(\frac{1}{1-xq}\right)}}.$$

Thus,

$$\psi(u) \underset{u\to+\infty}{\sim} \operatorname{erfc}\left(\frac{\alpha}{2\sqrt{\theta_0}}\right) + \frac{\alpha}{2\sqrt{\pi}}\theta_0^{-3/2} e^{-\frac{\alpha^2}{4\theta_0}} \frac{1}{u+2} + o\left(\frac{1}{u+2}\right),$$

with $\theta_0 = -\log(1-q)$. Similar results for the ruin probability as in Figure 1 can be obtained in the discrete-time setting.

3.3. Tail claim distributions

In this subsection, we analyze the tail of the claim distribution, i.e. $P(X > x)$ for large values of x. For the present model by mixing, the survival function is the following Stieltjes integral

$$P(X > x) = \int_0^{+\infty} P(X > x|\Theta = \theta) dF_\Theta(\theta).$$

In the continuous time framework (where the conditional claim distribution is exponential), this leads to

$$P(X > x) = \int_0^{+\infty} e^{-\theta x} dF_\Theta(\theta),$$

which is the Laplace transform of the random variable Θ. Here too, one can see that a similar argument works only when Θ has a light-tailed distribution. In fact, we cannot obtain interesting results by applying the integration by part directly on this Stieltjes integral (as for the ruin probability). So, we assume that Θ has a continuous distribution and, similarly to the first subsection, we are going to derive the asymptotic survival function.

Proposition 3.3. *Let us consider the continuous time framework of Subsection 2.1 and assume Θ has a continuous distribution with density f_Θ.*

(i) If f_Θ is almost everywhere differentiable on $\mathbb{R}_+$ with f'_Θ being a Lebesgue-integrable, then for $x > 0$,

$$P(X > x) = \frac{f_\Theta(0)}{x} + o\left(\frac{1}{x}\right).$$

(ii) If f_Θ is C^∞ in the neighborhood of the origin, then for $x > 0$,

$$P(X > x) \underset{x \to +\infty}{\sim} \sum_{k=0}^{+\infty} \frac{f_\Theta^{(k)}(0)}{x^k}.$$

(iii) If f_Θ can be expanded in the neighborhood of the origin as

$$f_\Theta(t) \underset{t \to 0}{\sim} \sum_{k=0}^{+\infty} f_k t^{\frac{k+\eta}{\mu} - 1},$$

for $\eta, \mu > 0$, then for $x > 0$,

$$P(X > x) \underset{x \to +\infty}{\sim} \sum_{k=0}^{+\infty} \Gamma\left(\frac{k+\eta}{\mu}\right) \frac{f_k}{x^{\frac{k+\eta}{\mu}}}.$$

Proof. (i) f_Θ satisfies the minimum requirement for an application of the integration by parts. We get

$$P(X > x) = \left[f_\Theta(t) \frac{e^{-\theta x}}{-x} \right]_0^{+\infty} + \frac{1}{x} \int_0^{+\infty} e^{-\theta x} f'_\Theta(\theta) d\theta = \frac{f_\Theta(0)}{x} + o\left(\frac{1}{x}\right).$$

(ii) and (iii) are direct applications of Propertiy 2.3(i) and the Watson lemma 2.3(ii) of Olver et al. (2010), respectively. □

Remark 3. *Parts (i) and (ii) of this proposition may be not applicable when the density is not defined or zero at the origin. This justifies the part (iii).*

Remark 4. *The reason why the behavior of the integrand function f_Θ at the origin matters is explained by Laplace's method. In Laplace's method, one studies the asymptotics of the following integral*

$$I(x) = \int_a^b e^{xp(t)} q(t) dt,$$

where p and q are continuous functions around the point a, assumed to be the minimum of p in $[a, b[$. In our case, $p(t) = t$, hence the minimum of the exponent on $\mathbb{R}_+$ is attained at the origin. See e.g. 2.3(iii) of Olver et al. (2010).

Let us see if the two special cases studied in the previous section fall within the framework of the previous proposition. Firstly, assume that Θ follows a gamma distribution $\mathcal{G}a(\alpha, \lambda)$. Using the integral representation of the exponential function, the density function can be expanded as

$$f_\Theta(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{+\infty} \frac{(-\lambda)^k}{k!} t^{\alpha+k-1}.$$

Thus, we get

$$P(X > x) \underset{x \to +\infty}{\sim} \sum_{k=0}^{+\infty} (-1)^k \frac{\Gamma(k+\alpha)}{\Gamma(\alpha) k!} \left(\frac{\lambda}{x}\right)^{k+\alpha}.$$

with $\eta = \alpha$ and $\mu = 1$. This (asymptotic) polynomial decrease of the survival function is consistent with the fact that X is Pareto type II distributed

$$P(X > x) = \frac{1}{\left(1 + \frac{x}{\lambda}\right)^\alpha}.$$

When Θ follows a Lévy distribution $\mathcal{L}e(\alpha)$,

$$f_\Theta(\theta) = \frac{\alpha}{2\sqrt{\pi \theta^3}} e^{-\alpha^2/4\theta}.$$

Although this function is not defined at zero, the density converges to zero, since we have

$$f_\Theta(1/t) = \frac{\alpha t^{3/2}}{2\sqrt{\pi}} e^{-\alpha^2 t/4} \underset{t \to +\infty}{\longrightarrow} 0.$$

However, we cannot find a valid series expansion of $f_\Theta(1/t)$ at $+\infty$, or equivalently of $f_\Theta(\theta)$ at 0, based on the series expansion of the exponential function. Therefore, the preceding proposition is of limited use in the Lévy case, where we already know that

$$P(X > x) = e^{-\alpha\sqrt{x}}, x \geq 0.$$

More generally, Proposition 3.3 is difficult to apply when the density function of Θ is not defined.

Now, we look at the tail of the claim distribution in the discrete time framework. We have

$$P(X > u) = \int_0^{+\infty} (1-q) \left(1 - e^{-\theta}\right)^u dF_\Theta(\theta).$$

One way to deal with such an integral is to use an integration by part directly on the integral. But, it does not lead to satisfying results as for the ruin probability. Even if we assume Θ has a continuous distribution, we do not get a Laplace transform of a certain function as in the continuous time:

$$P(X > u) = \int_0^{+\infty} (1-q) f_\Theta(\theta) \left(1 - e^{-\theta}\right)^u d\theta.$$

To get a term easily integrable, one can try to make a change of variable, e.g. $x = 1-e^{-\theta}$ or $x = -\log(1-e^{-\theta})$. The latter is not possible on $\mathbb{R}_+$ because the derivative of the function $\theta \mapsto -\log(1-e^{-\theta})$ is unbounded near 0. Let us try $x = 1-e^{-\theta}$. We get

$$P(X > u) = \int_0^1 (1-q)\frac{f_\Theta(-\log(1-x))}{1-x}x^u dx.$$

Using an integration by part theorem requires that the following limit to exist

$$\lim_{x\to 1}\frac{f_\Theta(-\log(1-x))}{1-x} = \lim_{t\to+\infty} f_\Theta(t)e^t.$$

This requirement is strong and will be satisfied only for light-tailed distributions and a certain range of parameter values. For example, when Θ is exponentially distributed $\mathcal{E}(\lambda)$, the previous constraint imposes $\lambda > 1$. Another way to deal with such integral asymptotic is to apply the Pascal formula, assuming u is an integer. We get

$$P(X > u) = \sum_{k=0}^{u}\binom{u}{k}(1-q)(-1)^k\int_0^{+\infty} e^{-k\theta}dF_\Theta(\theta).$$

The integral is (once again) the Laplace transform L_Θ of the random variable Θ at k. This binomial alternating sum requires a special treatment because finding an asymptotic for $L_\Theta(k)$ will not help us to derive an asymptotic of the sum. This issue is studied in the next subsection.

3.4. Binomial alternating sum for claim tails

In the discrete time framework, the survival function can be expressed as

$$P(X > u) = \sum_{k=0}^{u}\binom{u}{k}(1-q)(-1)^k L_\Theta(k),$$

where L_Θ denotes the Laplace transform of the random variable Θ. This integral falls within the framework of the alternating binomial sum defined as

$$S_n(\phi) = \sum_{k=n_0}^{n}\binom{n}{k}(-1)^k\phi(k), \tag{11}$$

where $0 \le n_0 \le n$, ϕ is a real function and $n \in \mathbb{N}$ is large. n_0 can be used to exclude first few points of the sum that would not be defined. Letting

$$\phi(k) = (1-q)\frac{\lambda^\alpha}{(\lambda+j)^\alpha},$$

in Equation (11), we get the distribution function of X when Θ is gamma distributed of Subsection 2.2. Note that, having started with the integral representation $\int_0^{+\infty} P(X = k|\Theta = \theta)dF_\Theta(\theta)$, we know that the alternating sum is valued on $[0,1]$. This is not immediate without that integral representation. Let us point out that the probability $P(X = k)$ is a decreasing function of k. This is not easy to see by using the alternating binomial sum representation (11). Here is a simle proof. To indicate the dependence on the parameter λ, denote by $P(X = k)_\lambda$. From algebraic manipulation and using the binomial recurrence equation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we get

$$P(X = k+1)_\lambda = P(X = k)_\lambda - P(X = k)_{\lambda+1}\frac{\lambda^\alpha}{(\lambda+1)^\alpha}.$$

So, as announced, the probability mass function of X is decreasing. The binomial alternating sum representation of $P(X > k)$ is

$$P(X > k) = (1-q)\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\lambda^\alpha}{(\lambda+j)^\alpha}.$$

There should be an exponential canceling in the sum, since $\binom{k}{j}$ tends quickly to infinity for large values of k (we recall $\binom{n}{k} \sim e^{-k}n^k/k!$) and $P(X > k)$ is a decreasing function. A study of the alternating sum seems to be rather complex. Going back to the alternating sum with $n_0 = 0$, the first few terms can be expressed as

$$S_0(\phi) = \phi(0), S_1(\phi) = \phi(0) - \phi(1), S_2(\phi) = \phi(0) - 2\phi(1) + \phi(2).$$

Let Δ be the forward difference operator. Then we have $S_0 = \Delta^0\phi(0)$, $S_1 = -\Delta\phi(0)$ and $S_2 = \Delta^2\phi(0)$. More generally, the binomial alternating sum can be rewritten as

$$S_n(\phi) = \sum_{k=n_0}^{n} \binom{n}{k}(-1)^k\phi(k) = (-1)^n\Delta^n(\phi)(0).$$

3.4.1. Some complex analysis

To deal with such sums, a standard method consists in using complex analysis and contour integrals. Flajolet and Sedgewick (1995) provide a complete overview of this topic. In this subsection, we consider that the complex extension of ϕ of the sum $S_n(\phi)$. Their Lemma 1 gives the so-called Rice integral representation of $S_n(\phi)$

$$\sum_{k=n_0}^{n} \binom{n}{k}(-1)^k\phi(k) = (-1)^n \oint_\gamma \phi(z)\frac{n!}{z(z-1)\dots(z-n)}dz, \tag{12}$$

where ϕ is assumed to be analytic on a domain Ω containing $[n_0, n[$ and γ is a closed curve in Ω encircling $[n_0, n[$ but not $[0, n_0 - 1]$. Let f be the integrand of the right-hand side of (12). By the residue theorem (e.g. Chapter 10 of Bak and Newman (2010)), if the integrand is analytic except at a countable number of isolated singularities inside the domain γ, then the contour integral equals to the sum of the residues of the integrand taken at the singularities inside γ. The right-hand side of Equation (12) is still cumbersome to compute, since the integrand has at least $n + 1$ singularities at $0, 1, \dots, n$. So, in the present situation, the complex contour integration does not really simplify the problem. As we want to derive some asymptotics when n tends to infinity, the domain γ of the contour integration has to be extended to the entire positive real half-plane $\{z, Re(z) > 0\}$. However, we do not want to compute the sum of the residuals at integers $\{n_0, n_0+1, \dots, +\infty\}$. Furthermore, we do not know if $f(z)$ does not explode as $Re(z) \to +\infty$. Nevertheless, the solution does come by extending the contour of integration. Let $\gamma = C_R$ the circle of radius R centered at 0 excluding poles in $\mathbb{N}$. Assuming f is of polynomial growth towards infinity, the integral

$$\oint_{C_R} f(z)dz \underset{R\to+\infty}{\longrightarrow} 0.$$

By the residue theorem, the contour integral $\oint_{C_\infty} f(z)dz$ also equals to the sum of residuals of f at integers $\{0, \dots, n_0 - 1\}$ and $\{n_0, n_0 + 1, \dots, +\infty\}$. The first residual contribution is a finite sum, while the second contribution is the binomial alternating sum $S_n(\phi)$. Thus, in the particular case of polynomial growth, the binomial sum $S_n(\phi)$ reduces to the computation of a limited number of residuals at $\{0, \dots, n_0 - 1\}$, see the proof of Theorem 1 of Flajolet and Sedgewick (1995).

Theorem (Flajolet and Sedgewick)**.** *Let ϕ be a rational analytic function on $[n_0, \infty[$. Then we have*

$$\sum_{k=n_0}^{n} \binom{n}{k}(-1)^k\phi(k) = -(-1)^n \sum_s Res\left(\phi(s)\frac{n!}{s(s-1)\dots(s-n)}\right),$$

where the summation of residues is done over poles not on $[n_0, \infty[$.

Theorem 2(i) of Flajolet and Sedgewick (1995) applies the same approach when f is meromorphic (i.e. complex differentiable everywhere except at a countable number of points) and not necessarily of polynomial growth. The same argument applies when we replace the circle C_R by a semicircle $S_{(R,d)} = \{z \in \mathbb{C}, Re(z) > d, |z| < R\}$, see part (ii) of Theorem 2 of Flajolet and Sedgewick (1995). But this time, we only get

an asymptotic for the original problem. Furthermore, things are intrinsically more complicated when the function ϕ is not a rational function, because we consider the complex extension of ϕ. For instance, function $z \mapsto 1/z^2$ has a second-order pole at $z = 0$ but function $z \mapsto 1/z^{1.414}$ has algebraic singularity at $z = 0$. To deal with algebraic singularities, the integration contour γ must exclude the singularities. Flajolet and Sedgewick (1995) provide a keyhole structure approach (i.e. Hankel contour) when $\phi(z)$ has a (non polar) algebraic singularity $1/x^\lambda$, see proof of their Theorem 3. The keyhole structure captures the effect of the singularity at 0 by decreasing the radius of the hole in $1/\log(n)$ as n tends to infinity. Dealing with non-isolated singularities (i.e. branch points) need even more care than just skipping it as with a semicircle. Let us consider, for example, the complex square root $\sqrt{z}$, the branch point is the negative real axis $]-\infty, 0]$. The branch point is of finite order, compared to the complex logarithm for instance. Indeed we have

$$\sqrt{\rho e^{i(\theta+2k\pi)}} = \sqrt{\rho} e^{i(\theta/2+k\pi)} = \begin{cases} -\sqrt{\rho} e^{i(\theta/2)} & \text{if } k = 1 \\ \sqrt{\rho} e^{i(\theta/2)} & \text{if } k = 2 \end{cases}$$

Flajolet and Sedgewick (1995) consider a local approximation of $z^{1/2}$ around the origin only at the contour part of the keyhole structure surrounding the origin. Otherwise, we keep the polynomial growth of the square root for the rest of the contour, see Example 7. Handling branch points of infinity order, say with the complex logarithm function $\log(z)$, is similar except that the resulting asymptotic as n tends to infinity is different, see Example 8. We report below a table of correspondences between singularity and asymptotics.

Singular part $s_0 \notin \mathbb{N}$	Asymptotics
simple pole $(z - s_0)^{-1}$	$-\Gamma(-s_0) n^{s_0}$
multiple pole $(z - s_0)^{-m}$	$-\Gamma(-s_0) n^{s_0} \frac{(\log n)^{m-1}}{(m-1)!}$
algebraic singularity $(z - s_0)^\lambda$	$-\Gamma(-s_0) n^{s_0} \frac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}$
+ logarithmic singularity $(z - s_0)^\lambda (\log(z - s_0))^r$	$-\Gamma(-s_0) n^{s_0} \frac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)} (\log\log n)^r$

Table 1: Correspondences between singularity and asymptotic

3.4.2. Two simple illustrations

Let us consider for phi two particular functions of interest below. Firstly, we choose $f_1(z)$ be $1/(z+\beta)^\alpha$ where $z \in \mathbb{C}$. It has a singularity at $-\beta$, which is a multiple pole if $\alpha \in \mathbb{N}$. Using a keyhole structure centered at $-\beta$ and Table 1, we have an asymptotic of the form

$$S_n(f_1) \underset{n\to+\infty}{\sim} \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{(\log n)^{\alpha-1}}{n^\beta}.$$

Secondly, we choose $f_2(z)$ be $e^{-\alpha\sqrt{z}}$. The function f_2 has a branch point at $z = 0$, because of the complex square root. First, we use an infinitesimal asymptotic of the exponential around 0. That is $f_2(z) = 1 - \alpha\sqrt{z} + o(z)$. Since the contour integral of a sum is the sum of contour integrals and that the contour integral of an analytic function is zero, we can drop the constant 1. We use a right-oriented half-plane keyhole structure centered at 0 for $Re(z) > d$ (with $-\infty < d < 0$), similar to Theorem 3 of Flajolet and Sedgewick (1995), since the function f_2 has a exponential growth on the half-plane $Re(z) < d$ and cannot be integrated as the radius tends to infinity. We cannot use the singularity correspondence table for the square root, because the singularity is zero. But, the square root can be approximated by Theorem 3 of Flajolet and Sedgewick (1995). And the term $o(z)$ is controlled by the small circle of the keyhole structure on which $|z| < 1/\log(n)$. Thus we get the following asymptotic of the alternating sum

$$S_n(f_2) \underset{n\to+\infty}{\sim} \frac{\alpha}{\sqrt{\pi}\log(n)} - \frac{\alpha\gamma_e}{2\sqrt{\pi\log^3(n)}},$$

where $\gamma_e = 0.5772156649$ is the Euler-Mascheroni constant.

3.4.3. Claim tail asymptotics

Based on the previous subsections, we are able to derive tail asymptotics of a claim X given a mixing distribution Θ in the discrete time framework presented in Subsection 2.2. When Θ follows an exponential distribution $\mathcal{E}(\lambda)$, we use the asymptotic of the beta function $\beta(a,b)$ for large values of b, see Olver et al. (2010). We get that the tail of the distribution is asymptotically

$$P(X > k) \underset{k\to+\infty}{\sim} (1-q)\frac{\Gamma(\lambda+1)}{(k+1)^{\lambda}},$$

which decreases like a discrete Pareto distribution (i.e. a Zipf distribution). This tail behavior of a Yule-Simon distribution was already reported in Simon (1955). When Θ follows a gamma distribution $\mathcal{G}a(\alpha,\lambda)$, we use asymptotics of the alternating binomial sum with the function $f_1(z) = 1/(z+\lambda+1)^{\alpha}$. Therefore, the tail distribution is asymptotically

$$P(X > k) \underset{k\to+\infty}{\sim} (1-q)\frac{\lambda^{\alpha}\Gamma(\lambda+1)}{\Gamma(\alpha)}\frac{(\log k)^{\alpha-1}}{k^{\lambda+1}},$$

which decreases slightly slower than a Zipf distribution due to the logarithm in the numerator. When Θ follows a Lévy distribution $\mathcal{L}e(\alpha)$, again we use an asymptotic of alternating binomial sums with the function $f_2(z) = e^{-\alpha\sqrt{z}}$. Thus, the tail distribution is asymptotically

$$P(X > k) \underset{k\to+\infty}{\sim} (1-q)\alpha\left(\frac{1}{\sqrt{\pi}\log(k)} - \frac{\gamma_e}{2\sqrt{\pi\log^3(k)}}\right),$$

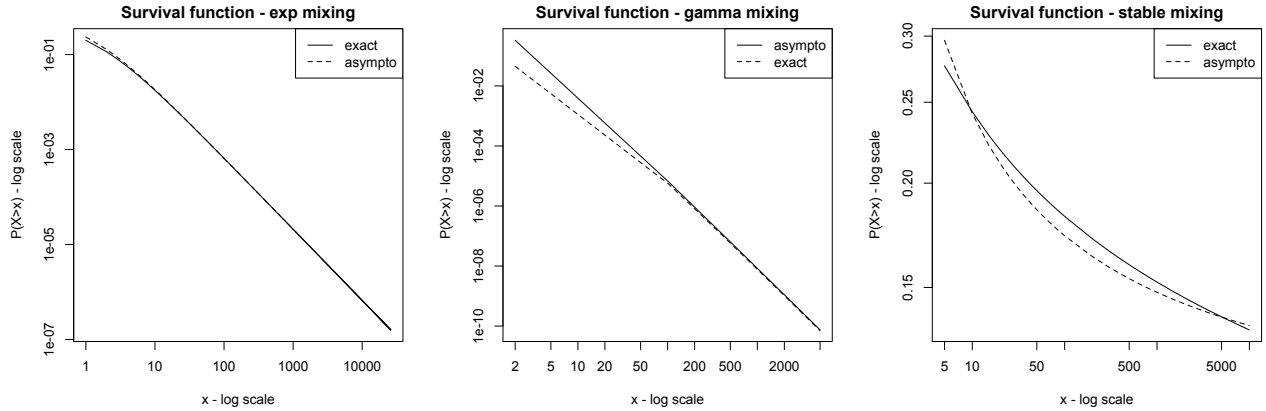
which decreases extremely slowly. Such a tail behaviour is heavier than for a Pareto distribution. With continuous distributions, a similar behaviour is obtained for the log-Cauchy distribution, for example.

3.4.4. Numerical illustrations

In Figure 2, we plot the survival functions derived above. The exponential-geometric distribution has a very tractable survival function, since incomplete Beta function is available in most softwares, e.g., in R via the `pbeta` function. Therefore, we can benchmark the asymptotic with the true value. However, for the two other distributions, we have to compute two binomial alternating sums. These sums are particularly unstable because the central term $C_n^{n/2}$ tends to infinity as n tends to infinity, which drives the binomial alternating sum between $+\infty$ or $-\infty$.

In modern computers, a real number is stored in eight bytes (i.e. 64 bits), but only 53 bits are reserved for the precision (see e.g. http://en.wikipedia.org/wiki/Double-precision_floating-point_format). In our numerical experiment, the alternating binomial sum $S_n(\phi)$ becomes unstable for $n \geq 48$ with the standard double precision. To compute the alternating sum for large n, we have no other way than to use high precision floating-point arithmetic libraries such as the GMP library of Grandlund Torbjoern & the GMP Devel. Team (2011) as well as the MPFR library of Fousse et al. (2011). Using MPFR libraries allows us to work with a high number of bits, say 500 or 1000. Using the Rmpfr package of Maechler (2012), we are then able to assess the survival function of the claim distribution.

We consider the three examples presented before with $q = 1/2$. Figures 2a, 2b and 2c correspond to a mixing distribution when Θ is Exponential, Gamma and Lévy-stable distributed, respectively. On Figures 2a, 2b, the distribution tail shows a Pareto-type behavior, as we observe a straight line. The Lévy stable mixing on Figure 2c clearly exhibits a heavier tail. The heavier the tail of the distribution of Θ, the larger the error. For Figure 2c, the asymptotic is not as reliable as for other cases. This small numerical experiment also illustrates the importance of high precision arithmetic libraries for some numerical issues in actuarial problems.



(a) Exponential $\mathcal{E}(3/2)$ (b) Gamma $\mathcal{G}a(3/2, 2)$ (c) Lévy $\mathcal{L}e(3/2)$

Figure 2: Survival functions

4. Focus on the dependence structure

This section studies the dependence structure of the dependent risk models described in Subsections 2.1 and 2.2, respectively for discrete-time and continuous-time settings. Let us start by recalling Property 2.1 of Albrecher et al. (2011) for the continuous-time model.

Proposition. *When claim sizes fulfill for each $n \geq 1$,*

$$P(X_1 > x_1, \ldots, X_n > x_n | \Theta = \theta) = \prod_{i=1}^{n} e^{-\theta x_i},$$

then, they have a dependence structure due to an Archimedean survival copula with generator $\phi = L_\Theta^{-1}$, the inverse Laplace transform of Θ.

Therefore, in continuous time, the dependence structure is simply an Archimedean copula. Regarding the discrete-time setting, things are more complicated: the dependence among discrete random variables is a complex topic. Genest and Nešlehová (2007) present issues linked to discrete copula. By the Sklar theorem, see e.g. Joe (1997), Nelsen (2006), we recall that the copula C of a random pair (X, Y) is not unique outside the support of the random variables X and Y. So, when X, Y are discrete variables in $\mathbb{N}$, C is only unique on $\mathbb{N}^2$ but not on $\mathbb{R}^2 \setminus \mathbb{N}^2$. The non-identifiability is a major source of issues. An example of discrete copulas is the empirical copula for observation sample $(X_i, Y_i)_{1 \leq i \leq n}$.

Let us introduce Genest and Nešlehová (2007)'s notation. Let $\mathcal{A}$ be the class of functions verifying

$$H(x, y) = C(F_X(x), F_Y(y)), \tag{13}$$

for all $x, y \in \mathbb{R}$ for given distribution functions F_X, F_Y and H. Let us define the function B as for all $u, v \in [0, 1], B(u, v) = (F_X^{-1}(u), F_Y^{-1}(v))$. We also denote by D the distribution function of the pair $(F_X(X), F_Y(Y))$. With a simple bivariate Bernoulli vector, Example 1 of Genest and Nešlehová (2007) shows that (i) functions B and D are different, (ii) B is not a distribution function despite both B and D belong to the class $\mathcal{A}$. Even in that simple support $\{0, 1\}^2$, the identifiability issue of the copula C cannot be discarded. Proposition 1 of Genest and Nešlehová (2007) extends to any bivariate pair (X, Y): B is not a distribution, whereas D is a distribution function but not a copula.

The function D is not a copula but only a distribution function. This leads to the question of the maximal differences between two copulas satisfying Equation (13) when X, Y take discrete values. The answer is given

by the Carley bounds C_H^-, C_H^+. For all copulas C_H verifying Equation (13), we have

$$C_H^- \leq C_H \leq C_H^+,$$

cf. Proposition 2 of Genest and Nešlehová (2007). The non-identifiability issue matters, since with discrete copulas the dependence measure, such as the tau of Kendall or the rho of Spearman, are no longer unique. Furthermore, if X and Y are independent, it does not imply that the copula of (X, Y) is the independent copula. In other words, the copula alone does not characterize the dependence: we need assumptions on margins. In the copula literature, efforts have been done to tackle the non-identifiability issue of dependence measure.

The current answer is the interpolated copula $C_{X,Y}^{\maltese}$, which is a bilinear interpolation of the distribution function D of the pair $(F(X), F(Y))$ on the discrete set $\mathrm{Im}(F_X) \times \mathrm{Im}(F_Y)$. This copula was already mentioned in Lemma 2.3.5 of Nelsen (2006) to prove the Sklar theorem. The interpolated copula can also be interpreted as the copula of $(X + U, Y + V)$ where U, V are two independent uniform random variables, see e.g. Section 4 of Denuit and Lambert (2005). This formulation is useful when doing random generation. The properties of the interpolated copula of the pair (X, Y) are: (i) Kendall's tau $\tau(X, Y) = \tau(C_{X,Y}^{\maltese})$ and Spearman's rho $\rho(X, Y) = \rho(C_{X,Y}^{\maltese})$, (ii) $C_{X,Y}^{\maltese}$ is absolutely continuous and (iii) $X \perp Y \Leftrightarrow C_{X,Y}^{\maltese} = \Pi$, the independent copula. Unfortunately, $C_{X,Y}^{\maltese}$ depends on marginals and $F_X = F_Y$ does not imply $C_{X,Y}^{\maltese}(u, v) = \min(u, v)$, as well as $F_X = 1 - F_Y \not\Rightarrow C_{X,Y}^{\maltese}(u, v) = (u + v - 1)_+$, see e.g. Genest and Nešlehová (2007).

Proposition 4.1. *Let W_1, W_2 be conditionally independent 0-modified geometric random variables, i.e. $W_i|\Theta = \theta \sim \mathcal{G}(q, e^{-\theta})$. The distribution function of (W_1, W_2) is given by*

$$D_{W_1,W_2}(u, v) = \begin{cases} 0 & \text{if } u < q \text{ or } v < q, \\ 1 - (1 - q)(L_\Theta(\lfloor l_u \rfloor) + L_\Theta(\lfloor l_v \rfloor)) + (1 - q)^2 L_\Theta(\lfloor l_u \rfloor + \lfloor l_v \rfloor) & \text{otherwise,} \end{cases}$$

where for $p \geq q$, $l_p = L_\Theta^{-1}((1 - p)/(1 - q))$ and $\lfloor x \rfloor$ denotes the floor function.

The interpolated copula $C_{W_1,W_2}^{\maltese}$ of (W_1, W_2) is expressed as

$$C_{W_1,W_2}^{\maltese}(u, v) = \bar{\lambda}_u \bar{\lambda}_v D_{W_1,W_2}(u_i, v_j) + \lambda_u \bar{\lambda}_v D_{W_1,W_2}(u_{i+1}, v_j) + \bar{\lambda}_u \lambda_v D_{W_1,W_2}(u_i, v_{j+1}) + \lambda_u \lambda_v D_{W_1,W_2}(u_{i+1}, v_{j+1}),$$

for $(u_i, v_j) \in Im(F_X) \times Im(F_Y)$ such that $(u, v) \in [u_i, u_{i+1}] \times [v_j, v_{j+1}]$ and where $\lambda_u = (u - u_i)/(u_{i+1} - u_i)$, $\lambda_v = (v - v_j)/(v_{j+1} - v_j)$. The density of the interpolated copula $C_{W_1,W_2}^{\maltese}$ is given by

$$c_{W_1,W_2}^{\maltese}(u, v) = \frac{D_{W_1,W_2}(u_i, v_j) - D_{W_1,W_2}(u_{i+1}, v_j) - D_{W_1,W_2}(u_i, v_{j+1}) + D_{W_1,W_2}(u_{i+1}, v_{j+1})}{(u_{i+1} - u_i)(v_{j+1} - v_j)},$$

for $(u, v) \in [u_i, u_{i+1}] \times [v_j, v_{j+1}]$.

Remark 5. *Note there is a jump when u or v equal q, since $D_{W_1,W_2}(q, v) = q - q(1 - q)L_\Theta(\lfloor l_v \rfloor)$ and $D_{W_1,W_2}(u, v) \geq q^2$ for $u, v \geq q$. W_i has the same distribution as $I_i \lceil Y_i \rceil$ where I_i (resp. Y_i) follows a Bernoulli distribution $\mathcal{B}(1 - q)$ (resp. an exponential distribution $\mathcal{E}(\theta)$) and the copula of (Y_1, Y_2) is an Archimedean copula with generator L_Θ^{-1}. Using floor functions on marginals of a continuous copula is also considered in many other articles, e.g. Joe (1997) and Frees and Wang (2006).*

Proof. If W follows a 0-modified geometric distribution $\mathcal{G}(q, 1 - e^{-\theta})$, then

$$P(W = k) = q\delta_{k,0} + (1 - q)(1 - \delta_{k,0})e^{-\theta(k-1)}(1 - e^{-\theta}).$$

Furthermore, $P(W \leq x) = q + (1 - q)(1 - e^{-\theta(\lfloor x \rfloor)})$ for all $x \in \mathbb{R}_+$. In particular for $x = k \in \mathbb{N}$, we have $F_W(k) = q + (1 - q)(1 - e^{-\theta k})$. Therefore, we have

$$P(W_i \leq x) = 1 - (1 - q)L_\Theta(\lfloor x \rfloor),$$

and

$$P(W_1 \leq x, W_2 \leq y) = 1 - (1 - q)(L_\Theta(\lfloor x \rfloor) + L_\Theta(\lfloor y \rfloor)) + (1 - q)^2 L_\Theta(\lfloor x \rfloor + \lfloor y \rfloor).$$

Let D_{W_1,W_2} be the distribution function of the pair $(F_{W_1}(W_1), F_{W_2}(W_2))$. Since $P(W_i = 0) > 0$, we get

$$D_{W_1,W_2}(u,v) = \begin{cases} 0 & \text{if } \ u < q \ \text{ or } \ v < q \\ P(W_1 \leq l_u, W_2 \leq l_v) & \text{otherwise} \end{cases} .$$

And by conditioning we get for $u, v \geq q$,

$$D_{W_1,W_2}(u,v) = 1 - (1-q)(L_\Theta(\lfloor l_u \rfloor) + L_\Theta(\lfloor l_v \rfloor)) + (1-q)^2 L_\Theta(\lfloor l_u \rfloor + \lfloor l_v \rfloor).$$

The derivation of $C^{\maltese}_{W_1,W_2}$ is the direct definition of a bilinear interpolation of D_{W_1,W_2}, whereas $c^{\maltese}_{W_1,W_2}(u,v)$ is the direct differentiation of $C^{\maltese}_{W_1,W_2}(u,v)$. □

Acknowledgements

This work is partially funded by the Research Chair *Actuariat Durable* sponsored by Milliman, the research chair *Management de la modélisation* sponsored by BNP Paribas Cardif and by the AXA research fund. C. Lefèvre is grateful to the FNRS Belge for a grant *Crédit aux Chercheurs*. C. Dutang gratefully acknowledges financial support from the Swiss National Science Foundation Project 200021-124635/1. The authors also thank Dominik Kortschak, the Editor and the Referee for useful comments and suggestions.

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Appendix A. For the continuous time model

The function $\Gamma(.,.;.)$ satisfies a recurrence on the a parameter,

$$\Gamma(a+1,x;b) = a\Gamma(a,x;b) + b\Gamma(a-1,x;b) + x^a e^{-x-b/x},$$

see Theorem 2.2 of Chaudry and Zubair (2002). Thus we deduce

$$\Gamma(-3/2,x;b) = \frac{1}{b}\left(\Gamma(1/2,x;b) + 1/2\Gamma(-1/2,x;b) - x^{-1/2}e^{-x-b/x}\right).$$

As reported in Theorem 2.6 and Corollary 2.7 of Chaudry and Zubair (2002), $\Gamma(a,x;b)$ has a simpler expresssion in terms of the error function when $a = 1/2, -1/2, \ldots,$

$$\Gamma(1/2,x;b) = \frac{\sqrt{\pi}}{2}\left[e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right) + e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)\right]$$

and

$$\Gamma(-1/2,x;b)=\frac{\sqrt{\pi}}{2\sqrt{b}}\left[-e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right)+e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)\right].$$

Therefore, we have

$$\Gamma(-3/2,x;b)=\frac{\sqrt{\pi}}{2b}\left[\left(1-\frac{1}{2\sqrt{b}}\right)e^{2\sqrt{b}}\mathrm{erfc}\left(d_+\right)+\left(1+\frac{1}{2\sqrt{b}}\right)e^{-2\sqrt{b}}\mathrm{erfc}\left(d_-\right)-\frac{2}{\sqrt{\pi x}}e^{-x-b/x}\right],$$

with $d_+=\sqrt{x}+\sqrt{b}/x$ and $d_-=\sqrt{x}-\sqrt{b}/x$. It yields

$$\Gamma(-3/2,\theta_0 u;\alpha^2u/4)=\frac{2\sqrt{\pi}}{\alpha^2u}\left[\left(1-\frac{1}{\alpha\sqrt{u}}\right)e^{\alpha\sqrt{u}}\mathrm{erfc}\left(d_+\right)+\left(1+\frac{1}{\alpha\sqrt{u}}\right)e^{-\alpha\sqrt{u}}\mathrm{erfc}\left(d_-\right)\right.$$
$$\left.-\frac{2}{\sqrt{\pi u\theta_0}}e^{-u\theta_0-\alpha^2/(4\theta_0)}\right],$$

with $d_+=\sqrt{u\theta_0}+\alpha/(2\sqrt{\theta_0})$ and $d_-=\sqrt{u\theta_0}-\alpha/(2\sqrt{\theta_0})$. We deduce that

$$I(u,\theta_0)=\frac{\theta_0\sqrt{u}}{\alpha}e^{u\theta_0}\left[\left(1-\frac{1}{\alpha\sqrt{u}}\right)e^{\alpha\sqrt{u}}\mathrm{erfc}\left(d_+\right)\right.$$
$$\left.+\left(1+\frac{1}{\alpha\sqrt{u}}\right)e^{-\alpha\sqrt{u}}\mathrm{erfc}\left(d_-\right)-\frac{2}{\sqrt{\pi u\theta_0}}e^{-u\theta_0-\alpha^2/(4\theta_0)}\right].$$

By reordering the terms, we get the formula of Albrecher et al. (2011), which is

$$I(u,\theta_0)=e^{-\alpha/(4\theta_0)}\left[e^{(c\alpha+2\lambda\sqrt{u})^2/4\lambda c}\left(-1+\alpha\sqrt{u}\right)\mathrm{erfc}\left(d_+\right)\right.$$
$$\left.+e^{(c\alpha-2\lambda\sqrt{u})^2/4\lambda c}\left(1+\alpha\sqrt{u}\right)\mathrm{erfc}\left(d_-\right)-\frac{2\alpha}{\sqrt{\pi\theta_0}}\right].$$

Appendix B. For the discrete time model

The geometric distribution $\mathcal{G}(p)$ has the probability mass function $P(X=k)=p(1-p)^k$, $k\in\mathbb{N}$, where $0\le p\le 1$. Note that $P(X>k)=(1-p)^{k+1}$. The geometric distribution parametrization depends one parameter, which can be restrictive. The 0-modified geometric distribution, $X\sim\mathcal{G}(q,\rho)$, has the probability mass function

$$P(X=k)=\begin{cases} q & \text{for } k=0,\\ (1-q)\rho(1-\rho)^{k-1} & \text{for } k\in\mathbb{N}^+,\end{cases}$$

with $P(X>k)=(1-q)(1-\rho)^k$.

Appendix C. Error function linked terms

We want to compute the following integral, linked to the error function

$$J(a,b,x)=\int_x^\infty e^{-ay^2-b/y^2}dy,$$

where $a,b,x>0$. The SAGE mathematical software (Stein et al. (2011)) suggests to do a change of variable in order to get $\int e^{-t^2}dt$. Since the equation $t^2=ay^2+b/y^2$ does not have a unique solution, we consider $t=\pm\sqrt{a}y+\sqrt{b}/y$. This leads to split the integral $J(a,b,x)$. With algebric manipulations, we get

$$2\sqrt{a}dy=\sqrt{a}dy+\frac{\sqrt{b}}{-y^2}dy+\sqrt{a}dy-\frac{\sqrt{b}}{-y^2}dy.$$

Therefore,

$$2\sqrt{a}J(a,b,x) = e^{2\sqrt{ab}}\int_{\tilde{x}_1}^{\infty} e^{-t^2}\,dt + e^{-2\sqrt{ab}}\int_{\tilde{x}_2}^{\infty} e^{-t^2}\,dt,$$

with $\tilde{x}_1 = \sqrt{a}x + \frac{\sqrt{b}}{x}$ and $\tilde{x}_2 = \sqrt{a}x - \frac{\sqrt{b}}{x}$. Hence

$$J(a,b,x) = \frac{\sqrt{\pi}}{4\sqrt{a}}\left[e^{2\sqrt{ab}}\mathrm{erfc}\left(\sqrt{a}x + \frac{\sqrt{b}}{x}\right) + e^{-2\sqrt{ab}}\mathrm{erfc}\left(\sqrt{a}x - \frac{\sqrt{b}}{x}\right)\right].$$

This result is in line with Theorem 3.1 of Chaudry and Zubair (2002) and is closely related to the generalized error function

$$\mathrm{erfc}(x;b) = \frac{\sqrt{\pi}}{4}e^{2\sqrt{b}}\left[e^{2\sqrt{b}}\mathrm{erfc}\left(x + \frac{\sqrt{b}}{x}\right) + e^{-2\sqrt{b}}\mathrm{erfc}\left(x - \frac{\sqrt{b}}{x}\right)\right].$$

If $x = 0$, we get

$$J(a,b,x) = \frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{ab}} \triangleq J(a,b).$$

If $b = -1$, then we replace $\sqrt{b}$ by the imaginary number i, hence

$$J(a,-1,x) = \frac{\sqrt{\pi}}{4\sqrt{a}}\left[e^{2i\sqrt{a}}\mathrm{erfc}\left(\sqrt{a}x + \frac{i}{x}\right) + e^{-2i\sqrt{a}}\mathrm{erfc}\left(\sqrt{a}x - \frac{i}{x}\right)\right].$$

This number is of type $z_1z_2 + \bar{z}_1\bar{z}_2$, where $z_1 = e^{2i\sqrt{a}}$ and $z_2 = \mathrm{erfc}\left(\sqrt{a}x + \frac{i}{x}\right)$. We check that $z_1z_2 + \bar{z}_1\bar{z}_2 = 2|z_1z_2|\cos(\arg(z_1)+\arg(z_2)) \in \mathbb{R}$. Thus, we extend the notation $J(a,b,x)$ for $b = -1$ by the above expression.