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To cite this version:

Christophe Dutang, Hansjoerg Albrecher, Stéphane Loisel. Competition among non-life insurers under solvency constraints: A game-theoretic approach. European Journal of Operational Research, 2013, 231 (3), pp.702-711. hal-00746245

HAL Id: hal-00746245

https://hal.science/hal-00746245v1

Submitted on 28 Oct 2012

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A game-theoretic approach to non-life insurance markets

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Abstract

In this paper, we formulate a noncooperative game to model a non-life insurance market. The aim is to analyze the effects of competition between insurers through different indicators: the market premium, the solvency level, the market share and the underwriting results. Resulting premium Nash equilibria are discussed and numerically illustrated.

Keywords: Non-life insurance; Market model; Game theory; Nash equilibrium

1. Introduction

Insurance pricing is a classical topic for both actuaries and academics. Standard actuarial approaches for non-life insurance typically suggest to use expectation, standard deviation, or quantiles of the underlying risk to derive a fair premium. For an overview of premium principles, see, e.g., Teugels and Sundt (2004). This actuarially-based premium, which is sometimes referred to as the technical premium, is then often altered by marketing and management departments, and deviations from technical premium can be considerable. Affordability by customers and mutualization across the portfolio are first reasons to explain the economic reality that policyholders do not necessarily pay the risk-based premium. But another major reason of such deviations from the fair premium is the dependency on market conditions. A market model is needed to study the economic interactions between insurers and policyholders.

Basic economic models suggest that the equilibrium premium is the marginal cost, as any upward deviation from this equilibrium will result in losing all the policies in the next period. This theory would imply that all insurers price at the market premium. However,

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in practice customers do not move from an insurer to a cheaper one as swiftly as economic models anticipate. There is an inertia of the insurance demand, preventing all policyholders to always look for the cheapest insurer when their premium is slightly higher than the market premium. So, the customer behavior is much more complicated. More refined economic models focus on moral hazard and adverse selection. The celebrated model of Rothschild and Stiglitz (see Rothschild and Stiglitz (1976)) deals with a utility-based agent framework where insureds have private information on their own risk. Insurers provide a menu of contracts (a pair of premium and deductible), and high-risk individuals choose full coverage, whereas low-risk individuals are more attracted by partial coverage. Note that the equilibrium price may not exist if all insurers offer just one type of contract. Picard (2009) considers an extension by allowing insurers to offer participating contracts (such as mutual-type contracts). This feature guarantees the existence of an equilibrium, which forces (rational) insureds to reveal their risk level. An important source of applications of such models is health insurance, where moral hazard and adverse selection play a major role, see, e.g., Geoffard et al. (1998), Wambach (2000); Mimra and Wambach (2010) and Picard (2009).

However, the economic models mentioned above can not address the insurance market cycle dynamics, so that one has to look for further alternatives. Taylor (1986, 1987) deals with discrete-time underwriting strategies of insurers and provides first attempts to model strategic responses to the market, see also Kliger and Levikson (1998); Emms et al. (2007); Moreno-Codina and Gomez-Alvado (2008). The main pitfall of the optimal control approach is that it focuses on one single insurer and thus implicitly assumes that insurers are playing a game against an impersonal market player and the market price is independent of their own actions.

In this paper, we want to investigate the suitability of game theory for insurance market modelling. The use of game theory in actuarial science has a long history dating back to K. Borch and J. Lemaire, who mainly used cooperative games to model risk transfer between insurer and reinsurer, see, e.g., Borch (1960, 1975), Lemaire and Quairière (1986). Bühlmann (1984) and Golubin (2006) also studied risk transfer with Pareto optimality. Among the articles using noncooperative game theory to model the non-life insurance market, Bertrand oligopoly models are studied by Polborn (1998), Rees et al. (1999), Hardelin and de Forge (2009). Powers and Shubik (1998, 2006) also study scale effects of the number of insurers and the optimal number of reinsurers in a market model having a central clearing house. More recently, Taksar and Zeng (2011) study non-proportional reinsurance with zero-sum stochastic continuous-time games. Demgne (2010) seems to be the first study from a game theory point of view of (re)insurance market cycles. She uses well known economic concets: pure monopoly, Cournot's oligopoly (i.e. war of quantity), Bertrand's oligopoly (i.e. war of price) and the Stackelberg equilibrium (leader/follower game). For all these, she tests various scenarios and checks the consistency of model outputs with reinsurance reality.

Finally, in many ruin theory models, one assumes that the portfolio size remains constant over time (see, e.g., Asmussen and Albrecher (2010) for a recent survey). Non-homogeneous claim arrival processes have usually been studied in the context of modelling catastrophe events. More recently, non-constant portfolio size has been considered, see, e.g., Trufin et al.

(2009) and the references therein. Malinovskii (2010) uses a ruin framework to analyze different situations for an insurer in its behavior against the market.

This paper aims to model competition in non-life insurance markets with noncooperative game theory in order to extend the player-vs-market reasoning of Taylor (1986, 1987)'s models. The main contribution is to show that incorporating competition when setting premiums leads to a significant deviation from the actuarial premium and from a one-player optimized premium. The rest of the paper is organized as follows. Section 2 introduces a one-period noncooperative game. Existence and uniqueness of a premium equilibrium are established. Section 3 relaxes assumptions on objective and constraint components of the one-period model. The existence of a premium equilibrium is still guaranteed, but uniqueness may not hold. A reasonable choice of an equilibrium is proposed in this situation. Section 4 presents numerical illustrations of the two games. A conclusion and perspectives are given in Section 5.

2. A one-period model

Consider I insurers competing in a market of n policyholders with one-year contracts (where n is considered constant). The "game" for insurers is to sell policies to the policyholders by setting the premium. Let $(x_1, \ldots, x_I) \in \mathbb{R}^I$ be a price vector, with x_j representing premium of insurer j. Once the premium is set by all insurers, the insureds choose to renew or to lapse from their current insurer. Then, insurers pay claims, according to their portfolio size, during the coverage year. At the end of the period, underwriting results are determined, and insurer capital is updated: some insurers may be bankrupt. As we deal with a one-period model, we ignore for simplicity investment results.

In the next subsections, we present the four components of the game: a lapse model, a loss model, an objective function and a solvency constraint function. In the sequel, a subscript $j \in \{1, \ldots, I\}$ will always denote a player index, whereas a subscript $i \in \{1, \ldots, n\}$ denotes an insured index.

2.1. Lapse model

Being with current insurer j, the insurer choice C_i of insured i for the next period follows an I-dimensional multinomial distribution $\mathcal{M}_I(1, p_{j\to})$ with probability vector $p_{j\to} = (p_{j\to 1}, \ldots, p_{j\to I})$ summing to 1. The probability mass function is given by $P(C_i = k \mid j) = p_{j\to k}$. It seems natural and it has been verified empirically that the probability to choose an insurer is highly influenced by the previous period choice. In other words, the probability to lapse $p_{j\to k}$ with $k \neq j$ is generally much lower than the probability to renew $p_{j\to j}$. To our knowledge, only the UK market shows lapse rates above 50%. Those probabilities have to depend on the premium x_j, x_k proposed by insurer j and k, respectively.

Assume at the beginning of the game that the insurer portfolio sizes are n_j (such that

$\sum_{j=1}^{I} n_j = n$). The portfolio size $N_j(x)$ of insurer j for the next period is a random variable determined by the sum of renewed policies and businesses coming from other insurers. Hence,

$$N_j(x) = B_{jj}(x) + \sum_{k=1, k\neq j}^{I} B_{kj}(x).$$

$N_j(x)$ is a sum of I independent binomial variables $(B_{kj})_k$ where B_{kj} has parameters $\mathcal{B}(n_k, p_{k\to j}(x))$.

In the economics literature, $p_{j\to k}$ is considered in the framework of discrete choice models. In the random utility maximization setting, McFadden (1981) or Anderson et al. (1989) propose multinomial logit and probit probability choice models. In this paper, we choose a multinomial logit model, since the probit link function does not really enhance the choice model despite its additional complexity. Working with unordered choices, we arbitrarily set the insurer reference category for $p_{j\to k}$ to j, the current insurer. We define the probability for a customer to go from insurer j to k given the price vector x by the following multinomial logit model

$$p_{j\to k} = \mathrm{lg}_j^k(x) = \begin{cases} \frac{1}{1+\sum_{l\neq j} e^{f_j(x_j,x_l)}} & \text{if } \ j = k, \\ \frac{e^{f_j(x_j,x_k)}}{1+\sum_{l\neq j} e^{f_j(x_j,x_l)}} & \text{if } \ j \neq k, \end{cases} \tag{1}$$

where the sum is taken over the set $\{1, \ldots, I\}$ and f_j is a price sensitivity function. In the following, we consider two types of price functions

$$\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j \frac{x_j}{x_l} \ \text{ and } \ \tilde{f}_j(x_j, x_l) = \tilde{\mu}_j + \tilde{\alpha}_j(x_j - x_l).$$

The first function $\bar{f}_j$ assumes a price sensitivity with the ratio of the proposed premium x_j and competitor premium x_l, whereas $\tilde{f}_j$ works with the premium difference $x_j - x_l$. Parameters μ_j, α_j represent a base lapse level and price sensitivity. We assume that insurance products display positive price elasiticity of demand $\alpha_j > 0$. One can check that $\sum_k \mathrm{lg}_j^k(x) = 1$.

The above expression can be rewritten as

$$\mathrm{lg}_j^k(x) = \mathrm{lg}_j^j(x) \left(\delta_{jk} + (1-\delta_{jk}) e^{f_j(x_j,x_k)}\right),$$

with δ_{ij} denoting the Kronecker product. It is difficult to derive general properties of the distribution of a sum of binomial variables with different probabilities, except when the size parameters n_j are reasonably large, in which case the normal approximation is appropriate. With this insurer choice probability, the expected portfolio size of insurer j reduces to

$$\hat{N}_j(x) = n_j \times \mathrm{lg}_j^j(x) + \sum_{l\neq j} n_l \times \mathrm{lg}_l^j(x),$$

where n_j denotes the last year portfolio size of insurer j.

2.2. Loss model

Let Y_i be the aggregate loss of policy i during the coverage period. We assume no adverse selection among insured of any insurers, i.e. Y_i are independent and identically distributed (i.i.d.) random variables, $\forall i = 1, \ldots, n$. Let us assume a simple frequency – average severity loss model

$$Y_i = \sum_{l=1}^{M_i} Z_{i,l},$$

where the claim number M_i is independent from the i.i.d. claim severities $(Z_{i,l})_l$. Therefore, the aggregate claim amount for insurer j is

$$S_j(x) = \sum_{i=1}^{N_j(x)} Y_i = \sum_{i=1}^{N_j(x)} \sum_{l=1}^{M_i} Z_{i,l},$$

where $N_j(x)$ is the portfolio size of insurer j given the price vector x. We consider two claim number distributions: (i) M_i follows a Poisson distribution $\mathcal{P}(\lambda)$ and (ii) M_i follows a negative binomial distribution $\mathcal{NB}(r, p)$. These instances of the frequency-average severity model are such the aggregate claim amount $S_j(x) = \sum_{i=1}^{N_j(x)} Y_i$ is still a compound distribution of the same kind, since Y_i are assumed i.i.d. random variables. Hence, the insurer aggregate claim amount $S_j(x)$ is a compound distribution

$$S_j(x) = \sum_{l=1}^{\widetilde{M}_j(x)} Z_l,$$

such that the claim number $\widetilde{M}_j(x)$ and claim severity Z_l follow

- a Poisson-lognormal with $\widetilde{M}_j(x) \sim \mathcal{P}(N_j(x)\lambda)$ and $Z_l \overset{i.i.d.}{\sim} \mathcal{LN}(\mu_1, \sigma_1^2)$,
- a negative binomial-lognormal with $\widetilde{M}_j(x) \sim \mathcal{NB}(N_j(x)r, p)$ and $Z_l \overset{i.i.d.}{\sim} \mathcal{LN}(\mu_1, \sigma_1^2)$.

In the numerical applications, these two loss models are denoted PLN and NBLN, respectively.

2.3. Objective function

In the two previous subsections, we presented two components of the insurance markets: the lapse model (how insureds react to premium changes) and the loss model (how insureds face claims). We now turn our attention to the underwriting strategy of insurers, i.e. on how they set premiums.

In Subsection 2.1, we assumed that price elasticity of demand for the insurance product is positive. Thus, if the whole market underwrites at a loss, any actions of a particular

insurer to get back to profitability will result in a reduction of his business volume. This has two consequences for possible choice of objective functions: (i) it should use a decreasing demand function of price x_j given the competitors price $x_{-j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_I)$ and (ii) it should depend on an assessment of the insurer break-even premium per unit of exposure π_j.

We suppose that insurer j maximizes the expected profit of renewing policies defined as

$$O_j(x) = \frac{n_j}{n}\left(1 - \beta_j\left(\frac{x_j}{m_j(x)} - 1\right)\right)(x_j - \pi_j)\,, \tag{2}$$

where π_j is the break-even premium j and $m_j(x)$ is a market premium proxy. The objective function O_j defined as the product of a demand function and an expected profit per policy represents a company-wide expected profit. O_j targets renewal business and does not take into account new business explicitly. In addition to focusing on renewal business only, the objective function locally approximates the true insurer choice probability lg_j^j presented in Subsection 2.1. However, since the demand function $D_j(x) = n_j/n(1 - \beta_j(x_j/m_j(x) - 1))$ is not restricted to [0,1], demand D_j can exceed the current market share n_j/n, but profit per policy will decline when the premium decreases. Thus, maximising the objective function O_j leads to a trade-off between increasing premium to favour higher projected profit margins and decreasing premium to defend the current market share. Note that O_j has the nice property to be infinitely differentiable. The parameter π_j corresponds to the estimated mean loss of insurer j and is expressed as

$$\pi_j = \omega_j \overline{a}_{j,0} + (1 - \omega_j)\overline{m}_0$$

where $\overline{a}_{j,0}$ is the actuarial premium based on the past loss experience of insurer j, $\overline{m}_0$ is the market premium, available for instance, via rating bureaus or through insurer associations and $\omega_j \in [0, 1]$ is the credibility factor of insurer j. If insurer j is the market leader, then ω_j should be close to 1, whereas when insurer j is a follower, ω_j should be close to 0. Note that π_j takes into account expenses implicitly via the actuarial and the market premiums.

The market proxy used in Equation (2) is the mean price of the other competitors

$$m_j(x) = \frac{1}{I-1}\sum_{k \neq j} x_k.$$

The market proxy aims to assess other insurer premiums without specifically targeting one competitor. By excluding the price x_j to compute the market proxy $m_j(x)$, we suppose insurer j is not dominant in the market. If, for example, insurer j underwrites 80% of the total premium available in the market, $m_j(x)$ will not be appropriate, but in such cases the market competition is low. We could have used the minimum of the competitors' premium, but then $m_j(x)$ would not have been a continuous function of the price vector x. Furthermore, insurer j does not necessarily take into account to be the cheapest insurer.

2.4. Solvency constraint function

In addition to maximizing a certain objective function, insurers must satisfy a solvency constraint imposed by the regulator. Currently, European insurers report their solvency

margin in the Solvency I framework, based on the maximum of a percentage of gross written premium and aggregate claim mean. According to Derien (2010), a non-life insurer computes its solvency margin as

$$\text{SM} = \max(18\% \times \text{GWP}, 26\% \times \text{AC}) \times \max(50\%, \text{AC net of reins/AC gross of reins}),$$

where GWP denotes the gross written premium and AC the aggregate claim mean[1]. Discarding reinsurance, the Solvency I framework leads to a solvency margin

$$\text{SM} = \max(9\% \times \text{GWP}, 13\% \times \text{AC}).$$

This approach is not really satisfactory, as it does not take into account the risk volality of underwritten business. Since 2005, actuaries are well busy with the upcoming Solvency II framework. In this new framework, the quantitative part leads to the computation of two capital values, both based on the difference between a certain quantile and the mean of the aggregate loss. The solvency capital requirement (SCR) is based on the 99.5%-quantile, whereas the minimum capital requirement (MCR) is based on the 85%-quantile.

In our game context, we want to avoid the simplistic Solvency I framework, but still want to keep the tractablity for the SCR computation rule. We recall that the aggregate claim amount is assumed to be a frequency – average severity model, i.e. Cat-losses are ignored. A simplification is to approximate a q-quantile $Q(n, q)$ of aggregate claim amount of n i.i.d. risks by a bilinear function of n and $\sqrt{n}$

$$Q(n, q) = E(Y)n + k_q \sigma(Y)\sqrt{n}, \tag{3}$$

where the coefficient k_q has to be determined and Y is the generic individual claim severity variable. The first term corresponds to the mean of the aggregate claim amount, while the second term is related to standard deviation.

Three methods have been tested to compute the solvency coefficient k_q: (i) a normal approximation $k_q^N = \Phi^{-1}(q)$, where Φ is the distribution function of the standard normal distribution, (ii) a simulation procedure with 10^5 sample size to get k_q^S as the empirical quantile and (iii) the Panjer recursion to compute the aggregate claim quantile k_q^{P2}.

While the normal approximation is based on the first two moments of the distribution only, simulation and Panjer methods need to have assumptions on claim frequency and claim severity distributions: we use the PLN and NBLN models defined in Subsection 2.2. We also need a risk number n. In Table 1, we report solvency coefficients for $n = 1000$ risks. Panjer and simulation methods appear twice since two loss models (PLN and NBLN) are tested.

Numerical experiments show that the normal approximation is less conservative for high quantiles (i.e. $k_q^N < k_q^P$) when the claim number follows a negative binomial distribution,

[1]The percentages 18% and 26% are replaced respectively by 16% and 23% when the GWP exceeds 57.5 Meur or AC exceeds 40.3 Meur.

[2]See, e.g., Theorem 12.4.3 of Bowers et al. (1997). Panjer recursion requires that the claim distribution is discrete. So before using Panjer algorithm, we use a lower discretization of the lognormal claim distribution.

prob q	k_q^N	k_q^P-PLN	k_q^P-NBLN	k_q^S-PLN	k_q^S-NBLN
0.75	0.674	1.251	0.913	0.649	0.627
0.8	0.842	1.431	1.104	0.829	0.812
0.85	1.036	1.642	1.332	1.029	1.03
0.9	1.282	1.912	1.627	1.299	1.312
0.95	1.645	2.321	2.083	1.695	1.759
0.99	2.326	3.117	2.997	2.475	2.633
0.995	2.576	3.419	3.352	2.777	2.976

Table 1: Solvency coefficient k

and the reverse for the Poisson distribution. Based on this study, we choose to approximate quantiles at 85% and 99.5% levels with coefficients $k_{85} = 1$ and $k_{995} = 3$.

Thus, using the approximation (3), the solvency capital requirement SCR is deduced as

$$\mathrm{SCR}_q \approx k_q \sigma(Y)\sqrt{n},$$

which is more complex than the Solvency I framework. Numerical investigations show that the Solvency I requirement corresponds to a 75% quantile. Therefore, we decide to choose the adapted solvency constraint function

$$g_j^1(x_j) = \frac{K_j + n_j(x_j - \pi_j)(1 - e_j)}{k_{995}\sigma(Y)\sqrt{n_j}} - 1, \tag{4}$$

where k_{995} is the solvency coefficient and e_j denotes the expense rate as a percentage of gross written premium. The numerator corresponds to the sum of current capital K_j and expected profit on the in-force portfolio (without taking into account new business). It is easy to see that the constraint $g_j^1(x) \geq 0$, is equivalent to $K_j + n_j(x_j - \pi_j)(1 - e_j) \geq k_{995}\sigma(Y)\sqrt{n_j}$, but g_j^1 is normalized with respect to capital, providing a better numerical stability.

In addition to the solvency constraint, we need to impose bounds on the possible premium. A first choice could be simple linear constraints as $x_j - \underline{x} \geq 0$ and $\overline{x} - x_j \geq 0$, where $\underline{x}$ and $\overline{x}$ represent the minimum and the maximum premium, respectively. But the following reformulation is equivalent and numerically more stable:

$$g_j^2(x_j) = 1 - e^{-(x_j - \underline{x})} \geq 0 \quad \text{and} \quad g_j^3(x_j) = 1 - e^{-(\overline{x} - x_j)} \geq 0.$$

The minimum premium $\underline{x}$ could be justified by a prudent point of view of regulators while the maximum premium $\overline{x}$ could be set, e.g., by a consumer right defense association. In the sequel, we set $\underline{x} = E(Y)/(1 - e_{min}) < \overline{x} = 3E(Y)$, where e_{min} is the minimum expense rate.

Overall, the constraint function $g_j(x_j) \geq 0$ is equivalent to

$$\{x_j, g_j(x_j) \geq 0\} = \left\{x_j \in [\underline{x}, \overline{x}], \quad K_j + n_j(x_j - \pi_j)(1 - e_j) \geq k_{995}\sigma(Y)\sqrt{n_j}\right\}. \tag{5}$$

2.5. Game sequence

For noncooperative games, there are two main solution concepts, Nash equilibrium and Stackelberg equilibrium: the Nash equilibrium assumes player actions are taken simultaneously while for the Stackelberg equilibrium actions take place sequentially, see, e.g., Fudenberg and Tirole (1991); Osborne and Rubinstein (2006). In our setting, we consider the Nash equilibrium as the most appropriate concept. We give below the definition of a generalized Nash equilibrium extending the Nash equilibrium with constraint functions.

Definition. *For a game with I players, with payoff functions O_j and constraint function g_j, a generalized Nash equilibrium is a vector $x^\star = (x_1^\star, \dots, x_I^\star)$ such that for all $j = 1, \dots, I$, $x_j^\star$ solves the subproblem*

$$\max_{x_j} O_j(x_j, x_{-j}^\star) \quad s.t. \quad g_j(x_j, x_{-j}^\star) \geq 0.$$

where x_j and x_{-j} denote action of player j and the other players' action, respectively.

A (generalized) Nash equilibrium is interpreted as a point at which no player can profitably deviate, given the actions of the other players. When each player's strategy set does not depend on the other players' strategies, a generalized Nash equilibrium reduces to a standard Nash equilibrium. Our game is a Nash equilibrium problem since our constraint functions g_j defined in Equation (4) depend on the price x_j only.

The game sequence is given as follows:

(i) Insurers set their premium according to a generalized Nash equilibrium $x^\star$, solving for all $j \in \{1, \dots, I\}$

$$x_{-j} \mapsto \underset{x_j, g_j(x_j) \geq 0}{\arg\max} \; O_j(x_j, x_{-j}).$$

(ii) Insureds randomly choose their new insurer according to probabilities $p_{k \to j}(x^\star)$: we get $N_j(x)$.

(iii) For the one-year coverage, claims are random according to a frequency-average severity model relative to the portfolio size $N_j(x^\star)$.

(iv) Finally the underwriting result is determined by $UW_j(x^\star) = N_j(x^\star) x_j^\star (1 - e_j) - S_j(x^\star)$, where e_j denotes the expense rate.

If the solvency requirement is not fullfilled, in Solvency I, the regulator response is immediate: depending on the insolvency severity, regulators can withdraw the authorisation to underwrite new business or even force the company to go run-off or to sell part of its portfolio. In Solvency II, this happens only when the MCR level is not met. There is a buffer between MCR and SCR where regulators impose some specific actions to help returning to the SCR level.

In our game, we choose to remove players which have a capital below MCR and to authorize players to continue underwriting when capital is between the MCR and the SCR. Note that the constraint function will be active when computing the Nash equilibrium, if the capital is between the MCR and SCR.

2.6. Properties of the premium equilibrium

In this subsection, we investigate properties of the premium equilibrium. We start by showing existence and uniqueness of a Nash equilibrium. Then, we focus on the sensitivity analysis on model parameters of such an equilibrium.

Proposition 2.1. *The I-player insurance game with objective function and solvency constraint function defined in Equations (2) and (5), respectively, admits a unique (Nash) premium equilibrium.*

Proof. The strategy set is $R = [\underline{x}, \overline{x}]^I$, which is nonempty, convex and compact. Given $x_{-j} \in [\underline{x}, \overline{x}]$, the function $x_j \mapsto O_j(x)$ is a quadratic function with second-degree term $-\beta_j x_j^2/m_j(x) < 0$ up to a constant n_j/n. Thus, this function is (strictly) concave. Moreover, for all players, the constraint functions g_j^1 are linear functions, hence also concave. By Theorem 1 of Rosen (1965), the game admits a Nash equilibrium, i.e. existence is guaranteed. By Theorem 2 of Rosen (1965), uniqueness is verified if we have the following inequality for all $x, y \in R$,

$$\sum_{j=1}^{I} r_j(x_j - y_j)\nabla_{x_j} O_j(y) + \sum_{j=1}^{I} r_j(y_j - x_j)\nabla_{x_j} O_j(x) > 0, \tag{6}$$

for some $r \in \mathbb{R}^I$ with strictly positive components $r_i > 0$. As the function $x_j \mapsto O_j(x)$ is a strictly concave and differentiable function for all x_{-j}, we have $\nabla_{x_j} O_j(x)(y_j - x_j) > O_j(y) - O_j(x)$ and equivalently $\nabla_{x_j} O_j(y)(x_j - y_j) > O_j(x) - O_j(y)$. Thus,

$$(x_j - y_j)\nabla_{x_j} O_j(y) + (y_j - x_j)\nabla_{x_j} O_j(x) > O_j(y) - O_j(x) + O_j(x) - O_j(y) = 0.$$

Taking $r = 1$, equation (6) is verified. □

Proposition 2.2. *Let $x^\star$ be the premium equilibrium of the I-player insurance game. For each player j, if $x_j^\star \in]\underline{x}, \overline{x}[$, the player equilibrium $x_j^\star$ depends on the parameters in the following way: it increases with break-even premium π_j, solvency coefficient k_{995}, loss volatility $\sigma(Y)$, expense rate e_j and decreases with sensitivity parameter β_j and capital K_j. When $x_j^\star = \underline{x}$ or $\overline{x}$, the premium equilibrium is independent of those parameters.*

Proof. The premium equilibrium $x_j^\star$ of insurer j solves the necessary Karush-Kuhn-Tucker conditions:

$$\begin{aligned} &\nabla_{x_j} O_j(x^\star) + \sum_{1 \le l \le 3} \lambda_l^{j\star} \nabla_{x_j} g_j^l(x_j^\star) = 0, \\ &0 \le \lambda^{j\star},\ g_j(x_j^\star) \ge 0,\ g_j(x_j^\star)^T \lambda^{j\star} = 0, \end{aligned} \tag{7}$$

where $\lambda^{j\star} \in \mathbb{R}^3$ are Lagrange multipliers, see, e.g., Facchinei and Kanzow (2009). In the last part of equation (7), $g_j(x_j^\star)^T \lambda^{j\star} = 0$ is the complementarity equation implying that the l constraint g_j^l is either active ($g_j^l(x_j^\star) = 0$) or inactive ($g_j^l(x_j^\star) > 0$), but $\lambda_l^{j\star} = 0$.

We suppose that $x_j^\star \in]\underline{x}, \overline{x}[$. Hence, $\lambda_2^{j\star} = \lambda_3^{j\star} = 0$. There are two cases: either the solvency constraint g_j^1 is active or not. Let us assume the solvency constraint is inactive. Insurer j's premium equilibrium verifies $\nabla_{x_j} O_j(x^\star) = 0$, i.e.

$$\frac{n_j}{n}\left(1 - 2\beta_j \frac{x_j^\star}{m_j(x^\star)} + \beta_j + \beta_j \frac{\pi_j}{m_j(x^\star)}\right) = 0. \tag{8}$$

Let x_y^j be the premium vector with the j component being y, i.e. $x_y^j = (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_I)$. We denote by z a parameter of interest and define the function F as

$$F_x^j(z, y) = \frac{\partial O_j}{\partial x_j}(x_y^j, z),$$

where the objective function depends (also) on the interest parameter z. Equation (8) can be rewritten as $F_{x^\star}^j(z, x_j^\star) = 0$.

By the continuous differentiability of F with respect to z and y and the fact that $F_x^j(z, y) = 0$ has at least one solution (z_0, y_0), we can invoke the implicit function theorem, see Appendix Appendix A.1. So there exists a function φ defined in a neighborhood of (z_0, y_0) such that $F_x^j(z, \varphi(z)) = 0$ and $\varphi(z_0) = y_0$. Furthermore, if $\frac{\partial F_x^j}{\partial y}(z_0, y_0) \neq 0$, the derivative of φ is given by

$$\varphi'(z) = -\left.\frac{\frac{\partial F_x^j}{\partial z}(z, y)}{\frac{\partial F_x^j}{\partial y}(z, y)}\right|_{y=\varphi(z)}.$$

In our case, we have

$$\frac{\partial F_x^j}{\partial y}(z, y) = \frac{\partial^2 O_j}{\partial x_j^2}(x_y^j, z) = -2\alpha_j \frac{n_j}{n m_j(x)} < 0.$$

As a consequence, the sign of φ' is simply

$$\operatorname{sign}(\varphi'(z)) = \operatorname{sign}\left(\frac{\partial F_x^j}{\partial z}(z, \varphi(z))\right).$$

Let us consider $z = \pi_j$. We have

$$\frac{\partial F_x^j}{\partial z}(z, y) = \frac{n_j \beta_j}{n m_j(x)} > 0.$$

Thus, the function $\pi_j \mapsto x_j^\star(\pi_j)$ is increasing.

Let z be the sensitivity coefficient β_j. We have

$$\frac{\partial F_x^j}{\partial z}(z, y) = \frac{n_j}{n}\left(-2\beta_j \frac{y}{m_j(x)} + 1 + \frac{\pi_j}{m_j(x)}\right).$$

Using $F_x^j(z, \varphi(z)) = 0$, it leads to

$$\frac{\partial F_x^j}{\partial z}(z, \varphi(z)) = \frac{n_j}{n}\frac{-1}{z} < 0.$$

Thus, the function $\beta_j \mapsto x_j^\star(\beta_j)$ is decreasing. In such a case of an inactive constraint, the premium equilibrium is independent of the initial portfolio size n_j.

When the solvency constraint is active, the premium equilibrium $x_j^\star$ verifies $g_j^1(x_j^\star) = 0$, i.e.

$$x_j^\star = \pi_j + \frac{k_{995}\sigma(Y)\sqrt{n_j} - K_j}{n_j(1 - e_j)}. \tag{9}$$

Here, the implicit function theorem is not necessary since $x_j^\star$ does not depend on $x_{-j}^\star$. We deduce that $x_j^\star$ is an increasing function of $\pi_j, k_{995}, \sigma(Y), e_j$ and a decreasing function K_j.

The function $n_j \mapsto x_j^\star(n_j)$ is not necessarily monotone. Let z be n_j. Differentiating Equation (9) with respect to z, we get

$$\varphi'(z) = \frac{1}{z^{3/2}(1 - e_j)}\left(-\frac{k\sigma(Y)}{2} + \frac{K_j}{\sqrt{z}}\right),$$

whose sign depends on the value of the other parameters.

□

3. Refinements of the one-period model

In this section, we propose refinements on the objective and constraint functions of the previous section.

3.1. Objective function

The objective function given in Subsection 2.3 is based on an approximation of the true demand function. For insurer j, the expected portfolio size is given by

$$\hat{N}_j(x) = n_j \times \mathrm{lg}_j^j(x) + \sum_{l \neq j} n_l \times \mathrm{lg}_l^j(x),$$

where lg_j^l's are lapse functions and lg_j^j the "renew" function. Note that the expected size $\hat{N}_j(x)$ contains both renewal and new businesses. So, a new objective function could be

$$\widetilde{O}_j(x) = \frac{\hat{N}_j(x)}{n}(x_j - \pi_j),$$

where π_j is the break-even premium as defined in Subsection 2.3. However, we do not consider this function, since the function $x_j \mapsto \widetilde{O}_j(x)$ does not verify some generalized convexity properties, which we will explain in Subsection 3.3. And also, the implicit assumption is that insurer j targets the whole market: this may not be true in most competitive insurance markets.

Instead, we will test the following objective function

$$\widetilde{O}_j(x) = \frac{n_j \, \mathrm{lg}_j^j(x)}{n}(x_j - \pi_j), \tag{10}$$

taking into account only renewal business. This function is infinitely differentiable. Using the definition lg_j^j in Equation (1), one can show that the function $x_j \mapsto \mathrm{lg}_j^j(x)$ is a strictly decreasing function, see Appendix Appendix A.3. As for the objective function O_j, maximising $\widetilde{O}_j$ is a trade-off between increasing premium for better expected profit and decreasing premium for better market share.

3.2. Constraint function

We also change the solvency constraint function $x_j \mapsto g_j^1(x_j)$ defined in Equation (4), which is a basic linear function of the premium x_j. We also integrate other insurer premium x_{-j} in the new constraint function, i.e. $x_j \mapsto \tilde{g}_j^1(x)$. We could use the following constraint function

$$\tilde{g}_j^1(x) = \frac{K_j + \hat{N}_j(x)(x_j - \pi_j)(1 - e_j)}{k_{995}\sigma(Y)\sqrt{\hat{N}_j(x)}} - 1,$$

the ratio of the expected capital and the required solvency capital. Unfortunately, this function does not respect a generalized convexity property, that we will define in the Subsection 3.3. So instead, we consider a simpler version

$$\tilde{g}_j^1(x) = \frac{K_j + n_j(x_j - \pi_j)(1 - e_j)}{k_{995}\sigma(Y)\sqrt{\hat{N}_j(x)}} - 1, \tag{11}$$

by removing the expected portfolio size $\hat{N}_j$ in the numerator. This function is also infinitely differentiable. The other two constraint functions g_j^2, g_j^3 are identical as in Subsection 2.4.

3.3. Properties of premium equilibrium

Conditions on the existence of a generalized Nash equilibrium can be found in Facchinei and Kanzow (2009) or Dutang (2012*b*). In our setting, we need to show (i) the objective function $O_j(x)$ is quasiconcave with respect to x_j, (ii) the constraint function $g_j(x)$ is quasiconcave with respect to x_j, (iii) the action set $\{x_j \in X_j, g_j(x_j, x_{-j}) \geq 0\}$ is nonempty.

Recall that a function $f : X \mapsto Y$ is concave if $\forall x, y \in X, \forall \lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$. Note that a convex and concave function is linear. If

inequalities are strict, we speak about concavity. A function $f : X \mapsto Y$ is quasiconcave if $\forall x, y \in X, \forall \lambda \in]0, 1[$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \min(f(x), f(y)).$$

Again, if inequalities are strict, we speak about strict quasiconcavity. As for concavity, there exist special characterizations when f is C^2.

Proposition. *When f is a differentiable function on an open convex $O \subset \mathbb{R}^n$, then f is quasiconcave if and only if $\forall x, y \in O, f(x) \geq f(y) \Rightarrow \nabla f(y)^T(x - y) \geq 0$.*
When f is a C^2 function on an open convex $O \subset \mathbb{R}^n$, then f is quasiconcave if and only if $\forall x \in O, \forall d \in \mathbb{R}^n, d^T \nabla f(x) = 0 \Rightarrow d^T \nabla^2 f(x) d \leq 0$.

Proof. See Theorems 2 and 5 of Diewert et al. (1981). □

From the last proposition, it is easy to see that for a C^2 univariate function, quasiconcavity implies unimodality. Furthermore, f is pseudoconcave if and only if $\forall x, y$, we have $f(x) > f(y) \Rightarrow \nabla f(y)^T(x - y) > 0$.

Proposition. *When f is a C^2-function on an open convex $O \subset \mathbb{R}^n$, then if $\forall x \in O, \forall d \in \mathbb{R}^n, d^T \nabla f(x) = 0 \Rightarrow d^T \nabla^2 f(x) d < 0$, then f is pseudoconcave, which in turn implies strict quasiconcavity.*

Proof. See Corollary 10.1 of Diewert et al. (1981). □

Examples of quasiconcave functions include monotone, concave or log-concave functions. A univariate quasiconcave function is either monotone or unimodal. More properties can be found in Diewert et al. (1981). Figure A.4 in Appendix Appendix A.5 relates the different concepts of convexity.

Proposition 3.1. *The I-player insurance game with objective function and solvency constraint function defined in Equations (10) and (11), respectively, admits a generalized Nash premium equilibrium, if for all $j = 1, \ldots, I$, $\tilde{g}_j^1(\overline{x}) > 0$.*

Proof. Properties of the expected portfolio size function have been established in Appendix Appendix A.3. The objective function can be rewritten as

$$\widetilde{O}_j(x) = \lg_j^j(x, f)(x_j - \pi_j),$$

up to a constant n_j/n. $\widetilde{O}_j$ has been built to be continuous on $\mathbb{R}_+^I$. Note that we stress the dependence on the price sensitivity function f. Using Appendix Appendix A.4, the gradient of the objective function is proportional to

$$\frac{\partial \widetilde{O}_j(x)}{\partial x_j} = \lg_j^j(x, f)(1 - S_j(x)(x_j - \pi_j)), \quad \text{where} \quad S_j(x) = \sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l(x, f).$$

The gradients cancel at $1 = S_j(x^{j\star})(x_j^\star - \pi_j)$, where $x^{j\star} = (x_1, \ldots, x_{j-1}, x_j^\star, x_{j+1}, \ldots, x_I)$.

The second-order derivative is given by

$$\begin{aligned}\frac{\partial^2 \widetilde{O}_j(x)}{\partial x_j^2} &= \lg_j^j(x,f)\left((x_j-\pi_j)2S_j^2(x) - 2S_j(x) - (x_j-\pi_j)\sum_{l\neq j} f'_{j1}(x_j,x_l)^2 \lg_j^l(x,f)\right)\\ &= \lg_j^j(x,f)2S_j(x)\left[(x_j-\pi_j)S_j(x)-1\right] - \lg_j^j(x,f)(x_j-\pi_j)\sum_{l\neq j} f'_{j1}(x_j,x_l)^2 \lg_j^l(x,f).\end{aligned}$$

The sign of the second order derivative at $x^{j\star}$ is

$$\operatorname{sign}\left(\frac{\partial^2 \widetilde{O}_j(x^{j\star})}{\partial x_j^2}\right) = -\lg_j^j(x^{j\star},f)(x_j^\star-\pi_j)\sum_{l\neq j} f'_{j1}(x_j^\star,x_l)^2 \lg_j^l(x^{j\star},f).$$

However, the root of the gradient is such that $x_j^\star - \pi_j = 1/S_j(x^{j\star}) > 0$. So we have

$$\operatorname{sign}\left(\frac{\partial^2 O_j(x^{j\star})}{\partial x_j^2}\right) < 0.$$

Hence, the function $x_j \mapsto O_j(x)$ is pseudoconcave, and thus strictly quasiconcave.

Functions g_j^2, g_j^3 are strictly concave since second-order derivatives are

$$\frac{\partial^2 g_j^2(x)}{\partial^2 x_j} = -e^{-(x_j-\underline{x})} < 0 \quad\text{and}\quad \frac{\partial^2 g_j^3(x)}{\partial^2 x_j} = -e^{-(\overline{x}-x_j)} < 0.$$

We verify quasiconcavity of the function $\tilde{g}_j^1$ with respect to x_j. The function $x_j \mapsto \tilde{g}_j^1(x)$ is monotone since its gradient

$$\frac{\partial \tilde{g}_j^1(x)}{\partial x_j} = \frac{K_j + n_j(x_j-\pi_j)(1-e_j)}{2k_{995}\sigma(Y)\hat{N}_j^{3/2}(x)}\left(-\frac{\partial \hat{N}_j(x)}{\partial x_j}\right) + \frac{n_j(1-e_j)}{k_{995}\sigma(Y)\sqrt{\hat{N}_j(x)}}$$

is positive for all $x \in \mathbb{R}_+^I$. Thus, function $x_j \mapsto \tilde{g}_j^1(x)$ is (strictly) quasiconcave.

Let $X_j = [\underline{x}, \overline{x}]$. The constraint set is $C_j(x_{-j}) = \{x_j \in X_j, \tilde{g}_j^1(x_j, x_{-j}) \geq 0\}$ where $x_j \mapsto \tilde{g}_j^1(x)$ is strictly increasing, continuous and by assumption, for all $j = 1, \ldots, I$, $\tilde{g}_j^1(\overline{x}) > 0$. Thus, $C_j(x_{-j})$ is a nonempty convex closed set. Furthermore, the point-to-set mapping C_j is upper semi-continuous by using Example 5.10 of Rockafellar and Wets (1997). Using Theorem 13 of Hogan (1973) and the continuity of $\tilde{g}_j^1$, the point-to-set mapping is also lower semi-continuous. By Theorem 4.1 of Facchinei and Kanzow (2009), there exists a generalized Nash equilibrium. □

3.3.1. Non-uniqueness issues

Uniqueness of a generalized Nash equilibrium is not guaranteed in general. Furthermore, there is no particular reason for a player to choose a certain Nash equilibrium rather than

another one. Rosen (1965) studied uniqueness of such an equilibrium in a jointly convex game (i.e. where objective functions are convex and the constraint function is common and convex). To deal with non-uniqueness, he studies a subset of generalized Nash equilibrium, where Lagrange multipliers resulting from the Karush-Kuhn-Tucker (KKT) conditions are normalized. Such a normalized equilibrium is unique given a scale of the Lagrange multiplier when the constraint function verifies additional assumptions. Other authors such as von Heusinger and Kanzow (2009) or Facchinei et al. (2007) define normalized equilibrium when Lagrange multipliers are set equal.

Another way is to look for generalized Nash equilibria having some specific properties, such as Pareto optimality. The selection of the equilibrium is particularly developed for games with finite action sets. In that setting, one can also use a mixed strategy, by playing ramdomly one among many equilibrium strategies.

3.3.2. Parameter sensitivity

Proposition. *Let $x^\star$ be a premium equilibrium of the I-player insurance game. For each player j, if $x_j^\star \in]\underline{x}, \overline{x}[$, player equilibrium $x_j^\star$ depends on parameter in the following way: it increases with break-even premium π_j, solvency coefficient k_{995}, loss volatility $\sigma(Y)$, expense rate e_j and decreases with lapse parameter μ_j, α_j and capital K_j. Otherwise when $x_j^\star = \underline{x}$ or $\overline{x}$, premium equilibrium is independent of any parameters.*

Proof. As explained in Appendix Appendix A.2, the KKT conditions at a premium equilibrium $x^\star$ are such there exist Lagrange multipliers $\lambda^{j\star}$,

$$\frac{\partial \widetilde{O}_j}{\partial x_j}(x) - \lambda_1^{j\star} \frac{\partial \tilde{g}_j^1}{\partial x_j}(x) = 0,$$

when assuming g_j^2, g_j^3 functions are not active. And the complementarity constraint is such that $\lambda_1^{j\star} \times \tilde{g}_j^1(x^\star) = 0$.

If the solvency constraint $\tilde{g}_j^1$ is inactive, then we necessarily have $\lambda_{j1}^\star = 0$. Let x_y^j be the premium vector with the j component being y, i.e. $x_y^j = (x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_I)$. We denote by z a parameter of interest, say for example e_j. We define the function F as

$$F_x^j(z, y) = \frac{\partial O_j}{\partial x_j}(x_y^j, z),$$

where the objective function depends on the interest parameter z. By the continuous differentiability of F with respect to z and y, we can invoke the implicit function theorem, see Appendix Appendix A.1. So there exists a function φ such that $F_x^j(z, \varphi(z)) = 0$, and the derivative is given by

$$\varphi'(x) = - \left. \frac{\frac{\partial F_x^j}{\partial z}(z, y)}{\frac{\partial F_x^j}{\partial y}(z, y)} \right|_{y=\varphi(z)}.$$

In our case[1], we have

$$F_x^j(z,y) = \frac{n_j}{n}\,\mathrm{lg}_j^j(x_y^j)[1 - S_j(x_y^j)(y-\pi_j)],$$

and

$$\frac{\partial F_x^j}{\partial z}(z,y) = \frac{\partial^2 O_j}{\partial z \partial x_j}(x_y^j, z), \quad \text{and} \quad \frac{\partial F_x^j}{\partial y}(z,y) = \frac{\partial^2 O_j}{\partial x_j^2}(x_y^j, z).$$

The first-order derivative is given by

$$\frac{\partial F_x^j}{\partial y}(z,y) = 2\frac{n_j}{n}\,\mathrm{lg}_j^j(x_y^j)S_j(x_y^j)\left[(y-\pi_j)S_j(x_y^j) - 1\right] - \frac{n_j}{n}\,\mathrm{lg}_j^j(x_y^j)(y-\pi_j)\sum_{l\neq j} f'_{j1}(y,x_l)^2\,\mathrm{lg}_j^l(x_y^j).$$

Using $F_x^j(z,\varphi(z)) = 0$ whatever z represents, it simplifies to

$$\frac{\partial F_x^j}{\partial y}(z,\varphi(z)) = -\frac{n_j}{n}\,\mathrm{lg}_j^j(x_{\varphi(z)}^j)(\varphi(z)-\pi_j)\sum_{l\neq j} f'_{j1}(\varphi(z),x_l)^2\,\mathrm{lg}_j^l(x_{\varphi(z)}^j).$$

Let z now be the insurer's break-even premium $z = \pi_j$. We have

$$\frac{\partial F_x^j}{\partial z}(z,y) = n_j\,\mathrm{lg}_j^j(x_y^j)S_j(x_y^j).$$

Thus, the derivative of φ is

$$\varphi'(z) = \frac{S_j\left(x_{\varphi(z)}^j\right)}{(\varphi(z)-z)\sum_{l\neq j} f'_{j1}\left(\varphi(z),x_l\right)^2\,\mathrm{lg}_j^l\left(x_{\varphi(z)}^j\right)}.$$

By definition, $F_x^j(z,\varphi(z)) = 0$ is equivalent to

$$1 = S_j\left(x_{\varphi(z)}^j\right)(\varphi(z)-z).$$

Thus $\varphi(z) - z > 0$. We conclude that $\varphi'(z) > 0$, i.e. the function $\pi_j \mapsto x_j^\star(\pi_j)$ is increasing.

Let z be the intercept lapse parameter $z = \mu_j$. By differentiating the lapse probability, we have

$$\frac{\partial\,\mathrm{lg}_j^j}{\partial z}(x_y^j) = -\,\mathrm{lg}_j^j(x_y^j)\sum_{l\neq j}\mathrm{lg}_j^l(x_y^j) \quad \text{and} \quad \left.\frac{\partial\,\mathrm{lg}_j^k}{\partial z}(x_y^j)\right|_{j\neq k} = -\,\mathrm{lg}_j^k(x_y^j)\sum_{l\neq j}\mathrm{lg}_j^l(x_y^j) + \mathrm{lg}_j^k(x_y^j).$$

We get

$$\frac{\partial F_x^j}{\partial z}(z,y) = -n_j\,\mathrm{lg}_j^j(x_y^j)(1-\mathrm{lg}_j^j(x_y^j))\left[1 - S_j(x_y^j)(y-\pi_j)\right] - n_j\,\mathrm{lg}_j^j(x_y^j)^2 S_j(x_y^j).$$

[1] To simplify, we do not stress the dependence of lg_j^k and S_j on f.

Note the first term when $y = \varphi(z)$ since $F_x^j(z, \varphi(z)) = 0$. We finally obtain

$$\varphi'(x) = -\frac{S_j\left(x^j_{\varphi(z)}\right) \lg_j^j\left(x^j_{\varphi(z)}\right)}{(\varphi(z) - z) \sum\limits_{l \neq j} f'_{j1}\left(\varphi(z), x_l\right)^2 \lg_j^l\left(x^j_{\varphi(z)}\right)}.$$

Using $1 = S_j\left(x^j_{\varphi(z)}\right)(\varphi(z) - z)$, we have

$$\varphi'(x) = -\frac{S_j\left(x^j_{\varphi(z)}\right)^2 \lg_j^j\left(x^j_{\varphi(z)}\right)}{\sum\limits_{l \neq j} f'_{j1}\left(\varphi(z), x_l\right)^2 \lg_j^l\left(x^j_{\varphi(z)}\right)} < 0.$$

Thus, the function $\mu_j \mapsto x_j^\star(\mu_j)$ is decreasing.

Let z be the slope lapse parameter $z = \alpha_j$.

$$\frac{\partial \lg_j^j}{\partial z}(x_y^j) = -\lg_j^j(x_y^j) \sum_{l \neq j} \Delta_{j,l}(x_y^j) \lg_j^l(x_y^j)$$

and

$$\left.\frac{\partial \lg_j^k}{\partial z}(x_y^j)\right|_{j \neq k} = -\lg_j^k(x_y^j) \sum_{l \neq j} \Delta_{j,l}(x_y^j) \lg_j^l(x_y^j) + \lg_j^k(x_y^j)\Delta_{j,k}(x_y^j),$$

where $\Delta_{j,l}(x_y^j) = x_j/x_l$ if we use the premium ratio function f_j and $x_j - x_l$ if we use the premium difference function $\tilde{f}_j$. We get

$$\begin{aligned}\frac{\partial F_x^j}{\partial z}(z, y) = &-n_j \lg_j^j(x_y^j) S_j^\Delta(x_y^j)[1 - S_j(x_y^j)(y - \pi_j)] \\ &- n_j \lg_j^j(x_y^j)^2 S_j^\Delta(x_y^j) - n_j \lg_j^j(x_y^j) \sum_{l \neq j} f'_{j1}(y, x_l)\Delta_{j,l}(x_y^j) \lg_j^l(x_y^j),\end{aligned}$$

where $S_j^\Delta(x_y^j) = \sum_{l \neq j} \Delta_{j,l}(x_y^j) \lg_j^l(x_y^j)$. Again the first term cancels when $y = \varphi(z)$. Hence, we have

$$\varphi'(z) = -\frac{\lg_j^j\left(x^j_{\varphi(z)}\right) S_j^\Delta\left(x^j_{\varphi(z)}\right) + \sum_{l \neq j} f'_{j1}(\varphi(z), x_l)\Delta_{j,l} \lg_j^l\left(x^j_{\varphi(z)}\right)}{(\varphi(z) - z) \sum\limits_{l \neq j} f'_{j1}\left(\varphi(z), x_l\right)^2 \lg_j^l\left(x^j_{\varphi(z)}\right)}.$$

Using $1 = S_j\left(x^j_{\varphi(z)}\right)(\varphi(z) - z)$, we have

$$\varphi'(z) = -S_j\left(x^j_{\varphi(z)}\right) \frac{\lg_j^j\left(x^j_{\varphi(z)}\right) S_j^\Delta\left(x^j_{\varphi(z)}\right) + \sum_{l \neq j} f'_{j1}(\varphi(z), x_l)\Delta_{j,l}\left(x^j_{\varphi(z)}\right) \lg_j^l\left(x^j_{\varphi(z)}\right)}{\sum\limits_{l \neq j} f'_{j1}\left(\varphi(z), x_l\right)^2 \lg_j^l\left(x^j_{\varphi(z)}\right)}.$$

If we use the premium ratio function, we have $\Delta_{j,l}(.) = \varphi(z)/x_l > 0$ as well as $f'_{j1}(\varphi(z), x_l) > 0$. It is immediate that $\varphi'(z) < 0$. Otherwise when we use the premium difference function ($\Delta_{j,l}(.) = \varphi(z) - x_l$), we cannot guarantee that the numerator is positive.

If the solvency constraint $\tilde{g}^1_j$ is active, then we necessarily have $\lambda^\star_{j1} > 0$, $g^1_j(x^\star) = 0$. Let x^j_y be the premium vector with the j component being y as above. We denote by z a parameter of interest, then we define the function G as

$$G^j_x(z, y) = \tilde{g}^1_j(x^j_y, z),$$

where the objective function depends on the interest parameter z. Again, we apply the implicit function theorem with a function ϕ such that $G^j_x(z, \phi(z)) = 0$. The first-order derivative is given by

$$\frac{\partial G^j_x}{\partial y}(z, y) = \frac{\partial g^1_j}{\partial x_j}(z, y) > 0,$$

since $x_j \mapsto \tilde{g}^1_j$ is a strictly increasing function. Therefore, the sign of ϕ' is

$$\text{sign}(\phi'(z)) = -\text{sign}\left(\frac{\partial G^j_x}{\partial z}(z, \phi(z))\right).$$

Let $z = \pi_j$ be the actuarial premium. We have

$$\frac{\partial G^j_x}{\partial z}(z, y) = -\frac{n_j(1 - e_j)}{k_{995}\sigma(Y)\sqrt{\hat{N}_j(x^j_y)}} < 0,$$

independently of y or z. So, $\text{sign}(\phi'(z)) > 0$, i.e. the function $\pi_j \mapsto x^\star_j(\pi_j)$ is increasing as in the previous case.

Let $z = K_j$ be the capital. We have

$$\frac{\partial G^j_x}{\partial z}(z, y) = \frac{1}{k_{995}\sigma(Y)\sqrt{\hat{N}_j(x^j_y)}} > 0.$$

So $\text{sign}(\phi'(z)) < 0$, i.e. the function $K_j \mapsto x^\star_j(K_j)$ is decreasing.

Let $z = \sigma(Y)$ be the actuarial premium. We have

$$\frac{\partial G^j_x}{\partial z}(z, y) = -\frac{1}{z^2} \times \frac{K_j + n_j(y - \pi_j)(1 - e_j)}{k_{995}\sqrt{\hat{N}_j(x^j_y)}},$$

which simplifies to $\frac{\partial G^j_x}{\partial z}(z, \phi(z)) = -1/z < 0$ using the definition of G^j. Thus, the function $\sigma(Y) \mapsto x^\star_j(\sigma(Y))$ is decreasing. By a similar reasoning, we have for $z = k_{995}$, that ϕ is decreasing. □

4. Numerical illustration

All numerical applications are carried out with the R software, R Core Team (2012), cf. Appendix Appendix A.2 for computation details. We start with the one-period model of Section 2, referred to as the simple model. Then, we continue with the Section 3's model, referred to as the refined model.

4.1. The simple model

4.1.1. Base parameters

We consider a three-player game operating a 10 000-customer insurance market, i.e. $n = 10000$, $I = 3$. Insurer initial portfolio sizes are $(n_1, n_2, n_3) = (4500, 3200, 2300)$. The portfolio size is chosen such that player 1 is the leader, player 2 the challenger and player 3 the outsider with 45%, 32% and 23% market shares, respectively.

We consider two types of loss models: (i) loss $E(Y) = 1, \sigma(Y) = 4.472$, Poisson-Lognormal model, (ii) loss $E(Y) = 1, \sigma(Y) = 9.487$, Negative Binomial-Lognormal model. The loss history is such that the actuarially based premiums $\bar{a}_{j,0}$'s and the market premium $\bar{m}_0$ are given in Table 2.

	P1	P2	P3	market
PLN	1.129	1.227	1.029	1.190
NBLN	1.142	1.258	1.095	1.299

Table 2: Actuarially based premium $\bar{a}_{j,0}$ and market premium $\bar{m}_0$

The weight parameters ω_j used in the computation of the insurer break-even premium are $\omega = (1/3, 1/3, 1/3)$. Before giving the sensitivity parameters β_j, we present the lapse models. For customer behavior, we have two parameters μ_j, α_j per player given a price sensitivity function. At first, we consider the price function based on the premium ratio

$$\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j \frac{x_j}{x_l}.$$

The central lapse rate parameters (i.e. lapse rate when every insurers use the same premium) are set to 10%, 14% and 18% for $j = 1, 2$ or 3, respectively. In addition to this first constraint, we also impose that an increase of 5% compared to other players increases the total lapse rate by 5%. Let $x^1 = (1, 1, 1)$ and $x^{1.05} = (1.05, 1, 1)$. The two constraints are equivalent to

$$\mathrm{lg}_1^2(x^1) + \mathrm{lg}_1^3(x^1) = 10\% \quad \text{and} \quad \mathrm{lg}_1^2(x^{1.05}) + \mathrm{lg}_1^3(x^{1.05}) = 15\%$$

for Insurer 1. We get $\mu_1 = -12.14284$ and $\alpha_1 = 9.25247$. With this central lapse rate parameters, the expected numbers of lost policies when all insurers propose the same premium are 450.1, 448.0 and 414.0.

Secondly, we consider the price function based on the premium difference

$$\tilde{f}_j(x_j, x_l) = \tilde{\mu}_j + \tilde{\alpha}_j(x_j - x_l).$$

Calibration is done similarly as for f_j. In Figure 1, we plot the total lapse rate function of insurer j defined as

$$x_j \overset{r_j}{\mapsto} \sum_{l \neq j} \mathrm{lg}_j^l(x),$$

where $x_{-j} = 1.4$ for other player than j. On the left and right graphs, we plot total lapse rate functions r_j for the price functions $\bar{f}_j$ and $\tilde{f}_j$, respectively. In the central graph, we plot the total lapse rate function of player 1 with the two different price functions. The horizontal line corresponds to $x_{-j} = 1.4$ and its intersections with total lapse rate curves correspond to central lapse rates.

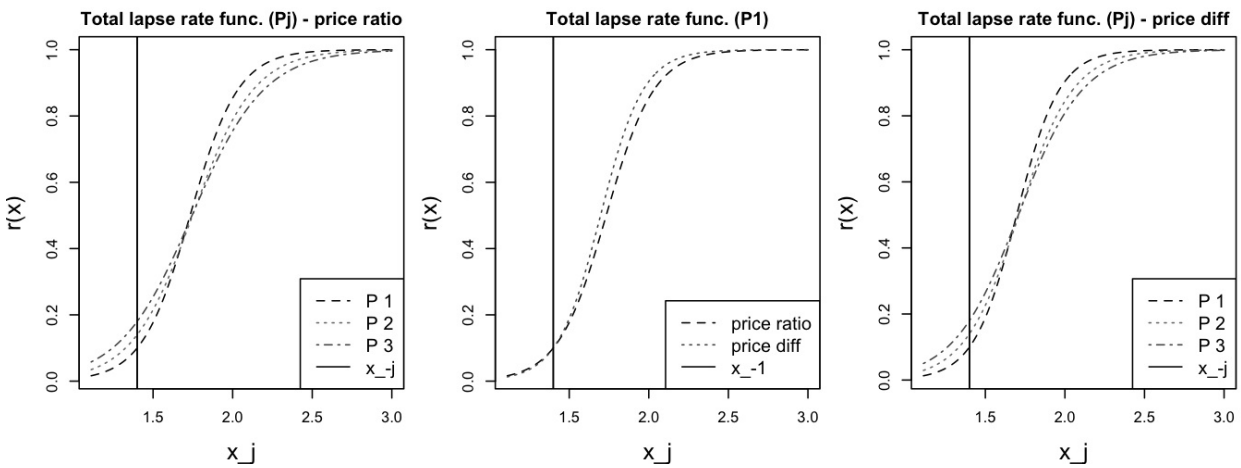


Figure 1: Total lapse rate functions

Price sensitivity parameters β_j of objective functions are fitted using $1-\beta_j\left(\frac{x_j}{m_j(x)} - 1\right) \approx \mathrm{lg}_j^j(x)$. With $x = (1.05, 1, 1)$, we get

$$\beta_j = \frac{1 - \mathrm{lg}_j^j(x)}{0.05}.$$

Using the premium ratio function $\bar{f}_j$, we have $(\beta_1, \beta_2, \beta_3) = (3.0, 3.8, 4.6)$.

The remaining parameters are capital values and the expense rates. Capital values (K_1, K_2, K_3) are set such that the solvency coverage ratio is 133%. Expense rates are $(e_1, e_2, e_3) = (15\%, 15\%, 15\%)$.

4.1.2. Base results

Since we consider two loss models (PLN, NBLN) and two price sensitivity functions $\bar{f}_j, \tilde{f}_j$ (denoted by 'ratio' and 'diff', respectively), we implicitly define four sets of parameters, which

differ on loss model and price sensitivity functions. In Table 3, we report premium equilibria of the four models (PLN-ratio, PLN-diff, NBLN-ratio and NBLN-diff), differences between equilibrium vector $x^\star$ and actuarial and average market premium, and expected difference in portfolio size ($\Delta\hat{N}_1$ negative means insurer 1 expects to lose customers).

	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
PLN-ratio	1.642	1.612	1.558	0.6914	0.5171	-258.7	-46.26	305
PLN-diff	1.600	1.577	1.527	0.5934	0.4316	-280.5	-63.11	343.6
NBLN-ratio	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
NBLN-diff	1.713	1.690	1.643	0.8119	0.443	-273.9	-55.78	329.7

Table 3: Base premium equilibrium

The premium equilibrium vector $x^\star$ is quite similar between the four different tested models. The change between price sensitivity functions $\bar{f}_j, \tilde{f}_j$ from an insurer point of view is a change in sensitivity parameter β_j in its objective function. The change between $\bar{f}_j, \tilde{f}_j$ results in a slight increase of premium equilibrium whereas the change between PLN or NBLN loss models is significantly higher. Unlike the sensitivity function change, the choice of loss model does not affect the objective function but the constraint function (an increase in $\sigma(Y)$).

4.1.3. Sensitivity analysis to parameters

In Tables 4 and 5, we perform a sensitivity analysis considering the NBLN-ratio model as the base model. Table 4 reports the analysis with respect to capital (K_3 decreases) and sensitvity parameter (β_j increases). Table 5 focuses on actuarially based premiums ($\bar{a}_{j,0}$ increases), average market premium ($\bar{m}_0$ increases) and credibility factors (ω_j increases). The results of this sensitivity analysis are in line with Proposition 2.2.

In Table 5, we also provide a comparison with a one-player optimization by testing the following objection function

$$O_j^F(x_j) = \frac{n_j}{n}\left(1 - \beta_j\left(\frac{x_j}{\overline{m}_0} - 1\right)\right)(x_j - \pi_j)\,.$$

This objective function O_j^F does not depend on other competitor premiums. Insurers optimize their premium x_j against the past market premium $\overline{m}_0$. We observe that the premium equilibrium $x^\star$ is well higher than "optimized" premiums $x^{F\star}$.

4.1.4. Loss uncertainty analysis

On Figure 2, we plot the histograms of insurer capitals at the game end for the base case, i.e. the NBLN loss with the price ratio function. Knowing that the initial capital values are $(K_1, K_2, K_3) = (2683, 2263, 1918)$, the premium equilibrium $x^\star = (1.758, 1.727, 1.676)$

	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
base	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
capital down	1.845	1.79	1.885	1.003	1.045	-70.3	225	-154.7
	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
base	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
sensitivity up	1.672	1.623	1.644	0.4655	0.4707	-174.2	119.9	54.35

Table 4: Sensitivity to capital and β_j parameters

	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
base	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
actuarial up	1.801	1.725	1.724	0.5672	0.752	-314.3	136.6	177.7
market up	1.933	1.876	1.873	1.224	0.6252	-223.4	83.93	139.5
	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
base	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
credibility up	1.758	1.702	1.718	0.6693	0.6794	-205.7	120.8	84.87
	$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$
base	1.758	1.727	1.676	0.9354	0.5362	-238.9	-37.78	276.7
follow	1.489	1.463	1.406	0.2587	0.07469	-278.1	-65.75	343.8

Table 5: Sensitivity to break-even premium

contains very high safety margins. From an insured point of view, $x^\star$ seems unfair compared to the actuarial premium $\overline{a} = (1.142, 1.258, 1.095)$. In practice, two scenarios seem natural: (i) some customers leave the market because they cannot afford such high premium and start a mutual fund or (ii) no customer notices this gap.

4.2. The refined model

Now, we consider the refined model of Subsection 3. We use the same set of parameters as in Subsection 4.1 for a 3-player game. As discussed above, a generalized premium equilibrium is not necessarily unique: in fact there are many of them. In Tables 6 and 7, we report generalized Nash equilibria found with different starting points (2^{10} feasible points randomly drawn in the hypercube $[\underline{x}, \overline{x}]^I$). The premium equilibria are sorted according to their difference to the average market premium $\bar{m}$.

In Table 7, this computation is done for the Negative Binomial-Lognormal loss model (NBLN), whereas Table 6 reports the computation for the Poisson-Lognormal model (PLN). Both tables use the price ratio function $\bar{f}_j$. The last column of those tables reports the number of optimization sequences converging to a given equilibrium. Most of the time, equilibriums found have one of their component hitting one of the barriers $\underline{x}, \overline{x}$. It may

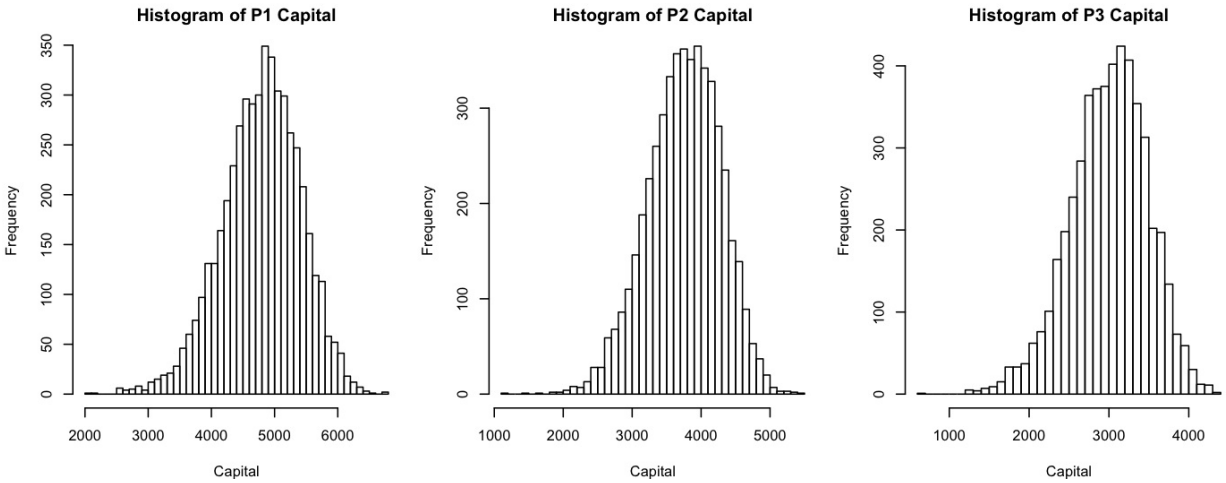


Figure 2: Histograms of capitals, sample size of 5000

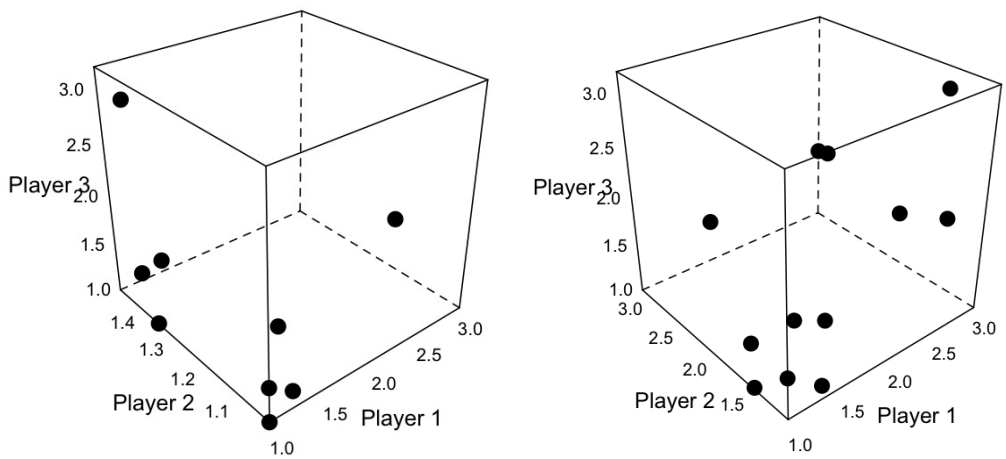
$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$	Nb
1	1	1	0.1132	0.0084	-19	-16	35	1
1.3041	1.025	1.0283	0.0497	0.0645	-2479	1264	1216	13
1	1.3183	1.0065	0.0964	0.0754	1001	-1899	898	4
1	1.0001	1.3427	0.1162	0.0896	722	701	-1423	3
1.0185	1.3694	1.3993	0.133	0.2215	2646	-1507	-1139	114
1.3856	1.0844	1.4501	0.1144	0.2696	-1729	2758	-1029	142
1.419	1.4541	1.1247	0.1121	0.3004	-1564	-1233	2797	111
1.0449	1.3931	3	3.4379	3.9075	3787	-1490	-2297	1
3	1.1738	1.5381	3.2767	4.0418	-4490	5412	-922	3

Table 6: Premium equilibria - PLN price ratio function

$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$	Nb
1.3644	1.0574	1.0661	0.1239	0.0611	-2635	1397	1239	10
1	1.3942	1.0208	0.1201	0.1003	1315	-2258	943	1
1	1.001	1.4206	0.1398	0.1192	851	818	-1670	3
1.0044	1.4216	1.4569	0.0887	0.1781	3333	-1923	-1411	109
1.4875	1.1726	1.5792	0.1836	0.2781	-1622	2696	-1075	116
1.555	1.6092	1.2508	0.2323	0.3598	-1369	-1210	2579	97
1.561	1.2526	3	3.0865	3.5394	-1405	3695	-2291	4
1.7346	3	1.4348	3.2546	3.7733	-955	-3174	4129	5
3	1.3699	1.7658	3.4794	3.7789	-4482	5299	-817	4
3	1.9041	1.5497	3.6941	4.0712	-4462	-743	5205	12
3	1.4664	3	6.226	6.8384	-4485	6746	-2261	2
3	3	1.7542	6.407	7.0956	-4354	-2970	7324	4

Table 7: Premium equilibria - NBLN price ratio function

appear awkward that such points are optimal in a sense, but one must not forget the Lagrange multipliers (not reported here). Those are not zero when a constraint g_j^i is active, (where $i = 1, 2, 3$ and $j = 1, \ldots, I$). Thus, it is not easy to understand the difference between two premium equilibria. On Figures 3a and 3b, we plot a 3-dimensional graph with premium equilibria of Tables 6 and 7. We observe that most premium equilibria are near the pure premium point $(E(Y), E(Y), E(Y)) = (1, 1, 1)$. Thus, these premium equilibria appear more fair than the premium equilibrium for the simple model of Subsection 4.1.



(a) PLN (b) NBLN

Figure 3: 3D plot of premium equilibria

Tables A.8 and A.9 in Appendix Appendix A.6 report the computation when we use the price difference function $\tilde{f}_j$. The number of different premium equilibria is similar as in the previous case, but most optimization sequences converge. For example for the PLN loss model, there were 392 converging sequences with the price ratio function $\bar{f}_j$ against 956 with the price difference function $\tilde{f}_j$. Premium equilibria seem easier to find for this price function.

This numerical application reveals that in our refined game, we have many generalized premium equilibria. In a game context, we can select one particular generalized equilibria for its properties or randomly draw one of them. In our insurance context, a possible way to deal with multiple equilibria is to choose as a premium equilibrium the generalized Nash equilibrium $x^\star$ that is closest to the average market premium $\bar{m}$. This option is motivated by the high level of competition present in most mature insurance markets (e.g. Europe and North America) where each insurer sets the premium with a view towards the market premium. However, this solution has drawbacks: while a single Nash equilibrium may be seen as a self-enforcing solution, multiple generalized Nash equilibria cannot be self-enforcing.

5. Conclusion

This paper assesses the suitability of noncooperative game theory of insurance market modelling. It extends the one-player models of Taylor (1986, 1987) and subsequent extensions based on optimal control theory. The game-theoretic approach proposed in this paper gives a first indicator of the effect of competition on the insurer solvency. The proposed game models a rational behavior of insurers in setting premiums taking into account other insurers. The ability of an insurer to sell contracts is essential for its survival. Unlike the classic risk theory where the collection of premiums is fixed per unit of time, the main source of risk for an insurance company in our game is a premium risk.

The game can be extended in various directions. A natural next step is to consider adverse selection among policyholders, since insurers do not propose the same premium to all customers. A second extension is to model investment results as well as loss reserves and reinsurance treaties. Furthermore, in practice, insurers play an insurance game over several years, gather new information on incurred losses, available capital and competition level. In addition from being dynamic, the market premium shows patterns of cycles with hard and soft phases, known as insurance market cycles, see, e.g., Feldblum (2001) for a recent survey. Hence, a dynamic game model for insurance markets to explain the occurence of market cycles could be of particular interest. This will be pursued in a future study.

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Appendix A. Appendix

Appendix A.1. Implicit function theorem

Below the implicit function theorem, see, e.g., (Zorich, 2000, Chap. 8).

Theorem. *Let F be a bivariate C^1 function on some open disk with center in (a, b), such that $F(a, b) = 0$. If $\frac{\partial F}{\partial y}(a, b) \neq 0$, then there exists an $h > 0$, and a unique function φ defined for $]a - h, a + h[$, such that*

$$\varphi(a) = b \quad \textit{and} \quad \forall |x - a| < h, F(x, \varphi(x)) = 0.$$

Moreover on $|x - a| < h$, the function φ is C^1 and

$$\varphi'(x) = - \left. \frac{\frac{\partial F}{\partial x}(x, y)}{\frac{\partial F}{\partial y}(x, y)} \right|_{y=\varphi(x)}.$$

Appendix A.2. Computation details

Computation is based on a Karush-Kuhn-Tucker (KKT) reformulation of the generalized Nash equilibrium problem (GNEP). We present briefly the problem reformulation and refer the interested readers to e.g. Facchinei and Kanzow (2009), Dreves et al. (2011) or Dutang (2012*a*). In our setting we have I players and three constraints for each player. For each j of the I subproblems, the KKT conditions are

$$\nabla_{x_j} O_j(x) - \sum_{1 \leq m \leq 3} \lambda_m^j \nabla_{x_j} g_j^m(x) = 0,$$
$$0 \leq \lambda^j \perp g_j(x) \geq 0.$$

The inequality part is called the complementarity constraint. The reformulation proposed uses a complementarity function $\phi(a,b)$ to reformulate the inequality constraints $\lambda^j, g_j(x) \geq 0$ and $\lambda^{jT} g_j(x) = 0$.

A point satisfying the KKT conditions is also a generalized Nash equilibrium if the objective functions are pseudoconcave and a constraint qualification holds. We have seen that objective functions are either strictly concave or pseudoconcave. Whereas constraint qualifications are always verified for linear constraints, or strictly monotone functions, see Theorem 2 of Arrow and Enthoven (1961), which is also verified.

By definition, a complementarity function is such that $\phi(a,b) = 0$ is equivalent to $a, b \geq 0$ and $ab = 0$. A typical example is $\phi(a,b) = \min(a,b)$ or $\phi(a,b) = \sqrt{a^2+b^2} - (a+b)$ called the Fischer-Burmeister function. With this tool, the KKT condition can be rewritten as

$$\begin{array}{c} \nabla_{x_j} L_j(x, \lambda^j) = 0 \\ \phi_{.}(\lambda^j, g_j(x)) = 0 \end{array},$$

where L_j is the Lagrangian function for the subproblem j and $\phi_{.}$ denotes the component wise version of ϕ. So, subproblem j reduces to solving a so-called nonsmooth equation. In this paper, we use the Fischer-Burmeister complementarity function. This method is implemented in the R package **GNE**[1].

Appendix A.3. Properties of multinomial logit function

We recall that the choice probability function is defined as

$$\mathrm{lg}_j^k(x) = \mathrm{lg}_j^j(x) \left(\delta_{jk} + (1-\delta_{jk}) e^{f_j(x_j, x_k)}\right),$$

and

$$\mathrm{lg}_j^j(x) = \frac{1}{1 + \sum\limits_{l \neq j} e^{f_j(x_j, x_l)}},$$

where the summation is over $l \in \{1, \ldots, I\} - \{j\}$ and f_j is the price function. The price function f_j goes from $(t,u) \in \mathbb{R}^2 \mapsto f_j(t,u) \in \mathbb{R}$. Partial derivatives are denoted by

$$\frac{\partial f_j(t,u)}{\partial t} = f'_{j1}(t,u) \quad \text{and} \quad \frac{\partial f_j(t,u)}{\partial u} = f'_{j2}(t,u).$$

Derivatives of higher order use the same notation principle.

The lg function has the good property to be infinitely differentiable. We have

$$\frac{\partial \, \mathrm{lg}_j^j(x)}{\partial x_i} = -\frac{\partial}{\partial x_i} \left(\sum_{l \neq j} e^{f_j(x_j, x_l)} \right) \frac{1}{\left(1 + \sum\limits_{l \neq j} e^{f_j(x_j, x_l)}\right)^2}.$$

[1] Dutang (2012*c*).

Since we have

$$\frac{\partial}{\partial x_i}\sum_{l\neq j} e^{f_j(x_j,x_l)} = \delta_{ji}\sum_{l\neq j} f'_{j1}(x_j,x_l)e^{f_j(x_j,x_l)} + (1-\delta_{ji})f'_{j2}(x_j,x_l)e^{f_j(x_j,x_i)},$$

we deduce

$$\frac{\partial \lg_j^j(x)}{\partial x_i} = -\delta_{ji}\sum_{l\neq j}\frac{f'_{j1}(x_j,x_l)e^{f_j(x_j,x_l)}}{\left(1+\sum\limits_{l\neq j} e^{f_j(x_j,x_l)}\right)^2} - (1-\delta_{ji})f'_{j2}(x_j,x_l)\frac{f'_{j1}(x_j,x_i)}{\left(1+\sum\limits_{l\neq j} e^{f'_{j1}(x_j,x_l)}\right)^2}.$$

This is equivalent to

$$\frac{\partial \lg_j^j(x)}{\partial x_i} = -\left(\sum_{l\neq j} f'_{j1}(x_j,x_l)\lg_j^l(x)\right)\lg_j^j(x)\delta_{ij} - f'_{j2}(x_j,x_l)\lg_j^i(x)\lg_j^j(x)(1-\delta_{ij}).$$

Furthermore,

$$\frac{\partial \lg_j^j(x)}{\partial x_i}\left(\delta_{jk}+(1-\delta_{jk})e^{f_j(x_j,x_k)}\right) = -\left(\sum_{l\neq j} f'_{j1}(x_j,x_l)\lg_j^l(x)\right)\lg_j^k(x)\delta_{ij} - f'_{j2}(x_j,x_i)\lg_j^i(x)\lg_j^k(x)(1-\delta_{ij}).$$

and also

$$\lg_j^j(x)\frac{\partial}{\partial x_i}\left(\delta_{jk}+(1-\delta_{jk})e^{f_j(x_j,x_k)}\right) = \lg_j^j(x)(1-\delta_{jk})\left(\delta_{ik}f'_{j2}(x_j,x_k)e^{f_j(x_j,x_k)} + \delta_{ij}f'_{j1}(x_j,x_k)e^{f_j(x_j,x_k)}\right).$$

Hence, we get

$$\begin{aligned}\frac{\partial \lg_j^k(x)}{\partial x_i} = -\delta_{ij}\left(\sum_{l\neq j} f'_{j1}(x_j,x_l)\lg_j^l(x)\right)\lg_j^k(x) - (1-\delta_{ij})f'_{j2}(x_j,x_i)\lg_j^i(x)\lg_j^k(x)\\ + (1-\delta_{jk})\left[\delta_{ij}f'_{j1}(x_j,x_k)\lg_j^k(x) + \delta_{ik}f'_{j2}(x_j,x_k)\lg_j^k(x)\right].\end{aligned}$$

Similarly, the second order derivative is given by[1]

$$\frac{\partial^2 \lg_j^k(x)}{\partial x_m \partial x_i} = -\delta_{ij}\left(\delta_{jm}\sum_{l\neq j} f''_{j11}(x_j,x_l)\lg_j^l + (1-\delta_{jm}) f''_{j12}(x_j,x_m)\lg_j^m + \sum_{l\neq j} f'_{j1}(x_j,x_l)\frac{\partial \lg_j^l}{\partial x_m}\right)\lg_j^k$$
$$-\delta_{ij}\left(\sum_{l\neq j} f'_{j1}(x_j,x_l)\lg_j^l\right)\frac{\partial \lg_j^k}{\partial x_m}$$
$$-(1-\delta_{ij})\left(\left(\delta_{jm} f''_{j21}(x_j,x_i) + \delta_{im} f''_{j22}(x_j,x_i)\right)\lg_j^i \lg_j^k + f'_{j2}(x_j,x_i)\frac{\partial \lg_j^i}{\partial x_m}\lg_j^k + f'_{j2}(x_j,x_i)\lg_j^i \frac{\partial \lg_j^k}{\partial x_m}\right)$$
$$+(1-\delta_{jk})\delta_{ij}\left(\left(f''_{j11}(x_j,x_k)\delta_{jm} + f''_{j12}(x_j,x_k)\delta_{km}\right)\lg_j^k + f'_{j1}(x_j,x_k)\frac{\partial \lg_j^k}{\partial x_m}\right)$$
$$+(1-\delta_{jk})\delta_{ik}\left(\left(f''_{j21}(x_j,x_k)\delta_{jm} + f''_{j22}(x_j,x_k)\delta_{im}\right)\lg_j^k + f'_{j2}(x_j,x_k)\frac{\partial \lg_j^k}{\partial x_m}\right).$$

Appendix A.4. Portfolio size function

We recall that the expected portfolio size of insurer j is defined as

$$\hat{N}_j(x) = n_j \times \lg_j^j(x) + \sum_{l\neq j} n_l \times \lg_l^j(x),$$

where n_j's denotes last year portfolio size of insurer j and $\lg_j^k$ is defined in equation (1).

The function $\phi_j : x_j \mapsto \lg_j^j(x)$ has the following derivative

$$\phi'_j(x_j) = \frac{\partial \lg_j^j(x)}{\partial x_j} = -\left(\sum_{l\neq j} f'_{j1}(x_j,x_l)\lg_j^l(x)\right)\lg_j^j(x).$$

For the two considered price function, we have

$$f'_{j1}(x_j,x_l) = \alpha_j \frac{1}{x_l} \quad \text{and} \quad \tilde{f}'_{j1}(x_j,x_l) = \tilde{\alpha}_j,$$

which are positive. So, the function ϕ_j will be a decreasing function.

For $l \neq j$, the function $\phi_l : x_j \mapsto \lg_l^j(x)$ has the following derivative

$$\phi'_l(x_j) = \frac{\partial \lg_l^j(x)}{\partial x_j} = -f'_{j2}(x_l,x_j)\lg_l^j(x)\lg_l^j(x) + f'_{j2}(x_l,x_j)\lg_l^j(x) = f'_{j2}(x_l,x_j)\lg_l^j(x)(1-\lg_l^j(x)).$$

For the two considered price function, we have

$$f'_{j2}(x_j,x_l) = -\alpha_j \frac{x_j}{x_l^2} \quad \text{and} \quad \tilde{f}'_{j2}(x_j,x_l) = -\tilde{\alpha}_j,$$

[1] We remove the variable x when possible.

which are negative. So, the function ϕ_l will also be a decreasing function.

Therefore the portfolio size $x_j \mapsto \hat{N}_j(x)$ function has the following derivative

$$\frac{\partial \hat{N}_j(x)}{\partial x_j} = -n_j \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l(x) \right) \lg_j^j(x) + \sum_{l \neq j} n_l f'_{j2}(x_l, x_j) \lg_l^j(x)(1 - \lg_l^j(x)).$$

Hence, it is decreasing from the total market size $\sum_l n_l$ to 0. So the function $x_j \mapsto \hat{N}_j$ is both a quasiconcave and a quasiconvex function.

Therefore, using the C^2 characterization of quasiconcave and quasiconvex functions, we have that

$$\frac{\partial \hat{N}_j(x)}{\partial x_j} = 0 \Rightarrow \frac{\partial^2 \hat{N}_j(x)}{\partial x_j^2} = 0.$$

Note that the function $\hat{N}_j(x)$ is horizontal (i.e. has gradient of 0) when $x_j \to 0$ and $x_j \to +\infty$ for fixed x_{-j}.

Finally, we also need

$$\frac{\partial^2 \lg_j^j(x)}{\partial x_j^2} = -\sum_{l \neq j} f'_{j1}(x_j, x_l) \frac{\partial \lg_j^l}{\partial x_j} \lg_j^j - \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l \right) \frac{\partial \lg_j^j}{\partial x_j},$$

as f''_{j11} is 0 for the two considered functions. Since,

$$\left. \frac{\partial \lg_j^l}{\partial x_j} \right|_{l \neq j} = -\lg_j^l \sum_{n \neq j} f'_{j1}(x_j, x_n) \lg_j^n + \lg_j^l f'_{j1}(x_j, x_l) \quad \text{and} \quad \frac{\partial \lg_j^j}{\partial x_j} = -\left(\sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l \right) \lg_j^j,$$

then we get

$$\frac{\partial^2 \lg_j^j(x)}{\partial x_j^2} = -\lg_j^j \sum_{l \neq j} \left(f'_{j1}(x_j, x_l) \right)^2 \lg_j^l + 2 \left(\sum_{l \neq j} f'_{j1}(x_j, x_l) \lg_j^l \right)^2 \lg_j^j .$$

Appendix A.5. Convexity concepts

Appendix A.6. Numerical applications for the refined one-period game

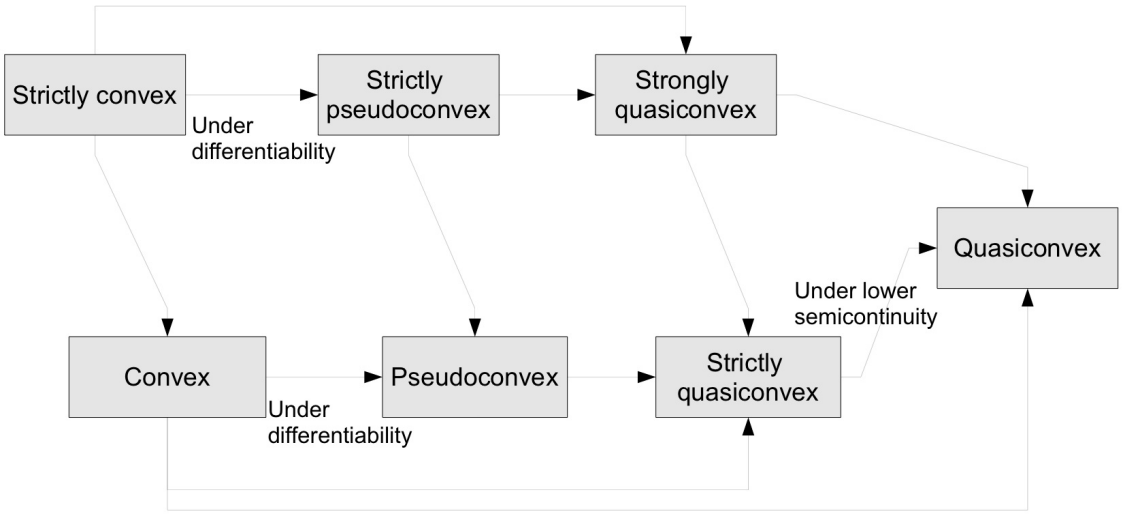

Figure A.4: Convexity and generalized convexity, from Bazaraa et al. (2006)

$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$	Nb
3	1.0705	1.085	3.1581	3.793	-4500	2390	2110	1
1.3535	1.0236	1.0273	0.0604	0.092	-2430	1231	1200	17
1.4282	1.084	1.4974	0.1587	0.3395	-1707	2746	-1039	108
1.4508	1.4873	1.1251	0.1444	0.3524	-1561	-1243	2804	117
3	2.7871	2.7716	8.2894	9.7537	-1494	652	842	30
1.4514	1.1173	3	3.4138	3.9547	-1379	3678	-2298	2
3	3	2.8892	9.4096	10.9558	-352	-272	623	3
2.3218	2.3114	2.2994	3.7533	4.7485	-104	-12	116	511
3	3	3	9.8033	11.3748	-19	-16	35	1
2.8669	2.8507	3	8.8286	10.3147	205	314	-519	13
2.8287	3	2.798	8.526	9.9912	314	-900	586	15
1	1	1.4139	0.1463	0.136	714	698	-1412	6
1	1.3819	1.0065	0.1178	0.1133	966	-1865	899	11
1.0168	1.4362	1.4698	0.2009	0.3221	2576	-1460	-1117	118
1.0396	1.4538	3	3.4692	3.9526	3553	-1254	-2299	3

Table A.8: Premium equilibria - PLN with price difference function

$x_1^\star$	$x_2^\star$	$x_3^\star$	$\|\|x^\star - \bar{a}\|\|^2$	$\|\|x^\star - \bar{m}_0\|\|^2$	$\Delta\hat{N}_1$	$\Delta\hat{N}_2$	$\Delta\hat{N}_3$	Nb
3	1.7906	1.5134	3.5479	3.8892	-4498	-357	4856	9
3	3	1.7839	6.436	7.1316	-4487	-3167	7654	1
1.4048	1.0537	1.0642	0.1413	0.0802	-2583	1357	1226	11
1.507	1.1725	1.5968	0.2059	0.3061	-1609	2695	-1086	128
1.5527	1.6031	1.2504	0.2267	0.3525	-1359	-1217	2576	106
3	2.8239	2.8087	7.9441	8.838	-1191	509	683	38
1.5495	1.2359	3	3.0797	3.5277	-1182	3479	-2297	2
1.6611	3	1.3993	3.172	3.6768	-616	-3197	3813	6
3	3	2.9022	8.812	9.7701	-305	-238	543	2
3	2.9326	3	8.912	9.877	-219	321	-102	6
2.4177	2.4107	2.3994	4.0024	4.6571	-83	-21	104	542
2.8877	2.874	3	8.3306	9.273	175	264	-439	11
2.8559	3	2.8279	8.0789	9.0088	259	-759	500	20
1	1.0214	3	3.065	3.4201	1277	1023	-2299	1
1	1.4896	1.5299	0.1449	0.2675	3127	-1790	-1338	56

Table A.9: Premium equilibria - NBLN with price difference function