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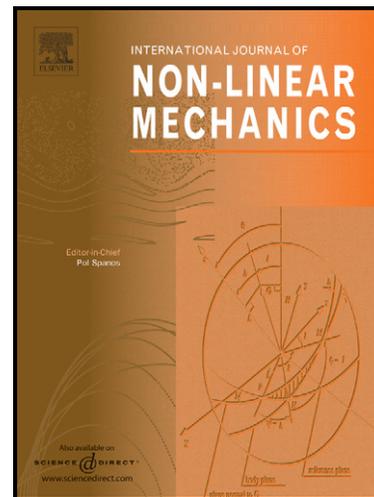
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**NONLINEAR FLEXURAL - TORSIONAL DYNAMIC
ANALYSIS OF BEAMS OF ARBITRARY CROSS
SECTION BY BEM**

by

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Key words: Flexural-Torsional Analysis; Dynamic analysis; Wagner's coefficients; Nonlinear analysis; Shortening Effect; Boundary element method

Abstract

In this paper, a boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary, simply or multiply connected, constant cross section, undergoing moderately large deflections and twisting rotations under general boundary conditions, taking into account the effects of rotary and warping inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading in both directions as well as to twisting

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and/or axial loading. Four boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the angle of twist and solved using the Analog Equation Method, a BEM based method. Application of the boundary element technique leads to a system of nonlinear coupled Differential – Algebraic Equations (DAE) of motion, which is solved iteratively using the Petzold-Gear Backward Differentiation Formula (BDF), a linear multistep method for differential equations coupled to algebraic equations. The geometric, inertia, torsion and warping constants are evaluated employing the Boundary Element Method. The proposed model takes into account, both the Wagner’s coefficients and the shortening effect. Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy, the range of applications of the developed method as well as the influence of the nonlinear effects to the response of the beam.

1. INTRODUCTION

In engineering practice the dynamic analysis of beam-like continuous systems is frequently encountered. Such structures often undergo arbitrary external loading, leading to the formulation of the flexural-torsional vibration problem. The complexity of this problem increases significantly in the case the cross section’s centroid does not coincide with its shear center (asymmetric beams). Furthermore when arbitrary torsional boundary conditions are applied either at the edges or at any other interior point of a beam due to construction requirements, the beam under the action of general twisting loading is leaded to nonuniform torsion. Moreover, since requirement of weight saving is a major aspect in the design of structures, thin-walled elements of arbitrary cross section and low flexural and/or torsional stiffness are extensively used. Treating displacements and angles of rotation of these elements as being small, leads

in many cases to inadequate prediction of the dynamic response; hence the occurring nonlinear effects should be taken into account. This can be achieved by retaining the nonlinear terms in the strain–displacement relations (finite displacement – small strain theory). When finite displacements are considered, the flexural-torsional dynamic analysis of bars becomes much more complicated, leading to the formulation of coupled and nonlinear flexural, torsional and axial equations of motion.

When the displacement components of a member are small, a wide range of linear analysis tools, such as modal analysis, can be used and some analytical results are possible. As these components become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. In such situations the possibility of an analytical solution method is significantly reduced and is restricted to special cases of beam boundary conditions or loading.

During the past few years, the nonlinear dynamic analysis of beams undergoing large deflections has received a good amount of attention in the literature. More specifically, Rozmarynowski and Szymczak in [1] studied the nonlinear free torsional vibrations of axially immovable thin-walled beams with doubly symmetric open cross section, employing the Finite Element Method. In this research only free vibrations are examined; the solution is provided only at points of reversal of motion (not in the time domain), no general axial, torsional or warping boundary conditions (elastic support case) are studied, while some nonlinear terms related to the finite twisting rotations as well as the axial inertia term are ignored. Crespo Da Silva in [2-3] presented the nonlinear flexural-torsional-extensional vibrations of Euler-Bernoulli doubly symmetric thin-walled closed cross section beams, primarily focusing on flexural vibrations and neglecting the effect of torsional warping. Pai and Nayfeh in [4-6] studied also the nonlinear flexural-torsional-extensional vibrations of metallic

and composite slewing or rotating closed cross section beams, primarily focusing to flexural vibrations and neglecting again the effect of torsional warping. Simo and Vu-Quoc in [7] presented a FEM solution to a fully nonlinear (small or large strains, hyperelastic material) three dimensional rod model including the effects of transverse shear and torsion-warping deformation based on a geometrically exact description of the kinematics of deformation. Qaisi in [8] obtained nonlinear normal modes of free vibrating geometrically nonlinear beams of various edge conditions employing the harmonic balance analytical method. Moreover, Pai and Nayfeh in [9] studied a geometrically exact nonlinear curved beam model for solid composite rotor blades using the concept of local engineering stress and strain measures and taking into account the in-plane and out-of-plane warpings. Di Egidio et al. in [10-11] presented also a FEM solution to the nonlinear flexural-torsional vibrations of shear undeformable thin-walled open beams taking into account in-plane and out-of-plane warpings and neglecting warping inertia. In this paper, the torsional-extensional coupling is taken into account but the inextensionality assumption leads to the fact that the axial boundary conditions are not general. Mohri et al. in [12] proposed a FEM solution to the linear vibration problem of pre- and post- buckled thin-walled open cross section beams, neglecting warping and axial inertia, considering geometrical nonlinearity only for the static loading and presenting examples of bars subjected to free vibrations and special boundary conditions. Machado and Cortinez in [13] presented also a FEM solution to the linear free vibration analysis of bisymmetric thin-walled composite beams of open cross section, taking into account initial stresses and deformations considering geometrical nonlinearity only for the static loading and presenting examples of bars subjected to special boundary conditions. Avramov et al. [14-15] studied the free flexural-torsional vibrations of

beams and obtained nonlinear normal modes by expansion of the equations of motion employing the Galerkin technique and neglecting the cross section warping. Lopes and Ribeiro [16] studied also the nonlinear flexural-torsional free vibrations of beams employing a FEM solution and neglecting the longitudinal and rotary inertia as well as the cross-section warping. Duan [17] presented a FEM formulation for the nonlinear free vibration problem of thin-walled curved beams of asymmetric cross-section based on a simplified displacement field. Finally, the boundary element method has also been used for the nonlinear flexural [18-20] and torsional [21] dynamic analysis of only doubly symmetric beams. To the authors' knowledge the general problem of coupled nonlinear flexural – torsional free or forced vibrations of asymmetric beams has not yet been presented.

In this paper, a boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary, simply or multiply connected, constant cross section, undergoing moderately large deflections and twisting rotations under general boundary conditions, taking into account the effects of rotary and warping inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading in both directions as well as to twisting and/or axial loading. Four boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the angle of twist and solved using the Analog Equation Method [22], a BEM based method. Application of the boundary element technique leads to a system of nonlinear coupled Differential – Algebraic Equations (DAE) of motion, which is solved iteratively using the Petzold-Gear Backward Differentiation Formula (BDF) [23], a linear multistep method for differential equations coupled to algebraic equations. The geometric, inertia, torsion and warping constants are evaluated employing the Boundary Element

Method. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

- i. The cross section is an arbitrarily shaped thin- or thick-walled one. The formulation does not stand on the assumption of a thin-walled structure and therefore the cross section's torsional and warping rigidities are evaluated "exactly" in a numerical sense.
- ii. The beam is subjected to arbitrarily distributed or concentrated transverse loading in both directions as well as to twisting and axial loading.
- iii. The beam is supported by the most general boundary conditions including elastic support or restraint.
- iv. The effects of rotary and warping inertia are taken into account on the nonlinear flexural-torsional dynamic analysis of asymmetric beams subjected to arbitrary loading and boundary conditions.
- v. The transverse loading can be applied at any arbitrary point of the beam cross section. The eccentricity change of the transverse loading during the torsional beam motion, resulting in additional torsional moment is taken into account.
- vi. The proposed model takes into account the coupling effects of bending, axial and torsional response of the beam as well as the Wagner's coefficients and the shortening effect.
- vii. The proposed method employs a BEM approach (requiring boundary discretization for the cross sectional analysis) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy, the range of applications of the developed method as well as the influence of the nonlinear effects to the response of the beam.

2. STATEMENT OF THE PROBLEM

Let us consider a prismatic beam of length l (Fig.1), of constant arbitrary cross section of area A . The homogeneous isotropic and linearly elastic material of the beam's cross section, with modulus of elasticity E , shear modulus G and Poisson's ratio ν , occupies the two dimensional multiply connected region Ω of the y, z plane and is bounded by the $\Gamma_j (j = 1, 2, \dots, K)$ boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1 CYZ is the principal bending coordinate system through the cross section's centroid C , while y_C, z_C are its coordinates with respect to the Syz shear system of axes through the cross section's shear center S , with axes parallel to those of the CYZ system. The beam is subjected to the combined action of the arbitrarily distributed or concentrated, time dependent and conservative axial loading $p_X = p_X(X, t)$ along X direction, twisting moment M along x direction and transverse loading $p_y = p_y(x, t)$, $p_z = p_z(x, t)$ acting along the y and z directions, applied at distances y_{p_y}, z_{p_y} and y_{p_z}, z_{p_z} , with respect to the Syz shear system of axes, respectively (Fig. 1b).

Under the action of the aforementioned loading, the displacement field of an arbitrary point of the cross section can be derived with respect to those of the shear center as [12]

$$\bar{u}(x, y, z, t) = u(x, t) - (y - y_C)\theta_Z(x, t) + (z - z_C)\theta_Y(x, t) + \theta_x'(x, t)\varphi_S^P(y, z) \quad (1a)$$

$$\bar{v}(x, y, z, t) = v(x, t) - z \sin(\theta_x(x, t)) - y [1 - \cos(\theta_x(x, t))] \quad (1b)$$

$$\bar{w}(x, y, z, t) = w(x, t) + y \sin(\theta_x(x, t)) - z [1 - \cos(\theta_x(x, t))] \quad (1c)$$

$$\theta_Y(x, t) = v'(x, t) \sin(\theta_x(x, t)) - w'(x, t) \cos(\theta_x(x, t)) \quad (1d)$$

$$\theta_Z(x, t) = v'(x, t) \cos(\theta_x(x, t)) + w'(x, t) \sin(\theta_x(x, t)) \quad (1e)$$

where \bar{u} , \bar{v} , \bar{w} are the axial and transverse beam displacement components with respect to the S_{yz} shear system of axes; $u(x, t) = \frac{I}{A} \int_A \bar{u}(x, y, z, t) dA$ denotes the average axial displacement of the cross section [24] and $v = v(x, t)$, $w = w(x, t)$ are the corresponding components of the shear center S ; $\theta_Y(x, t)$, $\theta_Z(x, t)$ are the angles of rotation of the cross section due to bending, with respect to its centroid; $\theta_x'(x, t)$ denotes the rate of change of the angle of twist $\theta_x(x, t)$ regarded as the torsional curvature and φ_S^P is the primary warping function with respect to the shear center S [25].

Employing the strain-displacement relations of the three-dimensional elasticity for moderate displacements [26, 27]

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial \bar{u}}{\partial x} \right)^2 + \left(\frac{\partial \bar{v}}{\partial x} \right)^2 + \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \right] \quad (2a)$$

$$\gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \left(\frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial y} \right) \quad (2b)$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \left(\frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial z} \right) \quad (2c)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \quad (2d)$$

the following strain components can be easily obtained

$$\varepsilon_{xx} \approx \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial \bar{v}}{\partial x} \right)^2 + \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \right] \quad (3a)$$

$$\gamma_{xy} \approx \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \left(\frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial y} \right) \quad (3b)$$

$$\gamma_{xz} \approx \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \left(\frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \frac{\partial \bar{w}}{\partial z} \right) \quad (3c)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \quad (3d)$$

where it has been assumed that for moderate displacements $\left(\frac{\partial \bar{u}}{\partial x} \right)^2 \ll \left(\frac{\partial \bar{u}}{\partial x} \right)$,

$\left(\frac{\partial \bar{u}}{\partial x} \right) \left(\frac{\partial \bar{u}}{\partial y} \right) \ll \left(\frac{\partial \bar{u}}{\partial x} \right) + \left(\frac{\partial \bar{u}}{\partial y} \right)$. Substituting $\left(\frac{\partial \bar{u}}{\partial x} \right) \left(\frac{\partial \bar{u}}{\partial z} \right) \ll \left(\frac{\partial \bar{u}}{\partial x} \right) + \left(\frac{\partial \bar{u}}{\partial z} \right)$ into

the displacement components (1a-1e) to the strain-displacement relations (3), the

strain components can be written as

$$\begin{aligned} \varepsilon_{xx} = & u' + (z - z_C)(v'' \sin \theta_x - w'' \cos \theta_x) - (y - y_C)(v'' \cos \theta_x + w'' \sin \theta_x) + \\ & + \theta_x'' \phi_S^P - \theta_x' \left(z_C (v' \cos \theta_x + w' \sin \theta_x) + y_C (v' \sin \theta_x - w' \cos \theta_x) \right) + \\ & + \frac{1}{2} \left(v'^2 + w'^2 + (y^2 + z^2) (\theta_x')^2 \right) \end{aligned} \quad (4a)$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{\partial \phi_S^P}{\partial y} - z \right) \theta_x' \quad (4b)$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \left(\frac{\partial \phi_S^P}{\partial z} + y \right) \theta_x' \quad (4c)$$

Considering strains to be small and employing the second Piola – Kirchhoff stress tensor, the non vanishing stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix} = \begin{bmatrix} E^* & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \quad (5)$$

where E^* is obtained from Hooke's stress-strain law as $E^* = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the

assumption of plane stress condition is made, the above expression is reduced in

$E^* = \frac{E}{1-\nu^2}$ [28], while in beam formulations, E is frequently considered instead of

E^* ($E^* \approx E$) [28, 29]. This last consideration has been followed throughout the

paper, while any other reasonable expression of E^* could also be used without any

difficulty. Substituting eqns. (4) into eqns. (5), the non vanishing stress components

are obtained as

$$\begin{aligned} S_{xx} = E & \left[u' + (z - z_C)(v'' \sin \theta_x - w'' \cos \theta_x) - (y - y_C)(v'' \cos \theta_x + w'' \sin \theta_x) + \right. \\ & + \theta_x'' \phi_S^P - \theta_x' (z_C (v' \cos \theta_x + w' \sin \theta_x) + y_C (v' \sin \theta_x - w' \cos \theta_x)) + \\ & \left. + \frac{1}{2} (v'^2 + w'^2 + (y^2 + z^2)(\theta_x')^2) \right] \end{aligned} \quad (6a)$$

$$S_{xy} = G \cdot \theta_x' \cdot \left(\frac{\partial \phi_S^P}{\partial y} - z \right) \quad (6b)$$

$$S_{xz} = G \cdot \theta_x' \cdot \left(\frac{\partial \phi_S^P}{\partial z} + y \right) \quad (6c)$$

In order to establish the nonlinear equations of motion, the principle of virtual work

$$\delta W_{int} + \delta W_{mass} = \delta W_{ext} \quad (7)$$

where

$$\delta W_{int} = \int_V (S_{xx} \delta \varepsilon_{xx} + S_{xy} \delta \gamma_{xy} + S_{xz} \delta \gamma_{xz}) dV \quad (8a)$$

$$\delta W_{mass} = \int_V \rho (\ddot{u} \delta \bar{u} + \ddot{v} \delta \bar{v} + \ddot{w} \delta \bar{w}) dV \quad (8b)$$

$$\delta W_{ext} = \int_l (p_x \delta u_C + p_y \delta \tilde{v}_{p_y} + p_z \delta \tilde{v}_{p_z} + m_x \delta \theta_x) dx \quad (8c)$$

under a Total Lagrangian formulation, is employed. In the above equations, $\delta(\cdot)$ denotes virtual quantities, $(\dot{\cdot})$ denotes differentiation with respect to time, V is the volume of the beam, u_C is the axial displacement of the centroid and \tilde{v}_{p_y} , \tilde{v}_{p_z} are the transverse displacements of the points where the loads p_y , p_z , respectively, are applied. It is worth here noting that the aforementioned expression of the external work (eqn. (8c)) takes into account the change of the eccentricity of the external conservative transverse loading, arising from the cross section torsional rotation, inducing additional (positive or negative) torsional moment. This effect may influence substantially the torsional response of the beam. Moreover, the stress resultants of the beam can be defined as

$$N = \int_{\Omega} S_{xx} d\Omega \quad (9a)$$

$$M_Y = \int_{\Omega} S_{xx} Z d\Omega \quad (9b)$$

$$M_Z = -\int_{\Omega} S_{xx} Y d\Omega \quad (9c)$$

$$M_t = \int_{\Omega} \left[S_{xy} \left(\frac{\partial \phi_S^P}{\partial y} - z \right) + S_{xz} \left(\frac{\partial \phi_S^P}{\partial z} + y \right) \right] d\Omega \quad (9d)$$

$$M_w = -\int_{\Omega} S_{xx} \phi_S^P d\Omega \quad (9e)$$

$$M_R = \int_{\Omega} S_{xx} (y^2 + z^2) d\Omega \quad (9f)$$

where M_t is the primary twisting moment [25] resulting from the primary shear stress distribution S_{xy} , S_{xz} , M_w is the warping moment due to torsional curvature and M_R is a higher order stress resultant. Substituting the expressions of the stress components (6) into equations (9a-9f), the stress resultants are obtained as

$$N = EA \left[u' + \frac{I}{2} (v'^2 + w'^2 + \frac{I_S}{A} \theta_x'^2) - \theta_x' \left(z_C (v' \cos \theta_x + w' \sin \theta_x) + y_C (v' \sin \theta_x - w' \cos \theta_x) \right) \right] \quad (10a)$$

$$M_Y = -EI_Y (w'' \cos \theta_x - v'' \sin \theta_x - \beta_Z \theta_x'^2) \quad (10b)$$

$$M_Z = EI_Z (v'' \cos \theta_x + w'' \sin \theta_x - \beta_Y \theta_x'^2) \quad (10c)$$

$$M_t = GI_t \theta_x' \quad (10d)$$

$$M_w = -EC_S (\theta_x'' + \beta_{\omega} \theta_x'^2) \quad (10e)$$

$$\begin{aligned}
M_R = N \frac{I_S}{A} - 2EI_Z \beta_Y (v'' \cos \theta_x + w'' \sin \theta_x) - 2EI_Y \beta_Z (w'' \cos \theta_x - v'' \sin \theta_x) + \\
+ 2EC_S \beta_\omega \theta_x'' + \frac{1}{2} E \left(I_R - \frac{I_S^2}{A} \right) \theta_x'^2
\end{aligned} \tag{10f}$$

where the area A , the polar moment of inertia I_S with respect to the shear center S , the principal moments of inertia I_Y , I_Z with respect to the cross section's centroid, the fourth moment of inertia I_R with respect to the shear center S , the torsion constant I_t and the warping constant C_S with respect to the shear center S , are given as

$$A = \int_{\Omega} d\Omega \tag{11a}$$

$$I_S = \int_{\Omega} (y^2 + z^2) d\Omega \tag{11b}$$

$$I_Y = \int_{\Omega} Z^2 d\Omega \tag{11c}$$

$$I_Z = \int_{\Omega} Y^2 d\Omega \tag{11d}$$

$$I_R = \int_{\Omega} (y^2 + z^2)^2 d\Omega \tag{11e}$$

$$C_S = \int_{\Omega} (\varphi_S^P)^2 d\Omega \tag{11f}$$

$$I_t = \int_{\Omega} \left(y^2 + z^2 + y \frac{\partial \varphi_S^P}{\partial z} - z \frac{\partial \varphi_S^P}{\partial y} \right) d\Omega \tag{11g}$$

while the Wagner's coefficients β_Z , β_Y and β_ω are given as

$$\beta_Z = \frac{1}{2I_Y} \int_{\Omega} (z - z_C) (y^2 + z^2) d\Omega \tag{12a}$$

$$\beta_Y = \frac{I}{2I_Z} \int_{\Omega} (y - y_C)(y^2 + z^2) d\Omega \quad (12b)$$

$$\beta_{\omega} = \frac{I}{2C_S} \int_{\Omega} (y^2 + z^2) \varphi_S^P d\Omega \quad (12c)$$

Using the expressions of strain obtained in equations (4), the definitions of the stress resultants given in equations (9) and applying the principle of virtual work (eqn. (7)), the equations of motion of the beam can be derived as

$$-N' + \rho A \ddot{u} = p_x \quad (13a)$$

$$\begin{aligned} & (-NF_{v1})' + (M_Z \cos \theta_x)'' + (M_Y \sin \theta_x)'' + \rho A \ddot{v} - \rho F_{v2} \ddot{\theta}_x + \rho F_{v3} \dot{\theta}_x^2 - (\rho F_{v4} \ddot{\theta}_x)' - \\ & - (\rho F_{v5} \dot{\theta}_x^2)' - [\rho F_{v6} (2\dot{\theta}_x \dot{v}' - \ddot{w}')] + [\rho F_{v7} (2\dot{\theta}_x \dot{w}' + \dot{v}')] = p_y \end{aligned} \quad (13b)$$

$$\begin{aligned} & (-NF_{w1})' - (M_Y \cos \theta_x)'' + (M_Z \sin \theta_x)'' + \rho A \ddot{w} - \rho F_{w2} \ddot{\theta}_x - \rho F_{w3} \dot{\theta}_x^2 - (\rho F_{w4} \ddot{\theta}_x)' + \\ & + (\rho F_{w5} \dot{\theta}_x^2)' - [\rho F_{w6} (2\dot{\theta}_x \dot{w}' + \dot{v}')] + [\rho F_{w7} (\ddot{w}' - 2\dot{\theta}_x \dot{v}')] = p_z \end{aligned} \quad (13c)$$

$$\begin{aligned} & NF_{\theta 1} + (NF_{\theta 2})' + (NF_{\theta 3})' + M_Y F_{\theta 4} + M_Z F_{\theta 5} - M_t' - M_w'' - (M_R \theta_x')' - \\ & - \rho (F_{\theta 6} \ddot{v} + F_{\theta 7} \ddot{w}) + \rho F_{\theta 8} \ddot{\theta}_x + \rho F_{\theta 9} (\dot{v}' + 2\dot{\theta}_x \dot{w}') + \rho F_{\theta 10} (\ddot{w}' - 2\dot{\theta}_x \dot{v}') - \rho F_{\theta 11} \dot{\theta}_x^2 - \\ & - (\rho C_S \ddot{\theta}_x')' = m_x + p_z y_{p_z} \cos \theta_x - p_y z_{p_y} \cos \theta_x - p_z z_{p_z} \sin \theta_x - p_y y_{p_y} \sin \theta_x \end{aligned} \quad (13d)$$

where the expressions of the stress resultants are given from equations (10) and F_{vi} ($i=1,2,\dots,7$), F_{wi} ($i=1,2,\dots,7$) and $F_{\theta i}$ ($i=1,2,\dots,11$) are functions of v , w , θ_x and their derivatives with respect to x , given in the Appendix A. Equations (13) are coupled and highly complicated. This set of equations can be simplified if the approximate expressions [12]

$$\cos \theta_x \approx 1 - \frac{\theta_x^2}{2!} = 1 - \frac{\theta_x^2}{2} \quad (14a)$$

$$\sin \theta_x \approx \theta_x - \frac{\theta_x^3}{3!} = \theta_x - \frac{\theta_x^3}{6} \quad (14b)$$

are employed. Thus, using the aforementioned approximations, neglecting the term $\rho A \ddot{u}$ of equation (13a) denoting the axial inertia of the beam, employing the expressions of the stress resultants (eqns. (10)) and ignoring the nonlinear terms of the fourth or greater order [12], the governing partial differential equations of motion for the beam at hand can be written as

$$-EA \left[u'' + w'w'' + v'v'' + \frac{I_S}{A} \theta_x' \theta_x'' - G_{u1} \right] = p_X \quad (15a)$$

$$\begin{aligned} EI_Z v'''' - NG_{v1} + (EI_Z - EI_Y)G_{v2} + EI_Z \beta_Y G_{v3} + EI_Y \beta_Z G_{v4} + \rho(A\ddot{v} + G_{v5} - G_{v6}) = \\ = p_y - G_{v7} p_X \end{aligned} \quad (15b)$$

$$\begin{aligned} EI_Y w'''' - NG_{w1} + (EI_Z - EI_Y)G_{w2} + EI_Z \beta_Y G_{w3} + EI_Y \beta_Z G_{w4} + \rho(A\ddot{w} + G_{w5} - G_{w6}) = \\ = p_z - G_{w7} p_X \end{aligned} \quad (15c)$$

$$\begin{aligned}
& EC_S \theta_x'''' - GI_t \theta_x'' - \frac{3}{2} E \left(I_R - \frac{I_S^2}{A} \right) \theta_x'^2 \theta_x'' - NG_{\theta 1} + (EI_Z - EI_Y) G_{\theta 2} + EI_Z \beta_Y G_{\theta 3} + \\
& + EI_Y \beta_Z G_{\theta 4} + \rho \left(G_{\theta 5} \ddot{\theta}_x - G_{\theta 6} \dot{v} + G_{\theta 7} \dot{w} + G_{\theta 8} - C_S \ddot{\theta}_x'' \right) = m_x + p_z y_{p_z} - p_y z_{p_z} + \\
& + G_{\theta 9} p_y + G_{\theta 10} p_z - G_{\theta 11} p_x
\end{aligned} \tag{15d}$$

where G_{ui} , G_{vi} ($i = 1, 2, \dots, 7$), G_{wi} ($i = 1, 2, \dots, 7$) and $G_{\theta i}$ ($i = 1, 2, \dots, 11$) are functions of v , w , θ_x and their derivatives with respect to x , t , given in the Appendix A, while the expression of the axial stress resultant N is given as

$$N = EA \left[u' + \frac{I}{2} \left(v'^2 + w'^2 + \frac{I_S}{A} \theta_x'^2 \right) - \theta_x' \left(z_C (w' \theta_x + v') + y_C (v' \theta_x - w') \right) \right] \tag{16}$$

The above governing differential equations (eqns. (15)) are also subjected to the initial conditions ($x \in (0, l)$)

$$u(x, 0) = u_0(x) \qquad \dot{u}(x, 0) = \dot{u}_0(x) \tag{17a,b}$$

$$v(x, 0) = v_0(x) \qquad \dot{v}(x, 0) = \dot{v}_0(x) \tag{18a,b}$$

$$w(x, 0) = w_0(x) \qquad \dot{w}(x, 0) = \dot{w}_0(x) \tag{19a,b}$$

$$\theta_x(x, 0) = \theta_{x0}(x) \qquad \dot{\theta}_x(x, 0) = \dot{\theta}_{x0}(x) \tag{20a,b}$$

together with the corresponding boundary conditions of the problem at hand, which are given as

$$\alpha_1 u(x,t) + \alpha_2 N(x,t) = \alpha_3 \quad (21)$$

$$\beta_1 v(x,t) + \beta_2 V_y(x,t) = \beta_3 \quad \bar{\beta}_1 \theta_Z(x,t) + \bar{\beta}_2 M_Z(x,t) = \bar{\beta}_3 \quad (22a,b)$$

$$\gamma_1 w(x,t) + \gamma_2 V_z(x,t) = \gamma_3 \quad \bar{\gamma}_1 \theta_Y(x,t) + \bar{\gamma}_2 M_Y(x,t) = \bar{\gamma}_3 \quad (23a,b)$$

$$\delta_1 \theta_x(x,t) + \delta_2 M_t(x,t) = \delta_3 \quad \bar{\delta}_1 \theta'_x(x,t) + \bar{\delta}_2 M_w(x,t) = \bar{\delta}_3 \quad (24a,b)$$

at the beam ends $x=0,l$, where V_y , V_z and M_Z , M_Y are the reactions and bending moments with respect to y , z or to Y , Z axes, respectively, given by the following relations (ignoring again the nonlinear terms of the fourth or greater order)

$$\begin{aligned} V_y = & N(v' - z_C \theta'_x - y_C \theta_x \theta'_x) + \\ & + EI_Z (v'' \theta_x'^2 - w'' \theta_x' - w''' \theta_x - v''' + 2v'' \theta_x \theta_x' + 2\beta_Y \theta_x' \theta_x'') + \\ & + EI_Y (w''' \theta_x + w'' \theta_x' - 2v'' \theta_x \theta_x' - v''' \theta_x'^2 - \beta_Z \theta_x'^3 - 2\beta_Z \theta_x \theta_x' \theta_x'') \end{aligned} \quad (25a)$$

$$\begin{aligned} V_z = & N(w' + y_C \theta'_x - z_C \theta_x \theta'_x) + \\ & + EI_Y (w''' \theta_x'^2 + v''' \theta_x - w''' + v'' \theta_x' + 2w'' \theta_x \theta_x' + 2\beta_Z \theta_x' \theta_x'') - \\ & - EI_Z (v'' \theta_x' + w''' \theta_x'^2 + v''' \theta_x + 2w'' \theta_x \theta_x' - 2\beta_Y \theta_x \theta_x' \theta_x'' - \beta_Y \theta_x'^3) \end{aligned} \quad (25b)$$

$$M_Z = EI_Z (w'' \theta_x - \beta_Y \theta_x'^2 + v'' - v'' \theta_x'^2) + EI_Y (-w'' \theta_x + v'' \theta_x'^2 + \beta_Z \theta_x'^2 \theta_x) \quad (25c)$$

$$M_Y = EI_Z (-w'' \theta_x'^2 + \beta_Y \theta_x'^2 \theta_x - v'' \theta_x) + EI_Y (\beta_Z \theta_x'^2 - w'' + w'' \theta_x'^2 + v'' \theta_x) \quad (25d)$$

while M_t and M_w are the torsional and warping moments at the boundaries of the bar, respectively, given as

$$M_t = GI_t \theta_x' - EC_S \theta_x''' + N \left(-z_C w' \theta_x + y_C w' - y_C v' \theta_x - z_C v' + \frac{I_S}{A} \theta_x' \right) + EI_Z \beta_Y \left(-2\theta_x' v'' - 2\theta_x' w'' \theta_x \right) + EI_Y \beta_Z \left(-2\theta_x' w'' + 2\theta_x' v'' \theta_x \right) + \frac{1}{2} E \left(I_R - \frac{I_S^2}{A} \right) \theta_x'^3 \quad (26a)$$

$$M_w = -EC_S \left(\theta_x'' + \beta_\omega \theta_x'^2 \right) \quad (26b)$$

Finally, $\alpha_k, \beta_k, \bar{\beta}_k, \gamma_k, \bar{\gamma}_k, \delta_k, \bar{\delta}_k$ ($k=1,2,3$) are time dependent functions specified at the boundaries of the bar ($x=0,l$). The boundary conditions (21)-(24) are the most general boundary conditions for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = \bar{\beta}_1 = \gamma_1 = \bar{\gamma}_1 = \delta_1 = \bar{\delta}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \delta_2 = \delta_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\delta}_2 = \bar{\delta}_3 = 0$).

3. INTEGRAL REPRESENTATIONS-NUMERICAL SOLUTION

3.1 For the axial displacement $u(x,t)$, the transverse displacements $v(x,t)$,

$w(x,t)$ and the angle of twist $\theta_x(x,t)$

According to the precedent analysis, the nonlinear flexural-torsional vibration problem of a beam reduces in establishing the axial displacement component $u(x,t)$ having continuous partial derivatives up to the second order and the transverse displacement components $v(x,t)$, $w(x,t)$ and the angle of rotation $\theta_x(x,t)$ having continuous partial derivatives up to the fourth order with respect to x and up to the second order with respect to t , satisfying the nonlinear initial boundary value problem described by the coupled governing differential equations of motion (eqns. (15)) along the beam, the initial conditions (eqns. (17)-(20)) and the boundary conditions (eqns. (21)-(24)) at the beam ends $x = 0, l$.

Eqns. (15) and (17)-(24) are solved using the Analog Equation Method [22] as it is developed for hyperbolic differential equations [30]. According to this method, let $u(x,t)$, $v(x,t)$, $w(x,t)$ and $\theta_x(x,t)$ be the sought solutions of the aforementioned problem. Setting as $u_1(x,t) = u(x,t)$, $u_2(x,t) = v(x,t)$, $u_3(x,t) = w(x,t)$ and $u_4(x,t) = \theta_x(x,t)$ and differentiating with respect to x these functions two and four times, respectively, yields

$$\frac{\partial^2 u_1}{\partial x^2} = q_1(x,t) \qquad \frac{\partial^4 u_i}{\partial x^4} = q_i(x,t), \quad (i = 2, 3, 4) \qquad (27a,b)$$

Eqns. (27) are quasi-static, i.e. the time variable appears as a parameter. They indicate that the solution of eqns. (15) and (17)-(24) can be established by solving eqns. (27) under the same boundary conditions (eqns. (21)-(24)), provided that the fictitious load distributions $q_i(x,t)$ ($i = 1, 2, 3, 4$) are first established. These distributions can be determined using BEM. Following the procedure presented in [30] and employing the

constant element assumption for the load distributions q_i along the L internal beam elements (as the numerical implementation becomes very simple and the obtained results are of high accuracy), the integral representations of the displacement components u_i ($i = 1, 2, 3, 4$) and their derivatives with respect to x when applied for the beam ends $(0, l)$, together with the boundary conditions (21)-(24) are employed to express the unknown boundary quantities $u_i(\zeta, t)$, $u_{i,x}(\zeta, t)$, $u_{i,xx}(\zeta, t)$ and $u_{i,xxx}(\zeta, t)$ ($\zeta = 0, l$) in terms of q_i . Thus, the following set of 28 nonlinear algebraic equations is obtained

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{44} \end{bmatrix} \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \\ \mathbf{d}_4 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{D}_1^{nl} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{D}_2^{nl} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{D}_3^{nl} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{D}_4^{nl} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{\alpha}_3 \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\beta} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\gamma} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\delta} \end{Bmatrix} \quad (28)$$

where

$$\mathbf{E}_{11} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{0} & \mathbf{D}_{22} & \mathbf{D}_{23} \end{bmatrix} \quad (29a)$$

$$\mathbf{E}_{22} = \begin{bmatrix} \mathbf{F}_3 & \mathbf{E}_{35} & \mathbf{E}_{36} & \mathbf{E}_{37} & \mathbf{E}_{38} \\ \mathbf{F}_4 & \mathbf{0} & \mathbf{E}_{46} & \mathbf{E}_{47} & \mathbf{E}_{48} \\ \mathbf{D}_{54} & \mathbf{D}_{55} & \mathbf{D}_{56} & \mathbf{D}_{57} & \mathbf{D}_{58} \\ \mathbf{D}_{64} & \mathbf{D}_{65} & \mathbf{D}_{66} & \mathbf{D}_{67} & \mathbf{D}_{68} \end{bmatrix} \quad (29b)$$

$$\mathbf{E}_{33} = \begin{bmatrix} \mathbf{F}_7 & \mathbf{E}_{79} & \mathbf{E}_{711} & \mathbf{E}_{712} & \mathbf{E}_{713} \\ \mathbf{F}_8 & \mathbf{0} & \mathbf{E}_{811} & \mathbf{E}_{812} & \mathbf{E}_{813} \\ \mathbf{D}_{99} & \mathbf{D}_{910} & \mathbf{D}_{911} & \mathbf{D}_{912} & \mathbf{D}_{913} \\ \mathbf{D}_{109} & \mathbf{D}_{1010} & \mathbf{D}_{1011} & \mathbf{D}_{1012} & \mathbf{D}_{1013} \end{bmatrix} \quad (29c)$$

$$\mathbf{E}_{44} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{E}_{1115} & \mathbf{E}_{1116} & \mathbf{E}_{1117} & \mathbf{E}_{1118} \\ \mathbf{F}_{12} & \mathbf{0} & \mathbf{E}_{1216} & \mathbf{E}_{1217} & \mathbf{E}_{1218} \\ \mathbf{D}_{1314} & \mathbf{D}_{1315} & \mathbf{D}_{1316} & \mathbf{D}_{1317} & \mathbf{D}_{1318} \\ \mathbf{D}_{1414} & \mathbf{D}_{1415} & \mathbf{D}_{1416} & \mathbf{D}_{1417} & \mathbf{D}_{1418} \end{bmatrix} \quad (29d)$$

$$\boldsymbol{\beta} = \{\boldsymbol{\beta}_3 \quad \bar{\boldsymbol{\beta}}_3\}^T, \quad \boldsymbol{\gamma} = \{\boldsymbol{\gamma}_3 \quad \bar{\boldsymbol{\gamma}}_3\}^T, \quad \boldsymbol{\delta} = \{\boldsymbol{\delta}_3 \quad \bar{\boldsymbol{\delta}}_3\}^T \quad (29e,f,g)$$

The matrices \mathbf{D}_{22} to \mathbf{D}_{1418} , are 2×2 rectangular known matrices including the values of the functions $a_j, \beta_j, \bar{\beta}_j, \gamma_j, \bar{\gamma}_j, \delta_j, \bar{\delta}_j$ ($j = 1, 2$) of eqns. (21)-(24); $\boldsymbol{\alpha}_3, \boldsymbol{\beta}_3, \bar{\boldsymbol{\beta}}_3, \boldsymbol{\gamma}_3, \bar{\boldsymbol{\gamma}}_3, \boldsymbol{\delta}_3, \bar{\boldsymbol{\delta}}_3$ are 2×1 known column matrices including the boundary values of the functions $a_3, \beta_3, \bar{\beta}_3, \gamma_3, \bar{\gamma}_3, \delta_3, \bar{\delta}_3$, of eqns. (21)-(24); \mathbf{E}_{12} to \mathbf{E}_{1218} are rectangular 2×2 known coefficient matrices resulting from the values of kernels at the bar ends; $\mathbf{F}_1, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_7, \mathbf{F}_8, \mathbf{F}_{11}, \mathbf{F}_{12}$ are $2 \times L$ rectangular known matrices originating from the integration of kernels on the axis of the beam; \mathbf{D}_j^{nl} is a 2×1 and \mathbf{D}_i^{nl} ($i = 2, 3, 4$) are 4×1 known column matrices containing the nonlinear terms included in the expressions of the boundary conditions (eqns. (21)-(24)). Finally

$$\mathbf{d}_l = \{\mathbf{q}_l \quad \hat{\mathbf{u}}_l \quad \hat{\mathbf{u}}_{l,x}\}^T \quad (30a)$$

$$\mathbf{d}_i = \{\mathbf{q}_i \quad \hat{\mathbf{u}}_i \quad \hat{\mathbf{u}}_{i,x} \quad \hat{\mathbf{u}}_{i,xx} \quad \hat{\mathbf{u}}_{i,xxx}\}^T, \quad (i = 2, 3, 4) \quad (30b)$$

are generalized unknown vectors, where

$$\hat{\mathbf{u}}_i = \{u_i(0,t) \quad u_i(l,t)\}^T, \quad (i = 1, 2, 3, 4) \quad (31a)$$

$$\hat{\mathbf{u}}_{i,x} = \left\{ \frac{\partial u_i(0,t)}{\partial x} \quad \frac{\partial u_i(l,t)}{\partial x} \right\}^T, \quad (i = 1, 2, 3, 4) \quad (31b)$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{ \frac{\partial^2 u_i(0,t)}{\partial x^2} \quad \frac{\partial^2 u_i(l,t)}{\partial x^2} \right\}^T, \quad (i = 2, 3, 4) \quad (31c)$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{ \frac{\partial^3 u_i(0,t)}{\partial x^3} \quad \frac{\partial^3 u_i(l,t)}{\partial x^3} \right\}^T, \quad (i = 2, 3, 4) \quad (31d)$$

are vectors including the two unknown time dependent boundary values of the respective boundary quantities and $\mathbf{q}_i = \{q_1^i \ q_2^i \ \dots \ q_N^i\}^T$ ($i = 1, 2, 3, 4$) are vectors including the L unknown time dependent nodal values of the fictitious loads.

Discretization of the integral representations of the unknown quantities u_i ($i = 1, 2, 3, 4$) inside the beam ($x \in (0, l)$) and application to the L collocation nodal points yields

$$\mathbf{u}_l = \mathbf{A}_l^0 \mathbf{q}_l + \mathbf{C}_0 \hat{\mathbf{u}}_l + \mathbf{C}_l \hat{\mathbf{u}}_{l,x} \quad (32a)$$

$$\mathbf{u}_{l,x} = \mathbf{A}_l^l \mathbf{q}_l + \mathbf{C}_0 \hat{\mathbf{u}}_{l,x} \quad (32b)$$

$$\mathbf{u}_{l,xx} = \mathbf{q}_l \quad (32c)$$

$$\mathbf{u}_i = \mathbf{A}_i^0 \mathbf{q}_i + \mathbf{C}_0 \hat{\mathbf{u}}_i + \mathbf{C}'_1 \hat{\mathbf{u}}_{i,x} + \mathbf{C}_2 \hat{\mathbf{u}}_{i,xx} + \mathbf{C}_3 \hat{\mathbf{u}}_{i,xxx}, \quad (i=2,3,4) \quad (32d)$$

$$\mathbf{u}_{i,x} = \mathbf{A}_i^1 \mathbf{q}_i + \mathbf{C}_0 \hat{\mathbf{u}}_{i,x} + \mathbf{C}'_1 \hat{\mathbf{u}}_{i,xx} + \mathbf{C}_2 \hat{\mathbf{u}}_{i,xxx}, \quad (i=2,3,4) \quad (32e)$$

$$\mathbf{u}_{i,xx} = \mathbf{A}_i^2 \mathbf{q}_i + \mathbf{C}_0 \hat{\mathbf{u}}_{i,xx} + \mathbf{C}'_1 \hat{\mathbf{u}}_{i,xxx}, \quad (i=2,3,4) \quad (32f)$$

$$\mathbf{u}_{i,xxx} = \mathbf{A}_i^3 \mathbf{q}_i + \mathbf{C}_0 \hat{\mathbf{u}}_{i,xxx}, \quad (i=2,3,4) \quad (32g)$$

$$\mathbf{u}_{i,xxxx} = \mathbf{q}_i, \quad (i=2,3,4) \quad (32h)$$

where $\mathbf{A}_i^j, \mathbf{A}_k^j$ ($i=0, 1$), ($j=0, 1, 2, 3$), ($k=2, 3, 4$) are $L \times L$ known matrices; $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}'_1, \mathbf{C}_2, \mathbf{C}_3$ are $L \times 2$ known matrices and $\mathbf{u}_i, \mathbf{u}_{i,x}, \mathbf{u}_{i,xx}, \mathbf{u}_{i,xxx}, \mathbf{u}_{i,xxxx}$ are time dependent vectors including the values of $u_i(x,t)$ and their derivatives at the L nodal points. Eqns. (32a, b, d, e, f, g) can be assembled more conveniently as

$$\mathbf{u}_l = \mathbf{H}_l^0 \mathbf{d}_l \quad (33a)$$

$$\mathbf{u}_{l,x} = \mathbf{H}_l^1 \mathbf{d}_l \quad (33b)$$

$$\mathbf{u}_i = \mathbf{H}_i^0 \mathbf{d}_i, \quad (i=2,3,4) \quad (33c)$$

$$\mathbf{u}_{i,x} = \mathbf{H}_i^1 \mathbf{d}_i, \quad (i=2,3,4) \quad (33d)$$

$$\mathbf{u}_{i,xx} = \mathbf{H}_i^2 \mathbf{d}_i, \quad (i=2,3,4) \quad (33e)$$

$$\mathbf{u}_{i,xxx} = \mathbf{H}_i^3 \mathbf{d}_i, \quad (i=2,3,4) \quad (33f)$$

where $\mathbf{H}_l^i, \mathbf{H}_k^j$ ($i=0, 1$), ($j=0, 1, 2, 3$), ($k=2, 3, 4$) are $L \times (L+4)$ and $L \times (L+8)$ known matrices, respectively arising from $\mathbf{A}_1^i, \mathbf{A}_2^j, \mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_1', \mathbf{C}_2, \mathbf{C}_3$.

Applying eqns. (15) to the L collocation points and employing eqns. (33), $4 \times L$ semidiscretized nonlinear equations of motion are formulated as

$$[\mathbf{M}] \begin{Bmatrix} \ddot{\mathbf{d}}_1 \\ \ddot{\mathbf{d}}_2 \\ \ddot{\mathbf{d}}_3 \\ \ddot{\mathbf{d}}_4 \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \\ \mathbf{d}_4 \end{Bmatrix} + \{\mathbf{m}^{nl}\} + \{\mathbf{k}^{nl}\} = \{\mathbf{f}\} \quad (34)$$

where $\mathbf{M}, \mathbf{K}, \mathbf{f}$ are generalized mass matrix, stiffness matrix and force vector, respectively, while $\mathbf{m}^{nl}, \mathbf{k}^{nl}$ are nonlinear generalized mass vector and stiffness vector, respectively, containing all the nonlinear terms of the semidiscretized equations of motion. It is noted that the coefficients of the mass matrix \mathbf{M} corresponding to the generalized vector $\ddot{\mathbf{d}}_1$ are equal to zero as the axial inertia of the beam has been neglected. Equations (34) with equations (28) form a system of $4 \times L + 28$ equations with respect to the generalized unknown vectors \mathbf{d}_i ($i=1, 2, 3, 4$).

Eqns. (33a,c) when combined with eqns. (17a)-(20a) yield the following $4 \times L$ linear equations with respect to $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4$ for $t=0$

$$\mathbf{H}_2^0 \mathbf{d}_1(0) = \mathbf{u}_0 \quad (35a)$$

$$\mathbf{H}_2^0 \mathbf{d}_2(0) = \mathbf{v}_0 \quad (35b)$$

$$\mathbf{H}_3^0 \mathbf{d}_3(0) = \mathbf{w}_0 \quad (35c)$$

$$\mathbf{H}_4^0 \mathbf{d}_4(0) = \boldsymbol{\theta}_{x0} \quad (35d)$$

The above equations, together with eqns. (28) written for $t=0$, form a set of $4 \times L + 28$ nonlinear algebraic equations which are solved to establish the initial conditions $\mathbf{d}_1(0)$, $\mathbf{d}_2(0)$, $\mathbf{d}_3(0)$, $\mathbf{d}_4(0)$. Similarly, eqns. (33a,c) when combined with eqns. (17b)-(20b) yield the following $4 \times L$ linear equations with respect to $\dot{\mathbf{d}}_1$, $\dot{\mathbf{d}}_2$, $\dot{\mathbf{d}}_3$, $\dot{\mathbf{d}}_4$ for $t=0$

$$\mathbf{H}_1^0 \dot{\mathbf{d}}_1(0) = \dot{\mathbf{u}}_0 \quad (36a)$$

$$\mathbf{H}_2^0 \dot{\mathbf{d}}_2(0) = \dot{\mathbf{v}}_0 \quad (36b)$$

$$\mathbf{H}_3^0 \dot{\mathbf{d}}_3(0) = \dot{\mathbf{w}}_0 \quad (36c)$$

$$\mathbf{H}_4^0 \dot{\mathbf{d}}_4(0) = \dot{\boldsymbol{\theta}}_{x0} \quad (36d)$$

The above equations, together with the 28 equations resulting after differentiating eqns. (28) with respect to time and writing them for $t=0$, form a set of $4 \times L + 28$ algebraic equations, from which the initial conditions $\dot{\mathbf{d}}_1(0)$, $\dot{\mathbf{d}}_2(0)$, $\dot{\mathbf{d}}_3(0)$, $\dot{\mathbf{d}}_4(0)$ are established.

The aforementioned initial conditions along with eqns. (28), (34) form an initial value problem of Differential-Algebraic Equations (DAE), which can be solved using any efficient solver. In this study, the Petzold-Gear Backward Differentiation Formula (BDF) [23], which is a linear multistep method for differential equations coupled to algebraic equations, is employed. For this case, the method is applied after

introducing new variables to reduce the order of the system [31] and after differentiating eqns. (28) with respect to time, in order to obtain an equivalent system with a value of system index $ind = 1$ [23].

3.2 For the primary warping function φ_S^P

The numerical solution for the evaluation of the displacement and rotation components assume that the warping C_S and torsion I_t constants given from eqns. (10e), (10d) are already established. Eqns. (11f), (11g) indicate that the evaluation of the aforementioned constants presumes that the primary warping function φ_S^P at any interior point of the domain Ω of the cross section of the beam is known. Once φ_S^P is established, C_S and I_t constants are evaluated by converting the domain integrals into line integrals along the boundary employing the following relations

$$C_S = -\int_{\Gamma} B \frac{\partial \varphi_S^P}{\partial n} ds \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \quad (37a)$$

$$I_t = \int_{\Gamma} \left[(yz^2 - z\varphi_S^P) n_y + (y^2z + y\varphi_S^P) n_z \right] ds \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \quad (37b)$$

and using constant boundary elements for the approximation of these line integrals. In eqns. (37a,b) n_y , n_z are the direction cosines, while $B(y,z)$ is a fictitious function defined as the solution of the following Neumann problem

$$\nabla^2 B = \varphi_S^P \quad \text{in } \Omega \quad (38a)$$

$$\frac{\partial B}{\partial n} = 0 \quad \text{on } \Gamma = \bigcup_{j=1}^{K+1} \Gamma_j \quad (38b)$$

The evaluation of the primary warping function φ_S^P and the fictitious function $B(y, z)$ is accomplished using BEM as this is presented in [25, 32, 33].

Moreover, since the torsion and warping constants of the arbitrary beam cross section are evaluated employing the boundary element method, using only boundary integration, the domain integrals for the evaluation of the area, the bending, the fourth and the polar moments of inertia and the Wagner's coefficients β_Z , β_Y and β_ω given from expressions (12) have to be converted to boundary line integrals. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the aforementioned quantities can be written as

$$A = \frac{1}{2} \int_{\Gamma} \left((y - y_C) n_y + (z - z_C) n_z \right) ds \quad (39a)$$

$$I_Y = \int_{\Gamma} \frac{1}{3} \left((z - z_C)^3 n_z \right) ds \quad (39b)$$

$$I_Z = \int_{\Gamma} \frac{1}{3} \left((y - y_C)^3 n_y \right) ds \quad (39c)$$

$$I_R = \int_{\Gamma} \frac{1}{5} \left(y^5 n_y + z^5 n_z \right) + \frac{1}{3} \left(y^2 z^2 \right) \left(y n_y + z n_z \right) ds \quad (39d)$$

$$I_S = \int_{\Gamma} \frac{1}{3} \left(y^3 n_y + z^3 n_z \right) ds \quad (39e)$$

$$\beta_Y = \frac{1}{2I_Z} \left[\int_{\Gamma} \left(\frac{(y - y_C)^4}{4} n_y + \frac{(y - y_C)(z - z_C)^3}{3} n_z \right) ds \right] + y_C \quad (39f)$$

$$\beta_Z = \frac{I}{2I_Y} \left[\int_{\Gamma} \left(\frac{(y-y_C)^3 (z-z_C)}{3} n_Y + \frac{(z-z_C)^4}{4} n_Z \right) ds \right] + z_C \quad (39g)$$

$$\beta_\omega = \frac{I}{2C_S} \int_{\Gamma} \left[\frac{I}{3} (y^3 n_Y + z^3 n_Z) \phi_S^P - \frac{I}{12} (y^4 + z^4) (z n_Y - y n_Z) \right] ds \quad (39h)$$

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the validation, the efficiency, wherever possible the accuracy and the range of applications of the developed method. The numerical results have been obtained employing 21 nodal points (longitudinal discretization) and 400 boundary elements (cross section discretization).

Example 1

In the first example, for comparison reasons, the forced vibration of a clamped beam (Fig. 2a), ($E = 2,1 \times 10^8 \text{ kN/m}^2$, $G = 8,0769 \times 10^7 \text{ kN/m}^2$, $\rho = 7,85 \text{ tn/m}^3$, $l = 4 \text{ m}$) of a hollow rectangular cross section, having geometric constants presented in Table 1, is examined. The beam is subjected to the suddenly applied uniformly distributed loading $p_y(t) = 250 \text{ kN/m}$, $p_z(t) = 500 \text{ kN/m}$ at its centroid, as this is shown in Fig. 2b. In Figs. 3, 4 the time histories of the midpoint displacements $v(l/2, t)$, $w(l/2, t)$ and angle of twist $\theta_x(l/2, t)$, respectively and in Table 2 the maximum values of these kinematical components are presented as compared with those obtained from a BEM solution [19], noting the accuracy of the proposed

method. As it can be observed from these figures and table, the strong coupling between flexure and torsion may lead beams of doubly symmetric cross section undergoing biaxial transverse loading applied at their centroid to develop torsional rotation, a phenomenon that cannot be predicted from linear analysis.

Example 2

In this example, in order to investigate the response of a monosymmetric beam and the influence of the loading point upon the cross section, in nonlinear flexural-torsional vibrations, the forced vibrations of a cantilever beam ($E = 2,164 \times 10^8 \text{ kN/m}^2$, $G = 8,0148 \times 10^7 \text{ kN/m}^2$, $\rho = 7,85 \text{ tn/m}^3$, $l = 1 \text{ m}$) of a thin-walled open shaped cross section (Fig. 5), under two load cases have been studied (its geometric constants are given in Table 3). More specifically, the beam is subjected to a suddenly applied concentrated force $P_Y(t) = 5 \text{ kN}$ either on the right (load case (i), Fig. 5b) or on the left (load case (ii), Fig. 5c) flange. In Figs. 6-9 the time histories of the axial displacement $u(l,t)$, the transverse displacements $v(l,t)$, $w(l,t)$ and the angle of twist $\theta_x(l,t)$ of the cantilever beam, respectively and in Table 4 the maximum values of these kinematical components, are presented. From the aforementioned figures and table, it can easily be observed that geometrical nonlinearity affects substantially the dynamic response of the beam inducing non vanishing axial displacement and displacement with respect to z axis, while in linear analysis these kinematical components vanish. From the obtained results it can also be verified that the loading position has significant influence on the response, altering substantially the magnitude of the kinematical components. This discrepancy can be explained by the fact that in load case (ii), the change of eccentricity of the transverse

load during torsional rotation increases the magnitude of the twisting moment, acting adversely compared with load case (i), where the applied twisting moment is reduced during torsional rotation.

Example 3

In order to demonstrate the range of applications of the developed method, in this final example the forced vibrations of a simply supported (free right end according to the axial boundary condition) steel L-shaped beam of unequal (asymmetric cross section) legs (Fig. 10a), ($E = 2,1 \times 10^8 \text{ kN/m}^2$, $\rho = 7,85 \text{ tn/m}^3$, $G = 8,0769 \times 10^7 \text{ kN/m}^2$, $l = 1 \text{ m}$), having the geometric constants presented in Table 5, is studied. The beam is subjected to a suddenly applied uniformly distributed twisting moment $m_x(x,t) = 8 \text{ kNm/m}$ (Fig. 10a,b). Due to lack of symmetry, apart from the angle of twist, the beam is expected to develop axial (axial displacement vanishes in linear analysis) and transverse displacements as well. In Figs. 11-14 the time histories of the axial displacement $u(l,t)$, the transverse displacements $v(l/2,t)$, $w(l/2,t)$ and the angle of twist $\theta_x(l/2,t)$, respectively, are presented, while in Table 6 the maximum values of these kinematical components, taking into account or ignoring rotary inertia effect are also shown. From these figures and table, it is observed that the geometrical nonlinearity leads to the increase of torsional stiffness decreasing the magnitude of angle of twist, while transverse displacements get significantly higher values compared with the linear ones. From Table 6 it is also noted that the influence of rotary inertia proves to be negligible on the magnitude of kinematical components.

Moreover, the response of a hinged-hinged beam (axially immovable ends), having the same cross section and length, under harmonic excitation is examined. More specifically, the beam is subjected to a uniformly distributed harmonic load $p_{\tilde{z}}(x,t) = p_0(x) \cdot \sin(\omega_{f,lin} \cdot t)$, as this is shown in Fig. 10a,c. The frequency $\omega_{f,lin}$ is considered as $\omega_{f,lin} = 2\pi \cdot f_{1,lin}$, where $f_{1,lin} = 118,200 \text{ Hz}$, is the first natural frequency of the examined beam, performing a linear analysis [34]. In Figs. 15-17 the time histories of the displacements $\tilde{v}(l/2,t)$, $\tilde{w}(l/2,t)$, with respect to the $S\tilde{y}\tilde{z}$ system of axes and the angle of twist $\theta_x(l/2,t)$ are presented, noting the significant difference in response between linear and nonlinear analysis. More specifically, it is observed that only in the linear response deformations continue to increase with time, while the beating phenomenon observed in the nonlinear one is explained from the fact that large kinematical components increase the bar's fundamental natural frequency ω_f (by increasing the stiffness of the bar due to the tensile axial force induced by the axially immovable ends), thereby causing a detuning of ω_f with the frequency of the external loading ($\omega_{f,lin}$). After the kinematical components reach their maximum values, the amplitude of deformations decreases, leading to the reversal of the previously mentioned effects.

5. CONCLUDING REMARKS

In this article a boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary cross section, undergoing moderately large displacements and twisting rotations and small deformations, taking into account the effect of rotary inertia, warping inertia and change of eccentricity of transverse loads during torsional rotation. The beam is subjected to arbitrarily

distributed conservative transverse loads, which can be applied on any point of the cross section, and/or axial loads and twisting moments, while its edges are restrained by the most general boundary conditions. The main conclusions that can be drawn from this investigation are:

- a. The numerical technique presented in this investigation is well suited for computer aided analysis of beams of arbitrary simply or multiply connected cross section, supported by the most general boundary conditions and subjected to the combined action of arbitrarily distributed or concentrated time dependent loading.
- b. Accurate results are obtained using a relatively small number of nodal points along the beam.
- c. The geometrical nonlinearity leads to strong coupling between the axial, torsional and bending equilibrium equations resulting in a significantly different response of the beam compared to the one obtained by linear analysis.
- d. The strong coupling between flexure and torsion may lead beams of doubly symmetric cross sections, undergoing biaxial transverse loading applied on their centroid, to develop torsional rotation.
- e. The eccentricity change of the transverse loading during the torsional beam motion, resulting in additional torsional moment influences the beam response.
- f. The influence of rotary inertia, as shown in the treated examples, on the dynamic response of the beams, proves to be negligible on the magnitude of kinematical components.
- g. The developed procedure retains most of the advantages of a BEM solution over a FEM approach, although it requires longitudinal domain discretization.

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APPENDIX A

Functions F_{vi} ($i = 1, 2, \dots, 7$), F_{wi} ($i = 1, 2, \dots, 7$) and $F_{\theta i}$ ($i = 1, 2, \dots, 11$) of the differential equations (13), concerning the nonlinear dynamic analysis of beams of arbitrary cross section are given as:

$$F_{v1} = v' - y_C \theta_x' \sin \theta_x - z_C \theta_x' \cos \theta_x \quad (\text{A.1a})$$

$$F_{v2} = A(y_C \sin \theta_x + z_C \cos \theta_x) \quad (\text{A.1b})$$

$$F_{v3} = A(z_C \sin \theta_x - y_C \cos \theta_x) \quad (\text{A.1c})$$

$$F_{v4} = (I_Y - I_Z) \sin \theta_x \cos \theta_x v' - [(I_Y - I_Z) \cos^2 \theta_x - I_Y] w' \quad (\text{A.1d})$$

$$F_{v5} = (I_Y - I_Z) \sin \theta_x \cos \theta_x w' + [(I_Y - I_Z) \cos^2 \theta_x - I_Y] v' \quad (\text{A.1e})$$

$$F_{v6} = (I_Y - I_Z) \cos \theta_x \sin \theta_x \quad (\text{A.1f})$$

$$F_{v7} = (I_Y - I_Z) \cos^2 \theta_x - I_Y \quad (\text{A.1g})$$

$$F_{w1} = w' + y_C \theta_x' \cos \theta_x - z_C \theta_x' \sin \theta_x \quad (\text{A.2a})$$

$$F_{w2} = A(-y_C \cos \theta_x + z_C \sin \theta_x) \quad (\text{A.2b})$$

$$F_{w3} = A(z_C \cos \theta_x + y_C \sin \theta_x) \quad (\text{A.2c})$$

$$F_{w4} = (I_Z - I_Y) \cos \theta_x \sin \theta_x w' + [(I_Z - I_Y) \cos^2 \theta_x - I_Z] v' \quad (\text{A.2d})$$

$$F_{w5} = (I_Z - I_Y) \cos \theta_x \sin \theta_x v' - [(I_Z - I_Y) \cos^2 \theta_x - I_Z] w' \quad (\text{A.2e})$$

$$F_{w6} = (I_Z - I_Y) \cos \theta_x \sin \theta_x \quad (\text{A.2f})$$

$$F_{w7} = (I_Z - I_Y) \cos^2 \theta_x - I_Z \quad (\text{A.2g})$$

$$F_{\theta1} = -y_C v' \theta_x' \cos \theta_x - y_C w' \theta_x' \sin \theta_x - z_C w' \theta_x' \cos \theta_x + z_C v' \theta_x' \sin \theta_x \quad (\text{A.3a})$$

$$F_{\theta2} = z_C (v' \cos \theta_x + w' \sin \theta_x) \quad (\text{A.3b})$$

$$F_{\theta3} = y_C (v' \sin \theta_x - w' \cos \theta_x) \quad (\text{A.3c})$$

$$F_{\theta4} = (w'' \sin \theta_x + v'' \cos \theta_x) \quad (\text{A.3d})$$

$$F_{\theta5} = (w'' \cos \theta_x - v'' \sin \theta_x) \quad (\text{A.3e})$$

$$F_{\theta6} = A(z_C \cos \theta_x + y_C \sin \theta_x) \quad (\text{A.3f})$$

$$F_{\theta7} = A(-z_C \sin \theta_x + y_C \cos \theta_x) \quad (\text{A.3g})$$

$$F_{\theta8} = [(I_Y - I_Z) \cos^2 \theta_x + I_Z] v'^2 - 2(I_Z - I_Y) v' w' \sin \theta_x \cos \theta_x + I_s +$$

$$+\left[(I_Z - I_Y)\cos^2 \theta_x + I_Y\right]w'^2 \quad (\text{A.3h})$$

$$F_{\theta 9} = \left[(I_Z - I_Y)\cos^2 \theta_x + I_Y\right]w' + (I_Y - I_Z)\sin \theta_x \cos \theta_x v' \quad (\text{A.3i})$$

$$F_{\theta 10} = \left[(I_Z - I_Y)\cos^2 \theta_x - I_Z\right]v' + (I_Z - I_Y)\sin \theta_x \cos \theta_x w' \quad (\text{A.3j})$$

$$F_{\theta 11} = (I_Y - I_Z)\sin \theta_x \cos \theta_x v'^2 + (I_Z - I_Y)\sin \theta_x \cos \theta_x w'^2 + \\ + \left[2(I_Z - I_Y)\cos^2 \theta_x - I_Z + I_Y\right]v'w' \quad (\text{A.3k})$$

Functions G_{u1} , G_{vi} ($i=1,2,\dots,7$), G_{wi} ($i=1,2,\dots,7$) and $G_{\theta i}$ ($i=1,2,\dots,11$) of the governing differential equations (15), concerning the nonlinear dynamic analysis of beams of arbitrary cross section are given as:

$$G_{u1} = z_C \left(\theta_x \theta_x' w'' + \theta_x \theta_x'' w' + \theta_x'' v' + \theta_x' v'' + \theta_x'^2 w' \right) + \\ + y_C \left(\theta_x \theta_x'' v' + \theta_x \theta_x' v'' - \theta_x'' w' - \theta_x' w'' + \theta_x'^2 v' \right) \quad (\text{A.4})$$

$$G_{v1} = -z_C \theta_x'' - y_C \left(\theta_x'^2 + \theta_x'' \theta_x \right) + v'' \quad (\text{A.5a})$$

$$G_{v2} = w'''' \theta_x + 2w''' \theta_x' + w'' \theta_x'' - v'''' \theta_x^2 - 4v''' \theta_x \theta_x' - 2v'' \theta_x \theta_x'' - 2v' \theta_x'^2 \quad (\text{A.5b})$$

$$G_{v3} = -2\theta_x' \theta_x''' - 2\theta_x''^2 \quad (\text{A.5c})$$

$$G_{v4} = 2\theta_x' \theta_x''' \theta_x + 2\theta_x''^2 \theta_x + 5\theta_x'^2 \theta_x'' \quad (\text{A.5d})$$

$$G_{v5} = \left[(I_Z - I_Y) \left(\theta_x v'' + \theta_x' v' \right) - I_Z w'' - A \left(y_C \theta_x + z_C - \frac{I}{2} z_C \theta_x^2 \right) \right] \ddot{\theta}_x + (I_Z - I_Y) \cdot \\ \cdot \left[2\theta_x \theta_x' \dot{v}' + \theta_x^2 \dot{v}'' + 2 \left(\theta_x' \dot{\theta}_x + \theta_x \dot{\theta}_x' \right) \dot{v}' + 2\theta_x \dot{\theta}_x v'' - \theta_x' \dot{w}' - \theta_x \dot{w}'' + v' \theta_x \ddot{\theta}_x' \right] +$$

$$+I_Z \cdot \left[-w' \ddot{\theta}_x' - \ddot{v}'' - 2\dot{\theta}_x \dot{w}'' + v'' \dot{\theta}_x^2 + 2v' \dot{\theta}_x \dot{\theta}_x' - 2\dot{\theta}_x' \dot{w}' \right] \quad (\text{A.5e})$$

$$G_{v6} = A(y_C - z_C \theta_x) \dot{\theta}_x^2 \quad (\text{A.5f})$$

$$G_{v7} = v' - y_C \theta_x' \theta_x - z_C \theta_x' \quad (\text{A.5g})$$

$$G_{w1} = w'' + y_C \theta_x'' - z_C (\theta_x'^2 + \theta_x \theta_x'') \quad (\text{A.6a})$$

$$G_{w2} = v'''' \theta_x + 2v'''' \theta_x' + v'' \theta_x'' + w'''' \theta_x^2 + 4w'''' \theta_x \theta_x' + 2w'' \theta_x \theta_x'' + 2w'' \theta_x'^2 \quad (\text{A.6b})$$

$$G_{w3} = -2\theta_x' \theta_x'' \theta_x - 5\theta_x'^2 \theta_x'' - 2\theta_x''^2 \theta_x \quad (\text{A.6c})$$

$$G_{w4} = -2\theta_x' \theta_x''' - 2\theta_x''^2 \quad (\text{A.6d})$$

$$G_{w5} = \left[(I_Z - I_Y) (-\theta_x w'' - \theta_x' w') + I_Y v'' + A \left(-z_C \theta_x + y_C - \frac{I}{2} y_C \theta_x^2 \right) \right] \ddot{\theta}_x - (I_Z - I_Y) \cdot \\ \cdot \left[2\theta_x \theta_x' \dot{w}' + \theta_x^2 \dot{w}'' + 2(\theta_x' \dot{\theta}_x + \theta_x \dot{\theta}_x') \dot{w}' + 2\theta_x \dot{\theta}_x \dot{w}'' + \theta_x' \dot{v}' + \theta_x \dot{v}'' + w' \theta_x \ddot{\theta}_x' \right] - I_Y \cdot \\ \cdot \left[-v' \ddot{\theta}_x' + \dot{w}'' - 2\dot{\theta}_x \dot{v}'' - w'' \dot{\theta}_x^2 - 2w' \dot{\theta}_x \dot{\theta}_x' - 2\dot{\theta}_x' \dot{v}' \right] \quad (\text{A.6e})$$

$$G_{w6} = A(z_C + y_C \theta_x) \dot{\theta}_x^2 \quad (\text{A.6f})$$

$$G_{w7} = w' + y_C \theta_x' - z_C \theta_x' \theta_x \quad (\text{A.6g})$$

$$G_{\theta1} = \frac{I_S}{A} \theta_x'' + y_C (w'' - v'' \theta_x) - z_C (v'' + w'' \theta_x) \quad (\text{A.7a})$$

$$G_{\theta2} = v'' w'' - v''^2 \theta_x + w''^2 \theta_x \quad (\text{A.7b})$$

$$G_{\theta3} = 2\theta_x'' w'' \theta_x + w'' \theta_x'^2 + 2\theta_x' v'' + 2\theta_x'' v'' + 2\theta_x' w'' \theta_x \quad (\text{A.7c})$$

$$G_{\theta4} = -2\theta_x'' v'' \theta_x + 2\theta_x'' w'' + 2\theta_x' w'' - v'' \theta_x'^2 - 2\theta_x' v'' \theta_x \quad (\text{A.7d})$$

$$G_{\theta5} = v'^2 I_Y + w'^2 I_Z + I_S \quad (\text{A.7e})$$

$$G_{\theta 6} = A \left(z_C + y_C \theta_x - \frac{1}{2} z_C \theta_x^2 \right) \quad (\text{A.7f})$$

$$G_{\theta 7} = A \left(y_C - z_C \theta_x - \frac{1}{2} y_C \theta_x^2 \right) \quad (\text{A.7g})$$

$$G_{\theta 8} = (I_Z - I_Y) \theta_x (w' \ddot{w}' - v' \ddot{v}') + I_Z (w' \ddot{v}' + 2 \dot{\theta}_x \dot{w}' w') + I_Y (-v' \ddot{w}' + 2 \dot{\theta}_x \dot{v}' v') \quad (\text{A.7h})$$

$$G_{\theta 9} = \frac{1}{2} \theta_x^2 z_{p_y} + \frac{1}{6} \theta_x^3 y_{p_y} - \theta_x y_{p_y} \quad (\text{A.7i})$$

$$G_{\theta 10} = \frac{1}{6} \theta_x^3 z_{p_z} - \frac{1}{2} \theta_x^2 y_{p_z} - z_{p_z} \theta_x \quad (\text{A.7j})$$

$$G_{\theta 11} = \frac{I_S}{A} \theta_x' - y_C v' \theta_x - z_C w' \theta_x - z_C v' + y_C w' \quad (\text{A.7k})$$

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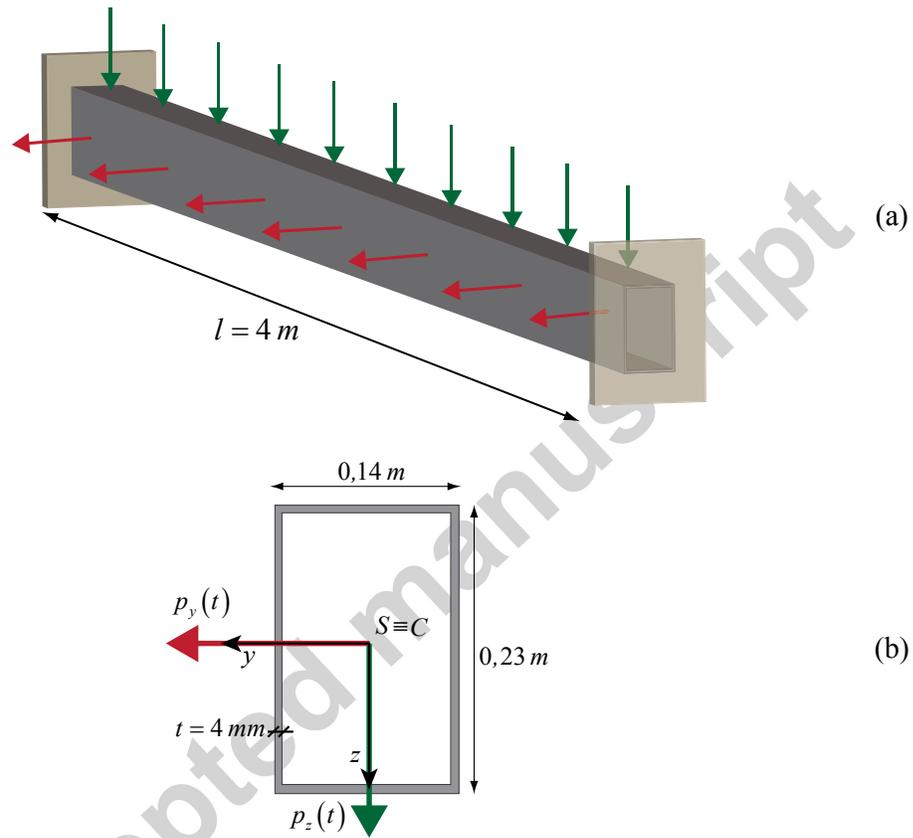


Fig. 2. Clamped beam of example 1 (a) and applied loading on the centroid of the cross section (b).

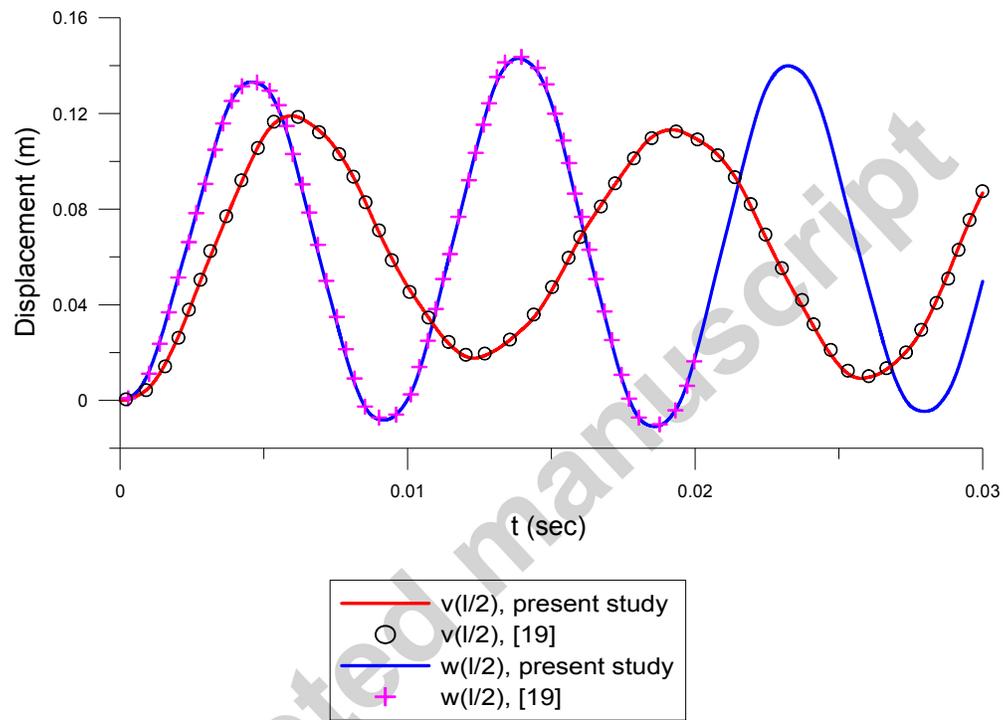


Fig. 3. Time history of the displacements v and w at the midpoint of the clamped beam of example 1.

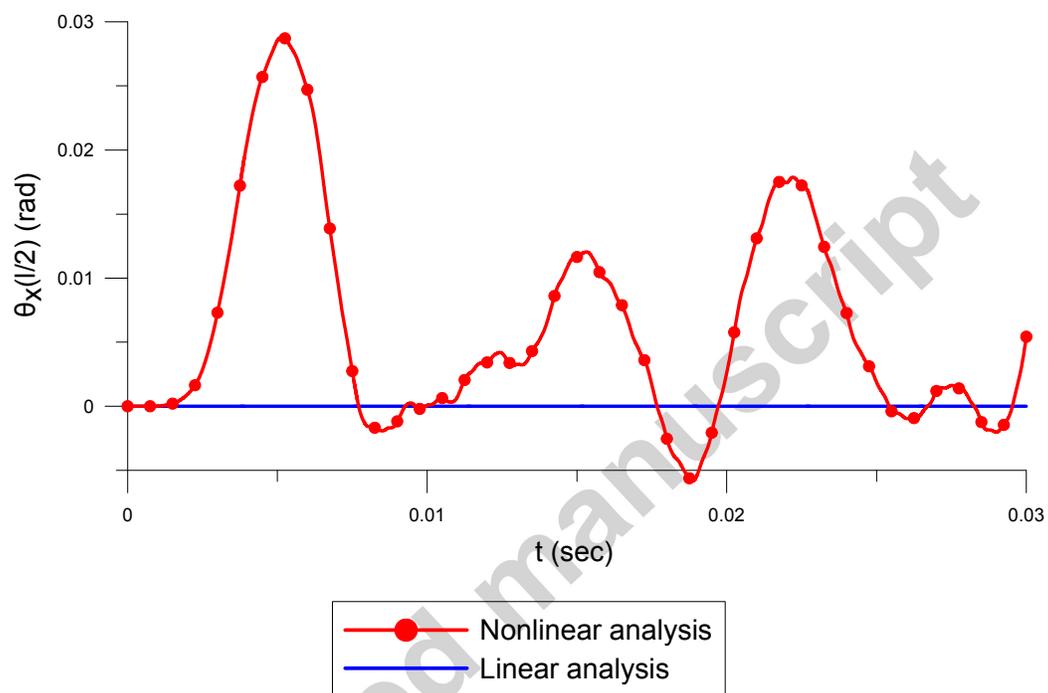


Fig. 4. Time history of the angle of twist θ_x at the midpoint of the clamped beam of example 1.

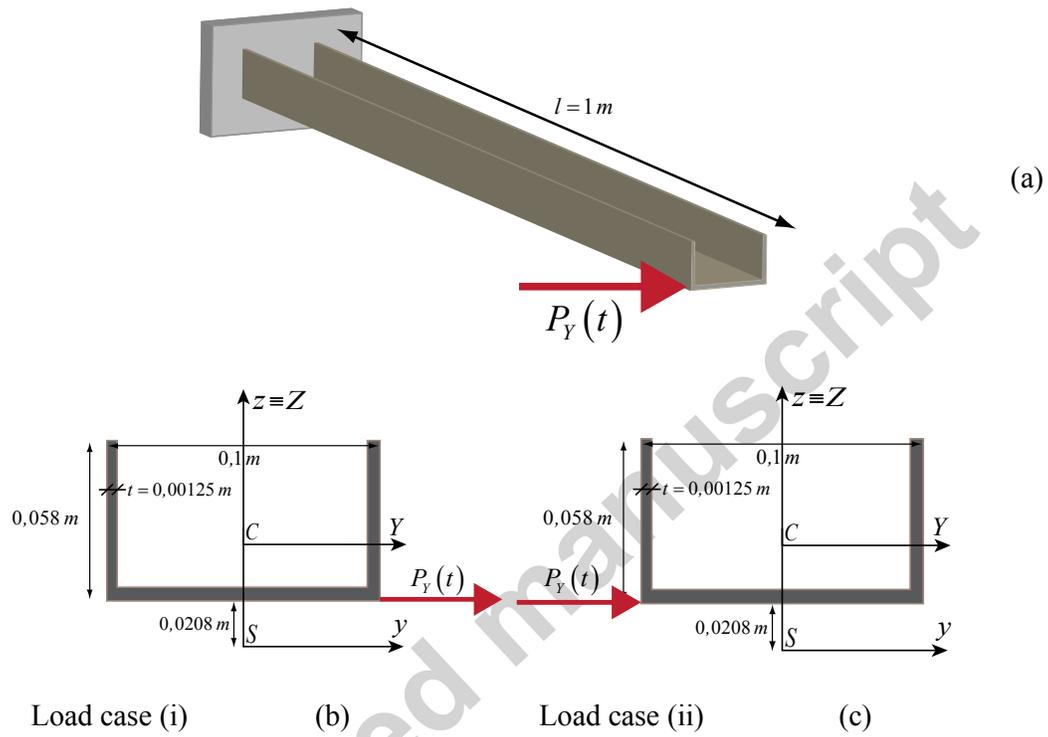


Fig. 5. Cantilever beam of example 2 (a). Transverse force applied on the right (b) or on the left (c) flange.

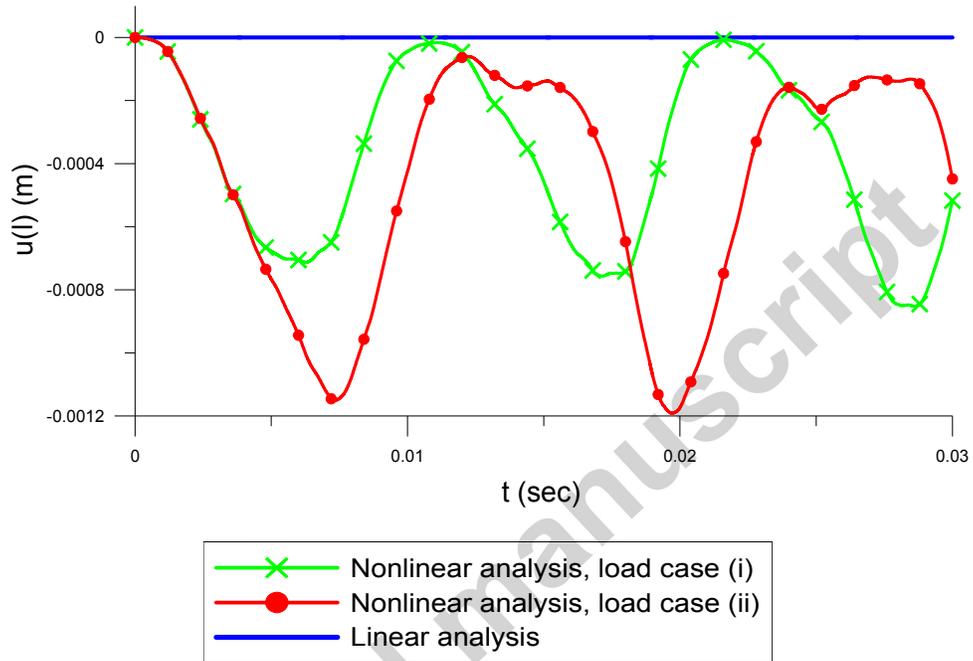


Fig. 6. Time history of the axial displacement u at the tip of the cantilever beam of example 2.

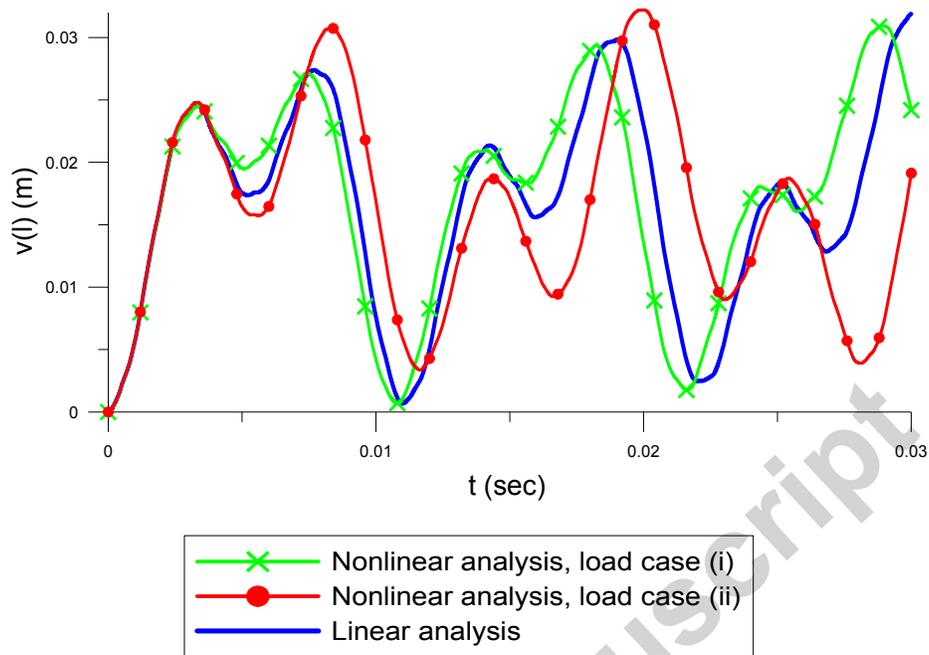


Fig. 7. Time history of the displacement v at the tip of the cantilever beam of example 2.

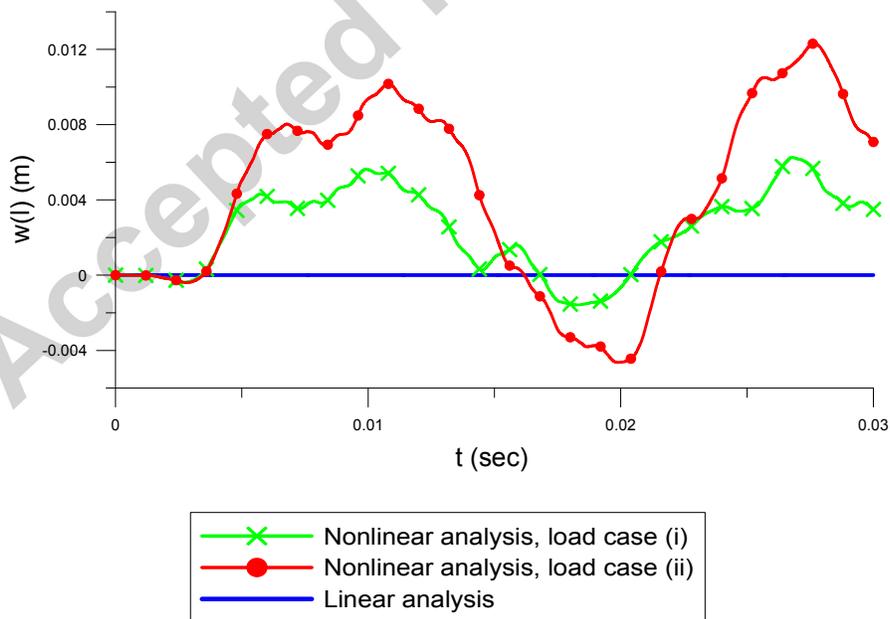


Fig. 8. Time history of the displacement w at the tip of the cantilever beam of example 2.

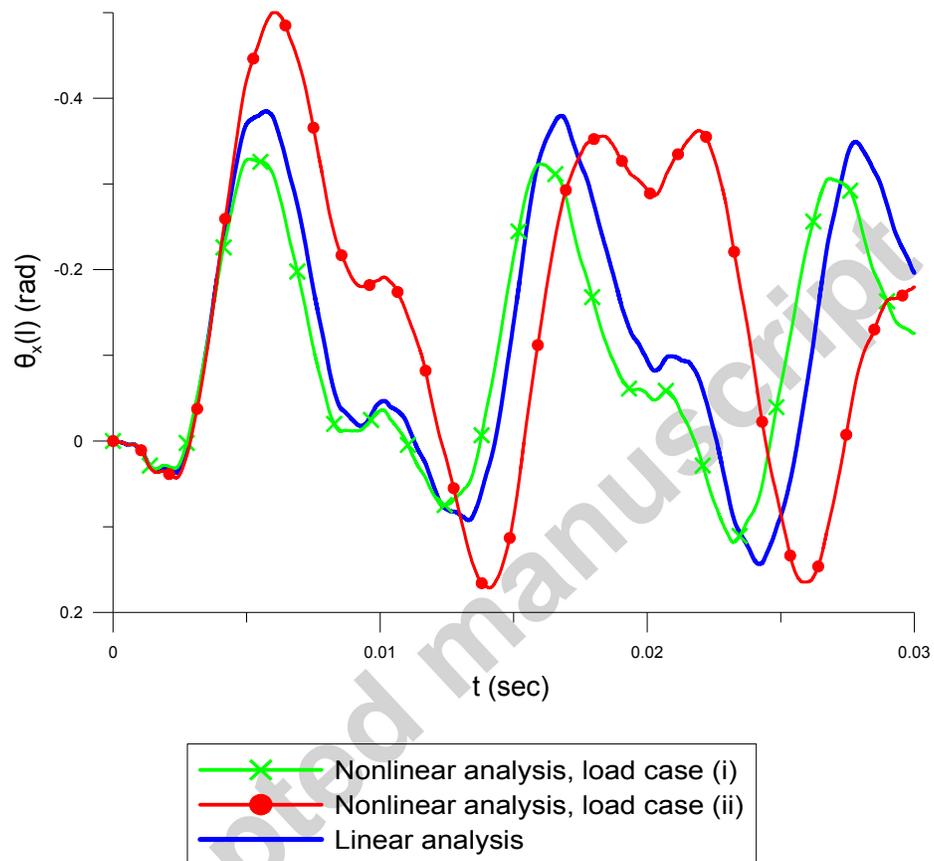


Fig. 9. Time history of the angle of twist θ_x at the tip of the cantilever beam of example 2.

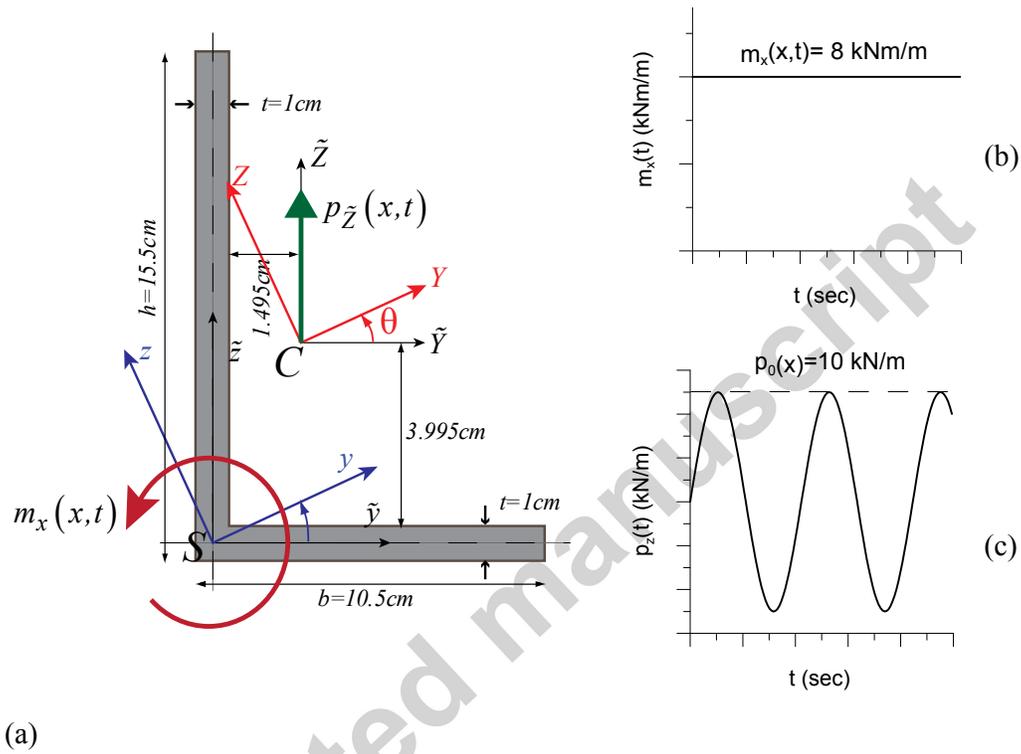


Fig. 10. L-shaped cross section of unequal legs of Example 3 (a). Applied distributed twisting moment (b) or transverse harmonic excitation (c).

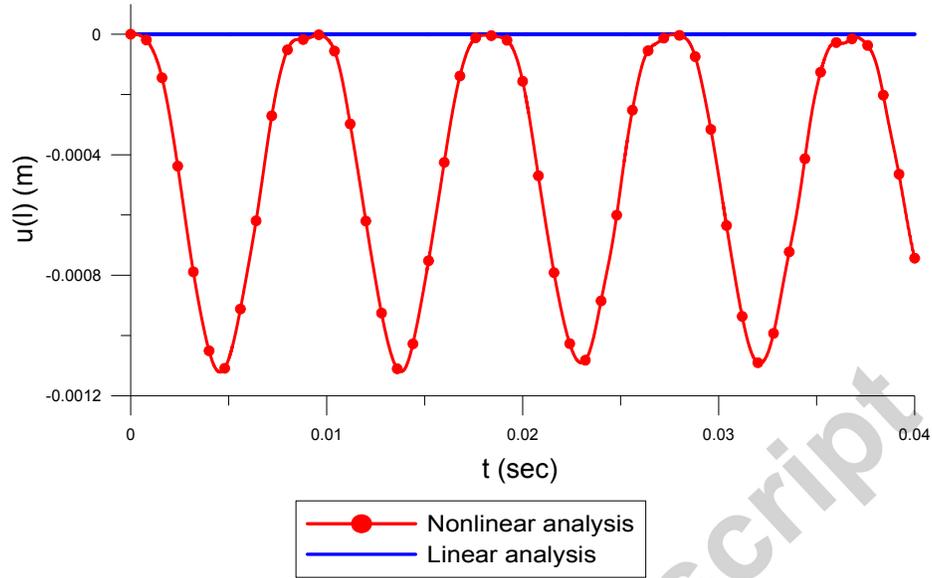


Fig. 11. Time history of the axial displacement u at the right end of the simply supported beam of example 3.

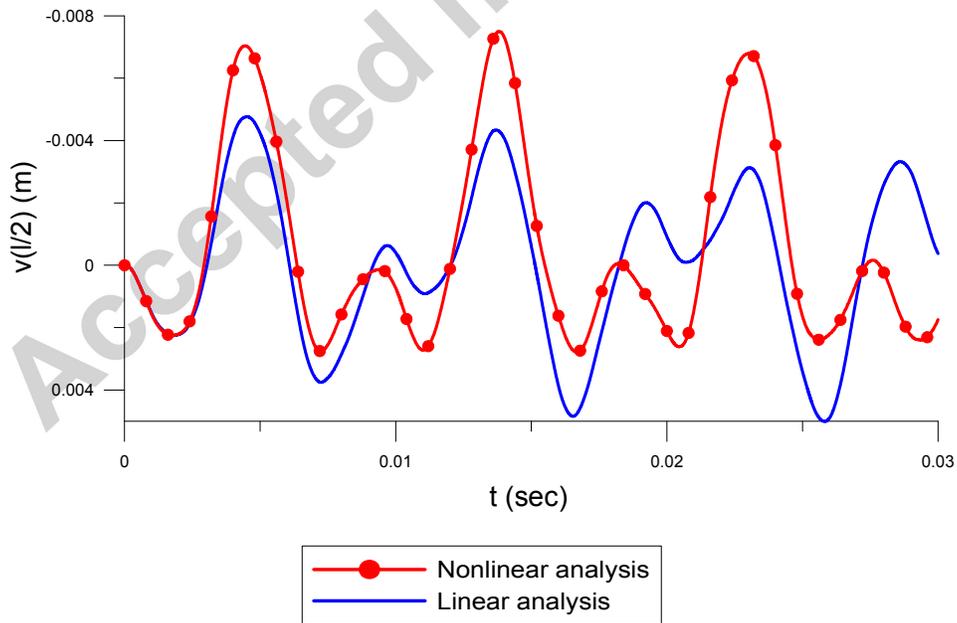


Fig. 12. Time history of the displacement v at the midpoint of the simply supported beam of example 3.

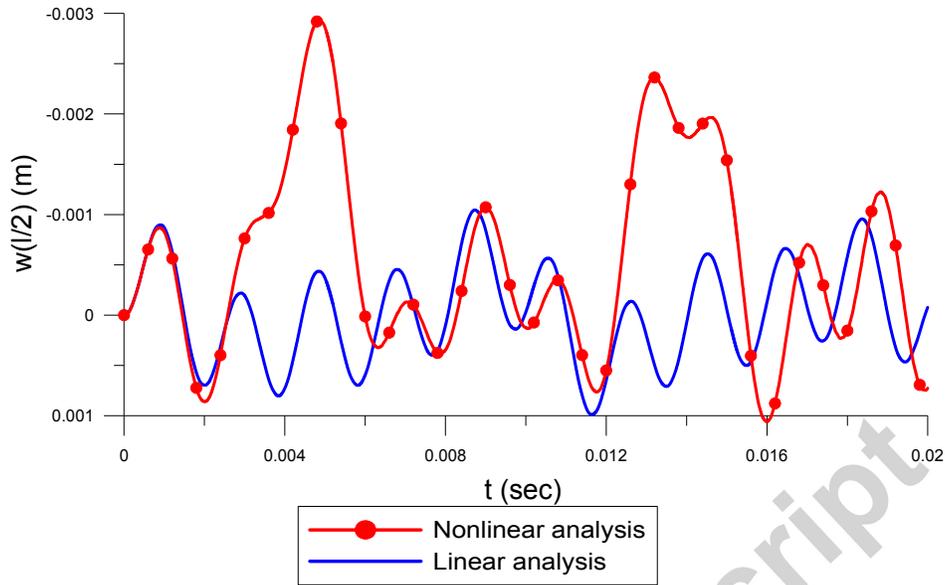


Fig. 13. Time history of the displacement w at the midpoint of the simply supported beam of example 3.

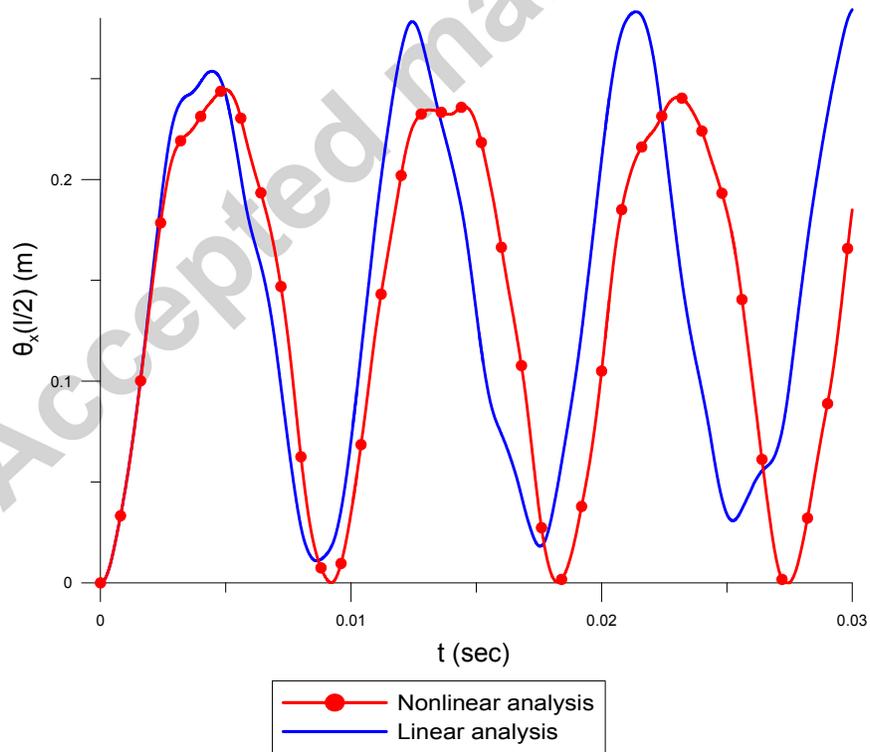


Fig. 14. Time history of the angle of twist θ_x at the midpoint of the simply supported beam of example 3.

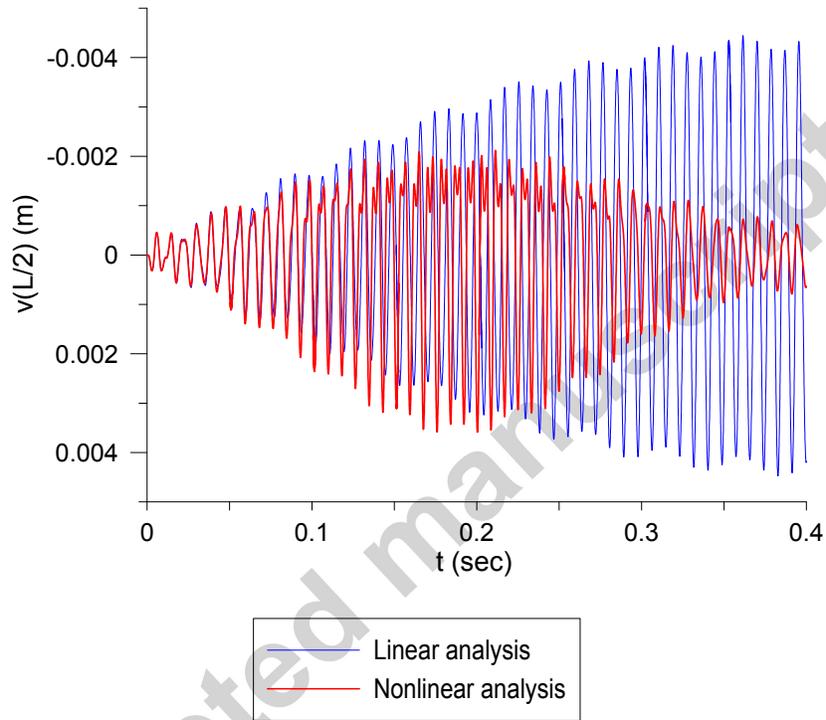


Fig. 15. Time history of the displacement \tilde{v} at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

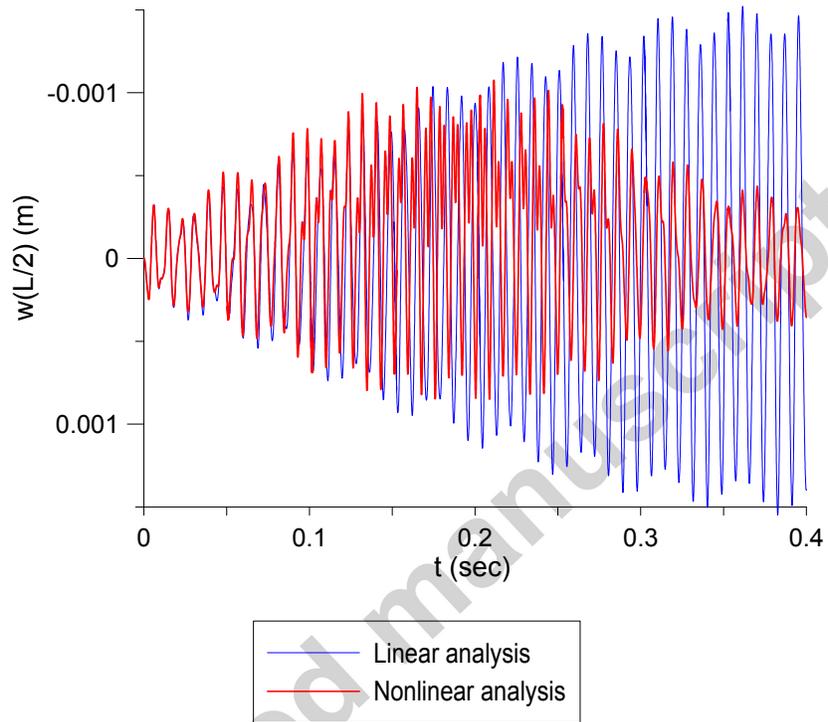


Fig. 16. Time history of the displacement \tilde{w} at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

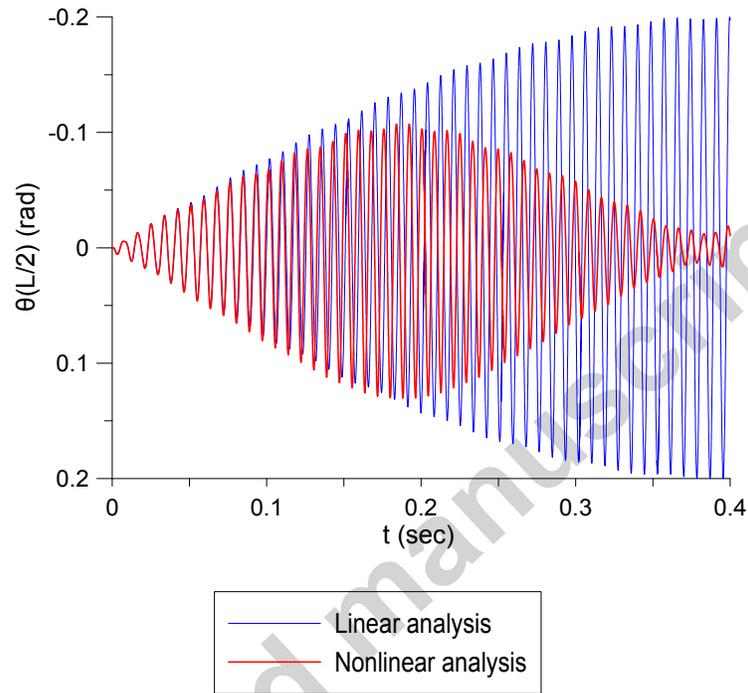


Fig. 17. Time history of the angle of twist θ_x at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

Table 1: Geometric constants of the beam of example 1.

$A = 2,896 \times 10^{-3} m^2$	$I_R = 3,94264 \times 10^{-7} m^6$
$I_Y = 2,15968 \times 10^{-5} m^4$	$I_t = 2,10844 \times 10^{-5} m^4$
$I_Z = 1,00439 \times 10^{-5} m^4$	$C_s = 3,59634 \times 10^{-9} m^6$
$I_S = 3,16407 \times 10^{-5} m^4$	

Table 2: Maximum values of the displacements $v(l/2, t)$ (m), $w(l/2, t)$ (m) (of the first cycle) and angle of rotation $\theta_x(l/2, t)$ (rad) (of the whole time history) of the clamped beam of example 1.

	Linear analysis		Nonlinear analysis	
	Present Study	Sapountzakis & Dourakopoulos [19]	Present Study	Sapountzakis & Dourakopoulos [19]
$v(l/2)_{\max}$	0,1590	0,1588	0,1190	0,1180
$w(l/2)_{\max}$	0,1480	0,1476	0,1330	0,1330
$\theta_x(l/2)_{\max}$	0,0000	-	0,0288	-

Table 3: Geometric constants of the beam of example 2.

$A = 2,66875 \times 10^{-4} m^2$	$I_t = 9,10243 \times 10^{-9} m^4$
$I_Y = 9,39789 \times 10^{-8} m^4$	$C_s = 1,31047 \times 10^{-10} m^6$
$I_Z = 4,50061 \times 10^{-7} m^4$	$\beta_z = 6,10287 \times 10^{-2} m$
$I_S = 9,06833 \times 10^{-7} m^4$	$z_c = 3,687 \times 10^{-2} m$
$I_R = 4,58807 \times 10^{-9} m^6$	

Table 4: Maximum values of the kinematical components $u(l,t)$ (m), $v(l,t)$ (m), $w(l,t)$ (m) and $\theta_x(l,t)$ (rad) of the cantilever beam of example 2 for load cases (i), (ii).

	Linear Analysis	Nonlinear Analysis	
		Load case (i)	Load case (ii)
$u(l)_{\max}$	0,00000	-0,00085	-0,00119
$v(l)_{\max}$	0,03190	0,03091	0,03217
$w(l)_{\max}$	0,00000	0,00626	0,01230
$\theta_x(l)_{\max}$	-0,38479	-0,32897	-0,50086

Table 5: Geometric constants of the beam of example 3

$A = 2,5 \times 10^{-3} m^2$	$\theta = 0,430 \text{ rad}$
$I_Y = 7,23593 \times 10^{-6} m^4$	$I_t = 8,3903 \times 10^{-8} m^4$
$I_Z = 1,32198 \times 10^{-6} m^4$	$C_s = 1,1937 \times 10^{-10} m^6$
$I_S = 1,45517 \times 10^{-5} m^4$	$\beta_y = 8,19154 \times 10^{-2} m$
$I_R = 1,68733 \times 10^{-7} m^6$	$\beta_z = 3,8866 \times 10^{-2} m$
$y_c = 3,662 \times 10^{-2} m$	$\beta_\omega = -0,1605$
$z_c = 3,190 \times 10^{-2} m$	

Table 6: Maximum values of the kinematical components $u(l,t)$ (m), $v(l/2,t)$ (m), $w(l/2,t)$ (m) and $\theta_x(l/2,t)$ (rad) for the simply supported beam of example 3.

	Without rotary inertia		With rotary inertia	
	Linear analysis	Nonlinear analysis	Linear analysis	Nonlinear analysis
$u(l)_{\max}$	0,00000	-0,00113	0,00000	-0,00112
$v(l/2)_{\max}$	-0,00473	-0,00752	-0,00476	-0,00750
$w(l/2)_{\max}$	-0,00101	-0,00296	-0,00105	-0,00293
$\theta_x(l/2)_{\max}$	0,28800	0,24592	0,28512	0,24467

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- Fig. 3. Time history of the displacements v and w at the midpoint of the clamped beam of example 1.
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- Fig. 5. Cantilever beam of example 2 (a). Transverse force applied on the right (b) or on the left (c) flange.
- Fig. 6. Time history of the axial displacement u at the tip of the cantilever beam of example 2.
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Fig. 13. Time history of the displacement w at the midpoint of the simply supported beam of example 3.

Fig. 14. Time history of the angle of twist θ_x at the midpoint of the simply supported beam of example 3.

Fig. 15. Time history of the displacement \tilde{v} at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

Fig. 16. Time history of the displacement \tilde{w} at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

Fig. 17. Time history of the angle of twist θ_x at the midpoint of the hinged-hinged beam of example 3 under harmonic excitation.

- Beams of arbitrary cross section under general boundary conditions and loading
- Rotary and warping inertia are included in the nonlinear dynamic analysis
- Wagner's coefficients and shortening effect are taken into account
- A BEM approach is employed and high accuracy is achieved
- Geometrical nonlinearity results in significantly different beam response

Accepted manuscript