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Weak Averaging of Semilinear Stochastic Differential Equations with Almost Periodic Coefficients

Mikhail KAMENSKII*, Omar MELLAH†and Paul RAYNAUD DE FITTE‡

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Abstract

An averaging result is proved for stochastic evolution equations with highly oscillating coefficients. This result applies in particular to equations with almost periodic coefficients. The convergence to the solution of the averaged equation is obtained in distribution, as in previous works by Khasminskii and Vrkoč.

Keywords : averaging methods, stochastic evolution equations, almost periodic solutions, Wasserstein distance

1 Introduction

Since the classical work of N.M. Krylov and N.N. Bogolyubov [17] devoted to the analysis, by the method of averaging, of the problem of the dependence on a small parameter $\varepsilon > 0$ of almost periodic solutions of ordinary differential equation containing terms of frequency of order $\frac{1}{\varepsilon}$, several articles and books have appeared, which develop this method for different kinds of differential equations. See the bibliography in the book of V.Sh. Burd [6], where a list of books related to this problem for deterministic differential

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equations is presented. We note here that the authors of these papers are greatly influenced by the books of N.N. Bogolyubov and A.Yu. Mitropolskii [5] and M.A. Krasnosel'skiĭ, V.Sh. Burd and Yu.S. Kolesov [15].

The method of averaging has been applied of course to stochastic differential equations, but in general it was applied to the initial problem in a finite interval, see for example [14]. Even in this case we can see a great difference with the deterministic case. To ensure the strong convergence in a space of stochastic processes, we must assume such convergence of the stochastic term when $\varepsilon \rightarrow 0$, which virtually excludes the consideration of high frequency oscillation of this term. R.Z. Khasminskii [14] has shown, in a finite dimensional setting, that it is possible to overcome this problem if one only looks for convergence *in distribution* to the solution of the averaged equation. Later Ivo Vrkoč [24] generalized this result in a Hilbert space setting, for which the initial problem was at this time already well developed (see for example the book of Da Prato and Zabczyk [10]).

During the last 20 years an intensive study of the problem of existence of almost periodic solutions of stochastic differential equations was performed by L. Arnold, C. Tudor, G. Da Prato (see in particular [9, 1]) and later by P.H. Bezandry and T. Diagana [2, 3, 4]. For the first group, an almost periodic solution means that the stochastic process generates an almost periodic measure on the paths space. The second group claims the existence of square mean almost periodic solutions, but square mean almost periodicity seems to be a too strong property for solutions of SDEs, see counterexamples in [18].

In this paper we propose the averaging principle for solutions to a family of semilinear stochastic differential equations in Hilbert space which are almost periodic in distribution. The second member of these equations contains a high frequency term. Under the Bezandry-Diagana conditions, we establish the convergence in distribution (actually, in Wasserstein distance) of the solutions of these equations to the solution of the averaged equation in the sense of Khasminskii-Vrkoč, with a weaker hypothesis than [24] on the linear evolution semigroup.

The paper is organized as follows: The next section is devoted to the notations and preliminaries. We then prove in Section 3 that the solutions of the equations we consider are almost periodic in distribution, when their coefficients are almost periodic. In section 4, we prove the fundamental averaging result of this paper.

2 Notations and Preliminaries

In the sequel, $(\mathbb{H}_1, \|\cdot\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|\cdot\|_{\mathbb{H}_2})$ denote separable Hilbert spaces and $L(\mathbb{H}_1, \mathbb{H}_2)$ (or $L(\mathbb{H}_1)$ if $\mathbb{H}_1 = \mathbb{H}_2$) is the space of all bounded linear operators from \mathbb{H}_1 to \mathbb{H}_2 , whose norm will be denoted by $\|\cdot\|_{L(\mathbb{H}_1, \mathbb{H}_2)}$. If $A \in L(\mathbb{H}_1)$ then A^* denotes its adjoint operator and if A is a nuclear operator,

$$|A|_{\mathcal{N}} = \sup \left\{ \sum_i | \langle Ae_i, f_i \rangle |, \{e_i\}, \{f_i\} \text{ orthonormal bases of } \mathbb{H}_1 \right\}$$

is the nuclear norm of A .

2.1 Almost periodic functions

Let (\mathbb{E}, d) be a separable metric space, we denote by $C_b(\mathbb{E})$ the Banach space of continuous and bounded functions $f : \mathbb{E} \rightarrow \mathbb{R}$ with $\|f\|_{\infty} = \sup_{x \in \mathbb{E}} |f(x)|$ and by $\mathcal{P}(\mathbb{E})$ the set of all probability measures onto σ -Borel field of \mathbb{E} . For $f \in C_b(\mathbb{E})$ we define

$$\|f\|_L = \sup \left\{ \frac{f(x) - f(y)}{d_{\mathbb{E}}(x, y)} : x \neq y \right\}$$

$$\|f\|_{\text{BL}} = \max\{\|f\|_{\infty}, \|f\|_L\}$$

and we define

$$\text{BL}(\mathbb{E}) = \{f \in C_b(\mathbb{E}); \|f\|_{\text{BL}} < \infty\}.$$

For $\mu, \nu \in \mathcal{P}(\mathbb{E})$ we define

$$d_{\text{BL}}(\mu, \nu) = \sup_{\|f\|_{\text{BL}} \leq 1} \left| \int_{\mathbb{E}} f d(\mu - \nu) \right|$$

which is a complete metric on $\mathcal{P}(\mathbb{E})$ and generates the narrow (or weak) topology, i.e. the coarsest topology on $\mathcal{P}(\mathbb{E})$ such that the mappings $\mu \mapsto \mu(f)$ are continuous for all bounded continuous $f : \mathbb{E} \rightarrow \mathbb{R}$.

Let (\mathbb{E}_1, d_1) and (\mathbb{E}_2, d_2) be separable and complete metric spaces. Let f be a continuous mapping from \mathbb{R} to \mathbb{E}_2 (resp. from $\mathbb{R} \times \mathbb{E}_1$ to \mathbb{E}_2). Let \mathcal{K} be a set of subsets of \mathbb{E}_1 . The function f is said to be *almost periodic* (respectively *almost periodic uniformly with respect to x in elements of \mathcal{K}*) if for every $\varepsilon > 0$ (respectively for every $\varepsilon > 0$ and every subset $K \in \mathcal{K}$), there exists a constant $l(\varepsilon, K) > 0$ such that any interval of length $l(\varepsilon, K)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} d_2(f(t + \tau), f(t)) < \varepsilon$$

(respectively $\sup_{t \in \mathbb{R}} \sup_{x \in K} d_2(f(t + \tau, x), f(t, x)) < \varepsilon$).

A characterization of almost periodicity is given in the following result, due to Bochner:

Theorem 2.1 *Let $f : \mathbb{R} \rightarrow \mathbb{H}_1$ be continuous. Then the following statements are equivalent*

- f is almost periodic.
- The set of translated functions $\{f(t + \cdot)\}_{t \in \mathbb{R}}$ is relatively compact in $\mathcal{C}(\mathbb{R}; \mathbb{E}_2)$ with respect to the uniform norm.
- f satisfies Bochner's double sequence criterion, that is, for every pair of sequences $\{\alpha'_n\} \subset \mathbb{R}$ and $\{\beta'_n\} \subset \mathbb{R}$, there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ respectively with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$(1) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} f(t + \alpha_n + \beta_m) \text{ and } \lim_{n \rightarrow \infty} f(t + \alpha_n + \beta_n),$$

exist and are equal.

Remark 2.2

- (i) A striking property of Bochner's double sequence criterion is that the limits in (1) exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on \mathbb{R} (with respect to $d_{\mathbb{E}}$). This criterion has thus the advantage that it allows to establish uniform convergence by checking pointwise convergence.
- (ii) The previous result holds for the metric spaces $(\mathcal{P}(\mathbb{E}), d_{\text{BL}})$ and $(\mathcal{P}(\mathcal{C}(\mathbb{R}, \mathbb{E})), d_{\text{BL}})$

2.2 Almost periodic stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \mathbb{R} \times \Omega \rightarrow \mathbb{H}_2$ be a stochastic process. We denote by $\text{law}(X(t))$ the distribution of the random variable $X(t)$. Following Tudor's terminology [22], we say that X has *almost periodic one-dimensional distributions* if the mapping $t \mapsto \text{law}(X(t))$ from \mathbb{R} to $(\mathcal{P}(\mathbb{H}_2), d_{\text{BL}})$ is almost periodic.

If X has continuous trajectories, we say that X is *almost periodic in distribution* if the mapping $t \mapsto \text{law}(X(t + \cdot))$ from \mathbb{R} to $\mathcal{P}(\mathcal{C}(\mathbb{R}; \mathbb{H}_2))$ is

almost periodic, where $C(\mathbb{R}; \mathbb{H}_2)$ is endowed with the uniform convergence on compact intervals and $\mathcal{P}(C(\mathbb{R}; \mathbb{H}_2))$ is endowed with the distance d_{BL} .

Let $L^2(\mathbb{P}, \mathbb{H}_2)$ be the space of \mathbb{H}_2 -valued random variables with a finite quadratic-mean. We say that a stochastic process $X : \mathbb{R} \rightarrow L^2(\mathbb{P}, \mathbb{H}_2)$ is *square-mean continuous* if, for every $s \in \mathbb{R}$,

$$\lim_{t \rightarrow s} \mathbf{E} \|X(t) - X(s)\|_{\mathbb{H}_2}^2 = 0.$$

We denote by $\text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$ the Banach space of square-mean continuous and uniformly bounded stochastic processes, endowed with the norm

$$\|X\|_{\infty}^2 = \sup_{t \in \mathbb{R}} (\mathbf{E} \|X(t)\|_{\mathbb{H}_2}^2).$$

A square-mean continuous stochastic process $X : \mathbb{R} \rightarrow L^2(\mathbb{P}, \mathbb{H}_2)$ is said to be *square-mean almost periodic* if, for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} \mathbf{E} \|X(t + \tau) - X(t)\|_{\mathbb{H}_2}^2 < \varepsilon.$$

The next theorem is interesting in itself, but we shall not use it in the sequel.

Theorem 2.3 *Let $F : \mathbb{R} \times \mathbb{H}_2 \rightarrow \mathbb{H}_2$ be an almost periodic function uniformly with respect to x in compact subsets of \mathbb{H}_2 such that*

$$\|F(t, x)\|_{\mathbb{H}_2} \leq C_1(1 + \|x\|_{\mathbb{H}_2}) \text{ and } \|F(t, x) - F(t, y)\|_{\mathbb{H}_2} \leq C_2\|x - y\|_{\mathbb{H}_2}.$$

Then the function

$$\tilde{F} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}_2) \rightarrow L^2(\mathbb{P}, \mathbb{H}_2)$$

(where $\tilde{F}(t, Y)(\omega) = F(t, Y(\omega))$ for every $\omega \in \Omega$) is square-mean almost periodic uniformly with respect to Y in compact subsets of $L^2(\mathbb{P}, \mathbb{H}_2)$.

Proof Let us prove that for each $Y \in L^2(\mathbb{P}, \mathbb{H}_2)$ the process $\tilde{F}_Y : \mathbb{R} \rightarrow L^2(\mathbb{P}, \mathbb{H}_2)$, $t \mapsto \tilde{F}(t, Y)$ is almost periodic.

For every $\delta > 0$, there exists a compact subset S of \mathbb{H}_2 such that

$$\mathbf{P}\{Y \notin S\} \leq \delta.$$

Let $\varepsilon > 0$, then there exist $\delta > 0$ and a compact subset S of \mathbb{H}_2 such that $\mathbf{P}\{Y \notin S\} \leq \delta$ and

$$\int_{\{Y \notin S\}} (1 + \|Y\|_{\mathbb{H}_2}^2) d\mathbf{P} < \frac{\varepsilon}{4C_1}.$$

Since F is almost periodic uniformly with respect to x in the compact subset S , there exists a constant $l(\varepsilon, S) > 0$ such that any interval of length $l(\varepsilon, S)$ contains at least a number τ for which

$$\sup_t \|F(t + \tau, x) - F(t, x)\|_{\mathbb{H}_2} < \frac{\sqrt{\varepsilon}}{\sqrt{2}} \text{ for all } x \in S.$$

We have

$$\begin{aligned} \mathbb{E}(\|F(t + \tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2) &= \int_{\{Y \in S\}} \|F(t + \tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2 d\mathbb{P} \\ &\quad + \int_{\{Y \notin S\}} \|F(t + \tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2 d\mathbb{P} \\ &< \frac{\varepsilon}{2} + 2C_1 \int_{\{Y \notin S\}} (1 + \|Y\|_{\mathbb{H}_2}^2) d\mathbb{P} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore the process \tilde{F}_Y is almost periodic. Since \tilde{F} is Lipschitz, it is almost periodic uniformly with respect to Y in compact subsets of $L^2(\mathbb{P}, \mathbb{H}_2)$ (see [12, Theorem 2.10 page 25]). \square

Proposition 2.4 *Let \mathcal{K} be a set of subsets of \mathbb{H}_2 . Let $F : \mathbb{R} \times \mathbb{H}_2 \rightarrow \mathbb{H}_2$, $(t, x) \mapsto F(t, x)$, be almost periodic, uniformly with respect to x in elements of \mathcal{K} . There exists a continuous function $F_0 : \mathbb{H}_2 \rightarrow \mathbb{H}_2$ such that*

$$(2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\kappa}^{\kappa+t} F(s, x) ds = F_0(x)$$

for every $\kappa \in \mathbb{R}$, uniformly with respect to x in elements of \mathcal{K} .

Furthermore, if $F(t, x)$ is Lipschitz in $x \in \mathbb{H}_2$ uniformly with respect to $t \in \mathbb{R}$, the mapping F_0 is Lipschitz too.

Proof Let $K \in \mathcal{K}$, and let $\mathbb{X} = \mathbb{H}_2^K$ be the Banach space of all mappings from K to \mathbb{H}_2 , endowed with the supremum norm. For every $t \in \mathbb{R}$, set

$$\widehat{F}(t) = (F(t, x))_{x \in K}, \quad \widehat{F}_0 = (F_0(x))_{x \in K}.$$

By [8, Theorem 6.11], we have, for the norm of \mathbb{X} ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{\kappa}^{\kappa+t} \widehat{F}(s) ds = \widehat{F}_0,$$

which proves (2). Alternatively, one may use the proof of [12, Theorem 3.1], though the result is given in a finite dimensional setting and for a compact set K .

The Lipschitz property of F_0 is trivial. \square

Let $Q \in L(\mathbb{H}_1)$ be a linear operator. Then Q is a bijection from $\text{range}(Q) = Q(\mathbb{H}_1)$ to $(\ker Q)^\perp$. We denote by Q^{-1} the *pseudo-inverse* of Q (see [20, Appendix C] or [10, Appendix B.2]), that is, the inverse of the mapping $(\ker Q)^\perp \rightarrow \text{range}(Q)$, $x \mapsto Q(x)$. Note that $\text{range}(Q)$ is a Hilbert space for the scalar product $\langle x, y \rangle_{\text{range}(Q)} = \langle Q^{-1}(x), Q^{-1}(y) \rangle$.

Proposition 2.5 *Let \mathcal{K} be a set of subsets of \mathbb{H}_2 . Let $G : \mathbb{R} \times \mathbb{H}_2 \rightarrow L(\mathbb{H}_1, \mathbb{H}_2)$, $t \mapsto G(t, x)$, be almost periodic uniformly with respect to x in elements of \mathcal{K} , and let $Q \in L(\mathbb{H}_1)$ be a self-adjoint nonnegative operator. Let $\mathbb{H}_0 = \text{range}(Q^{1/2})$, endowed with $\langle x, y \rangle_{\text{range}(Q^{1/2})} = \langle Q^{-1/2}(x), Q^{-1/2}(y) \rangle$. There exists a continuous function $G_0 : \mathbb{H}_2 \rightarrow L(\mathbb{H}_0, \mathbb{H}_2)$ such that*

$$(3) \quad \lim_{t \rightarrow \infty} \left| \frac{1}{t} \int_{\kappa}^{\kappa+t} G(s, x) Q G^*(s, x) ds - G_0(x) Q G_0^*(x) \right|_{\mathcal{N}} = 0$$

for all $\kappa \in \mathbb{R}$, uniformly with respect to x in elements of \mathcal{K} ,

where $G^*(s, x) = (G(s, x))^*$ and $G_0^*(x) = (G_0(x))^*$.

Proof Observe first that $G_0(x) Q G_0^*(x) = (G_0(x) Q^{1/2})(G_0(x) Q^{1/2})^*$, thus $G_0(x)$ does not need to be defined on the whole space \mathbb{H}_1 , it is sufficient that it be defined on \mathbb{H}_0 .

Since G is almost periodic, the function $H(s, x) = G(s, x) Q G^*(s, x)$ is almost periodic too, with positive self-adjoint nuclear values in $L(\mathbb{H}_2)$. Thus, reasoning as in the proof of Proposition 2.4, there exists a mapping $H_0 : \mathbb{H}_2 \rightarrow L(\mathbb{H}_2)$ such that, for every $\kappa \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{\kappa}^{\kappa+t} G(s, x) Q G^*(s, x) ds = H_0(x)$$

uniformly with respect to x in elements of \mathcal{K} . By e.g. [12, Theorem 3.1], H_0 is continuous. Thus the mapping

$$H_0^{1/2} : \begin{cases} \mathbb{H}_2 & \rightarrow L(\mathbb{H}_2) \\ x & \mapsto (H_0(x))^{1/2} \end{cases}$$

is continuous with positive self-adjoint values.

Let $G_0(x) = H_0^{1/2}(x)Q^{-1/2} : \mathbb{H}_0 \rightarrow \mathbb{H}_2$. We then have, for every $x \in \mathbb{H}_2$,

$$H_0(x) = G_0(x)Q(G_0(x))^*$$

and G_0 is continuous, which proves (3). \square

3 Solutions almost periodic in distribution

We consider the semilinear stochastic differential equation,

$$(4) \quad dX_t = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), t \in \mathbb{R}$$

Where $A : \text{Dom}(A) \subset \mathbb{H}_2 \rightarrow \mathbb{H}_2$ is a densely defined closed (possibly unbounded) linear operator, $F : \mathbb{R} \times \mathbb{H}_2 \rightarrow \mathbb{H}_2$, and $G : \mathbb{R} \times \mathbb{H}_2 \rightarrow L(\mathbb{H}_1, \mathbb{H}_2)$ are continuous functions. In this section, we assume that:

- (i) $W(t)$ is an \mathbb{H}_1 -valued Wiener process with nuclear covariance operator Q (we denote by $\text{tr } Q$ the trace of Q), defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$.
- (ii) $A : \text{Dom}(A) \rightarrow \mathbb{H}_2$ is the infinitesimal generator of a C_0 -semigroup $(S(t))_{t \geq 0}$ such that there exists a constant $\delta > 0$ with

$$\|S(t)\|_{L(\mathbb{H}_2)} \leq e^{-\delta t}, t \geq 0.$$

- (iii) There exists a constant K such that the mappings $F : \mathbb{R} \times \mathbb{H}_2 \rightarrow \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \rightarrow L(\mathbb{H}_1, \mathbb{H}_2)$ satisfy

$$\|F(t, x)\|_{\mathbb{H}_2} + \|G(t, x)\|_{L(\mathbb{H}_1, \mathbb{H}_2)} \leq K(1 + \|x\|_{\mathbb{H}_2})$$

- (iv) The functions F and G are Lipschitz, more precisely there exists a constant K such that

$$\|F(t, x) - F(t, y)\|_{\mathbb{H}_2} + \|G(t, x) - G(t, y)\|_{L(\mathbb{H}_1, \mathbb{H}_2)} \leq K\|x - y\|_{\mathbb{H}_2}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{H}_2$.

- (v) The mappings F and G are almost periodic in $t \in \mathbb{R}$ uniformly with respect to x in bounded subsets of \mathbb{H}_2 .

The assumptions in the following theorem are contained in those of Bezandry and Diagana [2, 3]. The result is similar to [9, Theorem 4.3], with different hypothesis and a different proof.

Theorem 3.1 *Let the assumptions (i) - (v) be fulfilled and the constant $\theta = \frac{K^2}{\delta} \left(\frac{1}{2\delta} + \text{tr } Q \right) < 1$. Then there exists a unique mild solution X to (4) in $\text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$. Furthermore, X has a.e. continuous trajectories, and $X(t)$ can be explicitly expressed as follows, for each $t \in \mathbb{R}$:*

$$(5) \quad X(t) = \int_{-\infty}^t S(t-s)F(s, X(s))ds + \int_{-\infty}^t S(t-s)G(s, X(s))dW(s).$$

If furthermore $\theta' = \frac{4K^2}{\delta} \left(\frac{1}{\delta} + \text{tr } Q \right) < 1$, then X is almost periodic in distribution.

To prove Theorem 3.1, we need several preliminary results. Let us first recall the following result, which is given in a more general form in [9]:

Proposition 3.2 *([9, Proposition 3.1-(c)]) Let $\tau \in \mathbb{R}$. Let $(\xi_n)_{0 \leq n \leq \infty}$ be a sequence of square integrable \mathbb{H}_2 -valued random variables. Let $(F_n)_{0 \leq n \leq \infty}$ and $(G_n)_{0 \leq n \leq \infty}$ be sequences of mappings from $\mathbb{R} \times \mathbb{H}_2$ to \mathbb{H}_2 and $L(\mathbb{H}_1, \mathbb{H}_2)$ respectively, satisfying (iii) and (iv) (replacing F and G by F_n and G_n respectively, and the constant K being independent of n). For each n , let X_n denote the solution of*

$$\begin{aligned} X_n(t) &= S(t-\tau)\xi_n \\ &+ \int_{\tau}^t S(t-s)F_n(s, X_n(s))ds + \int_{\tau}^t S(t-s)G_n(s, X_n(s))dW(s). \end{aligned}$$

Assume that, for every $(t, x) \in \mathbb{R} \times \mathbb{H}_2$,

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(t, x) &= F_{\infty}(t, x), \quad \lim_{n \rightarrow \infty} G_n(t, x) = G_{\infty}(t, x), \\ \lim_{n \rightarrow \infty} d_{\text{BL}}(\text{law}(\xi_n, W), \text{law}(\xi_{\infty}, W)) &= 0, \end{aligned}$$

(the last equality takes place in $\mathcal{P}(\mathbb{H}_2 \times C(\mathbb{R}, \mathbb{H}_1))$). Then we have in $\mathcal{C}([\tau, T]; \mathbb{H}_2)$, for any $T > \tau$,

$$\lim_{n \rightarrow \infty} d_{\text{BL}}(\text{law}(X_n), \text{law}(X_{\infty})) = 0.$$

We need also a variant of Gronwall's lemma, tailored for mild solutions.

Lemma 3.3 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that, for every $t \in \mathbb{R}$,*

$$(6) \quad 0 \leq g(t) \leq \alpha(t) + \beta_1 \int_{-\infty}^t e^{-\delta_1(t-s)}g(s) ds + \dots + \beta_n \int_{-\infty}^t e^{-\delta_n(t-s)}g(s) ds,$$

for some locally integrable function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$, and for some constants $\beta_1, \dots, \beta_n \geq 0$, and some constants $\delta_1, \dots, \delta_n > \beta$, where $\beta := \sum_{i=1}^n \beta_i$. We assume that the integrals in the right hand side of (6) are convergent. Let $\delta = \min_{1 \leq i \leq n} \delta_i$. Then, for every $\gamma \in]0, \delta - \beta]$ such that $\int_{-\infty}^0 e^{\gamma s} \alpha(s) ds$ converges, we have, for every $t \in \mathbb{R}$,

$$(7) \quad g(t) \leq \alpha(t) + \beta \int_{-\infty}^t e^{-\gamma(t-s)} \alpha(s) ds.$$

In particular, if α is constant, we have

$$(8) \quad g(t) \leq \alpha \frac{\delta}{\delta - \beta}.$$

Proof Let $\beta'_i = \beta_i/\beta$, $i = 1, \dots, n$. We have

$$\begin{aligned} & \frac{d}{dt} \left(e^{\gamma t} \sum_{i=1}^n \beta'_i \int_{-\infty}^t e^{-\delta_i(t-s)} g(s) ds \right) \\ &= \frac{d}{dt} \left(\sum_{i=1}^n \beta'_i e^{(\gamma - \delta_i)t} \int_{-\infty}^t e^{\delta_i s} g(s) ds \right) \\ &= \sum_{i=1}^n (\gamma - \delta_i) e^{(\gamma - \delta_i)t} \beta'_i \int_{-\infty}^t e^{\delta_i s} g(s) ds + \sum_{i=1}^n e^{(\gamma - \delta_i)t} e^{\delta_i t} \beta'_i g(t) \\ &= e^{\gamma t} \left(g(t) + \sum_{i=1}^n (\gamma - \delta_i) \beta'_i \int_{-\infty}^t e^{-\delta_i(t-s)} g(s) ds \right) \\ &\leq e^{\gamma t} \alpha(t). \end{aligned}$$

The last inequality holds because $\gamma - \delta_i \leq -\beta$ and $g \geq 0$. Integrating on $] -\infty, t]$, we get

$$e^{\gamma t} \sum_{i=1}^n \beta'_i \int_{-\infty}^t e^{-\delta_i(t-s)} g(s) ds \leq \int_{-\infty}^t e^{\gamma s} \alpha(s) ds$$

(because both terms go to 0 when $t \rightarrow -\infty$), *i.e.*

$$(9) \quad \sum_{i=1}^n \beta'_i \int_{-\infty}^t e^{-\delta_i(t-s)} g(s) ds \leq e^{-\gamma t} \int_{-\infty}^t e^{\gamma s} \alpha(s) ds.$$

Using (9) in (6) yields

$$g(t) \leq \alpha(t) + \beta \sum_{i=1}^n \beta'_i \int_{-\infty}^t e^{-\delta_i(t-s)} g(s) ds \leq \alpha(t) + \beta e^{-\gamma t} \int_{-\infty}^t e^{\gamma s} \alpha(s) ds.$$

Inequality (8) is a direct consequence of (7), with $\gamma = \delta - \beta$. \square

Lemma 3.3 will help us (among other things) state a result (Proposition 3.5) on estimation of the moments of solutions of (5), which will be useful in the proof of Theorem 3.1 as well as in the proof of Theorem 4.2. To prove this result, we need also one of the moment inequalities for stochastic integrals due to Novikov [19]. These inequalities are proved in a finite dimensional setting in [19], but their proofs extend easily in infinite dimension. For the sake of completeness, we give the proof of the inequality we need (for $p \geq 2$), in our setting.

Lemma 3.4 *Let $p \geq 2$, and let Y be an $L(\mathbb{H}_1, \mathbb{H}_2)$ -valued adapted stochastic process. We have, for every $t \geq 0$,*

$$(10) \quad \mathbb{E} \left\| \int_0^t Y(s) dW(s) \right\|^p \leq C_p \mathbb{E} \left(\int_0^t \text{tr}(Y(s)QY^*(s)) ds \right)^{p/2}$$

with

$$C_p = \frac{1}{(2c)^{p/2}} \left(\frac{2+2c}{p-1} - 2^{p/2} \right)$$

for any $c > (p-1)2^{p/2-1} - 1$. In particular, $C_2 = 1$ (and in that case, (10) is an equality).

Proof We denote

$$Z(t) = \int_0^t Y(s) dW(s), \quad V(t) = \int_0^t \text{tr}(Y(s)QY^*(s)) ds.$$

Let us first assume that $\|Y(t)\| \leq M$ a.e. for some constant M and for every $t \geq 0$. Let $\alpha, c \geq 0$. Let

$$X(t) = \alpha + cV(t) + \|Z(t)\|^2.$$

By Itô's formula (e.g. [10, Theorem 4.17]), we have

$$dX(t) = (1+c) dV(t) + 2 \langle Z(t), Y(t) dW(t) \rangle.$$

Denoting by $[X]$ the quadratic variation of X , we have thus

$$\begin{aligned} d(X(t))^{p/2} &= \frac{p}{2} X^{p/2-1}(t) dX(t) + \frac{1}{2} \frac{p}{2} \left(\frac{p}{2} - 1 \right) X^{p/2-2} d[X](t) \\ &= \left[\frac{p}{2} (1+c) \left(\alpha + cV(t) + \|Z(t)\|^2 \right)^{p/2-1} \text{tr}(Y(t)QY^*(t)) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{p}{2} (p-2) \left(\alpha + cV(t) + \|Z(t)\|^2 \right)^{p/2-2} \left\langle Z(t), Y(t)Q^{1/2} \right\rangle^2 \Big) dt \\
& + p \left(\alpha + cV(t) + \|Z(t)\|^2 \right)^{p/2-1} \langle Z(t), Y(t) dW(t) \rangle.
\end{aligned}$$

By the boundedness hypothesis on Y , the process Z is a continuous martingale and $e^{\|Z\|-\frac{1}{2}V}$ is a supermartingale (actually it is a martingale, see [21, Theorem IV.37.8]). We thus have

$$\mathbb{E} \left(e^{\|Z(t)\|} \right) = \mathbb{E} \left(e^{\frac{1}{2}V(t)} \right) \leq e^{\frac{1}{2} \text{tr} QM^2 t},$$

thus the moments of any order of $\|Z(t)\|$ are bounded. We deduce

$$\begin{aligned}
(11) \quad & \mathbb{E} \left(\alpha + cV(t) + \|Z(t)\|^2 \right)^{p/2} \\
& = \alpha^{p/2} + \frac{p}{2}(1+c) \mathbb{E} \int_0^t \left(\alpha + cV(s) + \|Z(s)\|^2 \right)^{p/2-1} \text{tr} (Y(s)QY^*(s)) ds \\
& \quad + \frac{p}{2}(p-2) \mathbb{E} \int_0^t \left(\alpha + cV(s) + \|Z(s)\|^2 \right)^{p/2-2} \left\langle Z(s), Y(s)Q^{1/2} \right\rangle^2 ds.
\end{aligned}$$

In particular, for $c = 0$,

$$\begin{aligned}
& \mathbb{E} \left(\alpha + \|Z(t)\|^2 \right)^{p/2} \\
& \leq \alpha^{p/2} + \frac{p}{2} \mathbb{E} \int_0^t \left(\alpha + \|Z(s)\|^2 \right)^{p/2-1} \text{tr} (Y(s)QY^*(s)) ds \\
& \quad + \frac{p}{2}(p-2) \mathbb{E} \int_0^t \left(\alpha + \|Z(s)\|^2 \right)^{p/2-2} \|Z(s)\|^2 \text{tr} (Y(s)QY^*(s)) ds
\end{aligned}$$

which yields

$$(12) \quad \mathbb{E} (\|Z(t)\|^p) \leq \frac{p}{2} (p-1) \mathbb{E} \int_0^t \left(\alpha + \|Z(s)\|^2 \right)^{p/2-1} \text{tr} (Y(s)QY^*(s)) ds.$$

On the other hand, (11) implies

$$\begin{aligned}
(13) \quad & 2^{p/2-1} \left(c^{p/2} \mathbb{E} (V(t))^{p/2} + \mathbb{E} \left(\alpha + \|Z(s)\|^2 \right)^{p/2} \right) \\
& \geq \alpha^{p/2} + \frac{p}{2}(1+c) \mathbb{E} \int_0^t \left(\alpha + \|Z(s)\|^2 \right)^{p/2-1} \text{tr} (Y(s)QY^*(s)) ds.
\end{aligned}$$

Substituting (12) in the right hand side of (13), we get that

$$\frac{(2c)^{p/2}}{2} \mathbb{E} (V(t))^{p/2} + 2^{p/2-1} \mathbb{E} \left(\alpha + \|Z(t)\|^2 \right)^{p/2} \geq \alpha^{p/2} + \frac{1+c}{p-1} \mathbb{E} \left(\alpha + \|Z(t)\|^2 \right)^{p/2}.$$

Taking the limit when α goes to 0, we then get the result for the case when Y is uniformly a.e. bounded.

In the general case, let us denote, for every integer $N \geq 1$,

$$Y^N(t) = \begin{cases} Y(t) & \text{if } \|Y(t)\| \leq N, \\ N \frac{Y(t)}{\|Y(t)\|} & \text{if } \|Y(t)\| \geq N. \end{cases}$$

By Itô's isometry, $X^N(t) := \int_0^t Y^N(s) dW(s)$ converges in quadratic mean to $X(t)$, thus there exists a subsequence (still denoted by $(X^N(t))$ for simplicity) which converges almost everywhere to X . On the other hand, if (e_k) is an orthonormal basis of \mathbb{H}_2 , we have also, for every s ,

$$\text{tr} (Y^N(s)Q(Y^N)^*(s)) = \sum_k \left\| Q^{1/2}(Y^N)^*(s)e_k \right\|^2 \nearrow \text{tr} (Y(s)QY^*(s)) \text{ as } N \rightarrow \infty.$$

Using Fatou's and Beppo Levi's lemmas, we thus obtain

$$\begin{aligned} \mathbb{E} \left\| \int_0^t Y(s) dW(s) \right\|^p &= \mathbb{E} \left(\liminf_N X^N(t) \right) \\ &\leq \liminf_N \mathbb{E} (X^N(t)) \\ &\leq C_p \liminf_N \mathbb{E} \left(\int_0^t \text{tr} (Y^N(s)Q(Y^N)^*(s)) ds \right)^{p/2} \\ &= C_p \mathbb{E} \left(\int_0^t \text{tr} (Y(s)QY^*(s)) ds \right)^{p/2}. \end{aligned}$$

□

Proposition 3.5 *With the notations of Theorem 3.1 and Lemma 3.4, assume that the process $X \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$ satisfies (5). Then, for every $p \geq 2$, if*

$$\theta'_p := \frac{2^{3p/2-1} K^p}{\delta^{p/2}} \left(\frac{2^{p/2-1}}{\delta^{p/2}} + C_p (\text{tr} Q)^{p/2} \right) < 1,$$

the family $(X_t)_{t \in \mathbb{R}}$ is bounded in L^p by a constant which depends only on p , K , δ and $\text{tr} Q$.

In particular, $\theta'_2 = \theta'$, where θ' is the constant given in Theorem 3.1. If $\theta' < 1$, we have, for every $t \in \mathbb{R}$,

$$\mathbb{E} \left(\|X(t)\|^2 \right) \leq \frac{\theta'}{1 - \theta'}.$$

Proof We have, using Lemma 3.4,

$$\begin{aligned} \mathbb{E} (\|X(t)\|^p) &\leq 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^t S(t-s) F(s, X(s)) ds \right\|^p \\ &\quad + 2^{p-1} \mathbb{E} \left\| \int_{-\infty}^t S(t-s) G(s, X(s)) dW(s) \right\|^p \\ &\leq 2^{p-1} \mathbb{E} \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s, X(s))\| ds \right)^p \\ &\quad + 2^{p-1} C_p \left(\text{tr } Q \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} \|G(s, X(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^2 ds \right)^{p/2} \\ &\leq 2^{p-1} \frac{1}{\delta^{p-1}} \mathbb{E} \int_{-\infty}^t e^{-\delta(t-s)} \|F(s, X(s))\|^p ds \\ &\quad + 2^{p-1} C_p (\text{tr } Q)^{p/2} \frac{1}{(2\delta)^{p/2-1}} \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} \|G(s, X(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^p ds \end{aligned}$$

(applying Jensen's inequality under the probabilities $\delta e^{-\delta(t-s)} ds$ and $2\delta e^{-2\delta(t-s)} ds$)

$$\begin{aligned} &\leq 2^{p-1} \frac{1}{\delta^{p-1}} K^p \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} (1 + \|X(s)\|)^p ds \\ &\quad + 2^{p/2} C_p (\text{tr } Q)^{p/2} \frac{1}{\delta^{p/2-1}} K^p \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} (1 + \|X(s)\|)^p ds \\ &\leq 2^{p-1} \left(2^{p-1} \frac{1}{\delta^{p-1}} K^p \int_{-\infty}^t e^{-\delta(t-s)} ds + 2^{p/2} C_p (\text{tr } Q)^{p/2} \frac{1}{\delta^{p/2-1}} K^p \int_{-\infty}^t e^{-2\delta(t-s)} ds \right) \\ &\quad + 2^{p-1} 2^{p-1} \frac{1}{\delta^{p-1}} K^p \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} (\|X(s)\|)^p ds \\ &\quad + 2^{p-1} 2^{p/2} C_p (\text{tr } Q)^{p/2} \frac{1}{\delta^{p/2-1}} K^p \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} (\|X(s)\|)^p ds \\ &= \alpha + \beta_1 \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} (\|X(s)\|^p) ds + \beta_2 \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} (\|X(s)\|^p) ds \end{aligned}$$

with

$$\alpha = \frac{2^{3p/2-2} K^p}{\delta^{p/2}} \left(\frac{2^{p/2}}{\delta^{p/2}} + C_p (\text{tr } Q)^{p/2} \right),$$

$$\beta_1 = \frac{2^{2p-2}K^p}{\delta^{p-1}}, \quad \beta_2 = \frac{2^{3p/2-1}K^p C_p (\text{tr } Q)^{p/2}}{\delta^{p/2-1}}.$$

The hypothesis $\theta'_p < 1$ is equivalent to $\delta > \beta$, with $\beta = \beta_1 + \beta_2$. We conclude by Lemma 3.3 that

$$\mathbb{E}(\|X(t)\|^p) \leq \alpha \frac{\delta}{\delta - \beta}.$$

In the case when $p = 2$, we have $C_2 = 1$, thus

$$\alpha = \frac{4K^2}{\delta} \left(\frac{1}{\delta} + \frac{\text{tr } Q}{2} \right) \text{ and } \beta = 4K^2 \left(\frac{1}{\delta} + \text{tr } Q \right)$$

and

$$\mathbb{E} \left(\|X(t)\|^2 \right) \leq \frac{4K^2 \left(\frac{1}{\delta} + \frac{\text{tr } Q}{2} \right)}{\delta - 4K^2 \left(\frac{1}{\delta} + \text{tr } Q \right)} \leq \frac{4K^2 \left(\frac{1}{\delta} + \text{tr } Q \right)}{\delta - 4K^2 \left(\frac{1}{\delta} + \text{tr } Q \right)} = \frac{\theta'}{1 - \theta'}.$$

□

Remark 3.6 We can choose C_p in Lemma 3.4 such that $\lim_{p \rightarrow 2+} C_p = 1$, which implies $\lim_{p \rightarrow 2+} \theta'_p = \theta'$. Thus, if $\theta' < 1$, and if $X \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$ satisfies (5), the family $(X_t)_{t \in \mathbb{R}}$ is bounded in L^p for some $p > 2$.

Proof of Theorem 3.1 Note that

$$X(t) = \int_{-\infty}^t S(t-s)F(s, X(s))ds + \int_{-\infty}^t S(t-s)G(s, X(s))dW(s)$$

satisfies

$$X(t) = S(t-s)X(s) + \int_s^t S(t-s)F(s, X(s))ds + \int_s^t S(t-s)G(s, X(s))dW(s)$$

for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X is a mild solution to (4).

We introduce an operator L by

$$LX(t) = \int_{-\infty}^t S(t-s)F(s, X(s))ds + \int_{-\infty}^t S(t-s)G(s, X(s))dW(s).$$

It can be seen easily that the operator L maps $\text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$ into itself.

First step. Let us show that L has a unique fixed point. We have, for any $t \in \mathbb{R}$,

$$\begin{aligned}
& \mathbb{E} \|(LX)(t) - (LY)(t)\|_{\mathbb{H}_2}^2 \\
& \leq 2 \mathbb{E} \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2} ds \right)^2 \\
& \quad + 2 \mathbb{E} \left(\left\| \int_{-\infty}^t S(t-s) [G(s, X(s)) - G(s, Y(s))] dW(s) \right\|_{\mathbb{H}_2} \right)^2 \\
& = I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 & \leq 2 \int_{-\infty}^t e^{-\delta(t-s)} ds \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} \|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2}^2 ds \\
& \leq 2K^2 \int_{-\infty}^t e^{-\delta(t-s)} ds \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2 ds \\
& \leq 2K^2 \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^2 \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2 \\
& \leq \frac{K^2}{2\delta^2} \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2.
\end{aligned}$$

For I_2 , using the isometry identity we get

$$\begin{aligned}
I_2 & \leq 2 \operatorname{tr} Q \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} \|G(s, X(s)) - G(s, Y(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^2 ds \\
& \leq 2 \operatorname{tr} Q K^2 \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2 ds \\
& \leq 2K^2 \operatorname{tr} Q \left(\int_{-\infty}^t e^{-2\delta(t-s)} ds \right) \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2 \\
& \leq \frac{K^2 \operatorname{tr} Q}{\delta} \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2.
\end{aligned}$$

Thus

$$\mathbb{E} \|(LX)(t) - (LY)(t)\|_{\mathbb{H}_2}^2 \leq I_1 + I_2 \leq \theta \sup_{s \in \mathbb{R}} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2.$$

Consequently, as $\theta < 1$, we deduce that L is a contraction operator, hence there exists a unique mild solution to (4) in $\text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_1))$.

Furthermore, by [10, Theorem 7.4], almost all trajectories of this solution are continuous.

Second step. We assume now that $\theta' < 1$. Let us show that X is almost periodic in distribution. We use Bochner's double sequences criterion. Let

(α'_n) and (β'_n) be two sequences in \mathbb{R} . We show that there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$(14) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mu(t + \alpha_n + \beta_m) \text{ and } \lim_{n \rightarrow \infty} \mu(t + \alpha_n + \beta_n),$$

exist and are equal, where $\mu(t) := \text{law}(X)(t)$ is the law or distribution of $X(t)$.

Since F and G are almost periodic, there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that

$$(15) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} F(t + \alpha_n + \beta_m, x) = \lim_{n \rightarrow \infty} F(t + \alpha_n + \beta_n, x) =: F_0(t, x)$$

and

$$(16) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} G(t + \alpha_n + \beta_m, x) = \lim_{n \rightarrow \infty} G(t + \alpha_n + \beta_n, x) =: G_0(t, x).$$

These limits exist uniformly with respect to $t \in \mathbb{R}$ and x in bounded subsets of \mathbb{H}_2 .

Set now $(\gamma_n) = (\alpha_n + \beta_n)$. For each fixed integer n , we consider

$$X^n(t) = \int_{-\infty}^t S(t-s)F(s+\gamma_n, X^n(s))ds + \int_{-\infty}^t S(t-s)G(s+\gamma_n, X^n(s))dW(s)$$

the mild solution of

$$(17) \quad dX^n(t) = AX^n(t)dt + F(t + \gamma_n, X^n(t))dt + G(t + \gamma_n, X^n(t))dW(t)$$

and

$$X^0(t) = \int_{-\infty}^t S(t-s)F_0(s, X^0(s))ds + \int_{-\infty}^t S(t-s)G_0(s, X^0(s))dW(s)$$

the mild solution of

$$(18) \quad dX^0(t) = AX^0(t)dt + F_0(t, X^0(t))dt + G_0(t, X^0(t))dW(t).$$

Make the change of variable $\sigma - \gamma_n = s$, the process

$$\begin{aligned} X(t + \gamma_n) &= \int_{-\infty}^{t+\gamma_n} S(t + \gamma_n - s)F(s, X(s))ds \\ &\quad + \int_{-\infty}^{t+\gamma_n} S(t + \gamma_n - s)G(s, X(s))dW(s) \end{aligned}$$

becomes

$$X(t + \gamma_n) = \int_{-\infty}^t S(t-s)F(s + \gamma_n, X(s + \gamma_n))ds \\ + \int_{-\infty}^t S(t-s)G(s + \gamma_n, X(s + \gamma_n))d\tilde{W}_n(s),$$

where $\tilde{W}_n(s) = W(s + \gamma_n) - W(\gamma_n)$ is a Brownian motion with the same distribution as $W(s)$. From the independence of the increments of W , we deduce that the process $X(t + \gamma_n)$ has the same distribution as $X^n(t)$.

Let us show that $X^n(t)$ converges in quadratic mean to $X^0(t)$ for each fixed $t \in \mathbb{R}$. We have

$$\begin{aligned} & \mathbf{E}\|X^n(t) - X^0(t)\|^2 \\ &= \mathbf{E}\left\| \int_{-\infty}^t S(t-s)[F(s + \gamma_n, X^n(s)) - F_0(s, X^0(s))]ds \right. \\ & \quad \left. + \int_{-\infty}^t S(t-s)[G(s + \gamma_n, X^n(s)) - G_0(s, X^0(s))]dW(s) \right\|^2 \\ &\leq 2\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[F(s + \gamma_n, X^n(s)) - F_0(s, X^0(s))]ds \right\|^2 \\ & \quad + 2\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[G(s + \gamma_n, X^n(s)) - G_0(s, X^0(s))]dW(s) \right\|^2 \\ &\leq 4\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[F(s + \gamma_n, X^n(s)) - F(s + \gamma_n, X^0(s))]ds \right\|^2 \\ & \quad + 4\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))]ds \right\|^2 \\ & \quad + 4\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[G(s + \gamma_n, X^n(s)) - G(s + \gamma_n, X^0(s))]dW(s) \right\|^2 \\ & \quad + 4\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[G(s + \gamma_n, X^0(s)) - G_0(s, X^0(s))]dW(s) \right\|^2 \\ &\leq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now, using (ii), (iv) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} I_1 &= 4\mathbf{E}\left\| \int_{-\infty}^t S(t-s)[F(s + \gamma_n, X^n(s)) - F(s + \gamma_n, X^0(s))]ds \right\|^2 \\ &\leq 4\mathbf{E}\left(\int_{-\infty}^t \|S(t-s)\| \|F(s + \gamma_n, X^n(s)) - F(s + \gamma_n, X^0(s))\| ds \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 4 \mathbf{E} \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s + \gamma_n, X^n(s)) - F(s + \gamma_n, X^0(s))\| ds \right)^2 \\
&\leq 4 \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right) \left(\int_{-\infty}^t e^{-\delta(t-s)} \mathbf{E} \|F(s + \gamma_n, X^n(s)) - F(s + \alpha_n, X^0(s))\|^2 ds \right) \\
&\leq \frac{4K^2}{\delta} \int_{-\infty}^t e^{-\delta(t-s)} \mathbf{E} \|X^n(s) - X^0(s)\|^2 ds.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
I_2 &= 4 \mathbf{E} \left\| \int_{-\infty}^t S(t-s) [F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))] ds \right\|^2 \\
&\leq 4 \mathbf{E} \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))\| ds \right)^2 \\
&\leq 4 \mathbf{E} \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right) \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2 ds \right) \\
&\leq 4 \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^2 \sup_s \mathbf{E} \|F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2 \\
&\leq \frac{4}{\delta^2} \sup_s \mathbf{E} \|F(s + \gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2,
\end{aligned}$$

which converges to 0 as $n \rightarrow \infty$ because $\sup_{t \in \mathbb{R}} \mathbf{E} \|X^0(t)\|^2 < \infty$ which implies that $\{X^0(t)\}_t$ is tight relatively to bounded sets.

Applying Itô's isometry, we get

$$\begin{aligned}
I_3 &= 4 \mathbf{E} \left\| \int_{-\infty}^t S(t-s) [G(s + \gamma_n, X^n(s)) - G(s + \gamma_n, X^0(s))] dW(s) \right\|^2 \\
&\leq 4 \operatorname{tr} Q \mathbf{E} \int_{-\infty}^t \|S(t-s)\|^2 \|G(s + \gamma_n, X^n(s)) - G(s + \gamma_n, X^0(s))\|^2 ds \\
&\leq 4 \operatorname{tr} Q \int_{-\infty}^t e^{-2\delta(t-s)} \mathbf{E} \|G(s + \gamma_n, X^n(s)) - G(s + \gamma_n, X^0(s))\|^2 ds \\
&\leq 4K^2 \operatorname{tr} Q \int_{-\infty}^t e^{-2\delta(t-s)} \mathbf{E} \|X^n(s) - X^0(s)\|^2 ds.
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= 4 \mathbf{E} \left\| \int_{-\infty}^t S(t-s) [G(s + \gamma_n, X^0(s)) - G_0(s, X^0(s))] dW(s) \right\|^2 \\
&\leq 4 \operatorname{tr} Q \mathbf{E} \left(\int_{-\infty}^t \|S(t-s)\|^2 \|G(s + \gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2 ds \right)
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \operatorname{tr} Q \left(\int_{-\infty}^t e^{-2\delta(t-s)} ds \right) \sup_{s \in \mathbb{R}} \mathbb{E} \|G(s + \gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2 \\
&\leq \frac{2 \operatorname{tr} Q}{\delta} \sup_{s \in \mathbb{R}} \mathbb{E} \|G(s + \gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2.
\end{aligned}$$

For the same reason as for I_2 , the right hand term goes to 0 as $n \rightarrow \infty$.

We thus have

$$\begin{aligned}
\mathbb{E} \|X^n(t) - X^0(t)\|^2 &\leq \alpha_n + \frac{4K^2}{\delta} \int_{-\infty}^t e^{-\delta(t-s)} \mathbb{E} \|X^n(s) - X^0(s)\|^2 ds \\
&\quad + 4K^2 \operatorname{tr} Q \int_{-\infty}^t e^{-2\delta(t-s)} \mathbb{E} \|X^n(s) - X^0(s)\|^2 ds
\end{aligned}$$

for a sequence (α_n) such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Furthermore, $\beta := \frac{4K^2}{\delta} + 4K^2 \operatorname{tr} Q < \delta$. We conclude by Lemma 3.3 that

$$\lim_{n \rightarrow \infty} \mathbb{E} \|X^n(t) - X^0(t)\|^2 = 0,$$

hence $X^n(t)$ converges in distribution to $X^0(t)$. But, since the distribution of $X^n(t)$ is the same as that of $X(t + \gamma_n)$, we deduce that $X(t + \gamma_n)$ converges in distribution to $X^0(t)$, i.e.

$$\lim_{n \rightarrow \infty} \mu(t + \alpha_n + \beta_n) = \operatorname{law}(X^0(t)) =: \mu_t^0.$$

By analogy and using (15), (16) we can easily deduce that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mu(t + \alpha_n + \beta_m) = \mu_t^0.$$

We have thus proved that X has almost periodic one-dimensional distributions. To prove that X is almost periodic in distribution, we apply Proposition 3.2: for fixed $\tau \in \mathbb{R}$, let $\xi_n = X(\tau + \alpha_n)$, $F_n(t, x) = F(t + \alpha_n, x)$, $G_n(t, x) = G(t + \alpha_n, x)$. By the foregoing, (ξ_n) converges in distribution to some variable $Y(\tau)$. We deduce that (ξ_n) is tight, and thus (ξ_n, W) is tight also. We can thus choose $Y(\tau)$ such that (ξ_n, W) converges in distribution to $(Y(\tau), W)$. Then, by Proposition 3.2, for every $T \geq \tau$, $X(\cdot + \alpha_n)$ converges in distribution on $C([\tau, T]; \mathbb{H}_2)$ to the (unique in distribution) solution to

$$Y(t) = S(t - \tau)Y(\tau) + \int_{\tau}^t S(t - s)F(s, Y(s)) ds + \int_{\tau}^t S(t - s)G(s, Y(s)) dW(s).$$

Note that Y does not depend on the chosen interval $[\tau, T]$, thus the convergence takes place on $C(\mathbb{R}; \mathbb{H}_2)$. Similarly, $Y_n := Y(\cdot + \beta_n)$ converges in distribution on $C(\mathbb{R}; \mathbb{H}_2)$ to a continuous process Z such that, for $t \geq \tau$,

$$Z(t) = S(t-\tau)Z(\tau) + \int_{\tau}^t S(t-s)F(s, Z(s))ds + \int_{\tau}^t S(t-s)G(s, Z(s))dW(s).$$

But, by (15) and (16), $X(\cdot + \gamma_n)$ converges in distribution to the same process Z . Thus X is almost periodic in distribution. \square

4 Weak averaging

In this section, we strengthen slightly the assumptions on the semigroup S but we replace the condition of almost periodicity on F and G by a weaker condition that keeps only the features of almost periodicity that are useful for averaging. More precisely, we assume Conditions (i), (iii), and (iv) of Section 3, but we replace Condition (ii) by the stronger Condition (ii') below and Condition (v) by the weaker Condition (v') below :

(ii') Condition (ii) is satisfied and the semigroup S is immediately norm continuous (see [11, Definition II.4.24]), i.e. the mapping $t \mapsto S(t)$ is continuous in operator norm on $]0, \infty[$.

(v') The mappings $F : \mathbb{R} \times \mathbb{H}_2 \rightarrow \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \rightarrow L(\mathbb{H}_1, \mathbb{H}_2)$ satisfy :

(a) For every compact subset K of \mathbb{H}_2 , the sets

$$\{F(t, x); t \in \mathbb{R}, x \in K\} \text{ and } \{G(t, x); t \in \mathbb{R}, x \in K\}$$

are compact.

(b) There exist continuous functions $F_0 : \mathbb{H}_2 \rightarrow \mathbb{H}_2$ and $G_0 : \mathbb{H}_2 \rightarrow L(\mathbb{H}_0, \mathbb{H}_2)$ satisfying (2) and (3) uniformly with respect to x in compact subsets of \mathbb{H}_2 .

Contrarily to [24], no condition of analyticity of S is required. Condition (ii') is satisfied by a broad class of semigroups, see [11] for details. Condition (v') is weaker than (v) thanks to Propositions 2.4 and 2.5.

Let us define the Hilbert space $\mathbb{H}_0 = \text{range}(Q^{1/2})$ as in Proposition 2.5, where Q is the covariance operator of the Wiener process W .

Lemma 4.1 *Under Hypothesis (i), (ii'), (iii), (iv), and (v'), for any continuous function $x : \mathbb{R} \rightarrow \mathbb{H}_2$, we have*

(19)

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\kappa}^{\kappa+t} S(\kappa+t-s)F\left(\frac{s}{\varepsilon}, x(s)\right) ds = \int_{\kappa}^{\kappa+t} S(\kappa+t-s)F_0(x(s)) ds$$

(20)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\kappa}^{\kappa+t} \left(S(\kappa+t-s)G\left(\frac{s}{\varepsilon}, x(s)\right)QG^*\left(\frac{s}{\varepsilon}, x(s)\right)S^*(\kappa+t-s) \right. \right. \\ \left. \left. - S(\kappa+t-s)G_0(x(s))QG_0^*(x(s))S^*(\kappa+t-s) \right) ds \right|_{\mathcal{N}} = 0 \end{aligned}$$

for all $\kappa \in \mathbb{R}$ and $t > 0$.

Proof Let $\kappa \in \mathbb{R}$ and $t > 0$. Let $\gamma > 0$, and let us choose $\alpha > 0$ such that

$$\left\| \int_{\kappa+t-\alpha}^{\kappa+t} S(\kappa+t-s)F\left(\frac{s}{\varepsilon}, x(s)\right) ds \right\| < \gamma$$

and

$$\left\| \int_{\kappa+t-\alpha}^{\kappa+t} S(\kappa+t-s)F_0(x(s)) ds \right\| < \gamma.$$

This is possible since the functions inside the integrals are bounded. By Condition (ii'), the semigroup S is uniformly continuous on $[\alpha, t]$. We can thus divide the interval $[\kappa, \kappa+t-\alpha]$ by a partition $(\kappa+ih)_{i=0, \dots, N}$, in such a way that $\|S(\kappa+t-s) - S(t+ih)\| < \gamma$ for $s \in [\kappa+ih, \kappa+(i+1)h]$, $i = 0, \dots, N-1$. We have

$$\int_{\kappa}^{\kappa+t-\alpha} S(\kappa+t-s)F\left(\frac{s}{\varepsilon}, x(s)\right) ds = \sum_{i=0}^{N-1} \int_{\kappa+ih}^{\kappa+(i+1)h} S(\kappa+t-s)F\left(\frac{s}{\varepsilon}, x(s)\right) ds$$

and

$$\int_{\kappa}^{\kappa+t-\alpha} S(\kappa+t-s)F_0(x(s)) ds = \sum_{i=0}^{N-1} \int_{\kappa+ih}^{\kappa+(i+1)h} S(\kappa+t-s)F_0(x(s)) ds.$$

But, for some constant C and $i = 0, \dots, N-1$,

$$\left\| \int_{\kappa+ih}^{\kappa+(i+1)h} S(\kappa+t-s)F\left(\frac{s}{\varepsilon}, x(s)\right) ds - \int_{\kappa+ih}^{\kappa+(i+1)h} S(t-ih)F\left(\frac{s}{\varepsilon}, x(s)\right) ds \right\| \leq C\gamma$$

and

$$\left\| \int_{\kappa+ih}^{\kappa+(i+1)h} S(\kappa+t-s)F_0(x(s))ds - \int_{\kappa+ih}^{\kappa+(i+1)h} S(t-ih)F_0(x(s))ds \right\| \leq C\gamma.$$

Now, by the Krasnoselski-Krein lemma [16], we have

$$\lim_{\varepsilon \rightarrow 0} S(t-ih) \int_{\kappa+ih}^{\kappa+(i+1)h} F\left(\frac{s}{\varepsilon}, x(s)\right) ds = S(t-ih) \int_{\kappa+ih}^{\kappa+(i+1)h} F_0(x(s)) ds,$$

which proves (19).

To prove (20), we need to show that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left(\int_{\kappa}^{\kappa+t} \left\langle S(\kappa+t-s)G\left(\frac{s}{\varepsilon}, x(s)\right)QG^*\left(\frac{s}{\varepsilon}, x(s)\right)S^*(\kappa+t-s)x, y \right\rangle ds \right. \\ \left. - \int_{\kappa}^{\kappa+t} \left\langle S(\kappa+t-s)G_0(x(s))QG_0^*(x(s))S^*(\kappa+t-s)x, y \right\rangle ds \right) = 0. \end{aligned}$$

uniformly with respect to x, y in the unit ball $B(0, 1)$ of \mathbb{H}_2 . As in the proof of (19), we have

$$\begin{aligned} \int_{\kappa}^{\kappa+t-\alpha} \left\langle G\left(\frac{s}{\varepsilon}, x(s)\right)QG^*\left(\frac{s}{\varepsilon}, x(s)\right)S^*(\kappa+t-s)x, S^*(\kappa+t-s)y \right\rangle ds \\ = \sum_{i=0}^{N-1} \int_{\kappa+ih}^{\kappa+(i+1)h} \left\langle G\left(\frac{s}{\varepsilon}, x(s)\right)QG^*\left(\frac{s}{\varepsilon}, x(s)\right)S^*(\kappa+t-s)x, S^*(\kappa+t-s)y \right\rangle ds \end{aligned}$$

and

$$\begin{aligned} \int_{\kappa}^{\kappa+t-\alpha} \left\langle G_0(x(s))QG_0^*(x(s))S^*(\kappa+t-s)x, S^*(\kappa+t-s)y \right\rangle ds \\ = \sum_{i=0}^{N-1} \int_{\kappa+ih}^{\kappa+(i+1)h} \left\langle G_0(x(s))QG_0^*(x(s))S^*(\kappa+t-s)x, S^*(\kappa+t-s)y \right\rangle ds. \end{aligned}$$

Replacing in the right hand sides $S^*(\kappa+t-s)$ by $S^*(t-ih)$, we reduce the proof to the equation

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\kappa+ih}^{\kappa+(i+1)h} \left\langle G\left(\frac{s}{\varepsilon}, x(s)\right)QG^*\left(\frac{s}{\varepsilon}, x(s)\right)S^*(t-ih)x, S^*(t-ih)y \right\rangle ds \\ - \int_{\kappa+ih}^{\kappa+(i+1)h} \left\langle G_0(x(s))QG_0^*(x(s))S^*(t-ih)x, S^*(t-ih)y \right\rangle ds = 0 \end{aligned}$$

uniformly with respect to x, y in $B(0, 1)$. But again this follows from the Krasnoselski-Krein lemma. \square

Recall that, if X and Y are two random vectors of a Banach space \mathbb{E} , the L^2 -Wasserstein distance $\mathbf{W}^2(X, Y)$ between the distributions of X and Y is

$$\mathbf{W}^2(X, Y) = \left(\inf \mathbb{E} \left(\left\| \widehat{X} - \widehat{Y} \right\|_{\mathbb{E}}^2 \right) \right)^{1/2}$$

where the infimum is taken over all joint distributions of random vectors \widehat{X} and \widehat{Y} satisfying $\text{law}(\widehat{X}) = \text{law}(X)$ and $\text{law}(\widehat{Y}) = \text{law}(Y)$.

By e.g. [23, Theorem 6.9]), if (X_n) is a sequence of random vectors of \mathbb{E} and if X is a random vector of \mathbb{E} , the sequence $(\text{law}(X_n))$ converges to $\text{law}(X)$ for \mathbf{W}^2 if and only if (X_n) converges to X in distribution and $(\|X_n\|_{\mathbb{E}}^2)$ is uniformly integrable.

If X and Y are continuous \mathbb{H}_2 -valued stochastic processes, for any interval $[a, b]$, we denote by $\mathbf{W}_{[a, b]}^2$ the L^2 -Wasserstein distance between the distributions of X and Y , seen as $\mathcal{C}([a, b], \mathbb{H}_2)$ -valued random variables.

We are now ready to state our main averaging result.

Theorem 4.2 *Let the assumptions (i), (ii'), (iii), (iv), and (v') be fulfilled and the constant $\theta' = \frac{4K^2}{\delta} \left(\frac{1}{\delta} + \text{tr } Q \right) < 1$. For each fixed $\varepsilon \in]0, 1[$, let X^ε be the mild solution to the equation*

$$(21) \quad dX^\varepsilon(t) = AX^\varepsilon(t)dt + F\left(\frac{t}{\varepsilon}, X^\varepsilon(t)\right)dt + G\left(\frac{t}{\varepsilon}, X^\varepsilon(t)\right)dW(t),$$

and let X^0 be the mild solution to

$$(22) \quad dX^0(t) = A(X^0(t))dt + F_0(X^0(t))dt + G_0(X^0(t))dW(t),$$

which is a stationary process. Then $\mathbf{W}_{[a, b]}^2(X^\varepsilon, X^0) \rightarrow 0$ as $\varepsilon \rightarrow 0+$, for any compact interval $[a, b]$.

Before we give the proof of this theorem, let us recall some well-known results.

Proposition 4.3 ([7]) *Let $(X_n)_{n \geq 0}$ be a sequence of centered Gaussian random variable on a separable Hilbert space \mathbb{H} with sequence of covariance operators $(Q_n)_{n \geq 0}$. Then $(X_n)_{n \geq 0}$ converges in distribution to X_0 in \mathbb{H} if and only if*

$$|Q_n - Q_0|_{\mathcal{N}} \rightarrow 0, n \rightarrow \infty$$

Let $\mathbb{U}, \mathbb{V}, \mathbb{H}$ be real separable Hilbert spaces, let W be a \mathbb{U} -valued (\mathcal{F}_t) -adapted Wiener process with nuclear covariance operator Q .

Proposition 4.4 ([24, Proposition 2.2]) *Let $\alpha : \mathbb{H} \rightarrow \mathbb{V}$ be a Lipschitz mapping and $\sigma : \mathbb{R} \times \mathbb{H} \rightarrow L(\mathbb{U}, \mathbb{V})$ be a measurable mapping such that $\|\sigma(r, x)\|_{L(\mathbb{U}, \mathbb{V})} \leq M(1 + \|x\|_{\mathbb{H}})$ and $\|\sigma(r, x) - \sigma(r, y)\|_{L(\mathbb{U}, \mathbb{V})} \leq M\|x - y\|_{\mathbb{H}}$ for a constant M and every $r \in [s, t], x, y \in \mathbb{H}$. Let $g \in \text{BL}(\mathbb{V})$, we define*

$$\psi(y) = \mathbb{E} g\left(\alpha(y) + \int_s^t \sigma(r, y) dW(r)\right), y \in \mathbb{H}.$$

Let $u : \Omega \rightarrow \mathbb{H}$ be a (\mathcal{F}_s) -measurable random variable with $\mathbb{E} \|u\|_{\mathbb{H}}^2 < \infty$. Then

$$\mathbb{E}\left[g\left(\alpha(u) + \int_s^t \sigma(r, u) dW(r)\right) \middle| \mathcal{F}_s\right] = \psi(u) \text{ P-a.s.}$$

Proof of Theorem 4.2 We denote $F_\varepsilon(s, x) := F\left(\frac{s}{\varepsilon}, x\right)$, $G_\varepsilon(s, x) := G\left(\frac{s}{\varepsilon}, x\right)$, and, for every $X \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$,

$$\begin{aligned} L_\varepsilon(X)(t) &:= \int_{-\infty}^t S(t-s) F_\varepsilon(s, X(s)) ds + \int_{-\infty}^t S(t-s) G_\varepsilon(s, X(s)) dW(s), \\ L_0(X)(t) &:= \int_{-\infty}^t S(t-s) F_0(X(s)) ds + \int_{-\infty}^t S(t-s) G_0(X(s)) dW(s). \end{aligned}$$

First step. Let $X \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$. Let us show that $L_\varepsilon(X) \rightarrow L_0(X)$ in distribution, as $\varepsilon \rightarrow 0$, in the space $\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$ endowed with the topology of uniform convergence on compact intervals of \mathbb{R} . This amounts to prove that, for any compact interval $[a, b]$, $L_\varepsilon(X) \rightarrow L_0(X)$ in distribution in the space $\mathcal{C}([a, b], \mathbb{H}_2)$ (see [25, Theorem 5]). Actually, we will prove a slightly stronger result, using the L^2 -Wasserstein distance.

By Conditions (ii) and (iii) we deduce that, for every $\eta > 0$, there exists κ such that, for every $\eta \geq 0$ and for each $s < \kappa$, $\mathbb{E} \|L_\varepsilon(X)(s)\|_{\mathbb{H}_2}^2 < \eta$. Thus, for the proof that $L_\varepsilon(X)$ converges in distribution to $L_0(X)$ on $\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$, it suffices to show the convergence in distribution on $\mathcal{C}([\kappa, T], \mathbb{H}_2)$, for every $T \geq \kappa$, of

$$Y^\varepsilon(t) := \int_\kappa^t S(t-s) F_\varepsilon(s, X(s)) ds + \int_\kappa^t S(t-s) G_\varepsilon(s, X(s)) dW(s)$$

to

$$Y^0(t) := \int_\kappa^t S(t-s) F_0(X(s)) ds + \int_\kappa^t S(t-s) G_0(X(s)) dW(s).$$

As $X \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$, it satisfies the following condition: For every $\eta > 0$, there exist a partition

$$\{\kappa = t_0 < t_1 < \dots < t_k = T\} \text{ of } [\kappa, T]$$

and an adapted process

$$\tilde{X}(t) = \sum_{i=0}^{k-1} X(t_i) 1_{[t_i, t_{i+1}]}(t)$$

such that

$$\sup_{t \in [\kappa, T]} \mathbb{E} \|X(t) - \tilde{X}(t)\|_{\mathbb{H}_2}^2 < \eta.$$

Using the fact that L_ε is Lipschitz, we can furthermore choose the partition (t_0, \dots, t_k) such that

$$(23) \quad \sup_{\varepsilon > 0} \sup_{t \in [\kappa, T]} \mathbb{E} \|L_\varepsilon X(t) - L_\varepsilon \tilde{X}(t)\|_{\mathbb{H}_2}^2 < \eta.$$

For $\varepsilon > 0$, we denote $F_\varepsilon(s, x) := F(\frac{s}{\varepsilon}, x)$, $G_\varepsilon(s, x) := G(\frac{s}{\varepsilon}, x)$, and we set

$$\begin{aligned} \tilde{X}^\varepsilon(t) &= \int_{\kappa}^t S(t-s) F_\varepsilon(s, \tilde{X}(s)) ds + \int_{\kappa}^t S(t-s) G_\varepsilon(s, \tilde{X}(s)) dW(s) \\ &= \sum_{i=0}^{k-1} \left(\int_{t_i \wedge t}^{t_{i+1} \wedge t} S(t-s) F_\varepsilon(s, X(t_i)) ds \right. \\ &\quad \left. + \int_{t_i \wedge t}^{t_{i+1} \wedge t} S(t-s) G_\varepsilon(s, X(t_i)) dW(s) \right). \end{aligned}$$

By [10, Theorem 6.10], each \tilde{X}^ε has a continuous modification.

Let us prove that, for each $l = 1, \dots, k$, $\tilde{X}_{t_1}^\varepsilon$ converges in distribution to $\tilde{X}_{t_1}^0$ as $\varepsilon \rightarrow 0$. We define a mapping

$$\gamma_\varepsilon : \mathbb{H}_2^l \rightarrow L^1(\Omega, \mathbb{H}_2)$$

by

$$\begin{aligned} \gamma_\varepsilon(y_0, y_1, \dots, y_{l-1}) &= \sum_{i=1}^l \left(\int_{t_{i-1}}^{t_i} S(t_i - s) F_\varepsilon(s, y_{i-1}) ds \right. \\ &\quad \left. + \int_{t_{i-1}}^{t_i} S(t_i - s) G_\varepsilon(s, y_{i-1}) dW(s) \right). \end{aligned}$$

Using Proposition 4.4, we get that

$$\text{law}(\tilde{X}_{t_l}^\varepsilon) = \text{law}(\gamma_\varepsilon(\tilde{X}_{t_0}, \tilde{X}_{t_1}, \dots, \tilde{X}_{t_{l-1}})).$$

Let

$$\mu_{t_0, t_1, \dots, t_{l-1}} = \text{law}(\tilde{X}_{t_0}, \tilde{X}_{t_1}, \dots, \tilde{X}_{t_{l-1}}).$$

Let $g \in \text{BL}(\mathbb{H}_2)$, and $h_\varepsilon(y) = \mathbb{E}[g(\gamma_\varepsilon(y))]$; $y \in \mathbb{H}_2^l$. We have

$$\mathbb{E}[g(\tilde{X}_{t_l}^\varepsilon)] = \mathbb{E}[h_\varepsilon(\tilde{X}_{t_0}, \tilde{X}_{t_1}, \dots, \tilde{X}_{t_{l-1}})] = \int_{\mathbb{H}_2^l} h_\varepsilon(y) d\mu_{t_0, t_1, \dots, t_{l-1}}(y)$$

and

$$\begin{aligned} |\mathbb{E}[g(\tilde{X}_{t_l}^\varepsilon)] - \mathbb{E}[g(\tilde{X}_{t_l}^0)]| &= \left| \int_{\mathbb{H}_2^l} h_\varepsilon(y) - h_0(y) d\mu_{t_0, t_1, \dots, t_{l-1}}(y) \right| \\ &\leq \int_{\mathbb{H}_2^l} |h_\varepsilon(y) - h_0(y)| d\mu_{t_0, t_1, \dots, t_{l-1}}(y). \end{aligned}$$

Let us show that $h_\varepsilon(y) \rightarrow h_0(y)$ as $\varepsilon \rightarrow 0$ for every $y \in \mathbb{H}_2^l$. We have

$$\begin{aligned} \gamma_\varepsilon(y) - \gamma_0(y) &= \sum_{i=1}^l \int_{t_{i-1}}^{t_i} S(t_l - s) (F_\varepsilon(s, y_{i-1}) - F_0(y_{i-1})) ds \\ &\quad + \sum_{i=1}^l \int_{t_{i-1}}^{t_i} S(t_l - s) (G_\varepsilon(s, y_{i-1}) - G_0(y_{i-1})) dW(s) \\ &= I_\varepsilon + J_\varepsilon. \end{aligned}$$

Lemma 4.1 implies that $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, and since

$$\sum_{i=1}^l \int_{t_{i-1}}^{t_i} S(t_l - s) G_\varepsilon(s, y_{i-1}) dW(s)$$

is a centered Gaussian random variable in \mathbb{H}_2 , we deduce by Lemma 4.1 and Proposition 4.3 that $J_\varepsilon \rightarrow 0$ in distribution as $\varepsilon \rightarrow 0$ hence $\gamma_\varepsilon(y) \rightarrow \gamma_0(y)$ in distribution as $\varepsilon \rightarrow 0$. Consequently

$$(24) \quad h_\varepsilon(y) \rightarrow h_0(y) \text{ for any } y \in \mathbb{H}_2^l.$$

For every $\eta' > 0$, there exists a compact set $\mathcal{K} \subset \mathbb{H}_2^l$ such that

$$\mu_{t_0, t_1, \dots, t_{l-1}}(\mathbb{H}_2^l \setminus \mathcal{K}) < \eta'.$$

We have

$$(25) \quad h_\varepsilon \in \text{BL}(\mathbb{H}_2^l) \text{ and } \sup_\varepsilon \|h_\varepsilon\|_{\text{BL}} < \infty$$

because, for all $y, z \in \mathbb{H}_2^l$, and for some constant K_1 ,

$$|h_\varepsilon(y) - h_\varepsilon(z)| \leq \|g\|_{\text{BL}} \mathbb{E} \|\gamma_\varepsilon(y) - \gamma_\varepsilon(z)\|_{\mathbb{H}_2} \leq K_1 \|g\|_{\text{BL}} \|y - z\|_{\mathbb{H}_2^l}.$$

From (24), (25) and the compactness of \mathcal{K} , we deduce that h_ε converges to h_0 uniformly on \mathcal{K} , hence

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{K}} |h_\varepsilon(y) - h_0(y)| d\mu_{t_0, t_1, \dots, t_{l-1}}(y) = 0$$

and, since g is a bounded function,

$$\begin{aligned} \int_{\mathbb{H}_2^l \setminus \mathcal{K}} |h_\varepsilon(y) - h_0(y)| d\mu_{t_0, t_1, \dots, t_{l-1}}(y) \\ \leq 2 \sup_\varepsilon \sup_y |h_n(y)| \eta' = 2 \sup_\varepsilon \sup_y |\mathbb{E}[g(\gamma_\varepsilon(y))]| \eta'. \end{aligned}$$

Thus $\tilde{X}^\varepsilon(t_l) \rightarrow \tilde{X}^0(t_l)$ in distribution as $\varepsilon \rightarrow 0$.

We now prove by induction that $(\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_k}^\varepsilon)$ converges in distribution to $(\tilde{X}_{t_0}^0, \tilde{X}_{t_1}^0, \dots, \tilde{X}_{t_k}^0)$ as $\varepsilon \rightarrow 0$. By construction, we have $\tilde{X}_{t_0}^\varepsilon \rightarrow \tilde{X}_{t_0}^0$ in distribution. Assume that for $0 \leq l \leq k-1$, $(\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_l}^\varepsilon)$ converges in distribution in \mathbb{H}_2^{l+1} . Let us define $\alpha_\varepsilon : \mathbb{H}_2^{l+1} \rightarrow \mathbb{H}_2^{l+2}$ by

$$\alpha_\varepsilon(y_0, y_1, \dots, y_l) = \left(y_0, y_1, \dots, y_l, S(t_{l+1}-t_l)y_l + \int_{t_l}^{t_{l+1}} S(t_{l+1}-s)F_\varepsilon(s, y_l) ds \right)$$

and $\beta_\varepsilon : \mathbb{H}_2^{l+1} \rightarrow L^1(\Omega, \mathbb{H}_2^{l+2})$ by

$$\beta_\varepsilon(y_0, y_1, \dots, y_l) = \left(0, \dots, 0, \int_{t_l}^{t_{l+1}} S(t_{l+1}-s)G_\varepsilon(s, y_l) dW(s) \right)$$

so that

$$\text{law}((\alpha_n + \beta_n)(\tilde{X}_{t_0}^n, \tilde{X}_{t_1}^n, \dots, \tilde{X}_{t_l}^n)) = \text{law}((\tilde{X}_{t_0}^n, \tilde{X}_{t_1}^n, \dots, \tilde{X}_{t_l}^n, \tilde{X}_{t_{l+1}}^n)).$$

We denote $u_\varepsilon = (\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_l}^\varepsilon)$ and $\mu_\varepsilon = \text{law}(u_\varepsilon)$. Let $g \in \text{BL}(\mathbb{H}_2^{l+2})$, and

$$h_\varepsilon(y) = \mathbb{E}g(\alpha_\varepsilon(y) + \beta_\varepsilon(y)), \quad y \in \mathbb{H}_2^{l+1}.$$

Proposition 4.4 yields

$$\mathbb{E} g(\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_{i+1}}^\varepsilon) = \mathbb{E} h_\varepsilon(u_\varepsilon) = \int_{\mathbb{H}_2^{l+1}} h_\varepsilon(y) d\mu_\varepsilon(y).$$

It follows that

$$\begin{aligned} & |\mathbb{E} g(\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_{i+1}}^\varepsilon) - \mathbb{E} g(\tilde{X}_{t_0}^0, \tilde{X}_{t_1}^0, \dots, \tilde{X}_{t_{i+1}}^0)| \\ & \leq \int_{\mathbb{H}_2^{l+1}} |h_\varepsilon(y) - h_0(y)| d\mu_\varepsilon(y) + \left| \int_{\mathbb{H}_2^{l+1}} h_0(y) d\mu_\varepsilon(y) - \int_{\mathbb{H}_2^{l+1}} h_0(y) d\mu_0(y) \right| \\ & \leq J_1(\varepsilon) + J_2(\varepsilon). \end{aligned}$$

As in the above reasoning, we can prove that

$$h_\varepsilon \in \text{BL}(\mathbb{H}_2^{l+1}), \quad \sup_\varepsilon \|h_\varepsilon\|_{\text{BL}} < \infty$$

and

$$\alpha_\varepsilon(y) + \beta_\varepsilon(y) \rightarrow \alpha_0(y) + \beta_0(y)$$

in distribution, for every $y \in \mathbb{H}_2^{l+1}$ thus $h_\varepsilon(y) \rightarrow h_0(y)$ as $\varepsilon \rightarrow 0$, for any $y \in \mathbb{H}_2^{l+1}$. On the other hand, by the induction hypothesis we have $\mu_\varepsilon \rightarrow \mu_0$ and since $h_0 \in \text{BL}(\mathbb{H}_2^{l+1})$ we have $J_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The convergence of μ_ε implies that $\{\mu_\varepsilon\}$ is tight, i.e. for each $\eta' > 0$ there exists a compact set $\mathcal{K} \in \mathbb{H}_2^{l+1}$ such that

$$(26) \quad \sup_\varepsilon \mu_\varepsilon(\mathbb{H}_2^{l+1} \setminus \mathcal{K}) < \eta'$$

Since for every $y \in \mathbb{H}_2^{l+1}$, $h_\varepsilon(y) \rightarrow h_0(y)$, from (26) and the compactness of \mathcal{K} the function h_ε converges to h_0 uniformly on \mathcal{K} , hence

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{K}} |h_\varepsilon(y) - h_0(y)| d\mu_\varepsilon(y) = 0$$

and

$$\int_{\mathbb{H}_2^l \setminus \mathcal{K}} |h_\varepsilon(y) - h_0(y)| d\mu_\varepsilon(y) \leq 2 \sup_\varepsilon \sup_y |h_\varepsilon(y)| \eta$$

So $J_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, consequently $(\tilde{X}_{t_0}^\varepsilon, \tilde{X}_{t_1}^\varepsilon, \dots, \tilde{X}_{t_{i+1}}^\varepsilon)$ converges in distribution in \mathbb{H}_2^{l+2} .

Now, we have

$$d_{\text{BL}}\left(\text{law}(Y^\varepsilon(t_0), \dots, Y^\varepsilon(t_k)), \text{law}(Y^0(t_0), \dots, Y^0(t_k))\right)$$

$$\begin{aligned}
&\leq d_{\text{BL}}\left(\text{law}(Y^\varepsilon(t_0), \dots, Y^\varepsilon(t_k)), \text{law}(\tilde{X}^\varepsilon(t_0), \dots, \tilde{X}^\varepsilon(t_k))\right) \\
&\quad + d_{\text{BL}}\left(\text{law}(\tilde{X}^\varepsilon(t_0), \dots, \tilde{X}^\varepsilon(t_k)), \text{law}(\tilde{X}^0(t_0), \dots, \tilde{X}^0(t_k))\right) \\
&\quad + d_{\text{BL}}\left(\text{law}(\tilde{X}^0(t_0), \dots, \tilde{X}^0(t_k)), \text{law}(Y^0(t_0), \dots, Y^0(t_k))\right) \\
&\leq \mathbb{E}\left(\|Y^\varepsilon(t_0) - \tilde{X}^\varepsilon(t_0)\| + \dots + \|Y^\varepsilon(t_k) - \tilde{X}^\varepsilon(t_k)\|\right) \\
&\quad + d_{\text{BL}}\left(\text{law}(\tilde{X}^\varepsilon(t_0), \dots, \tilde{X}^\varepsilon(t_k)), \text{law}(\tilde{X}^0(t_0), \dots, \tilde{X}^0(t_k))\right) \\
&\quad + \mathbb{E}\left(\|Y^0(t_0) - \tilde{X}^0(t_0)\| + \dots + \|Y^0(t_k) - \tilde{X}^0(t_k)\|\right).
\end{aligned}$$

By (23), the first and third terms can be made arbitrarily small. Thus $(Y^\varepsilon(t_0), \dots, Y^\varepsilon(t_k))$ converges in distribution to $(Y^0(t_0), \dots, Y^0(t_k))$. Now, for any finite sequence τ_1, \dots, τ_m in $[\kappa, T]$, we can refine if necessary the partition (t_1, \dots, t_k) such as to include the points τ_1, \dots, τ_m . This proves that the finite dimensional distributions of (Y^ε) converge to the corresponding finite dimensional distributions of Y^0 .

To show that (Y^ε) converges in distribution to Y^0 in $\mathcal{C}([\kappa, T], \mathbb{H}_2)$, we only need to prove that (Y^ε) is tight in $\mathcal{C}([\kappa, T], \mathbb{H}_2)$.

By Condition (v')-(a) and the equicontinuity of (F_ε) , the sequence $\left(\int_\kappa^t S(t-s)F_\varepsilon(s, X(s)) ds\right)$ is tight. Tightness of $\left(\int_\kappa^t S(t-s)G_\varepsilon(s, X(s)) dW(s)\right)$ follows from [13, Lemma 3.2] applied to the multifunction

$$\mathfrak{G}(t, x) = \overline{\text{co}} \{G_\varepsilon(t, x); \varepsilon > 0\},$$

(where $\overline{\text{co}}$ denotes the closed convex hull), which is compact-valued thanks to Condition (v')-(a). This result is given in [13] for p -integrable stochastic processes with $p > 2$, in order to use the stochastic convolution inequality. But the reasoning remains unchanged for $p = 2$ for a contractions semigroup, as is the case here, because then the convolution equality still holds true, see [10, Theorem 6.10].

Second step. We assume now that

$$(27) \quad \sup_{t \in [\kappa, T]} \mathbb{E} \|X(t)\|^p < +\infty$$

for some $p > 2$. Let us prove that $\lim_{\varepsilon \rightarrow 0} \mathbf{W}_{[\kappa, T]}^2(Y^\varepsilon, Y^0) = 0$ for any $T > 0$. We only need to prove that $\left(\|Y^\varepsilon\|_{\mathcal{C}([\kappa, T], \mathbb{H}_2)}^2\right)$ is uniformly integrable, which

is a consequence of (27): we have, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E} \left(\sup_{t \in [\kappa, T]} \|Y^\varepsilon(t)\|^p \right) &\leq 2^{p-1} \mathbb{E} \left(\sup_{t \in [\kappa, T]} \left\| \int_\kappa^t S(t-s) F_\varepsilon(s, X(s)) ds \right\|^p \right) \\
&\quad + 2^{p-1} \mathbb{E} \left(\sup_{t \in [\kappa, T]} \left\| \int_\kappa^t S(t-s) G_\varepsilon(s, X(s)) dW(s) \right\|^p \right) \\
&\leq 2^{p-1} \mathbb{E} \left(\sup_{t \in [\kappa, T]} \left(\int_\kappa^t e^{-\delta(t-s)} \|F_\varepsilon(s, X(s))\| ds \right)^p \right) \\
&\quad + 2^{p-1} \mathfrak{C}_{p, T-\kappa} \mathbb{E} \int_\kappa^T \|G_\varepsilon(s, X(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^p ds
\end{aligned}$$

(where $\mathfrak{C}_{p, T-\kappa}$ is given by the convolution inequality [10, Proposition 7.3])

$$\begin{aligned}
&\leq 2^{p-1} \mathbb{E} \left(\left(\int_\kappa^T e^{-\delta(t-s)} \|F_\varepsilon(s, X(s))\| ds \right)^p \right) \\
&\quad + 2^{p-1} \mathfrak{C}_{p, T-\kappa} \mathbb{E} \int_\kappa^T \|G_\varepsilon(s, X(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^p ds \\
&\leq 2^{p-1} (T-\kappa)^{p-1} \int_\kappa^T \|F_\varepsilon(s, X(s))\|^p ds \\
&\quad + 2^{p-1} \mathfrak{C}_{p, T-\kappa} \mathbb{E} \int_\kappa^T \|G_\varepsilon(s, X(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^p ds \\
&\leq 2^{p-1} K^p ((T-\kappa)^{p-1} + \mathfrak{C}_{p, T-\kappa}) \int_\kappa^T (1 + \|X(s)\|)^p ds.
\end{aligned}$$

This estimation is independent of ε , thus $(Y^\varepsilon)_{\varepsilon > 0}$ is bounded in $L^p(\mathcal{C}([\kappa, T], \mathbb{H}_2))$. Thus $\left(\sup_{t \in [\kappa, T]} \|Y^\varepsilon(t)\|^2 \right)$ is uniformly integrable, which entails that (Y^ε) converges to Y^0 for $\mathbf{W}_{[\kappa, T]}^2$.

It is then straightforward to deduce that, if $(X(t))_{t \in \mathbb{R}}$ is bounded in L^p ,

$$\lim_{\varepsilon \rightarrow 0+} \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X), L_0(X)) = 0$$

for any interval $[\kappa, T]$ of \mathbb{R} .

Third step. Now, let us show that $L_\varepsilon(X^\varepsilon)$ converges to $L_0(X^0)$ in L^2 -Wasserstein distance on compact intervals of \mathbb{R} on the space $\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$, which means that $\mathbf{W}_{[\kappa, T]}^2(X^\varepsilon, X^0) \rightarrow 0$ as $\varepsilon \rightarrow 0+$ for any compact interval $[\kappa, T]$.

We have shown in the proof of Theorem 3.1 that L_ε is θ -Lipschitz for the norm of $\text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$. This is not sufficient for our present purpose,

but the hypothesis $\theta' < 1$ allows a more precise calculation: For $X, Y \in \text{CUB}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}_2))$, we have

$$\begin{aligned}
& \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|(LX)(t) - (LY)(t)\|_{\mathbb{H}_2}^2 \right) \\
& \leq 2 \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left(\int_{-\infty}^t e^{-\delta(t-s)} \|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2} ds \right)^2 \right) \\
& \quad + 2 \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left(\left\| \int_{-\infty}^t S(t-s) [G(s, X(s)) - G(s, Y(s))] dW(s) \right\|_{\mathbb{H}_2} \right)^2 \right) \\
& = J_1 + J_2.
\end{aligned}$$

We have

$$\begin{aligned}
J_1 & \leq 2 \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left(\int_{-\infty}^t e^{-\delta(t-s)} \sup_{\sigma \in \mathbb{R}} \|F(\sigma, X(\sigma)) - F(\sigma, Y(\sigma))\|_{\mathbb{H}_2} ds \right)^2 \right) \\
& = 2 \mathbb{E} \left(\sup_{\sigma \in \mathbb{R}} \|F(\sigma, X(\sigma)) - F(\sigma, Y(\sigma))\|_{\mathbb{H}_2}^2 \right) \sup_{t \in \mathbb{R}} \left(\int_{-\infty}^t e^{-\delta(t-s)} ds \right)^2 \\
& \leq \frac{2}{\delta^2} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|F(t, X(t)) - F(t, Y(t))\|_{\mathbb{H}_2}^2 \right) \\
& \leq \frac{2K^2}{\delta^2} \sup_{t \in \mathbb{R}} \mathbb{E} \|X(t) - Y(t)\|_{\mathbb{H}_2}^2.
\end{aligned}$$

By (iii) and (iv), the process $\int_{-\infty}^{\cdot} S(\cdot - s) (G(s, X(s)) - G(s, Y(s))) dW(s)$ is a square integrable martingale. Using Doob's inequality and Itô's isometry identity, we get

$$\begin{aligned}
J_2 & = 2 \sup_{T \in \mathbb{R}} \mathbb{E} \left(\sup_{t \leq T} \left(\left\| \int_{-\infty}^t S(t-s) [G(s, X(s)) - G(s, Y(s))] dW(s) \right\|_{\mathbb{H}_2} \right)^2 \right) \\
& \leq 8 \sup_{T \in \mathbb{R}} \mathbb{E} \left(\left\| \int_{-\infty}^T S(T-s) [G(s, X(s)) - G(s, Y(s))] dW(s) \right\|_{\mathbb{H}_2} \right)^2 \\
& \leq 8 \sup_{T \in \mathbb{R}} \text{tr} Q \int_{-\infty}^T e^{-2\delta(T-s)} \mathbb{E} \|G(s, X(s)) - G(s, Y(s))\|_{L(\mathbb{H}_1, \mathbb{H}_2)}^2 ds \\
& \leq 8 \sup_{T \in \mathbb{R}} \text{tr} Q K^2 \int_{-\infty}^T e^{-2\delta(T-s)} \mathbb{E} \|X(s) - Y(s)\|_{\mathbb{H}_2}^2 ds \\
& \leq 8 \sup_{T \in \mathbb{R}} K^2 \text{tr} Q \left(\int_{-\infty}^T e^{-2\delta(t-s)} ds \right) \sup_{\sigma \in \mathbb{R}} \mathbb{E} \|X(\sigma) - Y(\sigma)\|_{\mathbb{H}_2}^2 \\
& \leq \frac{4K^2 \text{tr} Q}{\delta} \sup_{t \in \mathbb{R}} \mathbb{E} \|X(t) - Y(t)\|_{\mathbb{H}_2}^2.
\end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|(LX)(t) - (LY)(t)\|_{\mathbb{H}_2}^2 \right) &\leq J_1 + J_2 \\ &\leq \frac{2K^2}{\delta} \left(\frac{1}{\delta} + 2 \operatorname{tr} Q \right) \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|X(t) - Y(t)\|_{\mathbb{H}_2}^2 \right). \end{aligned}$$

with

$$\frac{2K^2}{\delta} \left(\frac{1}{\delta} + 2 \operatorname{tr} Q \right) < \frac{4K^2}{\delta} \left(\frac{1}{\delta} + \operatorname{tr} Q \right) = \theta'.$$

We deduce

$$\begin{aligned} \mathbf{W}_{[\kappa, T]}^2(X^\varepsilon, X^0) &= \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X^\varepsilon), L_0(X^0)) \\ &\leq \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X^\varepsilon), L_\varepsilon(X^0)) + \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X^0), L_0(X^0)) \\ &\leq \theta' \mathbf{W}_{[\kappa, T]}^2(X^\varepsilon, X^0) + \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X^0), L_0(X^0)). \end{aligned}$$

As $\theta' < 1$, this entails

$$(28) \quad \mathbf{W}_{[\kappa, T]}^2(X^\varepsilon, X^0) \leq \frac{1}{1 - \theta'} \mathbf{W}_{[\kappa, T]}^2(L_\varepsilon(X^0), L_0(X^0)).$$

But, as $\theta' < 1$, there exists $p > 2$ such that $(X^0(t))_{t \in \mathbb{R}}$ is bounded in L^p , see Remark 3.6 and Proposition 3.5. Thus the right hand side of (28) converges to 0 as $\varepsilon \rightarrow 0+$.

Finally, by [9, Theorem 4.1], the mild solution (X^0) of (22) is a stationary process. \square

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