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On martingale approximations and the quenched weak invariance principle

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Abstract

In this paper, we obtain sufficient conditions in terms of projective criteria under which the partial sums of a stationary process with values in \mathcal{H} (a real and separable Hilbert space) admits an approximation, in $\mathbb{L}^p(\mathcal{H})$, $p > 1$, by a martingale with stationary differences and we then estimate the error of approximation in $\mathbb{L}^p(\mathcal{H})$. The results are exploited to further investigate the behavior of the partial sums. In particular we obtain new projective conditions concerning the Marcinkiewicz-Zygmund theorem, the moderate deviations principle and the rates in the central limit theorem in terms of Wasserstein distances. The conditions are well suited for a large variety of examples including linear processes or various kinds of weak dependent or mixing processes. In addition, our approach suits well to investigate the quenched central limit theorem and its invariance principle via martingale approximation, and allows us to show that they hold under the so-called Maxwell-Woodroffe condition that is known to be optimal.

Running head: Martingale approximations in \mathbb{L}^p

Key words: martingale approximation, stationary process, quenched invariance principle, moderate deviations, Wasserstein distances, ergodic theorems.

AMS 2000 Subject Classification: Primary: 60F15 ; Secondary: 60F05

1 Introduction

Since the seminal paper of Gordin [13] in 1969, approximation via a martingale is known to be a nice method to derive limit theorems for stochastic processes. For instance, the martingale method has been used successfully by Heyde [18] and Gordin and Lifsic [14] to derive central limit theorems for the partial sums of a stationary sequence, and it has undergone substantial improvements. For recent contributions where the central limit theory and weak convergence problems are handled with the help of martingale approximations, let us mention the recent papers by Maxwell and Woodroffe [21], Wu and Woodroffe [33], Peligrad and Utev [26], Merlevède and Peligrad [22], Zhao and Woodroffe [35] and Gordin and Peligrad [15]. In all these papers, conditions are then imposed to be able to implement the martingale method; namely, to approximate in a suitable way the partial sums of a stationary process by a martingale. However to derive many other kinds of limit theorems from the martingale method, more precise estimates of the approximation error of partial sums by a martingale can be useful. We refer to the recent papers by Wu [32], Dedecker, Doukhan and Merlevède [7] and Merlevède, Peligrad and Peligrad [24] where almost sure behaviors of the partial sums process have been addressed with the help of estimates of this approximation error.

In order to say more about these papers and to present our results, let us first introduce the following notation giving a way to define stationary processes.

Notation 1.1 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\theta : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability \mathbb{P} . Let \mathcal{F}_0 be a σ -algebra of \mathcal{A} satisfying $\mathcal{F}_0 \subseteq \theta^{-1}(\mathcal{F}_0)$. We then define a nondecreasing filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ by $\mathcal{F}_i = \theta^{-i}(\mathcal{F}_0)$, and a stationary sequence $(X_i)_{i \in \mathbb{Z}}$ by $X_i = X_0 \circ \theta^i$ where X_0 is a real centered random variable (or possibly taking values in some real and separable Hilbert space). The sequence will be called adapted to the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ if X_0 is \mathcal{F}_0 -measurable. Define then the partial sum by $S_n = X_1 + X_2 + \dots + X_n$. The following notations will also be used: $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$, $\mathcal{P}_k(X) = \mathbb{E}_k(X) - \mathbb{E}_{k-1}(X)$, and when X is real its \mathbb{L}^p norm is denoted by $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$. We shall also use the notation $a_n \ll b_n$ to mean that there exists a numerical constant C not depending on n such that $a_n \leq Cb_n$, for all positive integers n .

In what follows the sequence $(X_i)_{i \in \mathbb{Z}}$ is assumed to be stationary and adapted to $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ and the variables are in \mathbb{L}^p .

In [32] and [7], it is assumed that $D = \sum_{i \geq 0} \mathcal{P}_0(X_i)$ converges in \mathbb{L}^p , $p > 1$, and estimates of $\|S_n - M_n\|_p$ where $M_n = \sum_{i=1}^n D \circ \theta^i$ are provided involving either the terms $\sum_{k \geq n} \|\mathcal{P}_0(X_k)\|_p$ (see [32]) or the terms $\|\mathbb{E}_0(S_n)\|_p$ and $\|\sum_{k \geq n} \mathcal{P}_0(X_k)\|_p$ (see [7]). They are then exploited to derive explicit rates in the almost sure invariance principle under projective conditions that are well adapted to a large variety of examples. The paper by Merlevède *et al* [24] addresses different questions about the almost sure behavior of S_n such as quenched invariance principles or almost sure central limit theorems. Their proof is based under a precise estimate of the \mathbb{L}^2 approximation error between the partial sums process and their constructed approximating stationary martingale, provided that the Maxwell-Woodroffe condition (1) holds. More precisely, in the case where $p = 2$, they proved that if

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|_2}{k^{3/2}} < \infty, \quad (1)$$

then there is a martingale M_n with stationary and square integrable differences such that

$$\|S_n - M_n\|_2 \ll n^{1/2} \sum_{k \geq n} \frac{\|\mathbb{E}_0(S_k)\|_2}{k^{3/2}}. \quad (2)$$

To implement a martingale method for other questions related to the behavior of the partial sums, as for instance rates in the strong laws of large numbers or in the central limit theorem in terms of Wasserstein distances, or also moderate deviations principles, the first question that our paper addresses is the construction of a stationary martingale M_n in \mathbb{L}^p ($p > 1$) in such a way that an estimate of $\|S_n - M_n\|_p$ can be given in the spirit of (2). Our Theorem 2.4 is in this direction. When $p \geq 2$, it states in particular that if

$$\sum_{k=1}^{\infty} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} < \infty, \quad (3)$$

then we can construct a stationary sequence $(D_k = D \circ \theta^k)_{k \in \mathbb{Z}}$ of martingale differences in \mathbb{L}^p adapted to $(\mathcal{F}_k)_{k \in \mathbb{Z}}$ such that setting $M_n = \sum_{k=1}^n D \circ \theta^k$,

$$\|S_n - M_n\|_2 \ll n^{1/2} \sum_{k \geq \lceil n^{p/2} \rceil} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} < \infty \text{ where } p'' = \max(p, 2). \quad (4)$$

Our approach is different than the one used in [24] to prove (2). In Theorem 2.4 we shall consider also the case when $p \in]1, 2[$. The main tools to prove the martingale approximation with the bound (4) being algebraic computations and Burkholder's inequality, the estimate also holds for variables taking values in a separable real Hilbert space. Hence Theorem 2.4 is stated in this setting. As we shall see, with the help of a new ergodic theorem (Theorem 4.7), it will give us new projective conditions to

obtain rates in the strong law of large numbers. An application of this result is given to get rates in the almost sure convergence of some statistic tests based on the empirical process (see Section 3.1.1). Theorem 2.4 is also used to get new projective conditions allowing results concerning the moderate deviations principle or also estimates of Wasserstein distances in the CLT (see Sections 3.2 and 3.3). Notice that the projective conditions assumed all along the paper are general enough to contain a wide class of dependent sequences.

Another interesting point of our approach and of the approximating martingale we consider here, is that they lead not only to a useful estimate of $\|S_n - M_n\|_p$, but, together with our ergodic theorem (see Theorem 4.7), they allow also to show that, under the Maxwell-Woodroofe condition (1), $\mathbb{E}_0[(S_n - M_n)^2] = o(n)$ \mathbb{P} -a.s. (see our Proposition 4.9). This allows to give a definitive positive answer to the question whether the quenched central limit theorem for $n^{-1/2}S_n$ holds true under (1). As we shall see, we can even say more since, using a maximal inequality from Merlevède and Peligrad [23], we establish in Theorem 2.8 that the functional form of the quenched central limit theorem also holds under Maxwell-Woodroofe condition.

Our paper is structured as follows. Section 2 contains our main results. More precisely, in Section 2.1 we construct an approximating martingale with stationary differences in \mathbb{L}^p that leads to estimates of the \mathbb{L}^p approximating error between the partial sums and the constructed martingale (see Theorem 2.4). In Section 2.2, we address the question of the quenched weak invariance principle under the Maxwell-Woodroofe's condition (1). Section 3 is devoted to some applications of the estimates given in Theorem 2.4 to various kind of limit behavior of the partial sums. In Section 4, we prove the results stated in Sections 2.1 and 2.2 and state a new ergodic theorem with rates (see Theorem 4.7) whose proof is postponed in Section A. Some technical results are given and proven in Section B.

2 Main results

In complement to Notation 1.1, we introduce additional notations used all along the paper.

Notation 2.1 *Let \mathcal{H} be a real and separable Hilbert space equipped with the norm $|\cdot|_{\mathcal{H}}$. For a random variable X with values in \mathcal{H} , we denote its norm in $\mathbb{L}^p(\mathcal{H})$ by $\|X\|_{p,\mathcal{H}} = (\mathbb{E}(|X|_{\mathcal{H}})^p)^{1/p}$, and we simply denote $\mathbb{L}^p = \mathbb{L}^p(\mathbb{R})$.*

Notation 2.2 *Let $p' = \min(2, p)$, $p'' = \max(2, p)$ and $q = p''/p'$.*

2.1 Martingale approximation in $\mathbb{L}^p(\mathcal{H})$

Let $p > 1$. In this section, we shall establish conditions in order for S_n to be approximated by a martingale M_n with stationary differences in $\mathbb{L}^p(\mathcal{H})$ in such a way that the approximation error $\|S_n - M_n\|_{p,\mathcal{H}}$ is explicitly controlled. Let us adopt the following definition:

Definition 2.3 *Let $(r_n)_{n \geq 1}$ be a nondecreasing sequence of positive numbers and let $p > 1$. Let $(X_n)_{n \in \mathbb{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. We say that $(X_n)_{n \in \mathbb{Z}}$ admits a (stationary) martingale approximation in $\mathbb{L}^p(\mathcal{H})$ with rate r_n if there exists $D \in \mathbb{L}^p(\mathcal{H})$ such that $M_n = \sum_{k=1}^n D \circ \theta^k$ is a martingale adapted to $(\mathcal{F}_n)_{n \geq 1}$ and*

$$\|S_n - M_n\|_{p,\mathcal{H}} = o(r_n).$$

Notice that to get a martingale approximation in $\mathbb{L}^p(\mathcal{H})$ with rate r_n , the following minimal condition is needed:

$$\|\mathbb{E}_0(S_n)\|_{p,\mathcal{H}} = o(r_n).$$

Let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. When

$$D = \sum_{n \geq 0} \sum_{k \geq n} \frac{\mathcal{P}_0(X_k)}{k+1}, \quad (5)$$

converges in $\mathbb{L}^p(\mathcal{H})$, then $(D_k = D \circ \theta^k)_{k \in \mathbf{Z}}$ forms a stationary sequence of martingale differences in $\mathbb{L}^p(\mathcal{H})$ adapted to $(\mathcal{F}_k)_{k \in \mathbf{Z}}$. Notice that the series in (5) converges in $\mathbb{L}^p(\mathcal{H})$ as soon as the series $\sum_{k \geq 0} \mathcal{P}_0(X_k)$ does.

Theorem 2.4 *Let $p > 1$ and let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. Assume that*

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_{p, \mathcal{H}}}{n^{1+1/p''}} < \infty. \quad (6)$$

Then $\sum_{n \geq 1} |\sum_{k \geq n} k^{-1} \mathcal{P}_0(X_{k-1})|_{\mathcal{H}}$ converges in \mathbb{L}^p and setting $M_n = \sum_{k=1}^n D \circ \theta^k$ where D is defined by (5), the following inequality holds:

$$\|S_n - M_n\|_{p, \mathcal{H}} \ll n^{1/p'} \sum_{k \geq [n^q]} \frac{\|\mathbb{E}_0(S_k)\|_{p, \mathcal{H}}}{k^{1+1/p''}}, \quad (7)$$

let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. Assume that where we recall that $q = p''/p'$.

Remark 2.5 *Let $p > 1$ and $\alpha \in]0, 1/p'']$. Let us introduce the following assumption:*

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_{p, \mathcal{H}}}{n^{1+\alpha}} < \infty. \quad (8)$$

Assume that (8) holds with $\alpha = \min(1/2, 2/p^2)$. By combining (7) with Corollary 22 of [23] (with the norm $|\cdot|_{\mathcal{H}}$ replacing the absolute values) we have

$$\left\| \max_{1 \leq k \leq n} |S_k - M_k|_{\mathcal{H}} \right\|_p = o(n^{1/p}). \quad (9)$$

Notice also that if $p > 2$ and (8) holds with $\alpha \in]2/p^2, 1/p]$, then (7) combined with the maximal inequality (7) of [23] (with the norm $|\cdot|_{\mathcal{H}}$ replacing the absolute values) implies that

$$\left\| \max_{1 \leq k \leq n} |S_k - M_k|_{\mathcal{H}} \right\|_p = o(n^{\alpha p/2}). \quad (10)$$

The fact that the maximal inequality (7) of [23] is still valid when the variables take values in a Hilbert space comes from the fact that its proof is only based on chaining arguments (still valid in functional spaces by replacing the absolute values by the corresponding norms) and on the Doob's inequality that also holds in Hilbert spaces. Since Corollary 22 of [23] is proved via their maximal inequality (7), it is still valid in the Hilbert spaces setting.

Comment 2.6 *Theorem 1 in [32] (still valid in the Hilbert space context) states the following martingale approximation: Let $p > 1$ and assume that $\sum_{k \geq 0} \|\mathcal{P}_0(X_k)\|_{p, \mathcal{H}} < \infty$. Then setting $D = \sum_{k \geq 0} \mathcal{P}_0(X_k)$ and $M_n = \sum_{i=1}^n D \circ \theta^i$,*

$$\|S_n - M_n\|_{p, \mathcal{H}}^{p'} \ll \sum_{k=1}^n \left(\sum_{i \geq k} \|\mathcal{P}_0(X_j)\|_{p, \mathcal{H}} \right)^{p'}. \quad (11)$$

The approximations (7) and (11) cannot be compared and cover distinct classes of dependent sequences.

Comment 2.7 Notice that the quantity $\|\mathbb{E}_0(S_k)\|_{p,\mathcal{H}}$ can be estimated in a large variety of examples such as linear processes or mixing sequences. To give an example, let us consider $p \geq 2$ and the so-called stationary ρ -mixing real sequences defined by the coefficient

$$\rho(n) = \rho(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty) \text{ where } \mathcal{F}_i^j = \sigma(X_i, \dots, X_j) \quad (12)$$

where

$$\rho(\mathcal{A}, \mathcal{B}) = \sup \left\{ \frac{\text{Cov}(X, Y)}{\|X\|_2 \|Y\|_2} : X \in L^2(\mathcal{A}), Y \in L^2(\mathcal{B}) \right\}.$$

Here $L^2(\mathcal{A})$ denotes the space of real-valued random variables in \mathbb{L}^2 that are \mathcal{A} -measurable. In the proof of Lemma 1 in [27], it has been proven that for any $p \geq 2$ and any $k \geq 0$,

$$\|\mathbb{E}_0(S_{2^{k+1}})\|_p \ll \sum_{i=0}^k 2^{i/2} \rho^{2/p}(2^i), \quad (13)$$

provided that $\sum_{k \geq 0} \rho^{2/p}(2^k) < \infty$. On an other hand, $(\|\mathbb{E}_0(S_n)\|_p)_{n \geq 1}$ being a subadditive sequence, it follows from Lemma 2.7 in [26] that, for any $\alpha > 0$, (8) is equivalent to $\sum_{k \geq 0} 2^{-\alpha k} \|\mathbb{E}_0(S_{2^k})\|_p < \infty$. By using (13), one can see that this last convergence holds provided that $\sum_{i \geq 0} 2^{i(1/2-\alpha)} \rho^{2/p}(2^i) < \infty$.

2.2 Martingale approximation under \mathbb{P}_0 and the quenched (weak) invariance principle

Limit theorems for stochastic processes that do not start from equilibrium are timely and motivated by evolutions in quenched random environment. Recent discoveries by Volný and Woodroffe [30] show that many of the central limit theorems satisfied by classes of stochastic processes in equilibrium, fail to hold when the processes are started from a point. In this section, we address the question whether the Maxwell-Woodroffe condition (1) is sufficient for the validity of the quenched central limit theorem since this condition is known to be optimal (see e.g. [26] or [29] where the optimality of this condition is discussed). This question starts with a result in Borodin and Ibragimov ([1], Ch 4) stating that if $\|\mathbb{E}_0(S_n)\|_2$ is bounded, then one has the CLT starting at a point in its functional form. Later, works by Derriennic and Lin (see [10], [11], [12]), Zhao and Woodroffe [35], Cuny [4], Merlevède, Peligrad and Peligrad [24] improved on this result by imposing weaker and weaker conditions on $\|\mathbb{E}_0(S_n)\|_2$, but always stronger than (1). Let us mention that a result in Cuny and Peligrad ([5]) shows that the condition $\sum_{k=1}^{\infty} \|\mathbb{E}_0(X_k)\|_2/k^{1/2} < \infty$, is sufficient for the quenched CLT. It is also sufficient for the quenched weak invariance principle by a recent result of Cuny and Volny [6].

As we shall see in the proof of Theorem 2.8 below, the approximating martingale that we defined in Section 2.1 also allows to show that, under (1), $\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}_0(|S_n - \mathbb{E}_0(S_n) - M_n|^2) = 0$ \mathbb{P} -a.s. Together with a new ergodic theorem with rate (see our Theorem 4.7) and a maximal inequality from Merlevède and Peligrad [23], this implies that the quenched CLT in its functional form holds under Maxwell-Woodroffe condition (1).

Theorem 2.8 *let $(X_n)_{n \in \mathbb{Z}}$ be an adapted real stationary sequence in \mathbb{L}^2 in the sense of Notation 1.1. Assume that (1) holds. Then $\sum_{n \geq 1} |\sum_{k \geq n} k^{-1} \mathcal{P}_0(X_{k-1})|$ converges in \mathbb{L}^2 and setting $M_n = \sum_{k=1}^n D \circ \theta^k$ where D is defined by (5), the following holds:*

$$\frac{\mathbb{E}_0(\max_{1 \leq k \leq n} |S_k - M_k|^2)}{n} \xrightarrow{n \rightarrow +\infty} 0 \quad \mathbb{P}\text{-a.s.} \quad (14)$$

In particular, (S_n) satisfies the following quenched invariance principle: for any continuous and bounded function f on $\mathcal{D}([0, 1])$ (the space of càdlàg functions on $[0, 1]$) endowed with uniform topology,

$$\mathbb{E}_0(f(S_{[nt]}/\sqrt{n})) \rightarrow \mathbb{E}(f(\sqrt{\eta}W(t))) \quad \mathbb{P}\text{-a.s.} \quad (15)$$

where $\eta = \lim_{n \rightarrow \infty} \mathbb{E}(n^{-1} S_n^2 | \mathcal{I})$ in \mathbb{L}^1 and W is a standard Brownian motion on $[0, 1]$ independent of η . Here \mathcal{I} is the invariant sigma field, i.e. $\mathcal{I} = \{A \in \mathcal{A} : \theta^{-1}(A) = A\}$.

Theorem 2.8 also improves Corollary 5.10 of [4] stated for Markov chains with normal Markov operator. It follows from Comment 2.7 that if the ρ -mixing coefficients of $(X_n)_{n \in \mathbb{Z}}$ satisfy $\sum_{k \geq 0} \rho(2^k) < \infty$, then the quenched invariance principle holds. Hence the CLT from Ibragimov [19] for ρ -mixing sequences that is known to be essentially optimal, is also quenched.

A careful analysis of the proof of Theorem 2.8, shows that if the random variables are assumed to be in $\mathbb{L}^2(\mathcal{H})$, then under (6) with $p = 2$, the almost sure convergence (14) still holds with the norm $|\cdot|_{\mathcal{H}}$ replacing the absolute values.

3 Applications

3.1 Strong laws of large numbers with rate

Let $p > 1$ and $(X_n)_{n \in \mathbb{Z}}$ be an adapted and centered stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. From our ergodic Theorem 4.7, it follows that if there exists $D \in \mathbb{L}^p(\mathcal{H})$ such that $M_n = \sum_{k=1}^n D \circ \theta^k$ is a martingale adapted to $(\mathcal{F}_n)_{n \geq 1}$, then the condition

$$\sum_{n \geq 2} \frac{\|S_n - M_n\|_{p, \mathcal{H}}}{n^{1+1/p}} < \infty, \quad (16)$$

implies that $|S_n - M_n|_{\mathcal{H}} = o(n^{1/p})$ a.s. When p is in $]1, 2[$, together with the Marcinkiewicz-Zygmund strong law of large numbers for martingales (see [31]), this implies that $|S_n|_{\mathcal{H}} = o(n^{1/p})$ a.s. Therefore, using the approximations (7) and (11), to take care of (16), we derive the following corollary:

Corollary 3.1 *Let $p \in]1, 2[$. Assume that*

$$\sum_{n \geq 2} \log n \frac{\|\mathbb{E}_0(S_n)\|_{p, \mathcal{H}}}{n^{3/2}} < \infty \quad \text{or} \quad \sum_{n \geq 2} \log n \|\mathcal{P}_0(X_n)\|_{p, \mathcal{H}} < \infty. \quad (17)$$

Then (X_n) admits a martingale approximation in $\mathbb{L}^p(\mathcal{H})$ with rate $n^{1/p}$. In addition $|S_n|_{\mathcal{H}} = o(n^{1/p})$ a.s.

Proof. To prove that (16) holds under the first part of (17) is direct by using (7). To prove that (16) holds under the second part of (17), we first notice that by (11), (16) is satisfied provided that

$$\sum_{k \geq 0} 2^{-k/p} \left(\sum_{j=1}^{2^k} \left(\sum_{\ell \geq j} \|\mathcal{P}_0(X_n)\|_{p, \mathcal{H}} \right)^p \right)^{1/p} \leq \sum_{k \geq 0} 2^{-k/p} \left(\sum_{j=0}^k 2^j \left(\sum_{\ell \geq 2^j} \|\mathcal{P}_0(X_n)\|_{p, \mathcal{H}} \right)^p \right)^{1/p} < \infty.$$

Now, using the subadditivity of $x \mapsto x^{1/p}$, this last condition holds if the second part of (17) does. \square

Notice that the second part of (17) is weaker than the condition required in Corollary 1 of [32].

Remark 3.2 *Notice that*

$$\sum_{n \geq 2} \log n \|\mathcal{P}_0(X_n)\|_{p, \mathcal{H}} \ll \sum_{\ell \geq 1} \ell \sum_{k=2^\ell}^{2^{\ell+1}-1} \|\mathcal{P}_{-k}(X_0)\|_{p, \mathcal{H}} \ll \sum_{\ell \geq 1} \ell 2^{\ell/2} \left\| \left(\sum_{k=2^\ell}^{2^{\ell+1}-1} |\mathcal{P}_{-k}(X_0)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_p.$$

Therefore using Burkholder's inequality for \mathcal{H} -valued martingales (see [3]), one see that both conditions in (17) are satisfied as soon as:

$$\sum_{n \geq 2} (\log n) \frac{\|\mathbb{E}_0(X_n)\|_{p, \mathcal{H}}}{n^{1/2}} < \infty.$$

3.1.1 Application to the empirical process

Let X_0 be a real-valued random variable and $X_i = X_0 \circ \theta^i$. Let F be the distribution function of X_0 and define the empirical distribution

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{X_i \leq t}.$$

Let μ be a σ -finite measure on \mathbb{R} and suppose that F satisfies

$$\int_{\mathbb{R}_-} (F(t))^2 \mu(dt) + \int_{\mathbb{R}_+} (1 - F(t))^2 \mu(dt) < \infty. \quad (18)$$

Under this assumption, $(F_n - F)$ may be seen as a sequence of (centered) random variable taking values in the Hilbert space $\mathbb{L}^2(\mu)$. Define then

$$D_n(\mu) = \left(\int (F_n(t) - F(t))^2 \mu(dt) \right)^{1/2}.$$

When $\mu = dF$, $D_n(\mu)$ is known as the Cramér-von Mises statistics, and is commonly used for testing goodness of fit.

Let us first define the dependence coefficients which naturally appear in this context.

Definition 3.3 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let X be a real-valued random variable and let \mathcal{F} be a sub- σ -algebra of \mathcal{A} . Denote by \mathbb{P}_X the distribution of X and by $\mathbb{P}_{X|\mathcal{F}}$ a regular distribution of X given \mathcal{F} . Let $F_X(t) = \mathbb{P}_X([-\infty, t])$ and $F_{X|\mathcal{F}}(t) = \mathbb{P}_{X|\mathcal{F}}([-\infty, t])$. For $1 \leq p \leq \infty$, define the coefficient

$$\tau_{\mu,2,p}(\mathcal{F}, X) = \left\| \left(\int |F_{X|\mathcal{F}}(t) - F_X(t)|^2 \mu(dt) \right)^{1/2} \right\|_p := \|F_{X|\mathcal{F}} - F_X\|_{2,\mu} \| \cdot \|_p.$$

Corollary 3.4 Assume that the distribution function F of X_0 satisfies (18). Define the function F_μ by: $F_\mu(x) = \mu([0, x])$ if $x \geq 0$ and $F_\mu(x) = -\mu([x, 0])$ if $x \leq 0$. Let $1 < p < 2$ and assume that $\mathbb{E}(|F_\mu(X_0)|)^{p/2} < \infty$ and that

$$\sum_{n \geq 2} \log n \frac{\tau_{\mu,2,p}(\mathcal{F}_0, X_n)}{n^{1/2}} < \infty. \quad (19)$$

Then $nD_n(\mu) = o(n^{1/p})$ almost surely.

We refer to [8] for examples of models for which the coefficients $\tau_{\mu,2,p}(\mathcal{F}_0, X_n)$ have been evaluated.

Proof of Corollary 3.4. Define the variable $Z_i = \{t \rightarrow \mathbf{1}_{X_i \leq t} - F(t), t \in \mathbb{R}\}$ which belongs to $\mathcal{H} = \mathbb{L}^2(\mu)$ as soon as (18) holds. In addition $\|Z_0\|_{p,\mathcal{H}} < \infty$ as soon as $\mathbb{E}(|F_\mu(X_0)|)^{p/2} < \infty$ (see the proof of Theorem 2 in [8]). To prove Corollary 3.4, it suffices then to apply Corollary 3.1 by taking into account Remark 3.2 and by noticing that $\|\mathbb{E}_0(Z_i)\|_{p,\mathcal{H}} = \tau_{\mu,2,p}(\mathcal{F}_0, X_k)$. \square

3.2 Moderate deviations

The aim of this section is to obtain asymptotic expansions for probabilities of moderate deviations for stationary adapted real processes under projective criteria; more precisely we want to study the asymptotic behavior of $\mathbb{P}(S_n \geq \sigma \sqrt{nr_n})$ where (r_n) is a sequence of positive numbers that diverges to

infinity at an appropriate rate and $\sigma = \lim_{n \rightarrow \infty} \|S_n\|_2 / \sqrt{n}$. Specifically, we aim to find the zone for x of the following moderate deviations principle:

$$\frac{\mathbb{P}(S_n \geq x\sigma\sqrt{nr_n})}{1 - \Phi(x)} = 1 + o(1), \quad (20)$$

where $\Phi(x)$ is the standard normal distribution function. If $r_n = r \geq 0$ is fixed, then (20) becomes the well-known central limit theorem. However, for the case when $r = r_n$ is allowed to tend to infinity, the problem of *moderate deviation* probabilities is to find all the possible speed of convergence of $r_n \rightarrow \infty$ such that (20) holds. It is a challenging problem to establish moderate deviations principle (MDP) for dependent variables. However starting from the deep results of Grama ([16]) and of Grama and Haeusler ([17]) for martingales, Wu and Zhao ([34]) showed that it is possible to obtain MDP results for a certain class of stationary processes such as functions of an iid sequence as soon as the partial sum process can be well approximated by a martingale. Using our approximation theorem 2.4, we shall give alternative conditions to the ones obtain by Wu and Zhao [34], in order for the MDP to hold.

Let us first start with some notations and definitions.

Let $p \in (2, 4]$. For $x > 1$, let $r_x > 0$ be the solution to the equation

$$x = (1 + r_x)^{\nu(p)} \exp(r_x^2/2) \text{ where } \nu(p) = \begin{cases} p + 1 & \text{if } 2 < p \leq 3 \\ 3p - 3 & \text{if } 3 < p \leq 4. \end{cases}$$

The function $\nu(p)$ results from the martingale MDP as obtained in Grama ([16]) and of Grama and Haeusler ([17]) (see also Theorem 2 and Remark 5 in [34]).

Let $\tau_n \rightarrow \infty$ be a positive sequence of numbers and (U_n) a sequence of real valued random variables such that $U_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$. We shall say that (U_n) satisfies the moderate deviations principle (MDP) with rate τ_n and exponent $p > 0$ if for every $a > 0$ there exists a positive constant $C = C_{a,p}$ not depending on x neither on n such that

$$\max \left\{ \left| \frac{\mathbb{P}(U_n \geq r_x)}{1 - \Phi(r_x)} - 1 \right|, \left| \frac{\mathbb{P}(U_n \leq -r_x)}{1 - \Phi(-r_x)} - 1 \right| \right\} \leq C \left(\frac{x}{\tau_n} \right)^{1/(1+p)},$$

holds uniformly in $x \in [1, a\tau_n]$. Therefore τ_n gives a range for which the MDP holds.

Theorem 3.5 *Let $2 < p \leq 4$ and let $(X_n)_{n \in \mathbf{Z}}$ be a real adapted stationary sequence in \mathbb{L}^p in the sense of Notation 1.1. Assume that*

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_p}{n^{1+2/p^2}} < \infty \text{ and } \sum_{n \geq 1} \frac{1}{n^{2/p}} \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^{3/2}} < \infty. \quad (21)$$

Assume in addition that

$$\sum_{k \geq 0} \frac{1}{2^{2k/p}} \|\mathbb{E}_0(S_{2^k}^2) - \mathbb{E}(S_{2^k}^2)\|_{p/2} < \infty. \quad (22)$$

Then $n^{-1}\mathbb{E}(S_n^2)$ converges to some nonnegative number σ^2 and if $\sigma > 0$, $\left(\frac{S_n}{\sigma\sqrt{n}}\right)_{n \geq 1}$ satisfies the MDP with rate $\tau_n = n^{p/2-1}$ and exponent p .

Proof. Analyzing the proof of Theorem 1 in [34], the theorem will be proven if we can show that there exists a \mathbb{L}^p stationary sequence $(D_i)_{i \in \mathbf{Z}}$ of martingale differences with respect to $(\mathcal{F}_i)_{i \in \mathbf{Z}}$ such that setting $M_n = \sum_{i=1}^n D_i$,

$$\|S_n - M_n\|_p = o(n^{1/p}) \quad (23)$$

and

$$\sum_{i=1}^n \|\mathbb{E}_{i-1}(D_i^2) - \mathbb{E}(D_i^2)\|_{p/2} = O(n^{2/p}). \quad (24)$$

According to Theorem 2.4 combined with Remark 2.5, the first part of condition (21) implies (23). On the other hand, since $1 < p/2 \leq 2$, according to Theorem 3 in [34] applied to the stationary sequence $(\mathbb{E}_{i-1}(D_i^2) - \mathbb{E}(D_i^2))_{i \geq 1}$ and using the fact that M_n is a martingale, (24) holds if

$$\sum_{k \geq 0} \frac{1}{2^{2k/p}} \|\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2)\|_{p/2} < \infty. \quad (25)$$

Using now Proposition B.2, we infer that (25) holds if (22) does and if

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_p^2}{n^{1+4/p^2}} < \infty \text{ and } \sum_{k \geq 0} \frac{2^k}{2^{2k/p}} \sum_{\ell \geq 2^k} \frac{\|\mathbb{E}_{-2^k}(S_\ell)\|_2}{\ell^{3/2}} < \infty. \quad (26)$$

To end the proof, it suffices to notice that since $(\|\mathbb{E}_0(S_n)\|_p)_{n \geq 1}$ is a subadditive sequence, the first part of (21) implies the first part of (26) (see item 3 of Lemma 37 in [23]) and that due to the monotonicity of the sequence $(\sum_{\ell \geq n} \ell^{-3/2} \|\mathbb{E}_{-2^k}(S_\ell)\|_2)_{n \geq 1}$, the second part of (26) is equivalent to the second part of (21). \square

Comment 3.6 *Theorem 3.5 also holds if condition (22) is replaced by*

$$\sum_{n \geq 1} \frac{1}{n^{1+2/p}} \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} < \infty. \quad (27)$$

Even if these two conditions cannot be compared (there is no subadditivity or monotonicity property of the sequence $(\|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2})_{n \geq 1}$), this comes from the fact that condition (22) is assumed in order to prove that (25) holds. However condition (25) is equivalent to

$$\sum_{n \geq 1} \frac{1}{n^{1+2/p}} \|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2} < \infty. \quad (28)$$

To see this it suffices to notice that by stationarity and since M_n is a martingale, the sequence $(\|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2})_{n \geq 1}$ is subadditive and to apply Lemma 2.7 in [26]. Therefore, by using Proposition B.2, we then infer that (28) holds as soon as (27) and (21) do. This proves the fact that in Theorem 3.5, (22) can be replaced by (27). Notice however that condition (22) is more tractable than (27) when we want to prove that it holds in the context of mixing sequences as it is done in the forthcoming corollary 3.8.

The quantities involved in conditions (21) and (27) can be handled by controlling norms of individual summands which involve terms such as $\mathbb{E}_0(X_i X_j)$ and $\mathbb{E}_0(X_i)$. The latter quantities can be then in turn controlled by using various mixing or dependent coefficients (see [7] for instance). For instance, as a corollary of Theorem 3.5, the following result holds (its proof is omitted since it follows the lines of the proof of Corollary 2.1 in [7]).

Corollary 3.7 *Let $2 < p \leq 4$ and let $(X_n)_{n \in \mathbf{Z}}$ be a real adapted stationary sequence in \mathbb{L}^p in the sense of Notation 1.1. Assume that there exists $\gamma \in]0, 1]$ such that*

$$\sum_{n > 0} \frac{n^{(p-2)/(\gamma p)}}{n^{1/p}} \|\mathbb{E}_0(X_n)\|_p < \infty \text{ and } \sum_{n > 0} \frac{n^\gamma}{n^{2/p}} \sup_{i \geq j \geq n} \|\mathbb{E}_0(X_i X_j) - \mathbb{E}(X_i X_j)\|_{p/2} < \infty.$$

Then the conclusion of Theorem 3.5 holds with $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$.

As in [7], this result can be used, for instance, to derive under which conditions the partial sum of a function f of the stationary Markov chain $(\xi_k)_{k \in \mathbf{Z}}$ with transition $Kf(x) = \frac{1}{2}(f(x+a) + f(x-a))$, when a is irrational in $[0, 1]$ and badly approximable by rationals, satisfy the conclusion of Theorem 3.5. For instance, one can prove that if f is three times differentiable, $\left(\frac{S_n(f)}{\sigma(f)\sqrt{n}}\right)_{n \geq 1}$ satisfies the MDP with rate $\tau_n = n$ and exponent 4 provided that $\sigma(f) > 0$. Here $S_n(f) = \sum_{k=1}^n f(\xi_k) - m(f)$ where m is the Lebesgue-Haar measure and $\sigma^2(f) = m((f - m(f))^2) + 2 \sum_{n>0} m(fK^n(f - m(f)))$.

Since in Theorem 3.5 the conditions are expressed in terms of the conditional expectation of the partial sum or of its square, it is also possible to obtain applications for mixing sequences. As an example, the following corollary gives conditions in terms of ρ -mixing coefficients as defined in Comment 2.7.

Corollary 3.8 *Let $p \in]2, 4]$ and $r \geq p$. Let $(X_n)_{n \in \mathbf{Z}}$ be a real adapted stationary sequence in \mathbb{L}^r in the sense of Notation 1.1. Let $(\rho(n))_{n \geq 1}$ be its associated rho-mixing coefficients as defined in (12). Assume that*

$$\sum_{k \geq 0} 2^{k(1/2-2/p^2)} \rho^{2/p}(2^k) < \infty \text{ and } \sum_{k \geq 0} 2^{k(1-2/p)} \rho^s(2^k) < \infty \text{ where } s = \min(1, 2(r-2)/r). \quad (29)$$

Then the conclusion of Theorem 3.5 holds with rate $\tau_n = n^{p/2-1}$ and exponent p .

Notice that if $r \geq 4$, condition (29) reduces to its first part.

The proof, being of technical nature is left to the appendix.

3.3 Rates of convergence for Wasserstein distances in the CLT

Let $\mathcal{L}(\mu, \nu)$ be the set of the probability laws on \mathbb{R}^2 with marginals μ and ν . Let us consider the Wasserstein distances of order $r \geq 1$ defined by

$$W_r(\mu, \nu) = \inf \left\{ \left(\int |x - y|^r P(dx, dy) \right)^{1/r} : P \in \mathcal{L}(\mu, \nu) \right\}.$$

Let $p \in]2, 3[$ and let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in \mathbb{L}^p in the sense of Notation (1.1). Denote by $P_{S_n/n^{1/2}}$ the law of $S_n/n^{1/2}$ and by G_{σ^2} the normal distribution $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \lim_{n \rightarrow \infty} n^{-1} \mathbb{E}(S_n^2)$ provided the limit exists. Starting from their Theorem 2.1 that gives upper bounds, under suitable conditions, for $W_r(P_{S_n/n^{1/2}}, G_{\sigma^2})$ when S_n is a martingale, and next using a coboundary decomposition, Dedecker, Merlevède and Rio ([9]) have proven in particular that if for $p \in]2, 3[$, (27) holds, and if

$$\sum_{n \geq 1} n^{-3+p/2} \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{1+\gamma} < \infty \text{ for some } \gamma > 0, \quad (30)$$

and

$$\sum_{n > 0} \mathbb{E}(X_n | \mathcal{F}_0) \text{ converge in } \mathbb{L}^p, \quad (31)$$

then $n^{-1} \mathbb{E}(S_n^2)$ converges to $\sigma^2 = \sum_{k \in \mathbf{Z}} \text{Cov}(X_0, X_k)$, and for any $r \in [1, p]$,

$$W_r^r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2}). \quad (32)$$

Notice that in their Theorem 3.1, Dedecker, Merlevède and Rio stated the result by assuming stronger conditions than (27) and (30), but following their proof and taking into account their remark 2.1, their

result also holds under the weaker conditions (27) and (30) replacing their condition (3.2). We would like also to mention that the conditions (27) and (30) are imposed in order to verify that the same are true with the approximating martingale M_n replacing S_n . According to Remark 2.6 in [9], it follows that (32) also holds if (27) and (30) are respectively replaced with (22) and

$$\sum_{k \geq 0} 2^{-k(2-p/2)} \|\mathbb{E}_0(S_{2^k}^2) - \mathbb{E}(S_{2^k}^2)\|_{1+\gamma} < \infty \text{ for some } \gamma > 0. \quad (33)$$

Using our Theorem 2.4, we shall prove that condition (31) can be weakened and that the following result holds:

Theorem 3.9 *Let $p \in]2, 3[$ and let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in \mathbb{L}^p in the sense of Notation 1.1. Assume that (27) and (30) (or (22) and (33)) hold. Assume in addition that*

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_p^2}{n^{1+4/p^2}} < \infty, \quad (34)$$

and that

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_2}{n^{(5-p)/2}} < \infty \quad \text{if } r \in]1, 2] \quad \text{and } \|\mathbb{E}_0(S_n)\|_r = O(n^{(3-p)/r}) \quad \text{if } r \in]2, p]. \quad (35)$$

Then $n^{-1}\mathbb{E}(S_n^2)$ converges to some nonnegative number σ^2 and for any $r \in [1, p]$, the bound (32) holds true.

When ρ -mixing sequences are considered, the above result gives the following corollary (its proof is omitted since it uses similar bounds as those obtained in the proof of Corollary 3.8, see the Appendix).

Corollary 3.10 *Let $p \in]2, 3[$ and $r \geq p$. Let $(X_n)_{n \in \mathbf{Z}}$ be a real adapted stationary sequence in \mathbb{L}^r in the sense of Notation 1.1. Let $(\rho(n))_{n \geq 1}$ be its associated rho-mixing coefficients as defined in (12). Assume that*

$$\sum_{k \geq 0} 2^{k(p-2)/2} \rho^s(2^k) < \infty \quad \text{where } s = \min(1, 2(r-2)/r).$$

Then the conclusion of Theorem 3.9 holds.

Proof of Theorem 3.9. Notice first that (34) implies in particular that $\|\mathbb{E}_0(S_n)\|_p = o(n^{2/p^2})$ (apply for instance Item 2 of Lemma 37 in [23] to the sequence $(\|\mathbb{E}_0(S_n)\|_p^2)_{n \geq 0}$). Now, since $p > 2$, (34) then entails that (6) holds true. Therefore, by Theorem 2.4, D defined by (5) is in \mathbb{L}^p . In addition, since $p > 2$, (6) implies that $\sum_{n > 0} n^{-3/2} \|\mathbb{E}_0(S_n)\|_2 < \infty$ which is a sufficient condition for $n^{-1}\mathbb{E}(S_n^2)$ to converge (see Theorem 1 in [26]).

Let now $M_n = \sum_{k=1}^n D \circ \theta^k$ and $R_n = S_n - M_n$. According to the proof of Theorem 3.1 in [9], the theorem will follow if we can prove that

$$\|R_n\|_r = O(n^{(3-p)/2}). \quad (36)$$

and also that

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{1+\gamma}}{n^{3-p/2}} < \infty \text{ for a } \gamma > 0 \text{ and } \sum_{n \geq 1} \frac{\|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2}}{n^{1+2/p}} < \infty, \quad (37)$$

or equivalently to (37),

$$\sum_{k \geq 0} \frac{\|\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2)\|_{1+\gamma}}{2^{k(2-p/2)}} < \infty \text{ for a } \gamma > 0 \text{ and } \sum_{n \geq 1} \frac{\|\mathbb{E}_0(M_{2^k}^2) - \mathbb{E}(M_{2^k}^2)\|_{p/2}}{2^{2k/p}} < \infty. \quad (38)$$

Using Proposition B.2 we infer that (37) (resp. (38)) holds provided that (27) and (30) do (resp. (22) and (33)) and that

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_p^2}{n^{1+4/p^2}} < \infty, \quad \sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_{2(1+\gamma)}^2}{n^{1+(4-p)/(2+2\gamma)}} < \infty \quad \text{and} \quad \sum_{n \geq 1} \frac{\|\mathbb{E}_0(S_n)\|_2}{n^{(5-p)/2}} < \infty. \quad (39)$$

Notice first that the third part of (39) holds provided that (35) does (notice that the second part of (35), for $r > 2$ implies the first part of (35)), whereas the first part of (39) is exactly condition (34). Notice now that for any $p \in]2, 3[$ and γ small enough, $(4-p)/(2+2\gamma) \geq 4/p^2$ and $p \geq 2+2\gamma$. Therefore the second part of (39) is implied by condition (34).

It remains to prove (36). By Lemma 2.7 of [26], the first part of (35) implies that $\|\mathbb{E}_0(S_n)\|_2 = o(n^{(3-p)/2})$. Therefore by using Theorem 2.4, we infer that for $r \in [1, 2]$, $\|R_n\|_r \leq \|R_n\|_2 = o(n^{(3-p)/2})$ under the first part of (35). Now, for $r \in]2, p]$, the second part of (35) implies that $\|R_n\|_r = O(n^{(3-p)/2})$ by Theorem 2.4. \square .

4 Proof of the martingale approximation results

We begin with some lemmas.

Lemma 4.1 *Let $X_0 \in \mathbb{L}^p(\mathcal{H})$, $p > 1$, then $\sum_{k \geq 0} (k+1)^{-1} \|\mathcal{P}_0(X_k)\|_{p, \mathcal{H}} < \infty$.*

The proof using similar arguments are those developed in the proof of Lemma 4.3 below is therefore omitted.

Lemma 4.2 *Let $p > 1$ and let $(X_n)_{n \in \mathbf{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. Assume that*

$$\sum_{n \geq 1} \sum_{k \geq 0} \frac{\|\mathcal{P}_0(S_n \circ \theta^{k-1})\|_{p, \mathcal{H}}}{(n+k)(n+k+1)} < \infty. \quad (40)$$

Then $\sum_{n \geq 0} \left| \sum_{k \geq n} \frac{\mathcal{P}_0(X_k)}{k+1} \right|_{\mathcal{H}}$ converges in \mathbb{L}^p and a.s. Moreover for all integer $m \geq 0$,

$$\left\| \sum_{n \geq m} \sum_{k \geq n} \frac{\mathcal{P}_0(X_k)}{k+1} \right\|_{p, \mathcal{H}} \leq \sum_{k \geq m} \sum_{n \geq 1} \frac{\|\mathcal{P}_0(S_n \circ \theta^{k-1})\|_{p, \mathcal{H}}}{(n+k)(n+k+1)}. \quad (41)$$

Proof.

By assumption,

$$\sum_{k \geq 0} \left| \sum_{n \geq 1} \frac{\sum_{l=0}^{n-1} \mathcal{P}_0(X_{l+k})}{(n+k)(n+k+1)} \right|_{\mathcal{H}}$$

converges a.s. and in \mathbb{L}^p . On the other hand, using Lemma 4.1, we have

$$\sum_{n \geq 1} \frac{\sum_{l=0}^{n-1} \mathcal{P}_0(X_{l+k})}{(n+k)(n+k+1)} = \sum_{l \geq 0} \sum_{n \geq l+1} \frac{\mathcal{P}_0(X_{k+l})}{(k+n)(k+n+1)} = \sum_{l \geq 0} \frac{\mathcal{P}_0(X_{k+l})}{k+l+1} = \sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1},$$

which gives the desired convergence. \square

Lemma 4.3 *For every integer $r \geq 0$,*

$$\sum_{k \geq r} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}}{(m+k)^2} \ll \sum_{k \geq r+1} \frac{\|\mathbb{E}_{-r}(S_k)\|_{p, \mathcal{H}}}{k^{1+1/p'}}. \quad (42)$$

Proof.

Let m be a positive integer. Assume first that $p \geq 2$. By Hölder inequality and using that $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^2}$, we have

$$\begin{aligned} \sum_{k \geq r} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}}{(m+k)^2} &\ll (m+r)^{-(1+1/p)} \left(\sum_{k \geq r} \|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}^p \right)^{1/p} \\ &\ll (m+r)^{-(1+1/p)} \left(\mathbb{E} \left(\sum_{k \geq r} |\mathcal{P}_{-k}(S_m)|_{\mathcal{H}}^2 \right)^{p/2} \right)^{1/p} \\ &\ll \frac{\|\mathbb{E}_{-r}(S_m)\|_{p, \mathcal{H}}}{(m+r)^{1+1/p}}, \end{aligned}$$

where we used Burkholder's inequality for \mathcal{H} -valued martingales (see Burkholder [3]), in the last step.

Assume now that $1 < p < 2$. We use Hölder inequality twice and once again the Burkholder's inequality for \mathcal{H} -valued martingales in the last step, to obtain

$$\begin{aligned} \sum_{k \geq r} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}}{(m+k)^2} &\ll \frac{1}{(m+r)^{1/p}} \left(\sum_{k \geq r} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}^p}{(m+k)^p} \right)^{1/p} \\ &\ll \frac{1}{(m+r)^{1/p}} \left(\frac{1}{(m+r)^{3p/2-1}} \mathbb{E} \left(\sum_{k \geq r} |\mathcal{P}_{-k}(S_m)|_{\mathcal{H}}^2 \right)^{p/2} \right)^{1/p} \\ &\ll \frac{\|\mathbb{E}_{-r}(S_m)\|_{p, \mathcal{H}}}{(m+r)^{3/2}}. \end{aligned}$$

From the above computations, we then obtain that

$$\begin{aligned} \sum_{k \geq r} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p, \mathcal{H}}}{(m+k)^2} &\ll \sum_{m \geq 1} \frac{\|\mathbb{E}_{-r}(S_m)\|_{p, \mathcal{H}}}{(m+r)^{1+1/p''}} \\ &\ll \frac{1}{(r+1)^{1/p''}} \max_{1 \leq m \leq r} \|\mathbb{E}_{-r}(S_m)\|_{p, \mathcal{H}} + \sum_{m \geq r+1} \frac{\|\mathbb{E}_{-r}(S_m)\|_{p, \mathcal{H}}}{m^{1+1/p''}}. \end{aligned}$$

The lemma then follows by using Lemma B.1 with $\gamma = 1/p''$ and $\ell = r$. \square

Lemma 4.4 *For every $r \geq 0$,*

$$X_0 = \sum_{k=0}^r \sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1} - \sum_{k=0}^r \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1} + (r+1) \sum_{l \geq r} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}. \quad (43)$$

In particular, if we assume (6), letting $r \rightarrow \infty$, we have

$$X_0 = \sum_{k \geq 0} \sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1} - \sum_{k \geq 0} \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1}.$$

Proof.

Let $m \geq k \geq 0$. We have

$$\sum_{l=k}^m \frac{\mathcal{P}_0(X_l)}{l+1} = \frac{\mathbb{E}_0(X_k)}{k+1} - \frac{\mathbb{E}_0(X_{m+1})}{m+2} + \sum_{l=k}^m \frac{\mathbb{E}_0(X_{l+1})}{l+2} - \sum_{l=k}^m \frac{\mathbb{E}_{-1}(X_l)}{l+1}.$$

Hence

$$\sum_{l=k}^m \frac{\mathcal{P}_0 X_l}{l+1} = \frac{\mathbb{E}_0(X_k)}{k+1} - \frac{\mathbb{E}_0(X_{m+1})}{m+2} + \sum_{l=k}^m \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1} - \sum_{l=k}^m \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}.$$

Since $\|\mathbb{E}_0(X_m)\|_{p,\mathcal{H}} \leq \|X_0\|_{p,\mathcal{H}}$, $\|\mathbb{E}_0(X_m)\|_{p,\mathcal{H}} \rightarrow 0$ and the series $\sum_{l \geq 0} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}$ converges in $\mathbb{L}^p(\mathcal{H})$. Hence, using Lemma 4.1, we may and do let $m \rightarrow \infty$, to obtain

$$\sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1} = \frac{\mathbb{E}_0(X_k)}{k+1} + \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1} - \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}.$$

Let $r \geq 0$. We then deduce that

$$\sum_{k=0}^r \sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1} = \sum_{k=0}^r \frac{\mathbb{E}_0(X_k)}{k+1} + \sum_{k=0}^r \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1} - \sum_{k=0}^r \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}.$$

Hence, interverting the order of summation in the last term,

$$\sum_{k=0}^r \sum_{l \geq k} \frac{\mathcal{P}_0(X_l)}{l+1} = X_0 + \sum_{k=0}^r \sum_{l \geq k} \frac{\mathbb{E}_0(X_{l+1}) - \mathbb{E}_{-1}(X_l)}{l+1} - (r+1) \sum_{l \geq r} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}.$$

Assume (6). In view of Lemmas 4.2 and 4.3, we see that the series on the left converges in $\mathbb{L}^p(\mathcal{H})$. On an other hand, Lemma B.1 (with $\gamma = 1$) implies that $n^{-1} \|\mathbb{E}_0(S_n)\|_{p,\mathcal{H}} \rightarrow 0$. Therefore by Abel summation, $\|(r+1) \sum_{l \geq r} \frac{\mathbb{E}_0(X_{l+1})}{(l+1)(l+2)}\|_{p,\mathcal{H}} \rightarrow 0$, when $r \rightarrow \infty$. \square

4.1 Proof of Theorem 2.4

By Lemma 4.4, we have

$$X_1 = D \circ \theta - \sum_{k \geq 0} \sum_{l \geq k+1} \frac{\mathbb{E}_1(X_{l+1}) - \mathbb{E}_0(X_l)}{l}.$$

Hence, using that $\mathbb{E}_1(X_{l+1}) = \mathbb{E}_0(X_l) \circ \theta$, we obtain that for any positive integer n ,

$$S_n - M_n = - \sum_{k \geq 0} \sum_{l \geq k+1} \frac{\mathbb{E}_0(X_l) \circ \theta^n - \mathbb{E}_0(X_l)}{l} = - \sum_{k \geq 0} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_l)}{l}.$$

Let N be a positive integer, fixed for the moment. Then writing

$$V_{n,N} = \sum_{k=0}^{N-1} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l}, \quad (44)$$

and

$$W_{n,N} = \sum_{k \geq N} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l}. \quad (45)$$

we obtain

$$S_n - M_n - \mathbb{E}_0(S_n) = - \sum_{k \geq 0} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l} = -(V_{n,N} + W_{n,N}). \quad (46)$$

We first deal with $V_{n,N}$. We have

$$\begin{aligned}
V_{n,N} &= \sum_{k=0}^{N-1} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l} \\
&= \sum_{l=1}^N \mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n}) + N \sum_{l \geq N+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l} \\
&= \mathbb{E}_0(S_N) \circ \theta^n - \mathbb{E}_0(S_N \circ \theta^n) + N \sum_{l \geq N+1} \frac{\mathbb{E}_n(X_l \circ \theta^n) - \mathbb{E}_0(X_l \circ \theta^n)}{l}.
\end{aligned} \tag{47}$$

Let $j \in \{0, n\}$. By (6) and Lemma B.1 with $\gamma = 1$,

$$\frac{\|\mathbb{E}_0(S_N)\|_{p,\mathcal{H}}}{N} \ll \sum_{l \geq N} \frac{\|\mathbb{E}_0(S_l)\|_{p,\mathcal{H}}}{l^2} = o(1). \tag{48}$$

Hence, using Abel summation (notice that the "end" term vanishes by (48)), we derive

$$\begin{aligned}
\sum_{l \geq N+1} \frac{\mathbb{E}_j(X_l \circ \theta^n)}{l} &= \sum_{l \geq N+1} \frac{\mathbb{E}_j(S_l \circ \theta^n - S_{l-1} \circ \theta^n)}{l} \\
&= -\frac{\mathbb{E}_j(S_N \circ \theta^n)}{N+1} + \sum_{l \geq N+1} \frac{\mathbb{E}_j(S_l \circ \theta^n)}{l(l+1)}.
\end{aligned} \tag{49}$$

Hence, using (48),

$$\begin{aligned}
\|V_{n,N}\|_{p,\mathcal{H}} &\leq 2 \frac{\|\mathbb{E}_0(S_N)\|_{p,\mathcal{H}}}{N} + N \left\| \sum_{l \geq N+1} \frac{\mathbb{E}_n(S_l \circ \theta^n) - \mathbb{E}_0(S_l \circ \theta^n)}{l(l+1)} \right\|_{p,\mathcal{H}} \\
&\ll N \sum_{l \geq N} \frac{\|\mathbb{E}_0(S_l)\|_{p,\mathcal{H}}}{l^2}.
\end{aligned} \tag{50}$$

It remains to deal with $W_{n,N}$. Since $\mathbb{E}_0(W_{n,N}) = 0$, we have

$$W_{n,N} = \sum_{r=1}^n \mathcal{P}_r(W_{n,N}) = \sum_{r=1}^n \sum_{k \geq N} \sum_{l \geq k+1} \frac{\mathbb{E}_r(X_{l+n}) - \mathbb{E}_{r-1}(X_{l+n})}{l}. \tag{51}$$

But, by Burkholder's inequality for \mathcal{H} -valued martingales (see Burkholder [3]),

$$\|W_{n,N}\|_{p,\mathcal{H}}^{p'} \ll \sum_{r=1}^n \|\mathcal{P}_r(W_{n,N})\|_{p,\mathcal{H}}^{p'}. \tag{52}$$

Notice that for any $r \in \{1, \dots, n\}$,

$$\mathcal{P}_r(W_{n,N}) = \left(\sum_{k \geq N} \sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})}{l+k} \right) \circ \theta^r.$$

Now, using Lemma 4.1,

$$\begin{aligned}
\sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})}{l+k} &= \sum_{l \geq 1} \mathcal{P}_0(X_{l+k+n-r}) \sum_{m \geq l} \frac{1}{(m+k)(m+k+1)} \\
&= \sum_{m \geq 1} \frac{\mathcal{P}_0(S_m \circ \theta^{k+n-r})}{(m+k)(m+k+1)},
\end{aligned}$$

so

$$\left| \sum_{k \geq N} \sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})}{l+k} \right| \leq \sum_{m \geq 1} \sum_{k \geq N} \frac{|\mathcal{P}_0(S_m \circ \theta^{k+n-r})|}{(m+k)^2}. \quad (53)$$

Hence, with $s = n - r$,

$$\|W_{n,N}\|_{p,\mathcal{H}} \ll n^{1/p'} \max_{0 \leq s \leq n-1} \sum_{k \geq N+s} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p,\mathcal{H}}}{(m+k-s)^2}.$$

Now we take $N = u_n \geq n$. We then obtain

$$\|W_{n,u_n}\|_{p,\mathcal{H}} \ll n^{1/p'} \sum_{k \geq u_n} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p,\mathcal{H}}}{(m+k)^2}. \quad (54)$$

Hence using (46) and (50) with $N = u_n$, we get that

$$\|S_n - M_n\|_{p,\mathcal{H}} \ll \|\mathbb{E}_0(S_n)\|_{p,\mathcal{H}} + u_n \sum_{m \geq u_n} \frac{\|\mathbb{E}_0(S_m)\|_{p,\mathcal{H}}}{m^2} + n^{1/p'} \sum_{k \geq u_n} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_{p,\mathcal{H}}}{(m+k)^2}. \quad (55)$$

Next using Lemma B.1 with $\gamma = 1$, we derive that

$$\|\mathbb{E}_0(S_n)\|_{p,\mathcal{H}} \leq \max_{1 \leq k \leq u_n} \|\mathbb{E}_0(S_k)\|_{p,\mathcal{H}} \ll u_n \sum_{m \geq u_n} \frac{\|\mathbb{E}_0(S_m)\|_{p,\mathcal{H}}}{m^2}. \quad (56)$$

Starting from (55) with $u_n = [n^q]$ and taking into account (56) and Lemma 4.3, Theorem 2.4 follows. \square

4.2 Proof of Theorem 2.8

Part of the proof relies on a new ergodic theorem with rate. Hence we first recall some facts from ergodic theory and state our ergodic theorem, while we leave its proof to the appendix.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and T be a Dunford-Schwartz operator on X , i.e. T is a contraction of \mathbb{L}^1 and \mathbb{L}^∞ . Let \mathbf{T} be the linear modulus of T (see e.g. Theorem 1.1, chapter 4 of [20]). Recall that \mathbf{T} is a positive Dunford-Schwartz operator such that $|Tf| \leq \mathbf{T}|f|$, for every $f \in \mathbb{L}^1$ and $|Tf|^p \leq \mathbf{T}(|f|^p)$, for every $f \in \mathbb{L}^p$.

We will make use, for $p \geq 1$, of the weak \mathbb{L}^p -spaces

$$\mathbb{L}^{p,w} := \{f \in \mathbb{L}^1 : \sup_{\lambda > 0} \lambda^p \mathbb{P}\{|f| \geq \lambda\} < \infty\}.$$

Recall that, when $p > 1$, there exists a norm $\|\cdot\|_{p,w}$ on $\mathbb{L}^{p,w}$ that makes $\mathbb{L}^{p,w}$ a Banach space and which is equivalent to the "pseudo"-norm $(\sup_{\lambda > 0} \lambda^p \mathbb{P}\{|f| \geq \lambda\})^{1/p}$.

We define, for every $l \geq 0$, a maximal operator as follows. Let $h \in \mathbb{L}^1$ be non-negative

$$\mathcal{M}_l(h) = \sup_{n \geq 1} \frac{h + \mathbf{T}^{2^l} h + \dots + (\mathbf{T}^{2^l})^{n-1} h}{n}.$$

By the Dunford-Schwartz (or Hopf) ergodic theorem (see e.g. Krengel [20], Lemma 6.1 page 51 and Corollary 3.8 p. 131)

$$\sup_{\lambda > 0} \lambda \mathbb{P}\{\mathcal{M}_l(h) \geq \lambda\} \leq \|h\|_1.$$

In particular, for every $p > 1$, there exists $C_p > 0$ and such that for every $f \in \mathbb{L}^p$

$$\|(\mathcal{M}_l(|f|^p))^{1/p}\|_{p,w} \leq C_p \|f\|_p. \quad (57)$$

Let \mathcal{B} be a Banach space with norm $|\cdot|_{\mathcal{B}}$. For every $p \geq 1$, we denote $\mathbb{L}^p(\mathcal{B})$ the Bochner space $\{f : \Omega \rightarrow \mathcal{B}, |f|_{\mathcal{B}} \in \mathbb{L}^p\}$. When T is induced by a measurable transformation θ preserving \mathbb{P} , $\mathcal{M}_l(|f|_{\mathcal{B}})$ is well-defined for every $f \in \mathbb{L}^1(\mathcal{B})$. We prove the following, where $U_n(f) = f + \dots + T^{n-1}f$.

Proposition 4.5 *Let T be a Dunford-Schwartz operator on $(\Omega, \mathcal{A}, \mathbb{P})$ and $f \in \mathbb{L}^1$. We have*

$$\max_{1 \leq n \leq 2^r} |U_n(f)| \leq 2^{r/p} \sum_{k=0}^r \frac{[\mathcal{M}_k(|U_{2^k}(f)|^p)]^{1/p}}{2^{k/p}}.$$

When T is induced by a measure preserving transformation θ and \mathcal{B} is a Banach space, the result holds also for $f \in \mathbb{L}^1(\mathcal{B})$, replacing $|\cdot|$ with $|\cdot|_{\mathcal{B}}$.

Proof. The proof follows from the following lemma, using that $U_{2^k m}(f) - U_{2^k(m-1)}(f) = T^{2^k(m-1)}f + \dots + T^{2^k m-1}f = (T^{2^k})^{(m-1)}U_{2^k}(f)$. \square

Lemma 4.6 *Let (a_n) be a sequence in a Banach space \mathcal{B} with norm $|\cdot|_{\mathcal{B}}$. Write $s_n = a_1 + \dots + a_n$. Let $p \geq 1$. For every $r \geq 0$, we have*

$$\max_{1 \leq n \leq 2^r} |s_n|_{\mathcal{B}} \leq \sum_{k=0}^r \left(\sum_{m=1}^{2^{r-k}} |s_{2^k m} - s_{2^k(m-1)}|_{\mathcal{B}}^p \right)^{1/p}. \quad (58)$$

Proof. We make the proof by induction on $r \geq 0$. The result is obvious for $r = 0$. Let $1 \leq n \leq 2^r$. We have (with the convention that $s_0 = 0$) $|s_{2n-1}|_{\mathcal{B}} \leq |s_{2n-2}|_{\mathcal{B}} + |a_{2n-1}|_{\mathcal{B}}$. Hence, writing $\tilde{a}_n = a_{2n-1} + a_{2n}$ and $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k = s_{2n}$, we have

$$\max_{1 \leq l \leq 2^{r+1}} |s_l|_{\mathcal{B}} \leq \max_{1 \leq n \leq 2^r} |\tilde{s}_n|_{\mathcal{B}} + \left(\sum_{n=1}^{2^r} |a_{2n-1}|_{\mathcal{B}}^p \right)^{1/p},$$

and the result follows. \square

Theorem 4.7 *Let T be a Dunford-Schwartz operator on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $f \in \mathbb{L}^p$, $p > 1$. Let ψ be a positive non-decreasing function, such that there exists $C > 1$ such that $\psi(2x) \leq C\psi(x)$, for every $x \geq 1$. Assume that*

$$\sum_n \frac{\|f + \dots + T^{n-1}f\|_p}{\psi(n)n^{1+1/p}} < \infty. \quad (59)$$

Then $\sup_{n \geq 1} \frac{|f + \dots + T^{n-1}f|}{\psi(n)n^{1/p}} \in \mathbb{L}^{p,w}$ and $\frac{|f + \dots + T^{n-1}f|}{\psi(n)n^{1/p}} \rightarrow 0$ \mathbb{P} -a.s.

If T is induced by a measure-preserving transformation and \mathcal{B} is a Banach space the result holds with $|\cdot|_{\mathcal{B}}$ instead of $|\cdot|$ for every $f \in \mathbb{L}^p(\mathcal{B})$ such that $\sum_n \frac{\|f + \dots + T^{n-1}f\|_{p,\mathcal{B}}}{\psi(n)n^{1+1/p}} < \infty$.

Comment 4.8 *Take $\psi \equiv 1$, which is the relevant case in our applications. Then condition (59) is weaker than condition (8) in [32] and also (slightly) improves condition (10) of [4] (obtained for $p = 2$). In [32] and [4], only the case where T is induced by a transformation is considered.*

We turn now to the proof of Theorem 2.8. It will follow from the next two propositions. Notice that the second one is a version of Corollary 22 of Merlevède-Peligrad [23] under \mathbb{E}_0 .

Proposition 4.9 *Assume (1). Then $\mathbb{E}_0[(S_n - M_n - \mathbb{E}_0(S_n))^2] = o(n)$ \mathbb{P} -a.s. and $\mathbb{E}_0(S_n) = o(\sqrt{n})$ \mathbb{P} -a.s. In particular*

$$\mathbb{E}_0[(S_n - M_n)^2] = o(n) \quad \mathbb{P}\text{-a.s.}$$

Proposition 4.10 *Assume (1) and that $\mathbb{E}_0(S_n^2) = o(n)$ \mathbb{P} -a.s. then*

$$\mathbb{E}_0\left(\max_{1 \leq k \leq n} S_k^2\right) = o(n) \quad \mathbb{P}\text{-a.s.} \quad (60)$$

Before proving the above propositions, we indicate how they lead to Theorem 2.8. Using Proposition 4.9, we apply Proposition 4.10 with $S_n - M_n$ in place of S_n . This proves (14). Now the convergence (15) follows from (14) together with the fact that by the quenched weak invariance principle for martingales (see for instance Deriennic and Lin [10] for the ergodic case) (15) holds for M_n with $\eta = \mathbb{E}(d_0^2|\mathcal{I})$. Now it is proven in Theorem 1 of Peligrad and Utev (2005) that $\mathbb{E}(d_0^2|\mathcal{I}) = \lim_{n \rightarrow \infty} \mathbb{E}(n^{-1}S_n^2|\mathcal{I})$ in \mathbb{L}^1 .

It remains to prove Propositions 4.9 and 4.10. With this aim, we will make use of the operator Q (also used in Cuny-Volný [6]) defined by

$$QZ = \mathbb{E}_0(Z \circ \theta) \quad \forall Z \in \mathbb{L}^1.$$

The operator Q is markovian, hence it is a Dunford-Schwartz operator. Notice that $Q^n Z = \mathbb{E}_0(Z \circ \theta^n)$.

By the Dunford-Schwartz ergodic theorem, for every $f \in \mathbb{L}^1$, $(Qf + \dots + Q^n f)/n$ converges \mathbb{P} -a.s. and in \mathbb{L}^1 to some $g \in \mathbb{L}^1$. Here $g = \mathbb{E}(f|\mathcal{I})$, as we shall prove now. Indeed by the usual ergodic theorem, $(f \circ \theta + \dots + f \circ \theta^n)/n$ converges \mathbb{P} -a.s. and in \mathbb{L}^1 to $\mathbb{E}(f|\mathcal{I})$. Now $\mathbb{E}(f|\mathcal{I}) = \mathbb{E}_0(\mathbb{E}(f|\mathcal{I}))$ (see Bradley [2]). Therefore $\|n^{-1} \sum_{i=1}^n \mathbb{E}_0(f \circ \theta^i) - \mathbb{E}(f|\mathcal{I})\|_1 \leq \|n^{-1} \sum_{i=1}^n f \circ \theta^i - \mathbb{E}(f|\mathcal{I})\|_1$ which converges to zero. This proves that $g = \mathbb{E}(f|\mathcal{I})$.

Let us prove Proposition 4.9. The fact that $\mathbb{E}_0(S_n) = o(\sqrt{n})$ \mathbb{P} -a.s. under (1) comes directly from an application of Theorem 4.7 with $T = Q$. We prove now that under (1), we have $\mathbb{E}_0[(S_n - M_n - \mathbb{E}_0(S_n))^2] = o(n)$ \mathbb{P} -a.s.

Let N be a positive integer fixed for the moment. Recall that by (46), we have

$$S_n - M_n - \mathbb{E}_0(S_n) = -(V_{n,N} + W_{n,N}), \quad (61)$$

where $V_{n,N}$ and $W_{n,N}$ are given respectively by (44) and (45).

Let $\varphi_n := \mathbb{E}_0(S_n)$ and $\psi_N = \sum_{l \geq N} \frac{\varphi_l}{l(l+1)}$, where ψ_N is well-defined in \mathbb{L}^2 , by (1).

Then, by (47) and (49)

$$|V_{n,N}| \ll (|\varphi_N \circ \theta^n| + |Q^n \varphi_N| + |\psi_N \circ \theta^n| + |Q^n \psi_N|).$$

Hence,

$$\mathbb{E}_0(V_{n,N}^2) \ll Q^n(\varphi_N^2) + Q^n(\psi_N^2) = o(n) \quad \mathbb{P}\text{-a.s.},$$

by the ergodic theorem for Q . Then, using that $\mathbb{E}_0(S_n) = o(\sqrt{n})$ \mathbb{P} -a.s. and (61), we obtain

$$\limsup_n \frac{\mathbb{E}_0((S_n - M_n)^2)}{n} \leq \limsup_n \frac{\mathbb{E}_0(W_{n,N}^2)}{n}. \quad (62)$$

It remains to deal with $W_{n,N}$. Recall that by (51)

$$W_{n,N} = \sum_{r=1}^n \mathcal{P}_r(W_{n,N}) = \sum_{r=1}^n \left(\sum_{k \geq N} \sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})}{l+k} \right) \circ \theta^r.$$

and, by orthogonality,

$$\mathbb{E}_0(W_{n,N}^2) = \sum_{r=1}^n \mathbb{E}_0(\mathcal{P}_r(W_{n,N})^2) = \sum_{r=1}^n Q^r \left(\sum_{k \geq N} \sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})^2}{l+k} \right).$$

But, using (53) and Cauchy-Schwartz, we have

$$\begin{aligned} \left| \sum_{k \geq N} \sum_{l \geq 1} \frac{\mathcal{P}_0(X_{l+k+n-r})}{l+k} \right| &\leq \sum_{m \geq 1} \frac{1}{(m+N)^{3/2}} \left(\sum_{k \geq 0} |\mathcal{P}_0(S_m \circ \theta^k)|^2 \right)^{1/2} \\ &= \sum_{m \geq 1} \frac{1}{(m+N)^{3/2}} \left(\sum_{k \geq 0} |\mathcal{P}_{-k}(S_m)|^2 \circ \theta^k \right)^{1/2}. \end{aligned}$$

Let $g_N := \sum_{m \geq 1} \frac{1}{(m+N)^{3/2}} \left(\sum_{k \geq 0} |\mathcal{P}_{-k}(S_m)|^2 \circ \theta^k \right)^{1/2}$. Then g_N is in \mathbb{L}^2 and

$$\|g_N\|_2 \leq \sum_{m \geq 1} \frac{\|\mathbb{E}_0(S_m)\|_2}{(m+N)^{3/2}} < \infty.$$

In particular, $\|g_N\|_2 \rightarrow 0$, as $N \rightarrow \infty$.

Finally, we have

$$\frac{\mathbb{E}_0(W_{n,N}^2)}{n} \leq \frac{\sum_{r=1}^n Q^r(g_N^2)}{n} \xrightarrow{n \rightarrow +\infty} \mathbb{E}(g_N^2 | \mathcal{I}) \quad \mathbb{P}\text{-a.s.}$$

by the ergodic theorem for Q . Since $\|\mathbb{E}(g_N^2 | \mathcal{I})\|_1 \leq \|g_N^2\|_1 \rightarrow 0$, there exists a sub-sequence (N_j) such that $\mathbb{E}(g_{N_j}^2 | \mathcal{I}) \rightarrow 0$ \mathbb{P} -a.s. as $j \rightarrow \infty$ and the result follows.

To prove Proposition 4.10, we will make use of the following maximal inequality from Merlevède and Peligrad (2012). They did not state the result exactly in that context but it may be proved exactly the same way, applying Doob's maximal inequality conditionally, so the proof is omitted.

Proposition 4.11 *Let $(X_n)_{n \in \mathbb{Z}}$ be a stationary sequence in \mathbb{L}^2 adapted to the filtration (\mathcal{F}_n) . We have,*

$$\begin{aligned} (\mathbb{E}_0(\max_{1 \leq i \leq 2^r} |S_i|^2))^{1/2} &\leq 2(\mathbb{E}_0(S_{2^r}^2))^{1/2} + 2 \sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l-1}} \mathbb{E}_0((\mathbb{E}_{k2^l}(S_{(k+1)2^l}) - S_{k2^l})^2) \right)^{1/2} \\ &= 2(\mathbb{E}_0(S_{2^r}^2))^{1/2} + 2 \sum_{l=0}^{r-1} \left(\sum_{k=1}^{2^{r-l-1}} Q^{k2^l} (\mathbb{E}_0(S_{2^l})^2) \right)^{1/2} \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (63)$$

Proof of Proposition 4.10.

Let $v \geq 0$ be an integer, fixed for the moment. Let $r > v$. Then we have

$$\max_{1 \leq k \leq 2^r} |S_k| \leq \max_{1 \leq s \leq 2^{r-v}} |S_{s2^v}| + 2^v \max_{1 \leq j \leq 2^r} |X_j|.$$

Let $K \geq 1$, be fixed for the moment. We have $\max_{1 \leq j \leq 2^r} |X_j|^2 \leq K^2 + \sum_{j=1}^{2^r} |X_j|^2 \mathbf{1}_{\{|X_j| \geq K\}}$.

Hence, applying Proposition 4.11 to the stationary sequence $(S_{(k+1)2^v} - S_{k2^v})_{k \geq 0}$ adapted to the filtration $(\mathcal{F}_{k2^v})_{k \geq 0}$, we obtain (with the convention that $S_0 = 0$)

$$\begin{aligned} \mathbb{E}_0(\max_{1 \leq i \leq 2^r} |S_i|^2) &\ll 4^v K^2 + 4^v \sum_{j=1}^{2^r} Q^j (|X_0|^2 \mathbf{1}_{\{|X_0| \geq K\}}) + \mathbb{E}_0(S_{2^r}^2) + \left(\sum_{l=0}^{r-v-1} \left(\sum_{k=1}^{2^{r-v-l-1}} Q^{k2^{l+v}} (\mathbb{E}_0(S_{2^{l+v}}))^2 \right)^{1/2} \right)^2 \\ &\ll 4^v K^2 + 4^v \sum_{j=1}^{2^r} Q^j (|X_0|^2 \mathbf{1}_{\{|X_0| \geq K\}}) + \mathbb{E}_0(S_{2^r}^2) + 2^r \left(\sum_{l=0}^{r-v-1} \frac{(\mathcal{M}_{l+v}(\mathbb{E}_0(S_{2^{l+v}}))^2)^{1/2}}{2^{(l+v)/2}} \right)^2. \end{aligned}$$

By assumption $\mathbb{E}_0(S_{2^r}^2) = o(2^r)$ a.s. and by the ergodic theorem $(\sum_{j=1}^{2^r} Q^j(|X_0|^2 \mathbf{1}_{\{|X_0| \geq K\}}))/2^r \rightarrow \mathbb{E}(|X_0|^2 \mathbf{1}_{\{|X_0| \geq K\}} | \mathcal{I})$ \mathbb{P} -a.s. Since $\|\mathbb{E}(|X_0|^2 \mathbf{1}_{\{|X_0| \geq K\}} | \mathcal{I})\|_1 \leq \|X_0^2 \mathbf{1}_{\{|X_0| \geq K\}}\|_1 \rightarrow \infty$, there exists a sub-sequence (K_j) such that $\mathbb{E}(|X_0|^2 \mathbf{1}_{\{|X_0| \geq K_j\}} | \mathcal{I}) \rightarrow 0$ \mathbb{P} -a.s. as $j \rightarrow \infty$. Hence taking the \limsup_r and letting $j \rightarrow \infty$, we obtain

$$\limsup_r \frac{\mathbb{E}_0(\max_{1 \leq i \leq 2^r} |S_i|^2)}{2^r} \ll \left(\sum_{l \geq v} \frac{(\mathcal{M}_l(\mathbb{E}_0(S_{2^l}^2)))^{1/2}}{2^{l/2}} \right)^2 \quad \mathbb{P}\text{-a.s.}$$

To finish the proof, it suffices to prove that the random variable defined by the series in right-hand side is a.s. finite. But it is in $\mathbb{L}^{2,w}$ since

$$\begin{aligned} \left\| \sum_{l \geq 0} \frac{(\mathcal{M}_l(\mathbb{E}_0(S_{2^l}^2)))^{1/2}}{2^{l/2}} \right\|_{2,w} &\leq \sum_{l \geq 0} \frac{\|(\mathcal{M}_l(\mathbb{E}_0(S_{2^l}^2)))^{1/2}\|_{2,w}}{2^{l/2}} \\ &\ll \sum_{l \geq 0} \frac{\|\mathbb{E}_0(S_{2^l})\|_2}{2^{l/2}} < \infty. \end{aligned}$$

A Proof of Theorem 4.7

We make the proof for T Dunford-Schwartz and f real-valued since the proof in the case where f is \mathcal{B} -valued is identical, replacing $|\cdot|$ with $|\cdot|_{\mathcal{B}}$ when necessary.

Write $U_n(f) = f + \dots + T^{n-1}f$. Since ψ is monotonic, it follows from the subadditivity of $(\|U_n(f)\|_p)$ (see for instance [26] Lemma 2.7 and [23] equation (92)) that (59) is equivalent to

$$\sum_n \frac{\|f + \dots + T^{2^n-1}f\|_p}{\psi(2^n)2^{n/p}} = \sum_n \frac{\|U_{2^n}(f)\|_p}{\psi(2^n)2^{n/p}} < \infty.$$

We proceed now as in the proof of Proposition 4.10; namely, we consider dyadic blocs. Let us give the hints. Let $v \geq 0$ be an integer. For $r > v$, write that

$$\max_{1 \leq k \leq 2^r} |U_k(f)| \leq \max_{1 \leq s \leq 2^{r-v}} |U_{s2^v}(f)| + 2^v \max_{1 \leq j \leq 2^r} |T^j f|.$$

Using Proposition 4.5 to take care of the first term in the right hand side, it follows that

$$\max_{1 \leq k \leq 2^r} |U_k(f)| \leq 2^v \max_{1 \leq j \leq 2^r} |T^j f| + 2^{r/p} \sum_{k \geq 0} \frac{[\mathcal{M}_{k+v}(|U_{2^{k+v}}(f)|^p)]^{1/p}}{2^{(k+v)/p}}.$$

We finish the proof by using arguments developed in the proof of Proposition 4.10. \square

B Technical results

Proof of Corollary 3.8. Let us prove that the first part of (21) holds. With this aim, we first notice that, due to the subadditivity of the sequence $(\|\mathbb{E}_0(S_n)\|_p)_{n \geq 1}$, this condition is equivalent to (see Lemma 2.7 in [26])

$$\sum_{k \geq 0} \frac{\|\mathbb{E}_0(S_{2^k})\|_p}{2^{2k/p^2}} < \infty. \quad (64)$$

Since $p > 2$, (29) implies that $\sum_{k \geq 0} \rho^{2/p}(2^k) < \infty$. Therefore, by using (13), it follows that (64) is satisfied as soon as $\sum_{k \geq 0} 2^{-2k/p^2} \sum_{i=0}^k 2^{i/2} \rho^{2/p}(2^i) < \infty$, which is the first part of condition (29).

We prove now that the second part of (21) holds. As we noticed in the proof of Theorem 3.5, this condition is equivalent to the second part of (26).

Let r and j be positive integers. Let $l \in [2^j, 2^{j+1} - 1]$ be an integer. There exists $k \in [2^{j-1}, 2^j - 1]$ such that either $l = 2^{j-1} + k$ or $l = 2^j + k$ according to the fact that either $l \in [2^j, 2^j + 2^{j-1} - 1]$ or $l \in [2^j + 2^{j-1}, 2^{j+1} - 1]$. Then we have

$$S_l = S_{2^{j-1}} \circ \theta^k + S_k \quad \text{or} \quad S_l = S_{2^j} \circ \theta^k + S_k.$$

For $l \in [2^j + 2^{j-1}, 2^{j+1} - 1]$, we have

$$\|\mathbb{E}_{-r}(S_l)\|_2 \leq \|\mathbb{E}_{-r-2^{j-1}}(S_{2^j})\|_2 + \|\mathbb{E}_{-r}(S_k)\|_2,$$

For $l \in [2^j, 2^j + 2^{j-1} - 1]$, using the convention that $\mathbb{E}_{-r-1/2} = \mathbb{E}_{-r}$, we obtain

$$\|\mathbb{E}_{-r}(S_l)\|_2 \leq \|\mathbb{E}_{-r-2^{j-1}}(S_{2^{j-1}})\|_2 + \|\mathbb{E}_{-r}(S_k)\|_2 \leq \|\mathbb{E}_{-r-2^{j-2}}(S_{2^{j-1}})\|_2 + \|\mathbb{E}_{-r}(S_k)\|_2.$$

Hence

$$\max_{2^j \leq l \leq 2^{j+1} - 1} \|\mathbb{E}_{-r}(S_l)\|_2 \leq \|\mathbb{E}_{-r-2^{j-1}}(S_{2^j})\|_2 + \|\mathbb{E}_{-r-2^{j-2}}(S_{2^{j-1}})\|_2 + \max_{2^{j-1} \leq k \leq 2^j - 1} \|\mathbb{E}_{-r}(S_k)\|_2,$$

and, by an easy induction over j , we obtain that

$$\max_{2^j \leq l \leq 2^{j+1} - 1} \|\mathbb{E}_{-r}(S_l)\|_2 \leq 2 \sum_{s=0}^j \|\mathbb{E}_{-r-2^{s-1}}(S_{2^s})\|_2.$$

Therefore, the second part of (26) will hold if

$$\sum_{k \geq 0} \frac{2^k}{2^{2k/p}} \sum_{j \geq k+1} 2^{-j/2} \sum_{s=0}^j \|\mathbb{E}_{-2^k-2^{s-1}}(S_{2^s})\|_2 < \infty. \quad (65)$$

But, since $\|\mathbb{E}_{-r}(S_t)\|_2 \ll \rho(r)\sqrt{t}$,

$$\begin{aligned} \sum_{j \geq k+1} 2^{-j/2} \sum_{s=0}^j \|\mathbb{E}_{-2^k-2^{s-1}}(S_{2^s})\|_p &\ll 2^{-k/2} \sum_{s=0}^{k-1} \|\mathbb{E}_{-2^k}(S_{2^s})\|_2 + \sum_{s \geq k} 2^{-s/2} \|\mathbb{E}_{-2^{s-1}}(S_{2^s})\|_2 \\ &\ll \rho(2^k) + \sum_{s \geq k} \rho(2^s) \ll \sum_{s \geq k} \rho(2^s), \end{aligned}$$

so we obtain,

$$\sum_{k \geq 0} \frac{2^k}{2^{2k/p}} \sum_{j \geq k+1} 2^{-j/2} \sum_{s=0}^j \|\mathbb{E}_{-2^k-2^{s-1}}(S_{2^s})\|_2 \ll \sum_{k \geq 0} 2^{k(1-2/p)} \rho(2^k). \quad (66)$$

Noticing that (29) implies in particular that

$$\rho(2^k) = o(2^{-k(p^2-4)/(4p)}) \text{ as } k \rightarrow \infty, \quad (67)$$

and taking into account that $p > 2$, we then infer that the sums in the right-hand side of (66) are finite under (29). This ends the proof of (65), hence the second part of (21) holds.

It remains to show that (22) is satisfied. With this aim we set for any integer $i \geq 0$, $\tilde{S}_{2^i} = S_{2^{i+1}} - S_{2^i}$ and $\tilde{S}_{2^{-1}} = S_1$. With this notation,

$$\|\mathbb{E}_0(S_{2^k}^2) - \mathbb{E}(S_{2^k}^2)\|_{p/2} \leq 2 \sum_{j=-1}^{k-1} \sum_{i=-1}^j \|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2}. \quad (68)$$

We bound now the quantity $\|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2}$ in two different ways. In what follows, we use the following convention $\rho(2^{-1}) = \rho(1)$. Note first that since $p \in [2, 4]$ and $r \geq p$,

$$\begin{aligned} \|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2} &\leq \|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{2 \wedge (r/2)} \\ &\leq \sup_{Z \in B^{2^{\vee(r/(r-2))}(\mathcal{F}_0)}} \text{Cov}(Z, \tilde{S}_{2^i} \tilde{S}_{2^j}), \end{aligned}$$

where $B^p(\mathcal{F}_0)$ be the set of \mathcal{F}_0 -measurable random variables such that $\|Z\|_p \leq 1$. Using then Theorem 4.12 in [2], we get that for any $-1 \leq i \leq j$,

$$\|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2} \leq 2^{1-s} \rho^s(2^i) \|\tilde{S}_{2^i} \tilde{S}_{2^j}\|_{2 \wedge (r/2)} \leq 2^{1-s} \rho^s(2^i) \|\tilde{S}_{2^i}\|_{4 \wedge r} \|\tilde{S}_{2^j}\|_{4 \wedge r},$$

where $s = \min(1, 2(r-2)/r)$. Now the first part of (29) implies $\sum_{k>0} \rho^{1/2}(2^k) < \infty$ (see also (67)), therefore $\|S_n\|_{4 \wedge r} \ll n^{1/2}$ (see [25] or [28]). Hence, for any $-1 \leq i \leq j$,

$$\|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2} \ll \rho^s(2^i) 2^{i/2} 2^{j/2}. \quad (69)$$

On an other hand, for any $-1 \leq i < j$,

$$\|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2} \leq 2 \|\tilde{S}_{2^i} \mathbb{E}_{2^{i+1}}(\tilde{S}_{2^j})\|_{p/2} \leq 2 \|\tilde{S}_{2^i}\|_p \|\mathbb{E}_{2^{i+1}}(\tilde{S}_{2^j})\|_p.$$

Using Theorem 4.12 in [2] together with stationarity, we get that

$$\|\mathbb{E}_{2^{i+1}}(\tilde{S}_{2^j})\|_p \leq 2^{1-2/p} \rho^{2/p}(2^j - 2^{i+1}) \|S_{2^j}\|_p.$$

Hence taking into account that under (29), $\|S_n\|_p \ll n^{1/2}$ (see [25] or [28] again), we get that for any $-1 \leq i < j$,

$$\|\mathbb{E}_0(\tilde{S}_{2^i} \tilde{S}_{2^j}) - \mathbb{E}(\tilde{S}_{2^i} \tilde{S}_{2^j})\|_{p/2} \ll \rho^{2/p}(2^j - 2^{i+1}) 2^{i/2} 2^{j/2}. \quad (70)$$

Therefore stating from (68) and using the bounds (69) and (70), we get that

$$\begin{aligned} \|\mathbb{E}_0(S_{2^k}^2) - \mathbb{E}(S_{2^k}^2)\|_{p/2} &\ll \sum_{j=-1}^{k-1} \sum_{i=-1}^{[2j/p]} \rho^{2/p}(2^j - 2^{i+1}) 2^{i/2} 2^{j/2} + \sum_{j=-1}^{k-1} \sum_{i=[2j/p]+1}^j \rho^s(2^i) 2^{i/2} 2^{j/2} \\ &\ll \sum_{j=0}^{k-1} \rho^{2/p}(2^j) 2^{j/p} 2^{j/2} + \sum_{j=0}^{k-1} \sum_{i=[2j/p]+1}^j \rho^s(2^i) 2^{i/2} 2^{j/2} \\ &\ll \sum_{j=0}^{k-1} \rho^{2/p}(2^j) 2^{j/p} 2^{j/2} + \sum_{i=1}^{[2k/p]} \rho^s(2^i) 2^{i/2} 2^{ip/4} + 2^{k/2} \sum_{i=[2k/p]+1}^k \rho^s(2^i) 2^{i/2}. \end{aligned}$$

Hence, since $2 < p \leq 4$,

$$\sum_{k \geq 0} \frac{1}{2^{2k/p}} \|\mathbb{E}_0(S_{2^k}^2) - \mathbb{E}(S_{2^k}^2)\|_{p/2} \ll \sum_{k \geq 0} \frac{2^{k/2}}{2^{k/p}} \rho^{2/p}(2^k) + \sum_{k \geq 0} (\mathbf{1}_{2 < p < 4} + k \mathbf{1}_{p=4}) \frac{2^k}{2^{2k/p}} \rho^s(2^k).$$

The series being convergent under (29), condition (22) is satisfied. This ends the proof of the corollary. \square

The next lemma is Lemma 19 in Merlevède, Peligrad and Peligrad [24]. In their paper the lemma is stated with $\ell = 0$ and with $\mathcal{H} = \mathbb{R}$ but with similar arguments as done in their proof, it works for any nonnegative integer ℓ and for adapted stationary sequences with values in a normed space by replacing the absolute values by the corresponding norms.

Lemma B.1 *Let $p \geq 1$ and let $(X_n)_{n \in \mathbb{Z}}$ be an adapted stationary sequence in $\mathbb{L}^p(\mathcal{H})$ in the sense of Notation 1.1. For every $\gamma > 0$, $n \geq 1$ and any integer $\ell \geq 0$,*

$$\frac{1}{n^\gamma} \max_{1 \leq k \leq n} \|\mathbb{E}_{-\ell}(S_k)\|_{p, \mathcal{H}} \leq 2^{3\gamma+3} \sum_{k=n+1}^{6n} \frac{1}{k^{\gamma+1}} \|\mathbb{E}_{-\ell}(S_k)\|_{p, \mathcal{H}}.$$

Proposition B.2 *Let $p \geq 2$ and let $(X_n)_{n \in \mathbb{Z}}$ be an adapted and real stationary sequence in \mathbb{L}^p in the sense of Notation 1.1. Assume that (6) holds. Then setting $M_n = \sum_{k=1}^n D \circ \theta^k$ where D is defined by (5), the following inequality holds:*

$$\begin{aligned} \|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2} &\ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|\mathbb{E}_0(S_{2n}^2) - \mathbb{E}(S_{2n}^2)\|_{p/2, \mathcal{H}} \\ &+ n \left(\sum_{k \geq \lceil n^{p/2} \rceil} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2 + n \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^{3/2}}. \end{aligned}$$

Proof.

Setting $R_n = S_n - M_n$, we start with the following inequality:

$$\begin{aligned} \|\mathbb{E}_0(M_n^2) - \mathbb{E}(M_n^2)\|_{p/2} &\ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + 2\|R_n\|_p^2 \\ &+ 2\|\mathbb{E}_0(S_n R_n) - \mathbb{E}(S_n R_n)\|_{p/2}. \end{aligned} \quad (71)$$

Using Theorem 2.4 with $p \geq 2$, we first get that

$$\|R_n\|_p^2 \ll n \left(\sum_{\ell \geq \lceil n^{p/2} \rceil} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^{1+1/p}} \right)^2. \quad (72)$$

Now, starting from (46) and using the decompositions (44), (45), (47) and (49) with $u_n = 2n$, we write that

$$R_n = \mathbb{E}_0(S_n) + \frac{\mathbb{E}_0(S_{2n} \circ \theta_n)}{2n+1} - \frac{\mathbb{E}_n(S_{2n} \circ \theta_n)}{2n+1} - A_n - B_n, \quad (73)$$

where

$$A_n = 2n \sum_{l \geq 2n+1} \frac{\mathbb{E}_n(S_l \circ \theta^n) - \mathbb{E}_0(S_l \circ \theta^n)}{l(l+1)}, \quad (74)$$

and

$$B_n = \sum_{k \geq 2n} \sum_{l \geq k+1} \frac{\mathbb{E}_n(X_{l+n}) - \mathbb{E}_0(X_{l+n})}{l}. \quad (75)$$

Notice first that

$$\begin{aligned} &\left\| \mathbb{E}_0 \left(S_n \left(\mathbb{E}_0(S_n) + \frac{\mathbb{E}_0(S_{2n} \circ \theta^n)}{2n+1} \right) \right) - \mathbb{E} \left(S_n \left(\mathbb{E}_0(S_n) + \frac{\mathbb{E}_0(S_{2n} \circ \theta^n)}{2n+1} \right) \right) \right\|_{p/2} \\ &\leq 2\|\mathbb{E}_0(S_n)\|_p^2 + 2(2n+1)^{-1} \|\mathbb{E}_0(S_n)\|_p \|\mathbb{E}_0(S_{2n})\|_p, \end{aligned}$$

which combined with (56) with $u_n = [n^{p/2}]$ implies that

$$\begin{aligned} & \left\| \mathbb{E}_0 \left(S_n \left(\mathbb{E}_0(S_n) + \frac{\mathbb{E}_0(S_{2n} \circ \theta^n)}{2n+1} \right) \right) - \mathbb{E} \left(S_n \left(\mathbb{E}_0(S_n) + \frac{\mathbb{E}_0(S_{2n} \circ \theta^n)}{2n+1} \right) \right) \right\|_{p/2} \\ & \ll n \left(\sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2. \end{aligned} \quad (76)$$

Now writing that $S_{2n} \circ \theta^n = S_{2n} \circ \theta^n - S_n \circ \theta^n + S_n \circ \theta^n$ and using the fact that S_n is \mathcal{F}_n -measurable, we get

$$\begin{aligned} & \left\| \mathbb{E}_0 \left(S_n \left(\frac{\mathbb{E}_n(S_{2n} \circ \theta^n)}{2n+1} \right) \right) - \mathbb{E} \left(S_n \left(\frac{\mathbb{E}_n(S_{2n} \circ \theta^n)}{2n+1} \right) \right) \right\|_{p/2} \\ & \leq n^{-1} \|\mathbb{E}_0(S_n(S_{2n} - S_n)) - \mathbb{E}(S_n(S_{2n} - S_n))\|_{p/2} + n^{-1} \|\mathbb{E}_0(S_n \mathbb{E}_n(S_{3n} - S_{2n}))\|_{p/2}. \end{aligned} \quad (77)$$

Using the identity $2ab = (a+b)^2 - a^2 - b^2$ and the stationarity, we first obtain that

$$\begin{aligned} \|\mathbb{E}_0(S_n(S_{2n} - S_n)) - \mathbb{E}(S_n(S_{2n} - S_n))\|_{p/2} & \leq 2\|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} \\ & \quad + \|\mathbb{E}_0(S_{2n}^2) - \mathbb{E}(S_{2n}^2)\|_{p/2}. \end{aligned} \quad (78)$$

To bound up the second term in (77), we write $C_n := n^{-1} \mathbb{E}_n(S_{3n} - S_{2n})$ and we follow the lines of the proof of Theorem 2.3 in [7] (see the displaylines between their equations (4.13) and (4.16)). Hence we first write that

$$\begin{aligned} \|\mathbb{E}_0(S_n C_n)\|_{p/2} & \leq \|\mathbb{E}_0^{1/2}(S_n^2) \mathbb{E}_0^{1/2}(C_n^2)\|_{p/2} \\ & \leq \|(\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2))^{1/2} \mathbb{E}_0^{1/2}(C_n^2)\|_{p/2} + (\mathbb{E}(S_n^2))^{1/2} \|\mathbb{E}_0^{1/2}(C_n^2)\|_{p/2} \\ & \leq \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|C_n\|_p^2 + 2(\mathbb{E}(S_n^2))^{1/2} \|\mathbb{E}_0^{1/2}(C_n^2)\|_{p/2}. \end{aligned}$$

Notice that since (6) holds, by Theorem 2.4, we have in particular that $\|S_n\|_2 = o(n^{1/2}) + \|M_n\|_2$, implying that

$$\|S_n\|_2 \ll n^{1/2}. \quad (79)$$

Using (79) and the fact that the function $x \mapsto |x|^{p/4}$ is concave, it follows that

$$\|\mathbb{E}_0(S_n C_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|C_n\|_p^2 + n^{1/2} \|C_n\|_2. \quad (80)$$

By stationarity and using (56) with $u_n = [n^{p/2}]$, we get that

$$\|C_n\|_{p, \mathcal{H}} \ll n^{-1} \|\mathbb{E}_{-n}(S_n)\|_{p, \mathcal{H}} \ll n^{-1/2} \sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}}. \quad (81)$$

On another hand, by using once again the stationarity and Lemma B.1

$$\|C_n\|_2 \ll n^{-1} \|\mathbb{E}_{-n}(S_n)\|_2 \ll \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_p}{k^2}. \quad (82)$$

Therefore starting from (77) and using (77), (80), (81) and (82), we infer that

$$\begin{aligned} & \left\| \mathbb{E}_0 \left(S_n \left(\frac{\mathbb{E}_n(S_{2n} \circ \theta^n)}{2n+1} \right) \right) - \mathbb{E} \left(S_n \left(\frac{\mathbb{E}_n(S_{2n} \circ \theta^n)}{2n+1} \right) \right) \right\|_{p/2} \leq \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} \\ & \quad + n^{-1} \|\mathbb{E}_0(S_{2n}^2) - \mathbb{E}(S_{2n}^2)\|_{p/2} + n^{-1} \left(\sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2 + n^{1/2} \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_p}{k^2}. \end{aligned} \quad (83)$$

We consider now the term $\|\mathbb{E}_0(S_n A_n) - \mathbb{E}(S_n A_n)\|_{p/2}$. With this aim, we first define

$$\tilde{A}_n = 2n\mathbb{E}_n(S_n \circ \theta^n) \sum_{l \geq 2n+1} \frac{1}{l(l+1)}.$$

Since S_n is \mathcal{F}_n -measurable, by using stationarity,

$$\|\mathbb{E}_0(S_n \tilde{A}_n) - \mathbb{E}(S_n \tilde{A}_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n(S_{2n} - S_n)) - \mathbb{E}(S_n(S_{2n} - S_n))\|_{p/2}.$$

Using then the identity $2ab = (a+b)^2 - a^2 - b^2$ and stationarity, it follows that

$$\|\mathbb{E}_0(S_n \tilde{A}_n) - \mathbb{E}(S_n \tilde{A}_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|\mathbb{E}_0(S_{2n}^2) - \mathbb{E}(S_{2n}^2)\|_{p/2}. \quad (84)$$

Let now

$$D_n := n \sum_{k \geq 2n+1} \frac{\mathbb{E}_n(S_k \circ \theta^n) - \mathbb{E}_n(S_n \circ \theta^n)}{k(k+1)}$$

and notice that, by stationarity,

$$\begin{aligned} \|\mathbb{E}_0(S_n(A_n - \tilde{A}_n)) - \mathbb{E}(S_n(A_n - \tilde{A}_n))\|_{p/2} &\ll n\|\mathbb{E}_0(S_n)\|_p \sum_{k \geq n+1} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^2} \\ &+ \|\mathbb{E}_0(S_n D_n)\|_{p/2}. \end{aligned} \quad (85)$$

Using (56) with $u_n = n$, we first get that

$$n\|\mathbb{E}_0(S_n)\|_p \sum_{l \geq n+1} \frac{\|\mathbb{E}_0(S_l \circ \theta^n)\|_p}{l^2} \ll n^2 \left(\sum_{\ell \geq n} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^2} \right)^2.$$

But, by using Lemma B.1 and the fact that $p \geq 2$,

$$\begin{aligned} n \sum_{\ell \geq n} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^2} &\leq \max_{1 \leq k \leq [n^{p/2}]} \|\mathbb{E}_0(S_k)\|_p + n \sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^2} \\ &\ll n^{p/2} \sum_{\ell \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^2} \ll n^{1/2} \sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}}. \end{aligned} \quad (86)$$

Therefore

$$n\|\mathbb{E}_0(S_n)\|_p \sum_{l \geq n+1} \frac{\|\mathbb{E}_0(S_l \circ \theta^n)\|_p}{l^2} \ll n \left(\sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2. \quad (87)$$

We bound now the the second term in the right-hand side of (85). Proceeding as to get (80), we infer that

$$\|\mathbb{E}_0(S_n D_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|D_n\|_p^2 + n^{1/2} \|D_n\|_2. \quad (88)$$

Stationarity and inequality (86) imply that

$$\|D_n\|_p \ll n \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_p}{k^2} \ll n^{1/2} \sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}}. \quad (89)$$

On another hand, using once again, stationarity

$$\|D_n\|_2 \ll n \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^2}. \quad (90)$$

Overall, starting from (85) and considering the bounds (87), (88), (89) and (90), it follows that

$$\begin{aligned} & \|\mathbb{E}_0(S_n(A_n - \tilde{A}_n)) - \mathbb{E}(S_n(A_n - \tilde{A}_n))\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} \\ & + n \left(\sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2 + n^{3/2} \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^2}. \end{aligned} \quad (91)$$

We consider now the term $\|\mathbb{E}_0(S_n B_n) - \mathbb{E}(S_n B_n)\|_{p/2}$. Proceeding as to get (80), we infer that

$$\|\mathbb{E}_0(S_n B_n) - \mathbb{E}(S_n B_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|B_n\|_p^2 + n^{1/2} \|B_n\|_2. \quad (92)$$

According to the bound (54) followed by an application of Lemma 4.3,

$$\|B_n\|_2 \ll n^{1/2} \sum_{k \geq n} \sum_{m \geq 1} \frac{\|\mathcal{P}_{-k}(S_m)\|_2}{(m+k)^2} \ll n^{1/2} \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^{3/2}}. \quad (93)$$

To bound $\|B_n\|_p$, we use (73). By stationarity, we then infer that

$$\|B_n\|_p \leq \|R_n\|_p + 3\|\mathbb{E}_0(S_n)\|_p + 2n \sum_{\ell \geq n+1} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^2}.$$

Hence using Theorem 2.4 and inequality (56) with $u_n = n$, we get that

$$\|B_n\|_p \ll n^{1/2} \sum_{\ell \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^{1+1/p}} + n \sum_{\ell \geq n} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^2},$$

which together with (86) implies that

$$\|B_n\|_p \leq \|B_n\|_p \ll n^{1/2} \sum_{\ell \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^{1+1/p}}. \quad (94)$$

Starting from (92) and using (93) and (94), we then obtain that

$$\begin{aligned} \|\mathbb{E}_0(S_n B_n) - \mathbb{E}(S_n B_n)\|_{p/2} & \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + n \left(\sum_{k \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_k)\|_p}{k^{1+1/p}} \right)^2 \\ & + n \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^{3/2}}. \end{aligned} \quad (95)$$

Taking into account the decomposition (73) together with the bounds (76), (83), (84), (91) and (95), we then derive that

$$\begin{aligned} & \|\mathbb{E}_0(S_n R_n) - \mathbb{E}(S_n R_n)\|_{p/2} \ll \|\mathbb{E}_0(S_n^2) - \mathbb{E}(S_n^2)\|_{p/2} + \|\mathbb{E}_0(S_{2n}^2) - \mathbb{E}(S_{2n}^2)\|_{p/2} \\ & + n \left(\sum_{\ell \geq [n^{p/2}]} \frac{\|\mathbb{E}_0(S_\ell)\|_p}{\ell^{1+1/p}} \right)^2 + n \sum_{k \geq n} \frac{\|\mathbb{E}_{-n}(S_k)\|_2}{k^{3/2}}. \end{aligned} \quad (96)$$

Starting from (71) and considering the inequalities (72) and (96), the proposition follows. \square

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