Fluctuations of the front in a one dimensional model for the spread of an infection

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FLUCTUATIONS OF THE FRONT IN A ONE DIMENSIONAL MODEL FOR THE SPREAD OF AN INFECTION

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Abstract. We study the following microscopic model of infection or epidemic reaction: red and blue particles perform independent nearest-neighbor continuous-time symmetric random walks on the integer lattice $\mathbb{Z}$ with jump rates D_R for red particles and D_B for blue particles, the interaction rule being that blue particles turn red upon contact with a red particle. The initial condition consists of i.i.d. Poisson particle numbers at each site, with particles at the left of the origin being red, while particles at the right of the origin are blue. We are interested in the dynamics of the front, defined as the rightmost position of a red particle. For the case $D_R = D_B$ (in fact, for a general $d-$dimensional version of it), Kesten and Sidoravicius established that the front moves ballistically, and more precisely that it satisfies a law of large numbers. In this paper, we prove that a central limit theorem for the front holds when $D_R = D_B$. Moreover, this result can be extended to the case where $D_R > D_B$, up to modifying the dynamics so that blue particles turn red upon contact with a site that has previously been occupied by a red particle. Our approach is based on the definition of a renewal structure, extending ideas developed by Comets, Quastel and Ramírez for the so-called frog model, where $D_B = 0$.

1. Introduction

Consider the following microscopic model of infection or epidemic reaction on the integer lattice $\mathbb{Z}$. There are two types of particles: red and blue, both moving as independent, continuous-time, symmetric, nearest-neighbor random walks, with total jump rate D_R for red particles and D_B for blue particles. The interaction rule between particles is the following: when a red particle jumps to a site where there are blue particles, all of them immediately become red particles; when a blue particle jumps to a site where there are red particles, it immediately becomes a red particle. The initial condition is the following: at time zero, each site in $x \in \mathbb{Z}$ bears a random number of particles whose distribution is Poisson with parameter $\rho > 0$, the

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numbers of particles at distinct sites being independent. Moreover, particles at the left of the origin (including the origin) are red, while particles at the right of the origin are blue. We are interested in the asymptotic behavior of the rightmost site r_t occupied by a red particle at time t, which we call the *front*. This is the one-dimensional version of a model studied on $\mathbb{Z}^d$ by Kesten and Sidoravicius in [6] and [8]. The case in which $D_R = D_B > 0$ will be referred to as the *single-rate KS infection model*, to emphasize the fact that red and blue particles share the same jump rate.

Such particle systems have received attention in the physical literature, as microscopic stochastic models which, in the limit of a large average number of particles per lattice site, yield reaction-diffusion equations describing the propagation of a front, the prototypical example being the Fisher-Kolmogorov-Petrovsky-Piscounov equation, see e.g. [11, 12, 13, 9]. We refer to [15] for an extensive review of the subject from a theoretical physics perspective.

On the other hand, according to [6], this model was suggested within the mathematics community by Frank Spitzer around 1980, but rigorous mathematical results describing the behavior of the front have been difficult to obtain. Indeed, only in the special case where $D_R > 0, D_B = 0$, called the *frog model* ([16, 1]), and for the single rate KS infection case ([6, 8]), has it been possible to prove that the front is ballistic and satisfies a law of large numbers. Furthermore, only in the frog model have the fluctuations of the front been described and a large deviations principle established [3, 2].

Specifically, in [6] it is shown that the front moves ballistically, in the sense that there exist two constants C_1, C_2 such that a.s.

$$0 < C_2 \leq \liminf_{t\to+\infty} t^{-1} r_t \leq \limsup_{t\to+\infty} t^{-1} r_t \leq C_1 < +\infty. \tag{1}$$

This result is strengthened in [8] where it is shown that there exists $0 < v_* < +\infty$ such that a.s.,

$$\lim_{t\to+\infty} t^{-1} r_t = v_*. \tag{2}$$

Analogous results hold on $\mathbb{Z}^d$ for arbitrary $d \geq 1$, with (2) being the one-dimensional version of a general shape theorem proved in [8]. Here we are interested in the fluctuations of r_t, and the first main result of this paper is the following.

Theorem 1. *For the single-rate KS infection model, there exists a (non-random) number* $0 < \sigma_*^2 < +\infty$ *such that, as* ϵ *goes to zero,*

$$B_t^\epsilon := \epsilon^{1/2} \left(r_{\epsilon^{-1}t} - \epsilon^{-1} v_* t \right), \qquad t \geq 0,$$

converges in law on the Skorohod space to a Brownian motion with variance σ_*^2.

Note that the method used to derive the above results also yields the convergence to an invariant distribution of the environment of particles as seen from the front.

For the general case in which D_R is not necessarily equal to D_B, an upper bound on the speed similar to the one in (1) is proved in [6], but no corresponding lower bound is available. We now introduce a slight variation upon this model for which, when $D_R > D_B$, it is indeed possible to derive results similar to those that hold for the single-rate model. This variation consists in making the infectious power of red particles *remanent*, in the sense that a blue particle turns red not only when it is in contact with a red particle, but as soon as it is located at a site that has previously been occupied by a red particle. We call this model the *remanent KS infection model.* In this context, it is natural to define the position of the front at time t as the rightmost position ever occupied by a red particle up to time t. We can then prove the two following results.

Theorem 2. *For the remanent KS infection model with* $0 < D_B \leq D_R$*, there exists* $0 < v_\star < +\infty$ *such that a.s.,*

$$\lim_{t \to +\infty} t^{-1} r_t = v_\star.$$

Theorem 3. *For the remanent KS infection model with* $0 < D_B \leq D_R$*, there exists a (non-random) number* $0 < \sigma_\star^2 < +\infty$ *such that, as* ϵ *goes to zero,*

$$B_t^\epsilon := \epsilon^{1/2} \left(r_{\epsilon^{-1}t} - \epsilon^{-1} v_\star t \right), \qquad t \geq 0,$$

converges in law on the Skorohod space to a Brownian motion with variance $\sigma_\star^2$.

Our approach for proving Theorems 1, 2, 3 is based on the definition of a renewal structure, extending an idea introduced by Comets, Quastel and Ramírez in [3] to study the frog model, where blue particles are motionless, while red particles perform random walks with a constant jump rate. Broadly speaking, the idea is to find random times κ_n such that (i) the history of the front after time κ_n does not depend (up to translation) on the future trajectories of particles located below r_{κ_n} at time κ_n and (ii) the distribution of particles located above r_{κ_n} at time κ_n is fixed (up to translation). We achieve (i) by recycling the idea, already used in [3], to consider times after which the front remains forever above a (space-time) straight line, while particles lying below the front at these times remain forever below the straight-line. For the frog model, (ii) is then automatically satisfied, since the distribution of blue particles above the front[1] is fixed, due to the fact that blue particles do not move. In our context where both red and blue particles move, the situation is more complex, and new ideas are required. We achieve (ii) by extending the trajectories of our random walks infinitely far in the past, looking at times before which the front always lies below a straight line, while particles lying above the front at these times have remained above the straight line for their whole past history. A key role in the corresponding argument is played by the invariance properties of the Poisson distribution of particles,

[1]Strictly speaking, this is true only when the front hits a position above its past record.

which allows the construction of the time-reversal of the random walk trajectories and the analysis of the distribution of the blue particles in terms of this time-reversal. Once the renewal structure is defined, it is necessary to obtain tail estimates for the random variables κ_1, r_{κ_1}, and $\kappa_{n+1} - \kappa_n$ and $r_{\kappa_{n+1}} - r_{\kappa_n}$ for $n \geq 1$. To this end, we recycle some of the techniques used in [3], especially the use of martingale methods to control the behavior of systems of independent random walks. In fact, some of the more involved steps in the proof given in [3], that were needed to control the accumulation of particles below the front, are replaced in the present paper by a softer and (hopefully) more transparent argument. Let us point out one important technical difference between the frog model and the infection models considered here: ballistic lower bounds for the position of the front are easy to obtain in the case of the frog model, while they seem to be very difficult[2] for infection models where both red and blue particles move. In fact, the lower bound part[3] of (1) is the main result of [6], and is obtained through a quite demanding multiple-scale renormalization argument. We do not provide an independent proof of ballisticity here, and instead have to rely on the estimate proved in [6]. Still, at least in the one-dimensional case, our approach provides an alternative way of deriving the law of large numbers (2) (proved in [8]) from the coarser ballisticity estimate obtained in [6]. Note also that the only missing ingredient to make our proofs of Theorems 2 and 3 work in the non-remanent case is a lower bound on the speed comparable to the one established in [6] for the single-rate model (specifically, we would need the conclusion of Proposition 13 below).

A natural question concerns our specific choice for the Poisson initial distribution of particles. One can take advantage of the fact that the random variables $(\kappa_{i+1} - \kappa_i, r_{\kappa_{i+1}} - r_{\kappa_i})_{i \geq 1}$ are independent from the initial configuration of particles at the right of the origin to show that our results are still valid if one starts with a Poisson distribution of particles conditioned upon a non-zero probability event concerning only the initial configuration of particles at the right of the origin. For instance, we can prescribe the initial numbers of particles below zero at any given finite number of sites. Still, it seems necessary to use the Poisson distribution of particles as a reference probability measure, so it is unclear how we could extend our results to, say, an arbitrary initial condition with suitable decay of the number of particles at infinity.

One should note that, strictly speaking, the initial distribution of particles we have described is not the same as the one considered by Kesten and Sidoravicius. Indeed, in [6, 8], the initial condition is obtained by adding a deterministic finite and non-zero number of red particles placed arbitrarily, to a configuration formed by an i.i.d. Poisson number of particles at each site of $\mathbb{Z}$. For the single-rate KS model on $\mathbb{Z}$, it is irrelevant for the value

[2]By contrast, ballistic upper bounds are relatively easy to obtain.

[3]More precisely, a quantitative version of it.

of r_t whether particles initially at the left of r_0 are red or blue, so the only difference lies in the added red particles. Using the previous remark on the possibility to condition the initial configuration by the numbers of particles at a finite set of sites, we see that our results in fact include the kind of initial configurations considered in [6, 8].

One should also note that the results of [6, 8] are stated in terms of $\sup_{s\in[0,t]} r_s$ rather than r_t (in the more general $d-$dimensional framework they consider). It clearly makes no difference for results on the scale of the law of large numbers, since particles move sub-ballistically. Although such an argument cannot be used for the central limit theorem, it turns out that, with our definitions of the renewal structure, $r_{\kappa_n} = \sup_{s\in[0,\kappa_n]} r_s$, so that the CLT holds for either r_t or $\sup_{s\in[0,t]} r_s$.

Finally, note that our results do not say anything on the case $D_R < D_B$. The only available results for a model of this kind are those of [7], where a version of the infection model with $0 = D_R < D_B$ is considered, and it is shown that, for sufficiently small ρ, the asymptotic velocity of the front is zero, while it is conjectured that a positive asymptotic velocity is obtained for sufficiently large ρ.

The rest of the paper is organized as follows. In Section 2, we give a formal construction of the random process associated with the single-rate KS infection model, together with statements of its main structural properties, most of the proofs being deferred to Appendix A. Section 3 provides the definition of the renewal structure, and its key structural properties are stated and proved, save for the estimates on the tail, which form the content of Section 4. Finally, Section 5 briefly explains how to extend the previous results to the case of the remanent KS infection model with $D_R > D_B$.

2. Formal construction of the single-rate process

In this section, we describe the formal construction of the single-rate process in two steps. First, we construct, on appropriate spaces, the dynamics of systems of independent random walks, without any reference to a possible interaction between them. We establish important structural properties of the dynamics, such as the strong Markov property, or the invariance with respect to space-time shifts of the Poisson distribution on the space of trajectories. Then we define the infection process as a function of these random walks, together with the corresponding notion of red and blue particles. Most of the proofs are deferred to Appendix A.

2.1. **Reference spaces.** It is convenient to assign a label to each particle in the system, so that a particle can be uniquely identified by its label. More precisely, we assume that each particle is labelled by an element of the interval $[0, 1]$, in such a way that no two particles share the same label. As a consequence, a configuration of particles at a given time can be represented by a family

$$w = (w(x),\ x \in \mathbb{Z}),$$

where, for all x, $w(x)$ is a (possibly empty) subset of $[0,1]$, representing the labels of the particles located at site x.

Given $\theta > 0$, introduce the space $\mathbb{S}_\theta$ of all configurations of labelled particles $w = (w(x),\ x \in \mathbb{Z})$ satisfying $w(x) \cap w(y) = \emptyset$ whenever $x \neq y$, and $\sum_{x\in\mathbb{Z}} |w(x)|e^{-\theta|x|} < +\infty$. Throughout this paper, $\mathbb{S}_\theta$ is our reference space for particle configurations, where θ is assumed to be a given positive real number. The specific value of θ used in the proofs is made precise later, see (24), and the construction we now develop is valid for any $\theta > 0$.

To define a distance on $\mathbb{S}_\theta$, we first define a distance on the set of all finite subsets of elements of $[0,1]$. Consider two such subsets $a = \{a_1 > \ldots > a_p\}$, and $b = \{b_1 > \ldots > b_q\}$. If $p < q$, define $a_i := 0$ for $p+1 \leq i \leq q$; if $p > q$, define $b_i := 0$ for $q+1 \leq i \leq p$. Then define the distance between a and b by

$$d(a,b) := |q-p| + \sum_{i=1}^{\max(p,q)} |b_i - a_i|.$$

We now define a distance d_θ on $\mathbb{S}_\theta$ by

$$d_\theta(w_1,w_2) := \sum_{x\in\mathbb{Z}} d(w_1(x),w_2(x))e^{-\theta|x|}.$$

Let us turn to the description of particle trajectories. A priori, the model consists only of particles moving after time zero. However, the definition of the regeneration structure involves the extension of their trajectories to negative time indices, so we start from the beginning with a space allowing the description of trajectories with a time-index in $\mathbb{R}$. A pair (W,u), where $W = (W_t)_{t\in\mathbb{R}}$ is a càdlàg function from $\mathbb{R}$ to $\mathbb{Z}$ with nearest-neighbor jumps (i.e. ± 1), and $u \in [0,1]$, is called a (labelled) particle path, with u being the label of the particle whose path is described by W. In the sequel, we often call such a pair (W,u) a particle, instead of a particle path.

Given a finite or countable set ψ of particle paths with pairwise distinct labels, and a time coordinate $t \in \mathbb{R}$, we define the configuration of labelled particles $X_t(\psi) = (X_t(\psi)(x))_{x\in\mathbb{Z}}$ by

$$X_t(\psi)(x) := \{u;\ W_t = x,\ (u,W) \in \psi\}.$$

In words, $X_t(\psi)(x)$ is the set of labels of particle paths that are located at x at time t. Our reference space for the trajectories of the particles in the system is the set Ω formed by all the sets ψ of particle trajectories such that $t \mapsto X_t(\psi)$ is a càdlàg function from $\mathbb{R}$ to $(\mathbb{S}_\theta, d_\theta)$, and such that no two particle paths jump at the same time. We endow Ω with the cylindrical $\sigma-$algebra $\mathcal{F}$ generated by all the maps $\psi \mapsto X_t(\psi)$ from Ω to $\mathbb{S}_\theta$ equipped with the Borel sets associated with the metric d_θ. For all $t \in \mathbb{R}$, we define $\mathcal{F}_t := \sigma(X_s, s \in]-\infty, t])$. For all $x \in \mathbb{Z}$ and $t \in \mathbb{R}$, the space-time shift $\pi_{x,t}$ on Ω is defined by the fact that $\pi_{x,t}(\psi)$ is the set of particle paths obtained from ψ by replacing each path $((W_s)_{s\in\mathbb{R}}, u)$ by $((W_{s-t} - x)_{s\in\mathbb{R}}, u)$. We also consider the space $\mathcal{D}$ of càdlàg maps from $\mathbb{R}$ to $\mathbb{S}_\theta$, and similarly define $\mathcal{D}_+$

as the space of càdlàg maps from $[0,+\infty[$ to $\mathbb{S}_\theta$. Both spaces are equipped with their respective cylindrical $\sigma-$algebras. Finally, we denote by Ψ the canonical map on Ω, i.e. $\Psi(\psi) := \psi$.

2.2. **Construction of $\mathbb{P}_w$.** To each $w \in \mathbb{S}_\theta$, we associate a probability measure $\mathbb{P}_w$ on $(\Omega, \mathcal{F})$ describing the evolution of a system of independent particles starting in configuration w at time 0. This section is devoted to the construction of $\mathbb{P}_w$.

Fix $w \in \mathbb{S}_\theta$, and, for all x, write $w(x)$ as an ordered tuple

$$w(x) = \{u(x,1) > \cdots > u(x, |w(x)|)\},$$

and define

$$A := \{(x,i);\ x \in \mathbb{Z},\ 1 \le i \le |w(x)|\}.$$

Consider an i.i.d. family of random walks $Z = (Z(x,i),\ (x,i) \in A)$ where, for every $(x,i) \in A$, $Z(x,i) = (Z_t(x,i))_{t\in\mathbb{R}}$ is a two-sided continuous-time random walk on $\mathbb{Z}$, starting at x at time zero, and evolving in both positive and negative time directions, with symmetric nearest-neighbor steps, and constant jump rate equal to 2. We view $Z(x,i)$ as a random variable taking values in the space of càdlàg paths from $\mathbb{R}$ to $\mathbb{Z}$ equipped with the cylindrical $\sigma-$algebra. For all $t \in \mathbb{R}$, we define $S_t = (S_t(x))_{x\in\mathbb{Z}}$ by

$$S_t(x) := \{u(y,j);\ Z_t(y,j) = x,\ (y,j) \in A\}.$$

Broadly speaking, the idea is to define $\mathbb{P}_w$ as the distribution of the set of paths $(Z(x,i), u(x,i))$, where $(x,i) \in A$. However, we have to take care of the regularity properties of the map $t \mapsto S_t$, so we work with truncated versions involving finite numbers of particles, taking the limit to recover the desired process. Given $K \in \mathbb{N}$, define $S_t^K = (S_t^K(x))_{x\in\mathbb{Z}}$ by

$$S_t^K(x) := \{u(y,j);\ Z_t(y,j) = x,\ (y,j) \in A,\ |y| \le K\}.$$

We also define

$$Q_t^K := \sum_{(x,i)\in A,\ x\ge K} \exp(-\theta Z_t(x,i)),\ \ R_t^K := \sum_{(x,i)\in A,\ x\le -K} \exp(\theta Z_t(x,i)).$$

In the sequel, we use the notation P to denote the reference probability measure for Z, and E to denote the expectation with respect to P.

Proposition 1. *For any $t \ge 0$, with $P-$probability one,*

$$\lim_{K\to+\infty} \sup_{s\in[-t,t]} Q_t^K = 0,\ \ \lim_{K\to+\infty} \sup_{s\in[-t,t]} R_t^K = 0.$$

By Proposition 1 there exists an event N such that $P(N) = 0$ and such that, on N^c, one has that, for all $n \ge 0$, $\lim_{K\to+\infty} \sup_{s\in[-n,n]} Q_s^K = 0$ and $\lim_{K\to+\infty} \sup_{s\in[-n,n]} R_s^K = 0$. We also require that, on N, no two random walks perform a jump at the same time. From now on, we consider a modified version of the random walks $Z(x,i)$, where the definition of $Z(x,i)$ on the set N is given by $Z_t(x,i) := x$ for all t. With this modification, by definition of N, one has in particular that $S_t \in \mathbb{S}_\theta$ for all t, using the fact that, for all $x \in \mathbb{Z}$, $\exp(-\theta|x|) = \min(\exp(\theta x), \exp(-\theta x))$.

Lemma 1. *One has the following inequality*

$$d_\theta(S_t, S_t^K) \le 2(Q_t^K + R_t^K).$$

Corollary 1. *The set* $\{(Z(x,i), u(x,i));\ (x,i) \in A\}$ *is a random variable taking values in* $(\Omega, \mathcal{F})$.

Proof. Note that, for every K, the map $t \mapsto S_t^K$ from $\mathbb{R}$ to $\mathbb{S}_\theta$ is càdlàg, since it involves only a finite number of particle paths. From Lemma 1 and the definition of N, we see that, as K goes to infinity, S^K converges uniformly to S on every bounded interval. As a consequence, S is càdlàg too, and we can in fact view S as a random variable taking values in $\mathcal{D}$ equipped with its cylindrical $\sigma-$algebra. □

We can now safely define

$$\mathbb{P}_w := \text{ distribution of } \{(Z(x,i), u(x,i));\ (x,i) \in A\} \text{ on } (\Omega, \mathcal{F}).$$

The expectation with respect to $\mathbb{P}_w$ is denoted by $\mathbb{E}_w$.

2.3. **Properties of $\mathbb{P}_w$.** This section is devoted to various structural and regularity properties of $\mathbb{P}_w$. The main points are the strong Markov property of the family $(\mathbb{P}_w, w \in \mathbb{S}_\theta)$, the definition of the Poisson initial distribution $\mathbb{P}_\nu$ and its invariance with respect to space-time shifts.

Proposition 2. *For any* $w_1, w_2 \in \mathbb{S}_\theta$, *there exists a coupling between a version of* S *starting from* w_1, *denoted* $S^{(1)}$, *and a version starting from* w_2, *denoted* $S^{(2)}$, *such that, for all* $t \ge 0$, *and all* $\lambda > 0$,

$$P\left(\sup_{s\in[0,t]} d_\theta(S_s^{(1)}, S_s^{(2)}) > \lambda\right) \le 2\lambda^{-1}\exp(2(\cosh\theta - 1)t) d_\theta(w_1, w_2).$$

Using Proposition 2, we can then prove the following stability property.

Proposition 3. *Let* $m \ge 1$. *If* $f_1, \ldots, f_m : \mathbb{S}_\theta \to \mathbb{R}$ *are bounded and uniformly continuous, then, for any* $0 \le t_1 \le \ldots \le t_m$, *the map from* $(\mathbb{S}_\theta, d_\theta)$ *to* $\mathbb{R}$ *defined by* $w \mapsto \mathbb{E}_w(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m}))$ *is bounded and uniformly continuous too.*

Proposition 3 is one of the key tools used to establish the Markov properties of the family $(\mathbb{P}_w, w \in \mathbb{S}_\theta)$, as stated in the following propositions.

Proposition 4. *For any bounded measurable function* $F : \mathbb{S}_\theta \to \mathbb{R}$, *the map* $w \mapsto \mathbb{E}_w(F)$ *is measurable.*

Proposition 5. *The simple Markov property holds for our process: for all* $w \in \mathbb{S}_\theta$, *all* $t \ge 0$, *and bounded measurable function* F *on* $\mathcal{D}_+$, *one has that*

$$\mathbb{E}_w(F((X_{t+s})_{s\ge0})|\mathcal{F}_t) = \mathbb{E}_{X_t}(F((X_s)_{s\ge0}))\ \mathbb{P}_w - a.s. \tag{3}$$

Proposition 6. *The strong Markov property holds for our process: for every $w \in \mathbb{S}_\theta$, every non-negative $(\mathcal{F}_t)_{t\geq 0}-$stopping time T, and bounded measurable function F on $\mathcal{D}_+$, one has that, on $\{T < +\infty\}$,*

$$\mathbb{E}_w(F((X_{T+t})_{t\geq 0})|\mathcal{F}_T) = \mathbb{E}_{X_T}(F((X_t)_{t\geq 0}))\ \mathbb{P}_w - a.s. \tag{4}$$

Now consider an i.i.d. family $N = (N_x)_{x\in\mathbb{Z}}$ of Poisson processes on $[0,1]$, with intensity ρ. With probability one, $(N_x)_{x\in\mathbb{Z}} \in \mathbb{S}_\theta$, and we call ν the probability distribution on $\mathbb{S}_\theta$ induced by N. The probability measure $\mathbb{P}_\nu$ defined by $\mathbb{P}_\nu(\cdot) := \int_{\mathbb{S}_\theta} \mathbb{P}_w(\cdot) d\nu(w)$ is the reference measure we use to describe the dynamics starting from a Poisson initial distribution of particles.

Proposition 7. *The probability distribution $\mathbb{P}_\nu$ on Ω is invariant with respect to the space-time shifts $\pi_{x,t}$.*

To prove the above proposition, we use the following lemma.

Lemma 2. *For any $t \geq 0$, under $\mathbb{P}_\nu$, (X_0, X_t) and (X_t, X_0) have the same distribution.*

Proof of Lemma 2. We use the notation $p_t(x,y)$ to denote the probability for a continuous-time simple symmetric random walk on $\mathbb{Z}$, with total jump rate 2, started at x at time zero, to be at y at time t.

Fix $t \geq 0$, and, for $x, y \in \mathbb{Z}$ and $0 \leq a < b \leq 1$, define

$$N(x,y,a,b) := |X_0(x) \cap X_t(y) \cap [a,b]|\,.$$

In terms of particles, $N(x,y,a,b)$ is the number of particles with label in $[a,b]$ that start at site x at time zero and are located at site y at time t. By standard properties of Poisson processes, one checks that the distribution of $N(x,y,a,b)$ is Poisson with parameter $\rho(b-a)p_t(x,y)$, that, given a family of pairwise distinct pairs $(x_1,y_1),\ldots,(x_m,y_m)$, the random variables $N(x_i,y_i,a,b)$ with $1 \leq i \leq m$ are mutually independent, and that, moreover, given a family of paiwise disjoint intervals $[a_1,b_1],\ldots,[a_m,b_m]$, the random variables $N(a_i,b_i)$ with $1 \leq i \leq m$ are mutually independent, where $N(a,b) := (N(x,y,a,b);\ x,y \in \mathbb{Z})$. Now note that, since $p_t(x,y) = p_t(y,x)$ for all $x,y \in \mathbb{Z}$, $N(x,y,a,b)$ has the same distribution as $N(y,x,a,b)$ for all $x,y \in \mathbb{Z}$ and $0 \leq a < b \leq 1$. This and the independence properties discussed above show that, for any finite family (x_i,y_i,a_i,b_i), $1 \leq i \leq m$, the joint distribution of the random variables $N(x_i,y_i,a_i,b_i)$ with $1 \leq i \leq m$ is identical to that of the random variables $N(y_i,x_i,a_i,b_i)$ with $1 \leq i \leq m$. In turn, this proves that (X_0,X_t) and (X_t,X_0) have the same distribution. □

Proof of Proposition 7. Invariance with respect to space shifts is a direct consequence of the corresponding invariance of the distribution of $(N_x)_{x\in\mathbb{Z}}$ and of the distribution of random walk paths. Invariance with respect to time shifts comes from Lemma 2, which proves the reversibility of the dynamics with respect to ν. See Proposition 5.3 Chap. II in [10] for more details. □

2.4. **Infection dynamics.** We now formally define the infection dynamics, through random variables defined on $(\Omega, \mathcal{F})$. First, let $T_0 := 0$, $\mathfrak{r}_0 := \sup\{x \leq 0;\ \exists (W,u) \in \Psi,\ W_0 = x\}$ (with the convention $\inf \emptyset = -\infty$) and define inductively the families of random variables $(T_\ell)_{\ell \geq 0}$ and $(\mathfrak{r}_\ell)_{\ell \geq 0}$ as follows. Consider $t > T_\ell$. We say that t is upward if there exists $(W,u) \in \Psi$ such that $W_{t-} = \mathfrak{r}_\ell$ and $W_t = \mathfrak{r}_\ell + 1$. We say that t is downward if there exists $(W,u) \in \Psi$ such that $W_{t-} = \mathfrak{r}_\ell$, $W_t = \mathfrak{r}_\ell - 1$, and $X_{t-}(\mathfrak{r}_\ell) = \{(W,u)\}$. Then let

$$T_{\ell+1} := \inf\{t > T_\ell;\quad t \text{ is upward or downward}\},$$

with the convention that $\inf \emptyset = +\infty$. By the fact that paths are càdlàg in $\mathbb{S}_\theta$, one must have that $T_{\ell+1} > T_\ell$ when $T_\ell < +\infty$. Provided that $T_{\ell+1} < +\infty$, one must also have that $T_{\ell+1}$ is indeed a upward or downward time[4]. In the upward case, we let $\mathfrak{r}_{\ell+1} := \mathfrak{r}_\ell + 1$. In the downward case, we let $\mathfrak{r}_{\ell+1} := \mathfrak{r}_\ell - 1$. Otherwise we let $\mathfrak{r}_{\ell+1} := \dagger$. Now r_t is defined on each interval $[T_\ell, T_{\ell+1}[$ by $r_t := \mathfrak{r}_\ell$. We also define $r_\infty = \dagger$. Note that we do not rule out possible explosions, meaning that $T_\infty := \sup_\ell T_\ell$ may be finite, in which case we let[5] $r_t := \dagger$ for $t \geq T_\infty$. From the results in [6], one has that, for all $k \geq 1$, $T_k < +\infty$, and $T_\infty = +\infty$, almost surely with respect to $\mathbb{P}_\nu$.

In the sequel, we say that a time $t > 0$ is a jump time for the front if it is one of the times $T_1, T_2, \ldots$ at which the position of the front either increases or decreases by one unit.

For all $0 < t < T_\infty$, we denote by B_t the subfamily of particle paths corresponding to particles that are blue at time t, i.e.

$$B_t := \{(W,u) \in \Psi;\ \forall s \in [0,t[,\ W_s > r_s\}.$$

Similarly, the subfamily of paths associated with particles that are red at time t is

$$R_t := \{(W,u) \in \Psi;\ \exists s \in [0,t[,\ W_s \leq r_s\}.$$

We extend the definition by setting $B_0 := \{(W,u) \in \Psi;\ W_0 \geq 0\}$ and $R_0 := \{(W,u) \in \Psi;\ W_0 < 0\}$. For $t \geq T_\infty$, we set $B_t := \emptyset$ and $R_t := \Psi$.

One checks that, with these definitions, for all $0 < t < T_\infty$, r_t corresponds to the position of the rightmost red particle at time t.

In the sequel, we shall use the following $\sigma-$algebras. First, given $t \geq 0$, $\mathcal{F}_t^R$ is defined by[6]

$$\mathcal{F}_t^R := \sigma((W_s, u);\ s \leq t,\ (W,u) \in R_t).$$

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[4]Note that, in the definition of Ω, we have ruled out the possibility of two distinct particle paths performing a jump at exactly the same time, so that $T_{\ell+1}$ is either upward or downward, but cannot be both.

[5]Whenever we compare r_t with a real number, we implicitly mean that $r_t \neq \dagger$. For instance, the event $r_t > a$ should be read as the event that $r_t \neq \dagger$ and $r_t > a$.

[6]Formally, $\mathcal{F}_t^R$ is generated by all the random variables of the form

$$\#(R_t \cap \{(W,u);\ W_s = k,\ u \in [a,b]\}),$$

where $k \in \mathbb{Z}$, $0 \leq a < b \leq 1$, and $s \leq t$.

Informally, $\mathcal{F}_t^R$ contains the information relative to the trajectories of particles that are red at time t, up to time t. If T is a non-negative random variable on $(\Omega, \mathcal{F})$, we also define[7]

$$\mathcal{F}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u);\ s \leq t,\ (W, u) \in R_T).$$

Similarly, we let

$$\mathcal{G}_t^R := \sigma((W_s, u);\ s \in \mathbb{R},\ (W, u) \in R_t).$$

Informally, $\mathcal{G}_t^R$ contains the information relative to the full trajectories of the particles that are red at time t. When T is a non-negative random variable, we also define

$$\mathcal{G}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u);\ s \in \mathbb{R},\ (W, u) \in R_T).$$

When working with these $\sigma-$algebras, we will have several occasions to apply the following lemma, that we quote now for future reference.

Lemma 3. *Let $(O, \mathcal{H})$ denote a measurable space, let I denote a set of indices, and let $(\zeta_i^1)_{i \in I}$ and $(\zeta_i^2)_{i \in I}$ be two families of random variables on $(O, \mathcal{H})$, each ζ_i^1 and ζ_i^2 taking values in a measurable space $(S_i, \mathcal{S}_i)$. Let U denote an event in $\mathcal{H}$ such that, on U, $\zeta_i^1 = \zeta_2^i$ for all $i \in I$. Then, for any $A_1 \in \sigma(\zeta_i^1,\ i \in I)$, there exists $A_2 \in \sigma(\zeta_i^2,\ i \in I)$ such that $A_1 \cap U = A_2 \cap U$.*

3. Regeneration structure

We now define the regeneration structure that is used to prove the central limit theorem. Remember that is based on straight lines drawn on the space-time plane. In the sequel, α is a strictly positive real number corresponding to the slope of these straight lines.

Consider an upward jump time $t > 0$. We say that t is a *backward sub-α time* if $r_t > \alpha t$ and if, for all $0 \leq s < t$, one has $r_s < r_t - \alpha(t-s)$. We say that t is a *backward super-α time* if, for any (W, u) in B_t, and for all $s < t$, one has $W_s \geq r_t - \alpha(t-s)$. If t is both a backward sub-α and super-α time, we say that t is a *backward α time.* We say that t is a *forward sub-α time* if, for all $(W, u) \in R_t$ such that $W_t \leq r_t - 1$, one has that $W_s \leq r_t - 1 + \alpha(t-s)$ for all $s > t$, and if the particle (W, u) making the front jump at time t remains at r_t during the time-interval $[t, t + \alpha^{-1}]$, and then satisfies the inequality $W_s \leq r_t - 1 + \alpha(t-s)$ for all $s \geq t + \alpha^{-1}$. We say that t is a *forward super-α time* if, for all $s > t$, one has $r_s \geq r_t + \lfloor \alpha(s-t) \rfloor$, and if, moreover, there exists $(W, u) \in B_t$ such that $W_s = r_t$ for all $s \in [t, t + \alpha^{-1}]$. If t is both a forward sub-α and super-α time, we say that a t is a *forward α time.* Finally, if t is both a forward and backward α time, we say that t is an *α-separation time.* We extend the definition of a backward super-α time and of a forward super-α time by allowing $t = 0$ in the above definitions.

[7] Formally, $\mathcal{F}_T^R$ is generated by all the random variables of the form

$$\mathbf{1}(s \leq T) \times \#(R_T \cap \{(W, u);\ W_s = k,\ u \in [a, b]\}),$$

where $k \in \mathbb{Z}$, $0 \leq a < b \leq 1$, and $s \in \mathbb{R}$.

Now let $\kappa_0 := 0$ and define inductively the sequence $(\kappa_i)_{i\geq 0}$ by

$$\kappa_{i+1} := \inf\{T_j > \kappa_i;\ T_j \text{ is an } \alpha\text{-separation time}\}.$$

The following propositions show that the sequence $(\kappa_n)_{n\ \geq 1}$ indeed provides a renewal structure for the position of the front.

Proposition 8. *For all $n \geq 1$, $\kappa_1, \ldots, \kappa_n$ and $r_{\kappa_1}, \ldots, r_{\kappa_n}$ are measurable with respect to $\mathcal{G}^R_{\kappa_n}$.*

Proposition 9. *On $\{\kappa_n < +\infty\}$, the conditional distribution of $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})$ with respect to $\mathcal{G}^R_{\kappa_n}$ is the distribution*[8] *of $(\kappa_1, r_{\kappa_1})(B_0)$ with respect to $\mathbb{P}_\nu$, conditioned on $t = 0$ being a backward and forward super-α time.*

Proposition 10. *For small enough α (depending on ρ), there exists $\theta > 0$ such that $\mathbb{E}_\nu(\kappa_1^2) < +\infty$ and $\mathbb{E}_\nu(r^2_{\kappa_1}) < +\infty$.*

Corollary 2. *With respect to $\mathbb{P}_\nu$, the random variables*

$$(\kappa_{i+1} - \kappa_i, r_{\kappa_{i+1}} - r_{\kappa_i})_{i\geq 0}$$

are mutually independent; the random variables

$$(\kappa_{i+1} - \kappa_i, r_{\kappa_{i+1}} - r_{\kappa_i})_{i\geq 1}$$

are identically distributed.

Given Corollary 2 and Proposition 10, it is more or less standard to derive Theorem 1, approximating r_t by $r_{\kappa_{n_t}}$, where $n_t := \sup\{n \geq 0;\ \kappa_n \leq t\}$. Note that, due to the definition of κ, one has $r_{\kappa_{n_t}} \leq r_t \leq r_{\kappa_{n_t+1}}$, which eases the corresponding approximation argument. We do not give the details here (see e.g. [3]). The rest of this section is devoted to the proof of Propositions 8, 9, and Corollary 2, the proof of Proposition 10 being the object of the next section.

Proof of Proposition 8. Consider $n \geq 1$. First note that the measurability of κ_n and r_{κ_n} with respect to $\mathcal{G}^R_{\kappa_n}$ is a direct consequence of the definition of $\mathcal{G}^R_{\kappa_n}$. Also, with our conventions, the result is obvious on $\{\kappa_n = +\infty\}$, so we work on $\{\kappa_n < +\infty\}$ throughout the rest of the proof. Observe that, from the definition of the infection dynamics, particle paths (W, u) outside R_{κ_n} have no influence on the front jumps between time 0 and κ_n, so that the history of the front up to time κ_n is exactly the same as the one that would be obtained if there were no other particle paths in the system besides those in R_{κ_n}. As a consequence, the jump times $T_1 < \ldots < T_\ell = \kappa_n$ that lie between time 0 and κ_n, are measurable with respect to $\mathcal{G}^R_{\kappa_n}$.

Thus, to prove the proposition, it is enough to prove that, for every jump time T_i such that $1 \leq i \leq \ell - 1$, it is possible to tell whether T_i is a backward/forward sub/super-α time, using only the information contained in $\mathcal{G}^R_{\kappa_n}$.

[8]This is a slight abuse of terminology, since, strictly speaking, B_0 is only $\mathbb{P}_\nu-$a.s. equal to a random variable from $(\Omega, \mathcal{F})$ to itself, see Lemma 34.

We have already noted that the history of the front up to time κ_n can be deduced from $\mathcal{G}^R_{\kappa_n}$, so that the fact that T_i is a backward sub-α time can indeed be told using $\mathcal{G}^R_{\kappa_n}$.

To find whether T_i is a backward super-α time, we have to look at those particle paths (W, u) that belong to B_{T_i}. Since κ_n is itself a backward super-α time, we know that, for any $(W, u) \in B_{\kappa_n}$, $W_s \geq r_{\kappa_n} - \alpha(\kappa_n - s)$ for all $s < \kappa_n$. Since κ_n is also a backward sub-α time, one has that $r_{T_i} \leq r_{\kappa_n} - \alpha(\kappa_n - T_i)$. As a consequence, for all $s < T_i$, one has that

$$W_s \geq r_{\kappa_n} - \alpha(\kappa_n - s) \geq r_{T_i} + \alpha(\kappa_n - T_i) - \alpha(\kappa_n - s) = r_{T_i} - \alpha(T_i - s).$$

As a consequence, to check whether T_i is a backward super-α time, we only have to look at paths (W, u) that belong to $B_{T_i} \cap R_{\kappa_n}$. Whether these paths satisfy $W_s \geq r_{T_i} - \alpha(T_i - s)$ for all $s < T_i$ can be told from $\mathcal{G}^R_{\kappa_n}$.

Now the fact that T_i is a forward sub-α time can be told from $\mathcal{G}^R_{\kappa_n}$, since it involves the trajectories of paths in $R_{T_i} \subset R_{\kappa_n}$ only.

To conclude the proof, we claim that T_i is a forward super-α time if and only if $r_s \geq r_{T_i} + \lfloor \alpha(s - T_i) \rfloor$ for all $s \in]T_i, \kappa_n]$ and there exists $(W, u) \in B_{T_i} \cap R_{\kappa_n}$ such that $W_s = r_{T_i}$ on $[T_i, T_i + \alpha^{-1}]$. The first condition is clearly necessary, and, since $T_i < \kappa_n$, a particle path located at site r_{T_i} at time T_i must belong to R_{κ_n}, so the second condition is necessary too. It remains to show that these two conditions are also sufficient. For $s > \kappa_n$, one has that $r_s \geq r_{\kappa_n} + \lfloor \alpha(s - \kappa_n) \rfloor$, since κ_n is a forward super-α time. On the other hand, our assumption on T_i applied at time κ_n-, combined with the fact that κ_n is an upward jump time, shows that $r_{\kappa_n} \geq r_{T_i} + \lfloor \alpha(\kappa_n - T_i)- \rfloor + 1$, and one then deduces that $r_s \geq r_{T_i} + \lfloor \alpha(s - T_i) \rfloor$.

□

Lemma 4. *For $n \geq 1$ and $k \geq 1$, on $\{T_k < +\infty\}$, one can write the event $\{\kappa_n = T_k\}$ as $\{\kappa_n = T_k\} = H_k \cap J_k$, where $H_k \in \mathcal{G}^R_{T_k}$ and where*

$$J_k := \{t = 0 \text{ is a backward and forward super-}\alpha \text{ time for } \pi_{\mathfrak{r}_k, T_k}(B_{T_k})\}.$$

Proof. Throughout the proof we work on $\{T_k < +\infty\}$. Define H_k as the event that, between time 0 and T_k, there exist exactly n jump times $t > 0$ that satisfy the following properties:

- (i) t is a backward sub-α time;
- (ii) for any $(W, u) \in B_t \cap R_{T_k}$, one has that $W_s \geq r_t - \alpha(t - s)$ for all $s < t$;
- (iii) t is a forward sub-α time;
- (iv) for every time $s \in]t, T_k]$, $r_s \geq r_t + \lfloor \alpha(s - t) \rfloor$;
- (v) there exists $(W, u) \in B_t \cap R_{T_k}$ such that $W_s = r_t$ for all $s \in [t, t + \alpha^{-1}]$,

and that T_k is one of these jump times (note that (ii) and (iv) are void conditions for T_k).

We first check that $H_k \in \mathcal{G}^R_{T_k}$. Indeed, observe that, from the definition, we know that particle paths (W, u) outside R_{T_k} have no influence on the front jumps between time 0 and T_k, so that the history of the front up to time

T_k is exactly the same as the one that would be obtained if there were no other particle paths in the system besides those in R_{T_k}. As a consequence, the jump times $T_1, \ldots, T_k$ are measurable with respect to $\mathcal{G}^R_{T_k}$. Then, it is readily checked that, given such a jump time T_i, the event that T_i satisfies the five conditions (i)-(v) listed in the definition of H_k, indeed belongs to $\mathcal{G}^R_{T_k}$ since these only involve particle paths lying in R_{T_k}.

We now check that $\{\kappa_n = T_k\} = H_k \cap J_k$. Assume that $H_k \cap J_k$ holds, and let us show that κ_n must be equal to T_k.

We first observe that, by the fact that $t = 0$ is a forward super-α time for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$ and the fact that, by (iii), T_k is a forward sub-α time, we have that, for all $s \geq T_k$, r_s coincides with $\mathfrak{r}_k + r_{s-T_k}(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))$. Indeed, during the time-interval $[t, t+\alpha^{-1}]$, we know that no particle in R_{T_k} goes strictly above r_t, while one of these particles remains at r_t. On the other hand, at least one particle in B_{T_k} also remains at r_t. As a result, both r_s and $\mathfrak{r}_k + r_{s-T_k}(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))$ must lie above r_t on the interval $[t, t+\alpha^{-1}]$, and these two fronts must in fact coincide. For $s \geq t+\alpha^{-1}$, no particle in R_{T_k} can ever touch the front, so that indeed r_s coincides with $\mathfrak{r}_k + r_{s-T_k}(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))$.

Using the definition of J_k and the preceding discussion, we deduce that T_k is a forward super-α time. Also, the fact that $t = 0$ is a backward super-α time for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$ shows that T_k is a backward super-α time. Finally, (i) shows that T_k is a backward sub-α time. We conclude that T_k is an $\alpha-$separation time. Now consider $1 \leq i \leq k-1$. We claim that, provided that T_k is an $\alpha-$separation time, T_i is an $\alpha-$separation time if and only if T_i satisfies (i)-(v). Observe first that the "only if" part of the claim is immediate from the definition. Now assume that T_k is an $\alpha-$separation time, that T_i satisfies (i)-(v), and let us show that T_i is an $\alpha-$separation time.

To show that T_i is a backward super-α time, we have to check that, for all particle paths (W, u) in B_{T_i}, one has that $W_s \geq r_{T_i} - \alpha(T_i - s)$ for all $s < T_i$. This property for $(W, u) \in B_{T_i} \cap R_{T_k}$ is precisely (ii). Consider $(W, u) \in B_{T_i} \cap B_{T_k}$. Since T_k is a backward super-α time, we have that, for all $s < T_i$, $W_s \geq r_{T_k} - \alpha(T_k - s)$. Since T_k is a backward sub-α time, we also have that $r_{T_i} \leq r_{T_k} - \alpha(T_k - T_i)$. We deduce that, for all $s < T_i$,

$$W_s \geq r_{T_k} - \alpha(T_k - s) \geq r_{T_i} + \alpha(T_k - T_i) - \alpha(T_k - s) = r_{T_i} - \alpha(T_i - s).$$

To show that T_i is a forward super-α time, since we already have (v), we only have to check that, for all $s > T_i$, one has $r_s \geq r_{T_i} + \lfloor \alpha(s - T_i) \rfloor$. By (iv), this inequality is satisfied when $s \in]T_i, T_k]$. If $s > \kappa_n$, the fact that T_k is a forward super-α time yields $r_s \geq r_{T_k} + \lfloor \alpha(s - T_k) \rfloor$. Also, (iv) applied at time T_k- combined with the fact that T_k is an upward jump time shows that $r_{T_k} \geq r_{T_i} + \lfloor \alpha(T_k - T_i)- \rfloor + 1$, and one deduces that $r_s \geq r_{T_i} + \lfloor \alpha(s - T_i) \rfloor$.

It is now clear that, on $H_k \cap J_k$, there are exactly n $\alpha-$separation times between time 0 and T_k, with T_k being one of them. In other words, $\kappa_n = T_k$. Conversely, if $\kappa_n = T_k$, T_k is an $\alpha-$separation time, J_k holds, and $\kappa_1, \ldots, \kappa_n$ are exactly those jump times between time 0 and T_k that satisfy (i)-(v), with T_k being one of them.

□

Lemma 5. *For $n \geq 1$ and $k \geq 1$, on $\{T_k < +\infty\}$, one has that, on $\{\kappa_n = T_k\}$,*

$$(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n}) = (\kappa_1, r_{\kappa_1})(\pi_{\mathfrak{r}_k, T_k}(B_{T_k})).$$

Proof. First note that, as in the proof of Lemma 4, since T_k is a forward α time (sub- and super-), the history of the front posterior to T_k is just the history of the front for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$, up to shifting time indices by T_k and space by $\mathfrak{r}_k$. Now consider an upward jump time t posterior to $T_k = \kappa_n$. By the fact that T_k is a backward sub-α time, we have that $r_s < r_{T_k} - \alpha(T_k - s)$ for all $s < T_k$, while, since T_k is forward super-α time and t is an upward jump time, $r_t \geq r_{T_k} + \lfloor \alpha(t - T_k) - \rfloor + 1 \geq r_{T_k} + \alpha(t - T_k)$. We deduce that, for all $s < T_k$, one has $r_s < r_t - \alpha(t - s)$. By the fact that T_k is a backward sub-α time, we also have $\mathfrak{r}_k > \alpha T_k$. As a consequence, checking that t is a backward sub-α time turns out to be equivalent to checking that $r_s < r_t - \alpha(t - s)$ for all $T_k \leq s < t$. This implies that it is equivalent for t to be a backward sub-α time and for $t - T_k$ to be a backward sub-α time for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$. Then note that the condition for t to be a backward super-α time is equivalent to the corresponding condition for $t - T_k$ with respect to $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$, since, using the fact that $B_t \subset B_{T_k}$, exactly the same set of particle paths is involved in both cases, and since the condition $W_s \geq r_t - \alpha(t - s)$ for all $s < t$ is clearly invariant with respect to shifting space by $\mathfrak{r}_k$ and time by T_k. Now consider the condition for t to be a forward sub-α time. Since T_k is a forward sub-α time, a particle path in R_{T_k} can never violate the condition for t to be a forward sub-α time. To see why, assume first that $W_{T_k} \leq r_{T_k} - 1$. By definition, we have that, for $s \geq t$, $W_s \leq r_{T_k} - 1 + \alpha(s - T_k)$ since T_k is a forward sub-α time. Moreover, since T_k is also a forward super-α time and t is a jump time, one has that $r_t \geq r_{T_k} + \alpha(t - T_k)$, and we thus have that $W_s \leq r_t - 1 + \alpha(s - t)$. If $W_{T_k} = r_{T_k}$, we have the same condition for $s \geq T_k + \alpha^{-1}$, so we are also done if $t \geq T_k + \alpha^{-1}$. Finally, if $t < T_k + \alpha^{-1}$, we have $W_t = r_{T_k} \leq r_t - 1$. For $t \leq s \leq T_k + \alpha^{-1}$, we have that $W_s = r_{T_k} \leq r_t - 1 + \alpha(t - s)$, while, for $s \geq T_k + \alpha^{-1}$, we deduce that $W_s \leq r_t - 1 + \alpha(s - t)$ as above. We deduce that t is a forward sub-α time if and only if the corresponding condition is satisfied for any $(W, u) \in R_t \cap B_{T_k}$ and $s > t$, and this is equivalent to $t - T_k$ being a forward sub-α time for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$. Finally, the fact that the history of the front posterior to T_k coincides with the history of the front for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$, up to shifting time and space, shows that t is a super-α time if and only if $t - t_k$ is a super-α time for $\pi_{\mathfrak{r}_k,T_k}(B_{T_k})$. □

Given a càdlàg path $q = (q_s)_{t \leq s \leq 0}$ with values in $\mathbb{Z}$ and taking nearest-neighbor steps, such that $q_0 = 0$ and $q_{0-} = -1$, and containing a finite number of jumps, given $x \in \mathbb{Z}$ and given a bounded measurable function $F : \Omega \to \mathbb{R}$, we define $\xi(F, q, x)$ by

$$\xi(F, q, x) := \mathbb{E}_\nu(F(B_0)|G(q, x)),$$

where

$$G(q,x) := \{\forall (W,u) \in B_0, W_t > x, \ \ \forall t \le s < 0, \ W_s > q_s\}. \tag{5}$$

(Note that the definition makes sense since, as is easily checked, $\mathbb{P}_\nu(G(q,x)) > 0$ for all q.)

Proposition 11. *Let $F : \Omega \to \mathbb{R}$ denote a bounded measurable map. Then, for all $k \ge 1$, on the event that T_k is upward,*

$$\mathbb{E}_\nu(F(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))|\mathcal{G}^R_{T_k}) = \xi(F,(r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k) \ a.s.$$

Corollary 3. *Let $F : \Omega \to \mathbb{R}$ denote a bounded measurable map. Let*

$$G := \{t = 0 \textit{ is a backward super-}\alpha \textit{ time}\}. \tag{6}$$

Then, for all $k \ge 1$, on the event that T_k is a backward sub-α time, one has that

$$\mathbb{E}_\nu((F\mathbf{1}_G)(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))|\mathcal{G}^R_{T_k}) = \zeta(F,(r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k) \ a.s.,$$

where

$$\zeta(F,q,x) := \mathbb{E}_\nu(F(B_0)\mathbf{1}_G)\chi(q,x),$$
$$\chi(q,x) := \frac{1}{\mathbb{P}_\nu(G(q,x))}.$$

Proof. In view of Proposition 11, we have to prove that, on the event that T_k is a backward sub-α time,

$$\xi(F\mathbf{1}_G,(r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, \mathfrak{r}_k) = \zeta(F,(r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k) \ a.s.$$

To prove this identity, consider a path $q = (q_s)_{t \le s \le 0}$ such that $q_0 = 0$ and $q_s < \alpha s$ for all $t \le s < 0$, and x such that $x < \alpha t$. Observe that by definition, $\mathbf{1}_G(B_0) = \mathbf{1}_G$, and $\mathbf{1}_G(B_0) \le \mathbf{1}_{G(q,x)}$. As a consequence,

$$\xi(F\mathbf{1}_G,q,x) = \mathbb{E}_\nu(F(B_0)\mathbf{1}_G(B_0)|G(q,x)) = \frac{\mathbb{E}_\nu(F(B_0)\mathbf{1}_G(B_0))}{\mathbb{P}_\nu(G(q,x))} = \zeta(F,q,x).$$

□

Proof of Proposition 9. Consider $C \in \mathcal{G}^R_{\kappa_n}$ such that $C \subset \{\kappa_n < +\infty\}$, and a bounded measurable function $f : \mathbb{R} \times (\mathbb{Z} \cup \{\dagger\}) \to \mathbb{R}$.

Let $\Theta_n := f(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})$. We now write

$$\mathbb{E}_\nu(\Theta_n \mathbf{1}_C) = \sum_{k \ge 1} \mathbb{E}_\nu(\Theta_n \mathbf{1}(\kappa_n = T_k)\mathbf{1}_C). \tag{7}$$

By Lemma 3 there exists $D_k \in \mathcal{G}^R_{T_k}$ such that

$$C \cap \{\kappa_n = T_k\} = D_k \cap \{\kappa_n = T_k\}.$$

Now, from Lemma 4, one can write, on $\{T_k < +\infty\}$,

$$\{\kappa_n = T_k\} = H_k \cap J_k,$$

where $H_k \in \mathcal{G}^R_{T_k}$ and

$$J_k := \{t = 0 \text{ is a backward and forward super-}\alpha \text{ time for } \pi_{\mathfrak{r}_k,T_k}(B_{T_k})\}.$$

Moreover, by Lemma 5, on $\{T_k < +\infty\} \cap \{\kappa_n = T_k\}$, one has that

$$(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n}) = (\kappa_1, r_{\kappa_1})(\pi_{\mathfrak{r}_k,T_k}(B_{T_k})).$$

We deduce that

$$\mathbb{E}_\nu(\Theta_n \mathbf{1}(\kappa_n = T_k)\mathbf{1}_C) = \mathbb{E}_\nu((F\mathbf{1}_K\mathbf{1}_G)(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))\mathbf{1}_{D_k}\mathbf{1}_{H_k}),$$

where

$$F := f(\kappa_1, r_{\kappa_1}),\ K := \{t = 0 \text{ is a forward super-}\alpha \text{ time}\},$$

and where G is defined in (6). Using Corollary 3, we deduce that

$$\mathbb{E}_\nu(\Theta_n \mathbf{1}(\kappa_n = T_k)\mathbf{1}_C) = \mathbb{E}_\nu(\zeta(F\mathbf{1}_K, (r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k)\mathbf{1}_{D_k}\mathbf{1}_{H_k}), \quad (8)$$

whence, using (7) and the definition of ζ, the identity

$$\mathbb{E}_\nu(\Theta_n \mathbf{1}_C) = \mathbb{E}_\nu(F(B_0)\mathbf{1}_K(B_0)\mathbf{1}_G)\sum_{k\ge 1}\mathbb{E}_\nu(\chi((r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k)\mathbf{1}_{D_k}\mathbf{1}_{H_k}).$$

Using this identity for $f \equiv 1$ yields that

$$\mathbb{P}_\nu(C) = \mathbb{P}_\nu(\check{G})\sum_{k\ge 1}\mathbb{E}_\nu(\chi((r_{s+T_k} - \mathfrak{r}_k)_{-T_k \le s \le 0}, -\mathfrak{r}_k)\mathbf{1}_{D_k}\mathbf{1}_{H_k}),$$

where

$$\check{G} := \{\ t = 0 \text{ is a forward and backward super-}\alpha \text{ time for } B_0\ \}.$$

Lemma 13 states that $\mathbb{P}_\nu(\check{G}) > 0$. Returning to a general f, we deduce that

$$\mathbb{E}_\nu(\Theta_n \mathbf{1}_C) = \frac{\mathbb{E}_\nu(F(B_0)\mathbf{1}_{\check{G}})}{\mathbb{P}_\nu(\check{G})}\mathbb{P}_\nu(C),$$

whence the fact that, on $\{\kappa_n < +\infty\}$,

$$\mathbb{E}_\nu(f(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1} - r_{\kappa_n}})|\mathcal{G}^R_{T_k}) = \mathbb{E}_\nu(f(\kappa_1, r_{\kappa_1})(B_0)|\check{G})\ \mathbb{P}_\nu - a.s.$$

□

Before we prove Proposition 11, we need to introduce the following definitions. Consider a càdlàg path $q = (q_s)_{t \le s \le 0}$ with values in $\mathbb{Z}$ and taking nearest-neighbor steps, containing a finite number of jumps, and such that $q_0 = 0$. Define $t_0 := \inf\{s \in [t, 0];\ q \equiv 0 \text{ on } [s, 0]\}$. Assume in addition that $t_0 > t$ and that $q_{t_0-} = -1$. Given a bounded measurable function $F : \Omega \to \mathbb{R}$, we define

$$\xi'(F, q, x) := \mathbb{E}_\nu(F(B_0)|G'(q, x)),$$

where $G'(q, x)$ is the event that, for all $(W, u) \in B_0$, one has $W_t > x$ and, for all $s \in [t, t_0[$, $W_s > q_s$, and that, moreover, during the time-interval $]t_0, 0]$, no particle in B_0 hits or leaves 0. Let also $\chi'(q, x) := \frac{1}{\mathbb{P}_\nu(G'(q,x))}$, noting that, with our assumptions on q, we always have $\mathbb{P}_\nu(G'(q, x)) > 0$. We extend the definition for paths that do not satisfy these assumptions by setting $\xi'(F, q, x) := 0$.

Given real numbers $a \le b$, let $B_0(a, b)$ denote the subset of B_0 formed by excluding the particles at site zero whose labels lie within the interval $]a, b]$.

We define $\xi'(F,a,b,q,x)$, $G'(a,b,q,x)$ and $\chi'(a,b,q,x)$ just as $\xi'(F,q,x)$, $G'(q,x)$ and $\chi'(q,x)$, but with B_0 replaced by $B_0(a,b)$.

Proof of Proposition 11. We prove the result for F of the form $F = f_1(X_{t_1})\times \cdots \times f_p(X_{t_p})$, where $t_1 < \ldots < t_p$, and $f_1,\ldots,f_p$ are bounded and uniformly continuous. The result for a general bounded measurable F follows by a monotone class argument. For $x \in \mathbb{R}$ and $h \in \{1,2,\ldots,\}$, introduce the notation $[x]_h = 2^{-h}(\lceil 2^h x\rceil)$, and let $T_k^{(\ell)} := [T_k]_\ell$. Also, define U_k to be the label of the particle that makes the front jump at time T_k, and let $U_k^{(m)} := [U_k]_m$.

Remember that the history of the front up to time T_k is measurable with respect to $\mathcal{G}^R_{T_k}$. We want to prove that, for any event $C \in \mathcal{G}^R_{T_k}$ such that $C \subset \{T_k \text{ is upward}\}$, one has the following identity:

$$\mathbb{E}_\nu(F(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))\mathbf{1}_C) = \mathbb{E}_\nu\left(\xi(F,(r_{s+T_k}-\mathfrak{r}_k)_{-T_k\le s\le 0},-\mathfrak{r}_k)\mathbf{1}_C\right). \quad (9)$$

Introduce the notation

$$\Theta_\ell := F(\pi_{\mathfrak{r}_k,T_k^{(\ell)}}(B_{T_k})).$$

Our first remark is that, due to the càdlàg character of the paths, the fact that $T_k^{(\ell)} \ge T_k$ for all ℓ and $\lim_{\ell\to+\infty} T_k^{(\ell)} = T_k$ imply, using dominated convergence and the specific form of F, that

$$\lim_{\ell\to+\infty}\mathbb{E}_\nu(\Theta_\ell\mathbf{1}_C) = \mathbb{E}_\nu(F(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))\mathbf{1}_C). \quad (10)$$

We now introduce the events A_ℓ and V_m. First, A_ℓ is defined by the fact that, during the time interval $]T_k,T_k^{(\ell)}]$, no particle hits or leaves $\mathfrak{r}_k$. Then, V_m is defined by the fact that B_{T_k} contains no particle whose label u is such that $[u]_m = U_k^{(m)}$ and that also has $W_{T_k} = \mathfrak{r}_k$. By dominated convergence, one has that

$$\lim_{\ell\to+\infty}\mathbb{P}_\nu(A_\ell) = 1 \text{ and } \lim_{m\to+\infty}\mathbb{P}_\nu(V_m) = 1. \quad (11)$$

Moreover, since $|F|$ is bounded by say M, one has that

$$|\mathbb{E}_\nu(\Theta_\ell\mathbf{1}_C) - \mathbb{E}_\nu(\Theta_\ell\mathbf{1}_C\mathbf{1}_{A_\ell}\mathbf{1}_{V_m})| \le M(\mathbb{P}_\nu(A_\ell^c)+\mathbb{P}_\nu(V_m^c)). \quad (12)$$

We now prove the following identity

$$\mathbb{E}_\nu(\Theta_\ell\mathbf{1}_C\mathbf{1}_{A_\ell}\mathbf{1}_{V_m}) = \mathbb{E}_\nu(\Lambda_{\ell,m}\mathbf{1}_C\mathbf{1}_{A_\ell}\mathbf{1}_{V_m}), \quad (13)$$

with

$$\Lambda_{\ell,m} := \xi'(F,a_m,b_m,(r_{s+T_k^{(\ell)}}-\mathfrak{r}_k)_{-T_k^{(\ell)}\le s\le 0}),$$
$$a_m := U_k^{(m)} - 2^{-m},\ b_m := U_k^{(m)}.$$

Introduce the event

$$J(d,e,y) := \{T_k^{(\ell)} = d,\ U_k^{(m)} = e,\ \mathfrak{r}_k = y\}.$$

Clearly, to prove (13), it is enough to prove that, for every d,e,y, one has

$$\mathbb{E}_\nu(\Theta_\ell\mathbf{1}_C\mathbf{1}_{A_\ell}\mathbf{1}_{V_m}\mathbf{1}_{J(d,e,y)}) = \mathbb{E}_\nu(\Lambda_{\ell,m}\mathbf{1}_C\mathbf{1}_{A_\ell}\mathbf{1}_{V_m}\mathbf{1}_{J(d,e,y)}). \quad (14)$$

We now fix d, e, y, and work on $J(d, e, y)$. Introduce the set Δ formed by those particle paths (W, u) such that $W_d \geq y$ and $[u]_m \neq e$ if $W_d = y$. Define

$$\mathcal{H}^+_{d,e,y} := \sigma((W_s, u);\ s \in \mathbb{R},\ (W, u) \in \Delta),$$

$$\mathcal{H}^-_{d,e,y} := \sigma((W_s, u);\ s \in \mathbb{R},\ (W, u) \notin \Delta).$$

Consider the front $(r'_s)_{s\geq 0}$ generated by the particles in Δ^c up to time d. Define the event E_- that the $k-$th jump of this front is an upward jump to y at a time t' such that $[t']_\ell = d$, that the particle making the front jump at time t' has a label u satisfying $[u]_m = e$, remains at y during the time-interval $[t', d]$, and that no other particle in Δ^c is located at y during the interval $[t', d]$. Define also the event E_+ that the particles (W, u) in Δ do not hit or leave y during the time-interval $[t', d]$, and satisfy $W_s > r'_s$ for all $s \in [0, t'[$, and $W_0 > 0$. We claim that

$$A_\ell \cap V_m \cap J(d, e, y) \cap \{T_k \text{ is upward}\} = E_- \cap E_+. \tag{15}$$

Let us assume that $A_\ell \cap V_m \cap J(d, e, y) \cap \{T_k \text{ is upward}\}$ holds. We first observe that Δ and B_{T_k} coincide. Indeed, by definition, at time T_k, the blue particles are exactly those (W, u) satisfying $W_{T_k} \geq \mathfrak{r}_k = y$ and $u \neq U_k$. Thanks to A_ℓ, no particle in B_{T_k} can move from y to $y-1$ during the time-interval $]T_k, d]$, so that, at time d, all particles in B_{T_k} must have a position $\geq y$. Moreover, thanks to V_m, the only particles (W, u) in B_{T_k} whose label is such that $[u]_m = e$, have to satisfy $W_{T_k} \geq y + 1$, and, by A_ℓ, these particles are not allowed to hit y during $]T_k, d]$. We conclude that no particle (W, u) in B_{T_k} such that $W_d = y$ can have $[u]_m = e$. We have shown that $B_{T_k} \subset \Delta$. Conversely, thanks to A_ℓ, no particle in R_{T_k} except the one that made the front climb at y at time T_k, can have a location $\geq y$ at time d. Thanks to A_ℓ again, this specific particle (whose label is U_k) has to remain at y during $]T_k, d]$. This proves that $B^c_{T_k} \subset \Delta^c$. So we have proved that $B_{T_k} = \Delta$. We now prove that, for all $s \in [0, d]$, $r'_s = r_s$. Since $\Delta = B_{T_k}$, we know that, up to time T_k, (r_s) coincides with the front generated by particles in Δ^c, which is exactly the definition of (r'_s). Then, thanks to A_ℓ, both fronts must remain at y from time T_k to time d since no particle is allowed to either hit or leave y during $]T_k, d]$. In particular $t' = T_k$. We now prove that E_+ holds. The condition on particles not hitting or leaving y is a direct consequence of A_ℓ (and on the fact that $\mathfrak{r}_k = y$ since $J(d, e, y)$ holds). Since (r_s) and (r'_s) coincide on $[0, T_k] = [0, t']$, and since B_{T_k} and Δ coincide, the fact that any $(W, u) \in \Delta$ is such that $W_s > r'_s$ for all $s \in [0, t'[$ is just a consequence of the fact that, by definition, any $(W, u) \in B_{T_k}$ is such that $W_s > r_s$ for all $s \in [0, T_k[$. The same remark holds for the condition $W_0 > 0$. Similarly, E_- is a restatement in terms of r', t', Δ^c of properties holding for r, t, R_{T_k}.

Conversely, let us assume that $E_- \cap E_+$ holds. Our first claim is that $r_s = r'_s$ for all $s \in [0, t']$. To see this, let us show that a particle in Δ always lies strictly above $(r_s)_s$ for $s \in [0, t'[$. Assume that this is not the case, and define s_0 as the first time at which a particle in Δ lies below the front $(r_s)_s$. Due to the last condition defining E_+, one has $R_{0+} \subset \Delta^c$, so that necessarily

$s_0 > 0$. By definition, we have $s_0 < t'$ and $W_{s_0} \le r_{s_0}$ for some $(W, u) \in \Delta$. Prior to s_0, particles in Δ always lie strictly above $(r_s)_s$, so that $r_s = r'_s$ for all $s \le s_0$. Now the fact that $W_{s_0} \le r_{s_0}$ implies that also $W_{s_0} \le r'_{s_0}$, which is in contradiction with E_+. We deduce that particles in Δ always lies strictly above $(r_s)_s$ for $s \in [0, t'[$, so that one must have $r_s = r'_s$ for all $s \in [0, t']$. In particular, we deduce using E_- that $t' = T_k$, $U_k^{(m)} = e$, $T_k^{(\ell)} = d$, $\mathfrak{r}_k = y$, and the fact that T_k is upward. It remains to show that A_ℓ and V_m hold. We start with A_ℓ. Thanks to E_+, during $[t', d]$, particles in Δ cannot hit or leave y from above. Now, by E_-, all the particles in Δ^c save the one making the front climb at time t', have a location $\le y - 1$ at time t', and do not hit y during $[t', d]$. Again by E_-, the particle making the front climb at time t' remains at y during $[t', d]$. This proves that A_ℓ holds. Since T_k is an upward time, B_{T_k} coincides with the set of particles whose location at time T_k is $\ge y$, minus the particle that makes the front climb at T_k. On $E_- \cap E_+$, this coincides with Δ, and the definition of Δ precisely shows that V_m must then hold.

We retain from this discussion not only that (15) holds, but also that, on $E_- \cap E_+$, one has that r_s and r'_s coincide for all $s \in [0, d]$, and that B_{T_k} and Δ coincide. As a consequence, by Lemma 3, on $E_- \cap E_+$, we can find an event $D = D(d, e, y) \in \mathcal{H}^-_{d,e,y}$ that coincides with C. As a consequence, we have

$$\mathbb{E}_\nu(\Theta_\ell \mathbf{1}_C \mathbf{1}_{A_\ell} \mathbf{1}_{V_m} \mathbf{1}_{J(d,e,y)}) = \mathbb{E}_\nu(F(\pi_{y,d}(\Delta)) \mathbf{1}_D \mathbf{1}_{E_+} \mathbf{1}_{E_-}). \tag{16}$$

Now observe that, with respect to $\mathbb{P}_\nu$, $\mathcal{H}^+_{d,e,y}$ and $\mathcal{H}^-_{d,e,y}$ are independent. To see this, use the invariance of $\mathbb{P}_\nu$ with respect to the shift $\pi_{y,d}$, and observe that, for $d = 0$ and $y = 0$, the property is a direct consequence of the independence properties of the initial distribution of particle paths. Thus, conditioning by $\mathcal{H}^-_{d,e,y}$, and using the invariance of $\mathbb{P}_\nu$ with respect to the shift $\pi_{y,d}$, we obtain that

$$\mathbb{E}_\nu(F(\pi_{y,d}(\Delta)) \mathbf{1}_D \mathbf{1}_{E_+} \mathbf{1}_{E_-}) = \mathbb{E}_\nu(\gamma((r'_{s+d} - y)_{-d \le s \le 0}, -y) \mathbf{1}_D \mathbf{1}_{E_-}), \tag{17}$$

where

$$\gamma(q) := \frac{\xi'(F, e - 1/2^m, e, q, -y)}{\chi'(e - 1/2^m, e, q, -y)}.$$

To conclude the proof of (13), we repeat the above argument, starting with $\Lambda_{\ell,m}$ instead of Θ_ℓ. Using the fact that, on $E_- \cap E_+$, one has that r_s and r'_s coincide for all $s \in [0, d]$, we obtain that

$$\mathbb{E}_\nu(\Lambda_{\ell,m} \mathbf{1}_C \mathbf{1}_{A_\ell} \mathbf{1}_{V_m} \mathbf{1}_{J(d,e,y)}) = \mathbb{E}_\nu(\gamma((r'_{s+d} - y)_{-d \le s \le 0}, -y) \mathbf{1}_D \mathbf{1}_{E_-}). \tag{18}$$

Combining (16), (17) and (18) yields (14) for any d, e, y, from which we deduce the validity of (13).

Now note that, as for (12), we have that

$$|\mathbb{E}_\nu(\Lambda_{\ell,m} \mathbf{1}_C) - \mathbb{E}_\nu(\Lambda_{\ell,m} \mathbf{1}_C \mathbf{1}_{A_\ell} \mathbf{1}_{V_m})| \le M(\mathbb{P}_\nu(A^c_\ell) + \mathbb{P}_\nu(V^c_m)). \tag{19}$$

Our next claim is that, for fixed ℓ,

$$\lim_{m\to+\infty} \mathbb{E}_\nu(\Lambda_{\ell,m}\mathbf{1}_C) = \mathbb{E}_\nu(\Gamma_\ell \mathbf{1}_C), \tag{20}$$

with

$$\Gamma_\ell := \xi'(F, (r_{s+T_k^{(\ell)}} - \mathfrak{r}_k)_{-T_k^{(\ell)}\le s\le 0}, -\mathfrak{r}_k).$$

To see this, note that, for fixed q and F, one has that, for all c,

$$\lim_{a,b\to c} \xi'(F, a, b, q, x) = \xi'(F, q, x). \tag{21}$$

Indeed, given c, one has that, $\mathbb{P}_\nu-$a.s., $B_0 = B_0(a, b)$ for all a, b close enough to c, so (21) is a consequence of the dominated convergence theorem. Now one has that $\mathbb{P}_\nu-$a.s., $\lim_{m\to+\infty} a_m = \lim_{m\to+\infty} b_m = U_k$, so that (20) is a consequence of (21) and the dominated convergence theorem.

We now claim that

$$\lim_{\ell\to+\infty} \mathbb{E}_\nu(\Gamma_\ell \mathbf{1}_C) = \mathbb{E}_\nu\left(\xi(F, (r_{s+T_k} - \mathfrak{r}_k)_{-T_k\le s\le 0}, -\mathfrak{r}_k)\mathbf{1}_C\right). \tag{22}$$

To see this, consider a path $q = (q_s)_{t\le s\le 0}$ satisfying the requirements listed in the definition of ξ', and remember the definition of t_0. For $t_0 \le w \le 0$, define $q^{(w)}$ as the path defined for $t - w \le s \le 0$ by $q_s^{(w)} := q_{s+w}$. One then has that

$$\lim_{w\to t_0} \xi'(F, q^{(w)}, x) = \xi(F, q^{(t_0)}, x). \tag{23}$$

To prove (23), we note that, $\mathbb{P}_\nu-$a.s., for all w sufficiently close to t_0, $\mathbf{1}_{G'(q^{(w)},x)} = \mathbf{1}_{G(q^{(t_0)},x)}$. This would be immediate if B_0 consisted in a finite number of particle paths. Using the a.s. uniform convergence of approximations of the dynamics involving a finite number of particles over finite time intervals, this is indeed a.s. true with respect to $\mathbb{P}_\nu$, and (23) follows by dominated convergence. To prove (22), note that, $\mathbb{P}_\nu-$a.s., for sufficiently large ℓ, the path $(r_{s+T_k^{(\ell)}} - \mathfrak{r} - k)_{-T_k^{(\ell)}\le s\le 0}$ satisfies the assumptions of (23), so that (22) follows by dominated convergence.

Combining (10), (11), (12), (13), (19), (20) and (22), we see that, choosing ℓ large enough, then m large enough, we can make the difference

$$|\mathbb{E}_\nu(F(\pi_{\mathfrak{r}_k,T_k}(B_{T_k}))\mathbf{1}_C) - \mathbb{E}_\nu\left(\xi(F, (r_{s+T_k} - \mathfrak{r}_k)_{-T_k\le s\le 0}, -\mathfrak{r}_k)\mathbf{1}_C\right)|$$

as small as we want. This proves (9).

□

4. Estimates on the renewal structure

This section is devoted to the proof of Proposition 10. To control the tail of the regeneration time κ_1, we use a sequence of stopping times which can be viewed as successive attempts to produce an $\alpha-$separation time. [Extend the discussion]

We first introduce the following refinement of the notion of backward sub-α time: given $0 \le s < t$, we say that t is an $(s, \alpha)-$*crossing time* if there exists $k \in \{1, 2, \ldots\}$ such that $r_v < r_s + k + \alpha(v - s)$ for all $v \in [s, t[$ and

$r_t \geq r_s + k + \alpha(t-s)$. Note that if s is a backward sub-α time and if t is an $(s,\alpha)-$crossing time, then t is also a backward sub-α time.

We now define by induction the sequence of stopping times on which our estimates on the renewal structure are based. Besides α, the definition involves two integer parameters $\mathscr{C} \geq 1$ and $L \geq 1$. Let $D_0 := 0$ and $\Upsilon_0 := \emptyset$. For $n \geq 1$, assume that the random variables D_{n-1}, Υ_{n-1} have already been defined, and let S'_n be the infimum of the $t > D_{n-1}$ such that

- t is a backward sub-α time;
- $\Upsilon_{n-1} \subset R_t$;
- B_t contains at least $\mathscr{C}$ particles (W,u) such that $W_t = r_t$.

Then define S_n as the infimum of the $t > S'_n$ such that

- t is a backward sub-α time;
- $]S'_n, t[$ contains a number of $(S'_n,\alpha)-$crossing times at least equal to L;
- B_t contains at least $\mathscr{C}$ particles (W,u) such that $W_t = r_t$.

We use the notation (W^{*n}, u^{*n}) for the particle that makes the front jump at time S_n, and define the subset $R^*_{S_n} := R_{S_n} \setminus \{(W^{*n}, u^{*n})\}$. If S_n is a backward super-α time, then $\Upsilon_n := \emptyset$ and D_n is defined as the infimum of the $t > S_n$ such that *a least one* of the following five conditions holds:

(1) $r_t < r_{S_n} + \lfloor \alpha(t-S_n) \rfloor$
(2) $t \leq S_n + \alpha^{-1}$ and there is no $(W,u) \in B_{S_n}$ such that $W_{S_n} = r_{S_n}$ and W remains at r_{S_n} during $[S_n, S_n+t]$,
(3) $W_t > r_{S_n} - 1 + \alpha(t-S_n)$ for some $(W,u) \in R^*_{S_n}$,
(4) $t \leq S_n + \alpha^{-1}$ and $W^{*n}_t \neq r_{S_n}$,
(5) $t > S_n + \alpha^{-1}$ and $W^{*n}_t > r_{S_n} - 1 + \alpha(t-S_n)$,

Note that (1) and (2) detect the potential failure of S_n to be a forward super-α time, while (3)-(4)-(5) detect the potential failure of S_n to be a forward sub-α time.

On the other hand, if S_n is not a backward super-α time, consider the set of particle paths $(W,u) \in B_{S_n}$ such that there exists $t < S_n$ for which $W_t < r_{S_n} - \alpha(S_n - t)$. Among this set, consider the pair $(W^{(n)}, u^{(n)})$ such that (W_{S_n}, u) is the smallest with respect to the lexicographical order[9], and define $\Upsilon_n := \{(W^{(n)}, u^{(n)})\}$ and $D_n := S_n$.

The rest of this section is devoted to proving estimates on the random variables we have just introduced. The main result from [6] that is needed to obtain these estimates is the following.

Proposition 12. *There exists a constant $C_2(\rho) > 0$ such that, for all $K > 0$, there exists a constant c_1, depending on ρ and K, such that, for every $t > 0$,*

$$\mathbb{P}_\nu(r_t \leq C_2(\rho)t) \leq c_1 t^{-K}.$$

[9]Remember that (x_1, u_1) is smaller than (x_2, u_2) with respect to the lexicographical order if $x_1 < x_2$, or $x_1 = x_2$ and $u_1 < u_2$.

Note that, strictly speaking, Proposition 12 does not appear in [6], where a slightly different kind of initial condition is considered. It is however a direct consequence of the results in [6]. See Appendix B for a proof.

In the following proofs, an additional parameter β is used. Here are the assumptions on the various parameters that we assume to hold throughout the sequel:

$$\begin{cases} 0 < \alpha < \beta < (1/3)C_2(\rho/4) \\ 2(\cosh(\theta) - 1) < \alpha\theta \end{cases} \tag{24}$$

Such a choice of parameters is always possible by choosing first α and β, then θ close enough to zero, using the fact that $\cosh(\theta) = 1 + o(\theta)$ when θ goes to zero. In addition to (24), we shall have to assume that $\mathscr{C}$ is large enough, and also that L is large enough (depending on $\mathscr{C}$). These assumptions on $\mathscr{C}$ and/or L will always be made explicit in the sequel.

We now explain our convention for constants: what we call constants in the rest of this section may depend on $\rho, \alpha, \beta, \theta$, but, unless otherwise mentioned, not on $\mathscr{C}$ or L. As a rule, we use $c_1, c_2, \ldots$ to denote constants whose range of validity extends throughout the section, and are used in the statement of Propositions or Lemmas. On the other hand, we use $d_1, d_2, \ldots$ to denote constants that are purely local to proofs.

4.1. **Hitting of a straight line by random walks.** We introduce the notation

$$\mu := \alpha\theta - 2(\cosh\theta - 1). \tag{25}$$

Thanks to the assumption (24), we have that $\mu > 0$.

Lemma 6. *Let $(\zeta_s)_{s\geq 0}$ be a continuous-time simple symmetric random walk on $\mathbb{Z}$ with total jump rate 2 starting at $x \leq 0$, with respect to a probability measure P_x. Then for all $t \geq 0$*

$$P_x\left(\exists s \geq t;\ \zeta_s \geq \alpha s\right) \leq e^{\theta x} e^{-\mu t}.$$

Proof. For all $s \geq 0$, let $M_s := e^{\theta\zeta_s - 2(\cosh(\theta)-1)s}$, and $T := \inf\{s \geq t;\ \zeta_s \geq \alpha s\}$. Then $(M_s)_{s\geq 0}$ is a càdlàg martingale, and T is a stopping time, so that, for all finite $K > 0$, one has

$$E_x(M_{T\wedge K}) = E_x(M_0) = e^{\theta x}. \tag{26}$$

Now we have that $\liminf_{K\to+\infty} M_{T\wedge K} \geq M_T \mathbf{1}(T < +\infty)$, so that, by Fatou's lemma and (26),

$$E_x(M_T \mathbf{1}(T < +\infty)) \leq e^{\theta x}. \tag{27}$$

Now, by definition of T, one has that, on $\{T < +\infty\}$,

$$M_T \geq e^{\theta\alpha T - 2(\cosh(\theta)-1)T} = e^{\mu T} \geq e^{\mu t}, \tag{28}$$

where the last inequality comes from the fact that $\mu > 0$ and $T \geq t$. The result now follows from combining (27) and (28). $\square$

Let us now define the map ϕ_θ on $\mathbb{S}_\theta$ by

$$\phi_\theta(w) := \sum_{x \leq 0} \sum_{u \in w(x)} e^{\theta x}.$$

Lemma 7. *For all $K > 0$, there exists $g(K) > 0$ such that, for all $w \in \mathbb{S}_\theta$ such that $\phi_\theta(w) \leq K$, the following bound holds:*

$$\mathbb{P}_w\left(\forall (W, u) \text{ such that } W_0 \leq 0, \forall t > 0, \text{one has } W_t < \alpha t\right) \geq g(K).$$

Proof. We re-use the notations used in the statement of Lemma 6. Let us choose $\theta' > \theta$ such that $\mu' := \alpha\theta' - 2(\cosh(\theta') - 1) > 0$ (this is possible since $\mu > 0$), and observe that Lemma 6 holds with θ', μ' instead of θ, μ. We deduce that, for all $x < 0$, we have

$$\mathfrak{p}(x) \leq e^{\theta' x}, \tag{29}$$

where

$$\mathfrak{p}(x) := P_x\left(\exists s > 0;\ \zeta_s \geq \alpha s\right).$$

Moreover, we must have $\mathfrak{p}(0) < 1$, for otherwise we could prove that

$$P_0\left(\limsup_{t \to +\infty} \zeta_t / t \geq \alpha\right) = 1,$$

which would contradict the law of large numbers. Since all the random walks in our model evolve independently, we can rewrite the probability we want to bound from below as

$$\prod_{x \leq 0} (1 - \mathfrak{p}(x))^{|w(x)|}.$$

Now the inequality $\phi_\theta(w) \leq K$ implies that, for all $x \leq 0$, one has that

$$|w(x)| \leq e^{-\theta x} K. \tag{30}$$

As a consequence, we have the bound

$$\prod_{x \leq 0} (1 - \mathfrak{p}(x))^{|w(x)|} \geq \left(\prod_{x \leq 0} (1 - \mathfrak{p}(x))^{e^{-\theta x} K}\right).$$

In view of (29) and of the fact that $\theta' > \theta$, we have that $\sum_{x \leq 0} e^{-\theta x} e^{\theta' x} < +\infty$, so the r.h.s. of the above inequality is > 0, and depends only on K. □

Lemma 8. *For all $w \in \mathbb{S}_\theta$, and all $t \geq 0$, the following bound holds:*

$$\mathbb{P}_w\left(\exists (W, u)\ \exists s \geq t,\ W_0 \leq 0,\ W_s \geq \alpha s\right) \leq \phi_\theta(w) e^{-\mu t}.$$

Proof. Consequence of Lemma 6 and of the union bound. □

4.2. **Ballisticity estimates.** Remember that the initial distribution of particles ν on $\mathbb{S}_\theta$ is defined through an i.i.d. family $N = (N_x)_{x\in\mathbb{Z}}$ of Poisson processes on $[0,1]$, with intensity ρ. Now consider the configuration N^+ obtained from N by removing all the particles at the left of the origin, i.e. $N_x^+ := N_x$ if $x \geq 0$, $N_x^+ := 0$ if $x < 0$. Then define ν_+ as the distribution of $N^+ = (N_x^+)_{x\in\mathbb{Z}}$ on $\mathbb{S}_\theta$. Define also $\nu_{\mathscr{C},+}$ as the distribution of N^+ conditioned upon $|N_0^+| \geq \mathscr{C}$.

Our first task is to extend the ballisticity estimate given in Proposition 12 to the case where the initial distribution is $\nu_{\mathscr{C},+}$ instead of ν.

Proposition 13. *There exist constants $c_2, c_3 > 0$, with c_2 depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t \leq \beta t) \leq c_2 t^{-c_3\cdot\mathscr{C}}.$$

To prove Proposition 13, we need to introduce additional definitions and results. First, we define a modified version of the infection dynamics. In the modified version, the front is at zero at time zero and, after time zero, the dynamics is defined as the original one, with the difference that the front is never allowed to go below level zero (i.e. a jump that would make the front go below zero for the original dynamics has no effect on the front in the modified dynamics). We call $(\hat{r}_s)_{s\geq 0}$ the trajectory of the corresponding front.

Lemma 9. *Given $\psi_1, \psi_2 \in \Omega$, the fact that $\psi_1 \subset \psi_2$ implies that $\hat{r}_t(\psi_1) \leq \hat{r}_t(\psi_2)$ for all t such that $\hat{r}_t(\psi_1)$ and $\hat{r}_t(\psi_2)$ are distinct from †.*

Proof. Observe that, by definition, $\hat{r}_0(\psi_1) \leq \hat{r}_0(\psi_2)$. Moreover, due to our assumption on Ω, the two fronts cannot jump simultaneously unless they are at the same location prior to the jump. Since only nearest-neighbor steps can be performed, we see that the trajectories of our fronts must meet before crossing each other. As a consequence, to establish the conclusion of the lemma, it is enough to prove that, whenever the fronts are at the same location, the next step performed by any of the fronts, say at time t, is such that $\hat{r}_t(\psi_1) \leq \hat{r}_t(\psi_2)$. Assume that at time s one has $\hat{r}_s(\psi_1) = \hat{r}_s(\psi_2)$, and let t denote the next time at which any of the fronts jumps. Since $\psi_1 \subset \psi_2$, we see that if t is upward for ψ_1, it is also upward for ψ_2. On the other hand, if t is downward for ψ_2, by the fact that $\psi_1 \subset \psi_2$, it must also be downward for ψ_1. □

Lemma 10. *One has that $r_t \leq \hat{r}_t$ for all t such that both random variables are distinct from †.*

Proof. Arguing as in the proof of Lemma 9, all we have to prove is that, whenever the fronts are at the same location, the next step performed by any of the fronts, say at time t, is such that $r_t \leq \hat{r}_t$. Assume that at time s one has $\hat{r}_s(\psi_1) = \hat{r}_s(\psi_2)$, and consider the next time t at which any of the fronts jumps. By definition of the modified dynamics, both fronts must be at a location ≥ 0. Moreover, if t is upward for r, it must also be upward for

$\hat{r}$. On the other hand, if t is downward for $\hat{r}$ (in which case both fronts are at a location ≥ 1 at time s), it is also downward for r. $\square$

We now define a map $\mathscr{T} : \Omega \to \Omega$. Consider a pair (W, u). If $W_0 \geq 0$, then we let $W^{\times} := W$. On the other hand, if $W_0 < 0$, consider $\tau := \inf\{s > 0;\ W_s = 0\}$, and let $W^{\times}_s := -W_s$ on $]-\infty, \tau[$ and $W^{\times}_s := W_s$ on $[\tau, +\infty[$. Now we let

$$\mathscr{T}(\psi) := \{(W^{\times}, u);\ (W, u) \in \psi\}.$$

Lemma 11. *One has that $\hat{r}_t \leq \hat{r}_t \circ \mathscr{T}$ for all t such that both random variables are distinct from $\dagger$.*

Proof. First note that both fronts are at zero at time zero. Moreover, they cannot jump at the same time unless they are at the same location prior to the jump. Indeed, by definition of Ω, two distinct random walks cannot jump at the same time, and, moreover, since fronts for the modified dynamics always lie above zero, $W^{\times}$ and W can cause a jump simultaneously only from a location ≥ 0, in which case $W^{\times}$ coincides with W. As a consequence, as in the proof of Lemma 9, it is enough to prove that, if $\hat{r}_s = \hat{r}_s \circ \mathscr{T}$, and if t denotes the first time after s at which any of the fronts jumps, one has $\hat{r}_t \leq \hat{r}_t \circ \mathscr{T}$. Assume that t is upward for $\hat{r}$. Then by definition the corresponding random walk W is such that $W_s \geq 0$, so that $W^{\times}$ coincides with W on $[s, +\infty[$, and so t is also upward for $\hat{r} \circ \mathscr{T}$. On the other hand, if t is downward for $\hat{r} \circ \mathscr{T}$, then the common location of the fronts has to be ≥ 1, and there must be at least one $(W, u) \in \Psi$ such that $W_s = W^{\times}_s = \hat{r}_s$. In fact, there cannot be more than one such (W, u), since otherwise t could not be downward for $\hat{r} \circ \mathscr{T}$. As a consequence, there is only one such (W, u), and t must also be downward for $\hat{r}$. $\square$

Lemma 12. *There exist constants $c_4, c_5 > 0$, with c_4 depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C}},+}\left(\inf_{s \in [(2/3)t, t]} r_s \leq 0\right) \leq c_4 t^{-c_5 \cdot \mathscr{C}}.$$

Proof of Lemma 12. Define $t_0 := t/3$. Then fix a real number $0 < v < \sqrt{2/3}$, and define $y_{t_0} := \lfloor v(t_0 \log t_0)^{1/2} \rfloor$ and $\varepsilon(t_0) := \frac{t_0^{-v^2/4}}{v(\log t_0)^{1/2}}$. Let $(\zeta_s)_{s \geq 0}$ denote a continuous-time simple symmetric random walk with total jump rate 2 starting at site x, with respect to a probability measure P_x. By a standard local limit theorem[10], we have that, as t goes to infinity,

$$P_0(\zeta_{t_0} \leq -y_{t_0}) \sim d_1 \varepsilon(t_0), \tag{31}$$

[10] See e.g. [5] XVI.6 for the case of a discrete-time random walk. The continuous-time follows easily, by controlling the fluctuations of the number of steps performed by the walk.

where d_1 is a positive constant. Using the reflection principle, we deduce that there exists a strictly positive constant d_2 such that, for large t,

$$P_0\left(\inf_{s\in[0,t_0]}\zeta_s \le -y_{t_0}\right) \le d_2\varepsilon(t_0).$$

Now let $\mathfrak{z}_s$ denote the supremum of the positions at time s of the particle paths that are located at the origin at time zero, and let C_1 denote the event that $\mathfrak{z}_s > -y_{t_0}$ for all $s \in [0, t_0]$. Since the number of these particle paths is at least $\mathscr{C}$, we deduce that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_1^c) \le d_2^{\mathscr{C}}\,\varepsilon(t_0)^{\mathscr{C}}. \tag{32}$$

Now let $z_{t_0} := \lfloor \varepsilon(t_0)^{-3} \rfloor$, and consider the number $\mathfrak{N}$ of particle paths whose location at time zero lies in the interval $[0, z_{t_0}]$. Let C_2 denote the event that $\mathfrak{N}$ is at least equal to $\rho z_{t_0}/2$. By standard large deviations bounds for Poisson random variables (see e.g. [4]), we have that, for all large t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2^c) \le \exp(-d_3 z_{t_0}), \tag{33}$$

for some strictly positive constant d_3. Now define $\mathfrak{N}'$ to be the number of particle paths that:

(a) start at an initial position in $[0, z_{t_0}]$;
(b) hit $-y_{t_0}$ during the time-interval $[0, t_0]$;
(c) hit 0 after having hit $-y_{t_0}$ and before time $2t_0$.

For a particle starting in $[0, z_{t_0}]$, the probability to hit $-y_{t_0}$ during $[0, t_0]$ is larger than or equal to $q_{t_0} := P_{z_{t_0}}\left(\inf_{s\in[0,t_0]}\zeta_s \le -y_{t_0}\right)$. Moreover, using the symmetry of the walk, we see that, starting from $-y_{t_0}$, the probability for the walk to hit 0 before time t_0 is larger than or equal to q_{t_0}. As a consequence, given $\mathfrak{N}$, the distribution of $\mathfrak{N}'$ stochastically dominates a binomial distribution with parameters $\mathfrak{N}$ and $q_{t_0}^2$. Moreover, as t goes to infinity, $z_{t_0} = o(t_0^{1/2})$ and $y_{t_0} z_{t_0} = o(t_0)$ due to the fact that $v^2 < 2/3$, so that (31) is also valid for $P_{z_{t_0}}(\zeta_{t_0} \le -y_{t_0})$, from which we deduce that, for large t,

$$q_{t_0} \ge d_4\varepsilon(t_0),$$

where d_4 is a strictly positive constant. Define C_3 to be the event that $\mathfrak{N}' \ge \mathfrak{N} q_{t_0}^2/2$. Using standard (see e.g. [14]) large deviations bounds for binomial random variables, we deduce from the preceding discussion that for all large enough t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2 \cap C_3^c) \le \exp(-d_5\varepsilon(t_0)^{-1}), \tag{34}$$

for some strictly positive constant d_5. Now consider the intervals of the form $[2t_0 + k, 2t_0 + k + 1]$, for $0 \le k \le \lfloor t_0 \rfloor$. Then consider a random walk satisfying conditions (a) to (c) above, stopped at the first time it hits the origin after having hitted $-y_{t_0}$; by definition, this time is $\le 2t_0$. By symmetry, the probability that this walk is above 0 at time $2t_0 + k$ is $\ge 1/2$, and the probability that it then remains above 0 during the whole interval $[2t_0 + k, 2t_0 + k + 1]$ is larger than some strictly positive constant d_6. As a consequence, for each of the intervals we consider, the probability that none of the random walks that satisfy (a) to (c) lies above zero for the duration of the interval, is, conditional upon $\mathfrak{N}'$, bounded above by $(1-d_6)^{\mathfrak{N}'}$. Now define C_4 as the event that, for every $s \in [2t, 3t]$, there exists at least one random walk satisfying (a) to (c) whose position at time s is ≥ 0. Using the union bound over all the intervals, whose total number is $\leq t_0+1$, we obtain that for all large enough t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2 \cap C_3 \cap C_4^c) \leq (t_0+1)\exp\left(-d_7\varepsilon(t_0)^{-1}\right). \tag{35}$$

We now observe that, on $C_1 \cap C_2 \cap C_3 \cap C_4$, one must have $r_s \geq 0$ for all $s \in [2t_0, 3t_0] = [(2/3)t, t]$. Indeed, we know that the front always lies above the maximum position of the particles initially at zero. By C_1, the front lies above $-y_{t_0}$ during the interval $[0, t_0]$. As a consequence, any particle path satisfying (a) and (b) must hit the front before time t_0. For that reason, on C_4, the front lies above 0 during the interval $[2t_0, 3t_0]$. Now using (32), (33), (34), (35), we have that, for large enough t, the probability of the complement of $C_1 \cap C_2 \cap C_3 \cap C_4$ is bounded above by

$$d_2^{\mathscr{C}}\,\varepsilon(t_0)^{\mathscr{C}} + \exp(-d_3 z_{t_0}) + \exp(-d_5\varepsilon(t_0)^{-1}) + (t_0+1)\exp\left(-d_7\varepsilon(t_0)^{-1}\right),$$

and the first term dominates the others when t_0 is large. $\square$

Proof of Proposition 13. Denote $t_1 := (2/3)t$. Now, for $s \in [t_1, t]$, define $r_s^{(1)} := \hat{r}_{s-t_1} \circ \pi_{0,t_1}$, and let $C := \{r_s \geq 0 \text{ for all } s \in [t_1, t]\}$. Our first claim is that:

$$\text{on the event } C\text{, one has that } r_s^{(1)} \leq r_s \text{ for all } s \in [t_1, t]. \tag{36}$$

Indeed, on C, one has that $r_{t_1}^{(1)} \leq r_{t_1}$ since by definition $r_{t_1}^{(1)} = 0$. We argue as in the proofs of Lemmas 10, and assume that $s_0 \in [t_1, t]$ is such that $r_{s_0}^{(1)} = r_{s_0}$. Since, on C, the jumps that affect both fronts between time s_0 and time t are exactly the same, one must have that $r_s^{(1)} = r_s$ for all $s \in [s_0, t]$. This proves the claim.

We now define three distributions on $\mathbb{S}_\theta$, in addition to ν_+ and $\nu_{\mathscr{C},+}$ which were defined at the beginning of Section 4.2. Let $(N_x^{(1)})_{x\in\mathbb{Z}}$ denote an independent family of Poisson processes on $[0,1]$, where, for all $x \in \mathbb{Z}$, the rate of $N_x^{(1)}$ is equal to $\rho p_{t_1}(x, \mathbb{N})$, with $p_{t_1}(x,\mathbb{N}) := \sum_{y\in\mathbb{N}} p_{t_1}(x,y)$. Define also $(N_x^{(2)})_{x\in\mathbb{Z}}$ to be an independent family of Poisson processes on $[0,1]$, where the rate of $N_x^{(2)}$ is $\rho/2$ for $x \geq 1$, $\rho/4$ for $x = 0$, and 0 for $x < 0$. Denote ν_1 and ν_2 the distributions induced by $(N_x^{(1)})_{x\in\mathbb{Z}}$ and $(N_x^{(2)})_{x\in\mathbb{Z}}$ on $\mathbb{S}_\theta$. Finally, define ν_3 exactly as ν, with the difference that the constant value of the rate is equal to $\rho/4$ instead of ρ.

We now claim that

$$r_{t_0}(\mathbb{P}_{\nu_3}) \prec r_t^{(1)}(\mathbb{P}_{\nu_{\mathscr{C},+}}), \tag{37}$$

where $\prec$ denotes stochastic domination between probability measures on $\mathbb{R}$. We also use stochastic domination on $\mathbb{S}_\theta$ equipped with the order relation induced by inclusion between sets, i.e. $w_1 \leqslant w_2$ when $w_1(x) \subset w_2(x)$ for all $x \in \mathbb{Z}$.

To begin with, one checks that ν_+ is stochastically dominated by $\nu_{\mathscr{C},+}$. As a consequence, the distribution of X_{t_1} with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$ stochastically dominates ν_1. Using Lemma 9, we deduce that

$$r_t^{(1)}(\mathbb{P}_{\nu_+}) \prec r_t^{(1)}(\mathbb{P}_{\nu_{\mathscr{C},+}}). \tag{38}$$

Then, observe that ν_1 is the distribution of X_{t_1} with respect to $\mathbb{P}_{\nu_+}$, so that

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_1}) = r_t^{(1)}(\mathbb{P}_{\nu_+}). \tag{39}$$

Then, ν_2 is stochastically dominated by ν_1, since, for all $x \geq 0$, we have $p_{t_1}(x, \mathbb{N}) \geq 1/2$. By Lemma 9, we deduce that

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_2}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_1}). \tag{40}$$

We also have that the image of the probability measure $\mathbb{P}_{\nu_3}$ by the map $\mathscr{T}$ is $\mathbb{P}_{\nu_2}$, so that, by Lemma 11,

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_3}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_2}). \tag{41}$$

Using Lemma 10, we finally deduce that

$$r_{t_0}(\mathbb{P}_{\nu_3}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_3}). \tag{42}$$

Putting together (38), (39), (40), (41), (42), we see that (37) is proved.

We are now ready to prove the conclusion of the proposition. By (36), we have that, on C, $r_t^{(1)} \leq r_t$, so that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t \leq \beta t) \leq \mathbb{P}_{\nu_{\mathscr{C},+}}(r_t^{(1)} \leq \beta t) + \mathbb{P}_{\nu_{\mathscr{C},+}}(C^c).$$

Thanks to Lemma 12,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C^c) \leq a_1 t^{-a_2 \cdot \mathscr{C}}.$$

On the other hand, by (37), the distribution of $r_t^{(1)}$ with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$ stochastically dominates that of r_{t_0} with respect to $\mathbb{P}_{\nu_3}$. Using Proposition 12 with $K := \mathscr{C}$, and the fact that β is chosen such that $\beta < (1/3)C_2(\rho/4)$, we have that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t^{(1)} \leq \beta t) \leq \mathbb{P}_{\nu_3}(r_{t_0} \leq \beta t) \leq \mathbb{P}_{\nu_3}(r_{t_0} \leq C_2(\rho/4)t_0) \leq c_1 t_0^{-\mathscr{C}}.$$

□

We now derive some consequences of Proposition 13 (remember that the event G is defined in (6)).

Lemma 13. *For all large enough $\mathscr{C}$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap \{t = 0 \textit{ is a forward super } \alpha\text{–}\textit{time for } B_0\}) > 0.$$

We shall use the following lemma to prove the above result.

Lemma 14. *Given $\psi_1, \psi_2 \in \Omega$, the fact that $\psi_1 \subset \psi_2$ implies that $r_t(\psi_1) \leq r_t(\psi_2)$ for all t such that $r_t(\psi_1)$ and $r_t(\psi_2)$ are distinct from †.*

Proof. The proof is completely similar to the proof of Lemma 9. □

Proof of Lemma 13. Let us first note that $\mathbb{P}_{\nu_{\mathscr{C},+}}(G) > 0$, using Lemma 7 and the symmetry of the distribution of our random walks.

For $n \geq 1$, define $A_{n,1} := \{r_n \geq \beta n\}$ and let $A_{n,2}$ denote the event that the particle at the front at time n with the smallest label remains above level $\alpha(n+1)$ during the time-interval $[n, n+1]$. For $k \geq 1$, introduce

$$A^{(k)} := \{r_t \geq \alpha t \text{ for all } t \geq k\},$$

and note that

$$\bigcap_{n \geq k} (A_{n,1} \cap A_{n,2}) \subset A^{(k)}.$$

By Proposition 13, one has that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(A^c_{n,1}) \leq c_2 n^{-c_3 \cdot \mathscr{C}}.$$

Then, using e.g. a variance bound for the random walk, one has that, for large enough n,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(A_{n,1} \cap A^c_{n,2}) \leq d_1 n^{-2},$$

for some constant $d_1 > 0$. As a consequence, we have that, for all large enough $\mathscr{C}$, there exists $k \geq 1$ such that $\sum_{n \geq k} \mathbb{P}_{\nu_{\mathscr{C},+}}(A^c_{n,1} \cup A^c_{n,2}) < \mathbb{P}_{\nu_{\mathscr{C},+}}(G)$. We thus have that $\mathbb{P}_{\nu_{\mathscr{C},+}}\left(G \cap \bigcap_{n \geq k}(A_{n,1} \cap A_{n,2})\right) > 0$, whence the fact that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap A^{(k)}) > 0.$$

Let U_0 denote the largest label of a particle path (W, u) such that $W_0 = 0$ (if there is no such particle path, we set $U_0 := 0$). We deduce from the fact that $\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap A^{(k)}) > 0$ the existence of a $u_0 < 1$ such that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\left(G \cap A^{(k)} \cap \{U_0 \leq u_0\}\right) > 0. \tag{43}$$

Now let Ψ_0 denote the subset of Ψ obtained by removing all particle paths (W, u) such that $W_0 = 0$ and $u > u_0$. We deduce from (43) that

$$\mathbb{P}_{\nu_+}\left(G(\Psi_0) \cap A^{(k)}(\Psi_0) \cap \{|X_0(\Psi_0)| \geq \mathscr{C}\}\right) > 0,$$

with the convention that, for $D \in \mathcal{F}$, $D(\Psi_0)$ denotes the event that $\mathbf{1}_D(\Psi_0) = 1$. Now introduce the event A' that

- there exists a particle path (W, u) such that $u > u_0$ and $W_s = 0$ for $s \in [0, \alpha^{-1}]$, and another particle path (W, u) such that $u > u_0$, $W_s \geq \lfloor \alpha s \rfloor$ for all $s \in [0, k]$;
- every particle path (W, u) such that $W_0 = 0$ and $u > u_0$ satisfies $W_s > \alpha s$ for $s < 0$.

One clearly has that $\mathbb{P}_{\nu_+}(A') > 0$, and that the two events A' and $G(\Psi_0) \cap A^{(k)}(\Psi_0) \cap \{|X_0(\Psi_0)| \geq \mathscr{C}\}$ are independent with respect to $\mathbb{P}_{\nu_+}$, and (using Lemma 14), that $A^{(k)}(\Psi_0) \cap A'$ implies that 0 is a forward super $\alpha-$time for B_0. □

Lemma 15. *There exist strictly positive constants c_6, c_7, with c_6 depending on $\mathscr{C}$, such that*

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(\exists s \geq t;\ r_s \leq \alpha s) \leq c_6 t^{-c_7 \mathscr{C}}.$$

Proof. We re-use the notations of the proof of Lemma 13. One has the following inclusion for $k = \lfloor t \rfloor$:

$$\{\exists s \geq t;\ r_s \leq \alpha s\} \subset (A^{(k)})^c \subset \bigcup_{n \geq k} (A^c_{n,1} \cup A^c_{n,2}).$$

By the union bound, we deduce that

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(\exists s \geq t;\ r_s \leq \alpha s) \leq \sum_{n \geq \lfloor t \rfloor} \left(\mathbb{P}_{\nu_{\mathscr{C}},+}(A^c_{n,1}) + \mathbb{P}_{\nu_{\mathscr{C}},+}(A_{n,1} \cap A^c_{n,2}) \right). \quad (44)$$

By Proposition 13, one has that

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(A^c_{n,1}) \leq c_2 n^{-c_3 \cdot \mathscr{C}}.$$

On the other hand, using e.g. a moment bound of order $\lceil c_3 \cdot \mathscr{C} \rceil$ for the random walk, one has that, for large enough n,

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(A_{n,1} \cap A^c_{n,2}) \leq d_1 n^{-c_3 \cdot \mathscr{C}},$$

for some constant $d_1 > 0$. The result now follows from (44). □

So far, we have considered estimates bounding the speed of propagation of the front from below. We now consider bounds from above. We use the following result from [6].

Proposition 14. *There exist a constant $C_1(\rho) > 0$ and a constant c_8, depending on ρ and $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(r_t \geq C_1(\rho) t) \leq c_8 \exp(-t).$$

As for Proposition 13, Proposition 14 does not appear as such in [6], due to the different choice of initial conditions. It is a rather direct consequence of Theorem 1 in [6]. See Appendix B for a precise derivation.

We now derive two useful consequences of Proposition 14.

Lemma 16. *Let $C'_1(\rho) := C_1(\rho) + 1$. There exist strictly positive constants c_9, c_{10}, with c_9 depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(\exists s \geq t;\ r_s \geq C'_1(\rho) s) \leq c_9 \exp(-c_{10} t).$$

Proof. Let A_1 denote the event that there exists $s \geq t$ for which $r_s \geq C'_1(\rho)s$, then let A_2 denote the event that, for the first such s, the particle located at the front with the lowest label remains above level $C_1(\rho)(s+1)$ during the time-interval $[s, s+1]$. We have that, on $A_1 \cap A_2$, $r_{\lceil s \rceil} \geq C_1(\rho)\lceil s \rceil$. As a consequence, using the union bound over all the possible values of $\lceil s \rceil$, and Proposition 14,

$$\mathbb{P}_{\nu_{\mathscr{C}},+}(A_1 \cap A_2) \leq \sum_{k=\lceil t \rceil}^{+\infty} c_8 \exp(-k).$$

On the other hand, using an exponential moment bound for the maximal displacement of a random walk during a time-interval of length 1, we see that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(A_1 \cap A_2^c) \leq d_1 \exp(-d_2 t),$$

for strictly positive constants d_1, d_2. □

Lemma 17. *There exist strictly positive constants c_{11}, c_{12}, with c_{11} depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(\exists s \leq t;\ r_s \geq C_1'(\rho)t) \leq c_{11} \exp(-c_{12} t).$$

Proof. Let A_1' denote the event that there exists $s \leq t$ for which $r_s \geq C_1'(\rho)t$, then let A_2' denote the event that, for the first such s, the particle located at the front with the lowest label remains above level $C_1(\rho)t$ during the time-interval $[s,t]$. We have that, on $A_1' \cap A_2'$, $r_t \geq C_1(\rho)t$. As a consequence, using Proposition 14,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(A_1' \cap A_2') \leq c_8 \exp(-t).$$

On the other hand, using a standard large deviations bound for the maximal displacement of a random walk during a time-interval of length t, we see that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(A_1' \cap {A_2'}^c) \leq d_1 \exp(-d_2 t),$$

for strictly positive constants d_1, d_2.

□

Finally, the following lemma shows how ballisticity results such as Proposition 13 can be used to bound from below the number of $\alpha-$crossing times.

Lemma 18. *Consider $t > 0$, and assume that $r_t \geq r_0 + \beta t$. Then there exist at least $\lfloor \frac{(\beta-\alpha)t}{2} \rfloor$ distinct $(0,\alpha)-$crossing times in $[0,t]$.*

Proof. Let $d := \lfloor \frac{(\beta-\alpha)t}{2} \rfloor$. Observe that, for all $x \in \{1, \ldots, d\}$, one has that $2x + \alpha t \leq \beta t$. The fact that $r_t \geq r_0 + \beta t$ implies that, for each such x, there exists a smallest $s \in]0,t]$ such that $r_s \geq r_0 + 2x + \alpha s$. Denote by s_x the corresponding s. By definition s_x is a $(0,\alpha)-$crossing time. Our next claim is that $s_x < s_y$ whenever $x < y$. Indeed, one must have $r_{s_x -} < r_0 + 2x + \alpha s_x$ by definition, and, since the maximum possible increment for the front at each step is $+1$, one has $r_{s_x} \leq r_0 + 2x + \alpha s_x + 1 < r_0 + 2(x+1) + \alpha s_x$. Since in addition $r_s < r_0 + 2x + \alpha s < r_0 + 2(x+1) + \alpha s$ for all $s \in [0, s_x[$, we must have $s_{x+1} > s_x$. □

4.3. **Conditional distribution of $\pi_{r_{S_n}, S_n}(B_{S_n})$.** We now describe the conditional distribution of $\pi_{r_{S_n}, S_n}(B_{S_n})$ given $\mathcal{G}^R_{S_n}$.

For $t \geq 0$, let Ξ_t denote the indicator function of the event that the number of particle paths in B_t that are at site r_t at time t is $\geq \mathscr{C}$. Consider $\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2)$, where $\mathfrak{q}_1$ is a finite list of the form

$$\mathfrak{q}_1 = \{(t_1, x_1, u_1), \ldots, (t_\ell, x_\ell, u_\ell)\},$$

with $t_1 < \cdots < t_\ell < 0$, $x_1, \ldots, x_\ell \in \mathbb{Z}_- = \{0, -1, -2, \ldots\}$, $u_1, \ldots, u_\ell \in [0, 1]$, while $\mathfrak{q}_2$ is a finite list of the form

$$\mathfrak{q}_2 = \{s_1, \ldots, s_m\},$$

with $s_1 < \cdots < s_m < 0$. Consider a càdlàg path $q = (q_s)_{t \le s \le 0}$ with values in $\mathbb{Z}$ and taking nearest-neighbor steps, such that $q_0 = 0$, $q_{0-} = -1$, and containing a finite number of jumps, and such that $t < \min(s_1, t_1)$, and $x_i \ge q_{t_i}$ for all $1 \le i \le \ell$. Now define $\mathfrak{G}(\mathfrak{q}, q)$ as the event that

- for all $1 \le i \le \ell$, for all $(W, u) \in B_0$ such that (W_{t_i}, u) is less than (x_i, u_i) with respect to the lexicographical order, one has $W_s \ge q_{t_i} - \alpha(t_i - s)$ for all $s < t_i$;
- for all $1 \le i \le m$, for all $(W, u) \in B_0$, one has $W_s \ge q_{s_i} - \alpha(s_i - s)$ for all $s < s_i$.

Finally, let

$$G(\mathfrak{q}, q, x) := \mathfrak{G}(\mathfrak{q}, q) \cap G(q, x),$$

where $G(q, x)$ has been defined in (5), and, for any bounded measurable map $F : \Omega \to \mathbb{R}$, define

$$\eta(F, \mathfrak{q}, q, x) := \mathbb{E}_\nu(F(B_0) | G(\mathfrak{q}, q, x) \cap \{\Xi_0 = 1\}). \tag{45}$$

Proposition 15. *For any bounded measurable map $F : \Omega \to \mathbb{R}$, and all $n \ge 1$, one has that, on $\{S_n < +\infty\}$,*

$$\mathbb{E}_\nu(F(\pi_{r_{S_n}, S_n}(B_{S_n})) | \mathcal{G}^R_{S_n}) = \eta(F, Q^{(n)}, (r_{s+S_n} - r_{S_n})_{-S_n \le s \le 0}, -r_{S_n}) \ a.s.,$$

where $Q^{(n)}$ is a $\mathcal{G}^R_{S_n}$−measurable random variable.

Proof of Proposition 15. Let

$$Q_1^{(n)} := \{(S_j - S_n, W^{(j)}_{S_j} - r_{S_n}, u^{(j)})\},$$

where j runs over those indices in $1 \le j \le n-1$ such that S_j is not a backward super-α time. (The notation $(W^{(j)}, u^{(j)})$ is defined at the beginning of Section 4.) Let also

$$Q_2^{(n)} := \{S_j - S_n;\ 1 \le j \le n-1,\ \ S_j \text{ is a backward super-}\alpha \text{ time}\},$$

and define

$$Q^{(n)} := (Q_1^{(n)}, Q_2^{(n)}).$$

Let us check that $Q^{(n)}$ is indeed measurable with respect to $\mathcal{G}^R_{S_n}$. We have already seen that the history of the front from time 0 to time S_n is indeed measurable with respect to $\mathcal{G}^R_{S_n}$. Given $v > 0$, let us say that t is a backward super-α time relative to R_v if, for any $(W, u) \in B_t \cap R_v$, and all $s < t$, one has $W_s \ge r_t - \alpha(t - s)$. Now we have that, given $1 \le j \le n-1$, it is equivalent for S_j to be a backward super-α time and to be a backward super-α time relative to R_{S_n}. Indeed, if S_j fails to be a backward super-α time, by definition one must have that $(W^{(j)}, u^{(j)}) \in R_{S'_{j+1}} \subset R_{S_n}$, so that it is possible to check whether S_j is a backward super-α time just by checking that every particle path (W, u) in $B_{S_j} \cap R_{S_n}$ is such that $W_s \ge$

$r_{S_j} - \alpha(S_j - s)$ for all $s < S_j$. Moreover, when S_j fails to be a backward super-α time, $(W^{(j)}, u^{(j)})$ is precisely the smallest particle path in $B_{S_j} \cap R_{S_n}$ (with respect to the lexicographical order) such that there exists $s < S_j$ for which $W_s < r_{S_j} - \alpha(S_j - s)$.

Consider $C \in \mathcal{G}^R_{S_n}$, and write

$$\mathbb{E}_\nu(F(\pi_{r_{S_n}, S_n}(B_{S_n}))\mathbf{1}_C) = \sum_{k \geq 1} \mathbb{E}_\nu(\Theta_k \mathbf{1}_C \mathbf{1}(S_n = T_k)), \tag{46}$$

where

$$\Theta_k := F(\pi_{\mathfrak{r}_k, T_k}(B_{T_k})).$$

Now introduce a new set of random variables, depending on R_{T_k}, and defined inductively as follows. We start with $\tilde{D}_0 := 0$ and $\tilde{\Upsilon}_0 := \emptyset$. Then, for all $n \geq 1$, $\tilde{S}'_n$ is defined from $\tilde{D}_{n-1}$ and $\tilde{\Upsilon}_{n-1}$ exactly as S'_n is defined from D_{n-1} and Υ_{n-1}. Similarly, $\tilde{S}_n$ is defined from $\tilde{S}'_n$ exactly as S_n is defined from S'_n. Then if S_n is a backward super-α time relative to R_{T_k}, let $\tilde{\Upsilon}_n := \emptyset$, and let $\tilde{D}_n$ be defined from $\tilde{S}_n$ exactly as D_n is defined from S_n. If S_n is not a backward super-α time relative to R_{T_k}, let $(\tilde{W}^{(n)}, \tilde{u}^{(n)})$ denote the particle path such that $(W_{\tilde{S}_n}, u)$ is the smallest with respect to the lexicographical order among those paths in $B_{S_n} \cap R_{T_k}$ such that there exists $s < \tilde{S}_n$ for which $W_s < r_{\tilde{S}_n} - \alpha(\tilde{S}_n - s)$. Then let $\tilde{\Upsilon}_n := \{(\tilde{W}^{(n)}, \tilde{u}^{(n)})\}$, $\tilde{D}_n := \tilde{S}_n$. Finally, $\tilde{Q}^{(n)}$ is defined from $\tilde{S}, \tilde{W}, \tilde{u}$ just as $Q^{(n)}$ is defined from S, W, u. We now return to (46). First, by Lemma 3, we can find $C' \in \mathcal{G}^R_{T_k}$ such that C and C' coincide on $\{S_n = T_k\}$. On the other hand, we have that

$$\{S_n = T_k\} = \{\tilde{S}_n = T_k\} \cap \mathfrak{H}_k, \tag{47}$$

where $\mathfrak{H}_k$ is the event that, for all $1 \leq j \leq n-1$,

- if $\tilde{S}_j$ fails to be a backward super-α time relative to R_{T_k}, for all $(W, u) \in B_{T_k}$ such that $(W_{\tilde{S}_j}, u)$ is less than $(\tilde{W}^{(j)}_{\tilde{S}_j}, \tilde{u}^{(j)})$ with respect to the lexicographical order, one has $W_s \geq r_{\tilde{S}_j} - \alpha(\tilde{S}_j - s)$ for all $s < \tilde{S}_j$;
- if $\tilde{S}_j$ is indeed a backward super-α time relative to R_{T_k} for all $(W, u) \in B_{T_k}$, one has $W_s \geq r_{\tilde{S}_j} - \alpha(\tilde{S}_j - s)$ for all $s < \tilde{S}_j$.

To check (47), one checks that, on $\{S_n = T_k\}$ as well as on $\{\tilde{S}_n = T_k\} \cap \mathfrak{H}_k$, all the random variables $\tilde{D}, \tilde{\Upsilon}, \tilde{S}', \tilde{S}$ up to $\tilde{S}_n$ coincide with D, Υ, S', S up to S_n. Indeed, on $\{S_n = T_k\}$, it is equivalent for S_j to be a backward super-α time or a backward super-α time relative to R_{T_k}, and $(W^{(j)}, u^{(j)})$ belongs to R_{T_k} when S_j is not a backward super-α time. Similarly, on $\{\tilde{S}_n = T_k\} \cap \mathfrak{H}_k$, it is equivalent for $\tilde{S}_j$ to be a backward super-α time or a backward super-α time relative to R_{T_k}, and $(\tilde{W}^{(j)}, \tilde{u}^{(j)})$ belongs to R_{T_k} when $\tilde{S}_j$ is not a backward super-α time.

Now observe that we can write

$$\{\tilde{S}_n = T_k\} = \mathfrak{D}_k \cap \{\Xi_{T_k} = 1\},$$

where $\mathfrak{D}_k$ is the event that: $\tilde{S}'_n < T_k$, $\tilde{S}_n \geq T_k$, T_k is a backward sub-α time, and $]\tilde{S}'_n, T_k[$ contains a number of $(\tilde{S}'_n, \alpha)-$crossing times at least equal to L. Thus for all $k \geq 1$,

$$\mathbb{E}_\nu(\Theta_k \mathbf{1}_C \mathbf{1}(S_n = T_k)) = \mathbb{E}_\nu(\Theta_k \mathbf{1}_{C'} \mathbf{1}_{\mathfrak{D}_k} \mathbf{1}(\Xi_{T_k} = 1) \mathbf{1}_{\mathfrak{H}_k}).$$

Introduce the notation $f_k := (r_{s+T_k} - \mathfrak{r}_k)_{-T_k \leq s \leq 0}$. We observe that

$$\mathbf{1}_{\mathfrak{H}_k} = \mathbf{1}_{\mathfrak{G}(\tilde{Q}^{(n)}, f_k)}(\pi_{\mathfrak{r}_k, T_k}(B_{T_k})),$$

so that

$$\mathbb{E}_\nu(\Theta_k \mathbf{1}_C \mathbf{1}(S_n = T_k)) = \mathbb{E}_\nu((F \mathbf{1}_{\mathfrak{G}(\tilde{Q}^{(n)}, f_k)} \mathbf{1}(\Xi_0 = 1))(\pi_{\mathfrak{r}_k, T_k}(B_{T_k})) \mathbf{1}_{C'} \mathbf{1}_{\mathfrak{D}_k}).$$

One checks that $\mathfrak{D}_k$ is measurable with respect to $\mathcal{G}^R_{T_k}$, and that, on $\mathfrak{D}_k$, $\tilde{Q}^{(n)}$ is measurable with respect to $\mathcal{G}^R_{T_k}$. As a consequence, using[11] Proposition 11, we have that

$$\mathbb{E}_\nu(\Theta_k \mathbf{1}_C \mathbf{1}(S_n = T_k)) = \mathbb{E}_\nu(\xi(F \mathbf{1}_{\mathfrak{G}(\tilde{Q}^{(n)}, f_k, -\mathfrak{r}_k)} \mathbf{1}(\Xi_0 = 1), f_k) \mathbf{1}_{C'} \mathbf{1}_{\mathfrak{D}_k}).$$

Introduce

$$\lambda(\mathfrak{q}, q, x) := \frac{\mathbb{P}_\nu(G(\mathfrak{q}, q, x) \cap \{\Xi_0 = 1\})}{\mathbb{P}_\nu(G(q, x))}.$$

By definition, we have that

$$\xi(F \mathbf{1}_{\mathfrak{G}(\mathfrak{q}, q)} \mathbf{1}(\Xi_0 = 1), q, x) = \eta(F, \mathfrak{q}, q, x) \lambda(\mathfrak{q}, q, x),$$

so that

$$\mathbb{E}_\nu(\Theta_k \mathbf{1}_C \mathbf{1}(S_n = T_k)) = \mathbb{E}_\nu(\eta(F, \tilde{Q}^{(n)}, f_k, -\mathfrak{r}_k) \lambda(\tilde{Q}^{(n)}, f_k, -\mathfrak{r}_k) \Lambda_k \mathbf{1}_{C'} \mathbf{1}_{\mathfrak{D}_k}). \tag{48}$$

On the other hand, repeating the previous argument with

$$\Lambda_k := \eta(F, Q^{(n)}, f_k, -\mathfrak{r}_k)$$

instead of Θ_k (remember that η is defined in (45)), using the fact that $\tilde{Q}^{(n)} = Q^{(n)}$ on $\{S_n = T_k\}$, we also find that

$$\mathbb{E}_\nu(\Lambda_k \mathbf{1}_C \mathbf{1}(S_n = T_k)) = \mathbb{E}_\nu(\eta(F, \tilde{Q}^{(n)}, f_k, -\mathfrak{r}_k) \lambda(\tilde{Q}^{(n)}, f_k, -\mathfrak{r}_k) \Lambda_k \mathbf{1}_{C'} \mathbf{1}_{\mathfrak{D}_k}). \tag{49}$$

In view of (46), the combination of (48) and (49) yields the conclusion of the proposition. □

Corollary 4. *For any non-negative bounded measurable map $F : \Omega \to \mathbb{R}$, the following bound holds for all $n \geq 1$, on $\{S_n < +\infty\}$,*

$$\mathbb{E}_\nu(F(\pi_{r_{S_n}, S_n}(B_{S_n})) | \mathcal{G}^R_{S_n}) \leq c_{13} \mathbb{E}_{\nu_{\mathscr{C},+}}(F) \text{ a.s.},$$

where c_{13} is a positive constant depending on $\mathscr{C}$.

[11] In fact, we are using a slight extension of Proposition 11, in which the map F is allowed to depend on $\mathcal{G}^R_{T_k}$. This extension is readily derived from the case where F is deterministic.

Proof. Observe that, for all $(\mathfrak{q}, q, x)$ such that $q_s < \alpha s$ for all $s \in [t, 0[$ and $x < \alpha t$, the fact that 0 is a backward super $\alpha-$time and $\Xi_0 = 1$ implies that $G(\mathfrak{q}, q, x)$ holds. We thus have that, for non-negative F,

$$\eta(F, \mathfrak{q}, q, x) \leq \frac{\mathbb{E}_\nu(F(B_0)\mathbf{1}_{\Xi_0=1})}{\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\})} \leq c_{13}\mathbb{E}_{\nu_{\mathscr{C},+}}(F),$$

with

$$c_{13} := \frac{\mathbb{P}_\nu(\Xi_0 = 1)}{\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\})},$$

using Lemma 7 to establish that $\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\}) > 0$. The conclusion now follows from Proposition 15, using the fact that S_n is a backward sub-α time. □

4.4. **Tail bounds.** We are now ready to prove the tail bounds that are necessary to control the regeneration times. Let us first observe that, thanks to Proposition 13 and to the fact that $\alpha < \beta$, we know that, for $n \geq 0$, when $D_n < +\infty$, one almost surely has that $S_{n+1} < +\infty$.

One key quantity we shall work with is the random variable $\mathcal{M}_n$, defined for all $n \geq 1$, on the event $\{S_n < +\infty\}$, by

$$\mathcal{M}_n := \sum_{(W,u)\in R^*_{S_n}} e^{-\theta(r_{S_n} - W_{S_n})}, \tag{50}$$

where $R^*_{S_n}$ is defined at the beginning of Section 4.

For $n \geq 1$, let $\mathcal{N}_n$ denote the number of $(S_n, \alpha)-$crossings contained in the time-interval $[S_n, S'_{n+1}]$.

Lemma 19. *One has the following bound: for all $n \geq 1$, for all $K > 0$, on $\{S_n < +\infty\}$,*

$$\mathbb{P}_\nu(\mathcal{N}_n \geq K,\ D_n < +\infty | \mathcal{F}^R_{S_n}) \leq e^\theta \mathcal{M}_n e^{-c_{14}K} + c_{15}K^{-c_{16}\mathscr{C}} \text{ a.s.},$$

where c_{14}, c_{15}, c_{16} are strictly positive constants, with c_{15} depending on $\mathscr{C}$.

Proof. Let S''_{n+1} denote the infimum of the $t > D_n$ such that

- t is a backward sub-α time;
- $\Upsilon_n \subset R_t$.

Let $\mathcal{N}^{(1)}_n$ and $\mathcal{N}^{(2)}_n$ denote respectively the numbers of $(S_n, \alpha)-$crossings contained in the time-interval $[S_n, S''_{n+1}[$ and in the time-interval $[S''_{n+1}, S'_{n+1}]$, so that

$$\mathcal{N}_n = \mathcal{N}^{(1)}_n + \mathcal{N}^{(2)}_n. \tag{51}$$

Our first claim is that there exists a constant $d_1 < 1$, depending on $\mathscr{C}$, such that, for all $\ell \geq 1$,

$$\mathbb{P}_\nu(\mathcal{N}^{(2)}_n \geq \ell,\ D_n < +\infty | \mathcal{G}^R_{S_n}) \leq d_1^\ell \text{ a.s.} \tag{52}$$

Assume that $D_n < +\infty$, and denote by $\tau_1, \tau_2, \dots$ the successive backward sub-α times posterior to S''_{n+1} (with $\tau_1 := S''_{n+1}$), and let $J := \inf\{j \geq 1;\ \Xi_{\tau_j} = 1\}$ (remember that $\Xi_t = 1$ means that there are at least $\mathscr{C}$ particles located at site r_t in B_t). By definition, we have $S'_{n+1} = \tau_J$. Since S_n is a backward sub-α time, any $(S_n, \alpha)-$crossing in $[S''_{n+1}, S'_{n+1}]$ is a backward sub-α time, so we have

$$\mathcal{N}_n^{(2)} \leq J. \tag{53}$$

Now using an argument similar to the proof of Proposition 15, we have that, for all $i \geq 1$, on $\{D_n < +\infty\}$, the distribution of $\pi_{r_{\tau_i},\tau_i}(B_{\tau_i})$ conditional upon $\mathcal{G}^R_{\tau_i}$ is that of B_0 conditioned upon an event containing G (remember that G is defined in (6)), so that, on $\{D_n < +\infty\}$, one has the bound

$$\mathbb{P}_\nu(\Xi_{\tau_i} = 1|\mathcal{G}^R_{\tau_i}) \geq \mathbb{P}_\nu(\{\Xi_0 = 1\} \cap G) \ a.s.$$

Since for all $i \geq 2$, the random variables $\Xi_{\tau_1}, \ldots, \Xi_{\tau_{i-1}}$ are measurable with respect to $\mathcal{G}^R_{\tau_i}$, we deduce that, on $\{D_n < +\infty\}$,

$$\mathbb{P}_\nu(J \geq \ell|\mathcal{G}^R_{S''_{n+1}}) \leq (1 - \mathbb{P}_\nu(\{\Xi_0 = 1\} \cap G))^\ell \ a.s. \tag{54}$$

Combining (53) and (54), we deduce (52), using also the fact that[12] $\mathcal{G}^R_{S_n} \subset \mathcal{G}^R_{S''_{n+1}}$ since $S_n \leq S''_{n+1}$ and S_n is $\mathcal{G}^R_{S''_{n+1}}-$measurable.

Now consider the event $\mathcal{N}_n^{(1)} > \ell$. Start with the case where S_n is not a backward super-α time, and call H_n the corresponding event. We first bound the probability that $W^{(n)}_{S_n} > r_{S_n} + \ell/2$. From Lemma 6, a random walk starting at $x \geq 0$ at time zero has a probability bounded above by $e^{-\theta x}$ to ever cross at a negative time the half-line of slope α starting at $(0,0)$. Using Corollary 4 and the union bound over all the particle paths in B_{S_n}, we deduce that

$$\mathbb{P}_\nu(H_n, W^{(n)}_{S_n} > r_{S_n} + \ell/2|\mathcal{G}^R_{S_n}) \leq c_{13} \sum_{x > \ell/2} e^{-\theta x}\rho,$$

We deduce that

$$\mathbb{P}_\nu(H_n, W^{(n)}_{S_n} > r_{S_n} + \ell/2|\mathcal{G}^R_{S_n}) \leq d_2 e^{-d_3\ell}, \tag{55}$$

where d_2 and d_3 are strictly positive constants, with d_2, d_3 depending on $\mathscr{C}$. On the other hand, assume that $W^{(n)}_{S_n} \leq r_{S_n} + \ell/2$. Assume also that $\mathcal{N}_n^{(1)} > \ell$, and let t denote the time of the $\ell-$th $(S_n, \alpha)-$crossing posterior to S_n. By definition of $\mathcal{N}_n^{(1)}$, we must have that $(W^{(n)}, u^{(n)}) \notin R_t$, whence $W^{(n)}_t \geq r_t \geq r_{S_n} + \ell + \alpha(t - S_n)$. Since $W^{(n)}_{S_n} \leq r_{S_n} + \ell/2$, this implies that $W^{(n)}_t \geq W^{(n)}_{S_n} + \ell/2 + \alpha(t - S_n)$. Using again Corollary 4, Lemma 6 and the union bound, we deduce that

$$\mathbb{P}_\nu(\mathcal{N}_n^{(1)} > \ell, H_n, W^{(n)}_{S_n} \leq r_{S_n} + \ell/2|\mathcal{G}^R_{S_n}) \leq c_{13} \left(e^{-\theta\ell/2}\rho_{\mathscr{C}} + \sum_{1 \leq x \leq \ell/2} e^{-\theta\ell/2}\rho \right),$$

[12]Note that this property is not obvious. It is a consequence that it is enough to look at trajectories in R_{S_n} to check the backward super-α time property for S_j where $j \leq n-1$. See the proof of Proposition 15.

where $\rho_{\mathscr{C}}$ denotes the expected value of a Poisson random variable of parameter ρ conditioned upon being $\geq \mathscr{C}$ (we have to use $\rho_{\mathscr{C}}$ since, under $\nu_{\mathscr{C},+}$, the number of particles at the origin has a Poisson distribution conditioned by taking a value $\geq \mathscr{C}$). We deduce that there exists a strictly positive constant d_4 depending on $\mathscr{C}$ and a strictly positive constant d_5 such that

$$\mathbb{P}_\nu(\mathcal{N}_n^{(1)} > \ell, H_n, W_{S_n}^{(n)} \leq r_{S_n} + \ell/2 | \mathcal{G}_{S_n}^R) \leq d_4 e^{-d_5 \ell}. \tag{56}$$

Now consider the case where S_n is a backward super-α time. In this case, $\Upsilon = \emptyset$ and, by definition of S''_{n+1}, $\mathcal{N}_n^{(1)}$ is also the number of $(S_n, \alpha)-$crossings contained in the time-interval $[S_n, D_n]$. Introduce $t_0 := \ell / C'_1(\rho)$ (remember that $C'_1(\rho)$ is defined in Proposition 18), assuming that ℓ is large enough so that $t_0 > \alpha^{-1}$, and consider the cases $D_n - S_n > t_0$ and $D_n - S_n \leq t_0$ separately. Assume first that $D_n - S_n \leq t_0$, and let t denote the time of the $\ell-$th $(S_n, \alpha)-$crossing posterior to S_n.The fact that $\mathcal{N}_n^{(1)} > \ell$ implies that $t < D_n$, while $r_t \geq r_{S_n} + \ell$. Moreover, since $t < D_n$, r_t is in fact equal to $r_{S_n} + r_{t-S_n}(\pi_{r_{S_n}, S_n}(B_{S_n}))$ since, by definition, particles in R_{S_n} cannot influence the front between time S_n and time D_n (see the proof of Lemma 4). As a consequence, $r_{t-S_n}(\pi_{r_{S_n}, S_n}(B_{S_n})) \geq \ell$, while $t - S_n \leq t_0$. Using Corollary 4 and Lemma 17, we deduce that

$$\mathbb{P}_\nu(\mathcal{N}_n^{(1)} \geq \ell, H_n^c, D_n - S_n \leq t_0 | \mathcal{G}_{S_n}^R) \leq c_{11} e^{-c_{12} t_0}. \tag{57}$$

On the other hand, using again the fact that particles in R_{S_n} cannot influence the front prior between time S_n and time D_n, we see that, if $D_n - S_n > t_0$, at least one of the three following events must occur, according to which of the five conditions defining D_n corresponds to the smallest time (note that our assumption that $t_0 > \alpha^{-1}$ rules out (2) and (4)): for some $t \geq S_n + t_0$, $r_t(\pi_{r_{S_n}, S_n}(B_{S_n})) < \lfloor \alpha(t - S_n) \rfloor$, or there exists a particle path $(W, u) \in R_{S_n}$ such that $W_{S_n} \leq r_{S_n} - 1$ and a $t \geq t_0$ such that $W_{S_n + t} \geq r_{S_n} - 1 + \alpha(t - S_n)$, or there exists a $t \geq t_0$ such that $W_t^{*n} > r_{S_n} - 1 + \alpha t$, while $W^{*n}_{S_n + \alpha^{-1}} = r_{S_n}$. Using Corollary 4, Lemma 15, Lemma 8, and Lemma 6, and the strong Markov property[13] at time S_n and $S_n + \alpha^{-1}$, we deduce by the union bound that

$$\mathbb{P}_\nu(\mathcal{N}_n^{(1)} \geq \ell, H_n^c, t_0 < D_n - S_n < +\infty | \mathcal{F}_{S_n}^R) \leq c_{13} c_6 t_0^{-c_7 \mathscr{C}} + e^\theta \mathcal{M}_n e^{-\mu t_0} + e^{-\mu(t_0 - \alpha^{-1})}. \tag{58}$$

Putting toghether (51), (52), (55), (56), (57) and (58), we deduce the conclusion of the lemma. □

Corollary 5. *One has the following bound: for all $n \geq 1$, for all $t > 0$, on $\{S_n < +\infty\}$,*

$$\mathbb{P}_\nu(S'_{n+1} - S_n \geq t \ , \ D_n < +\infty | \mathcal{F}_{S_n}^R) \leq e^\theta \mathcal{M}_n e^{-c_{17} t} + c_{18} t^{-c_{19} \mathscr{C}} \ a.s.,$$

where c_{17}, c_{18}, c_{19} are strictly positive constants, with c_{18} depending on $\mathscr{C}$.

[13] The Markov property of the dynamics holds with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, not $(\mathcal{F}_t^R)_{t \geq 0}$. Here we use the fact that $\mathcal{F}_{S_n}^R \subset \mathcal{F}_{S_n}$, and also that $\mathcal{F}_{S_n}^R \subset \mathcal{G}_{S_n}^R$.

Proof. Assume that $S'_{n+1} - S_n \geq t$. If $r_{S_n+t} \geq r_{S_n} + \beta t$, we deduce by Lemma 18 that there exist at least $\lfloor \frac{(\beta-\alpha)t}{2} \rfloor$ distinct $(S_n, \alpha)-$crossing times in $[S_n, S_n + t]$, whence the fact that $\mathcal{N}_n \geq \lfloor \frac{(\beta-\alpha)t}{2} \rfloor$. On the other hand, using Lemma 14, Proposition 13 and Corollary 4, we see that

$$\mathbb{P}_\nu(r_{S_n+t} - r_{S_n} \leq \beta t \ , \ D_n < +\infty | \mathcal{F}^R_{S_n}) \leq c_{13} c_2 t^{-c_3 \cdot \mathscr{C}} a.s.$$

The conclusion now follows from Lemma 19. □

Corollary 6. *One has the following bound: for all $n \geq 1$, for all $t > 0$, on $\{S_n < +\infty\}$,*

$$\mathbb{P}_\nu(r_{S'_{n+1}} - r_{S_n} \geq K \ , \ D_n < +\infty | \mathcal{F}^R_{S_n}) \leq \mathcal{M}_n e^{-c_{20} K} + c_{21} K^{-c_{22} \mathscr{C}} \ a.s.,$$

where c_{20}, c_{21}, c_{22} are strictly positive constants, with c_{21} depending on $\mathscr{C}$.

Proof. By definition of α-crossing times, we have

$$r_{S'_{n+1}} \leq r_{S_n} + \alpha(S'_{n+1} - S_n) + \mathcal{N}_n + 1.$$

The result follows from combining Corollary 6 and Lemma 19. □

Define $\mathcal{L}^{(1)}_n$ to be the number of particle paths in $B_{S_n} \cap R_{S'_{n+1}}$.

Lemma 20. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bound:*

$$\mathbb{E}_\nu(\mathcal{L}^{(1)}_n \mathbf{1}(D_n < +\infty) | \mathcal{F}^R_{S_n}) \leq c_{23} + c_{24} \mathcal{M}_n,$$

where c_{23} and c_{24} are strictly positive constants depending on $\mathscr{C}$.

Proof. Assume that $S'_{n+1} \leq S_n + t$ and that $r_{S'_{n+1}} \leq r_{S_n} + K$ for some $t, K > 0$. We can then bound $\mathcal{L}^{(1)}_n$ by counting the total number of particle paths (W, u) in B_{S_n} for which there exists $s \in [S_n, S_n + t]$ such that $W_s \in [r_{S_n}, r_{S_n} + K]$. This number includes all the particle paths (W, u) in B_{S_n} such that $W_{S_n} \in [r_{S_n}, r_{S_n} + K]$, plus the particle paths in B_{S_n} such that $W_{S_n} \geq r_{S_n} + K + 1$ that hit level $r_{S_n} + K$ during the time-interval $[S_n, S_n + t]$. Assume that we start with $\mathbb{P}_{\nu_{\mathscr{C},+}}$, and let $\mathcal{K}_1$ denote the number of particle paths (W, u) in B_0 such that $W_0 \in [0, K]$, while $\mathcal{K}_2$ denotes the number of particle paths in B_0 such that $W_0 \geq r_0 + K + 1$ that hit level K during the time-interval $[0, t]$. By standard properties of the Poisson distribution, we see that $\mathcal{K}_2$ is a Poisson random variable with distribution ρg, where

$$g := \sum_{x \geq K+1} P_x(\inf_{s \in [0,t]} \zeta_s \leq K).$$

Using the reflection principle, we see that $g \leq g'$, where

$$g' = 2 \sum_{x \geq K+1} P_x(\zeta_t \leq K).$$

Now using translation invariance, we can rewrite

$$g' = 2 \sum_{x \geq 1} P_x(\zeta_t \leq 0) = 2 \sum_{x \geq 1} P_0(x + \zeta_t \leq 0) = 2 E_0(-\zeta_t \mathbf{1}(\zeta_t \leq -1)).$$

Using Schwarz's inequality, we deduce that $g' \leq 2\sqrt{2t}$. On the other hand, $\mathcal{K}_1$ is the sum of Ξ_0, whose distribution is that of a Poisson random variable of parameter ρ conditioned to be $\geq \mathscr{C}$, and of a Poisson random variable of parameter ρK, these two variables being independent, and independent from $\mathcal{K}_2$. Using Corollary 4, we deduce that, for some strictly positive constant d_1 depending on $\mathscr{C}$,

$$\mathbb{P}_\nu(\mathcal{L}_n^{(1)} \geq \ell, S'_{n+1} \leq S_n + t, r_{S'_{n+1}} \leq r_{S_n} + K|\mathcal{G}^R_{S_n}) \leq d_1 a_{t,K}(\ell), \tag{59}$$

where $a_{t,K}(\ell)$ denotes the probability for a Poisson random variable with parameter $\rho(K + 1 + 2\sqrt{2t})$ to be $\geq \ell$. Now consider two strictly positive constants b_1, b_2 with $\rho b_1 < 1$. Note that, for $K := b_1\ell$ and $t := b_2\ell$, one has, by standard large deviations bounds for Poisson random variables (see e.g. [4]), that for all $\ell \geq 1$,

$$a_{t,K}(\ell) \leq d_2 e^{-d_3\ell}, \tag{60}$$

where d_2, d_3 are strictly positive constants. Combining Corollary 5 and Corollary 6, we deduce that, on $\{S_n < +\infty\}$,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(\mathcal{V}_n(\ell)^c, D_n < +\infty|\mathcal{F}^R_{S_n}) \leq e^\theta \mathcal{M}_n e^{-d_4\ell} + d_5 \ell^{-d_6\mathscr{C}} \, a.s., \tag{61}$$

with

$$\mathcal{V}_n(\ell) := \{S'_{n+1} \leq S_n + b_2\ell\} \cap \{r_{S'_{n+1}} \leq r_{S_n} + b_1\ell\},$$

and where d_4, d_5, d_6 are strictly positive constants, d_5 depending on $\mathscr{C}$. Combining (59), (60) and (61), we deduce the result. □

In the sequel, we use the random variable $\mathcal{N}_n^{(3)}$ defined as the total number of $(S'_{n+1}, \alpha)-$crossing times contained in the time-interval $[S'_{n+1}, S_{n+1}]$.

Lemma 21. *One has the following bound: for all $n \geq 1$, for all $t > 0$, on $\{S_n < +\infty\}$,*

$$\mathbb{P}_\nu(S_{n+1} - S'_{n+1} \geq t \ , \ D_n < +\infty|\mathcal{F}^R_{S'_{n+1}}) \leq c_{25} t^{-c_{26}\mathscr{C}} \ a.s.,$$

where c_{25}, c_{26} are strictly positive constants, with c_{25} depending on $\mathscr{C}$ and L.

Proof. Define S'''_{n+1} as the infimum of the $t > S'_{n+1}$ such that t is a backward sub-α time and $]S'_{n+1}, t[$ contains at least L $(S'_{n+1}, \alpha)-$crossing times, and let $\tau'_1, \tau'_2, \ldots$ denote the successive backward sub-α times posterior to S'''_{n+1} (with $\tau'_1 := S'''_{n+1}$), and let $I := \inf\{i \geq 1, \ \Xi_{\tau'_i} = 1\}$. We have by definition that $\mathcal{N}_n^{(3)} \leq L + I$. Arguing exactly as in the proof of Lemma 19, we can prove a bound of the form

$$\mathbb{P}_\nu(I \geq \ell, \ D_n < +\infty|\mathcal{G}^R_{S'_{n+1}}) \leq d_1{}^\ell \ a.s., \tag{62}$$

where $d_1 < 1$ is a constant depending on $\mathscr{C}$. Combining the resulting bound on the tail of $\mathcal{N}_n^{(3)}$ with Proposition 13 and the analog of Corollary 4 for S'_{n+1} as in the proof of Corollary 5, we deduce the result. □

Corollary 7. *One has the following bound: for all $n \geq 1$, for all $K > 0$, on $\{S_n < +\infty\}$,*

$$\mathbb{P}_\nu(r_{S_{n+1}} - r_{S'_{n+1}} \geq K \ , \ D_n < +\infty | \mathcal{F}^R_{S'_{n+1}}) \leq c_{27} K^{-c_{28}\mathscr{C}} \ a.s.,$$

where c_{27}, c_{28} are strictly positive constants with c_{27} depending on $\mathscr{C}$ and L.

Proof. By definition of α-crossing times, we have

$$r_{S_{n+1}} \leq r_{S'_{n+1}} + \alpha(S_{n+1} - S'_{n+1}) + \mathcal{N}^{(3)}_n,$$

then use Lemma 21 and the bound on the tail of $\mathcal{N}^{(3)}_n$ derived in its proof. □

Define $\mathcal{L}^{(2)}_n$ to be the number of particle paths in $B_{S'_{n+1}} \cap R_{S_{n+1}}$.

Lemma 22. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bound: on $\{S_n < +\infty\}$,*

$$\mathbb{E}_\nu(\mathcal{L}^{(2)}_n \mathbf{1}(D_n < +\infty) | \mathcal{F}^R_{S_n}) \leq c_{29} \ a.s.,$$

where c_{29} is a strictly positive constant depending on $\mathscr{C}$ and L.

Proof. Use the same argument as in the proof of Lemma 20, with Lemma 21 and Corollary 7 replacing Corollaries 5 and 6 respectively. □

Lemma 23. *Consider $w \in \mathbb{S}_\theta$ such that there is at least one particle at site 0. Let T be an $(\mathcal{F}_t)_{t\geq 0}$ stopping time such that T is a backward super-α time and $]0, T[$ contains a number of $(0, \alpha)-$crossing times at least equal to $m \geq 0$. Then one has the following bound:*

$$\mathbb{E}_w \left(\sum_{(W,u)\in R_{0+}} e^{-\theta(r_T - W_T)} \mathbf{1}(T < +\infty) \right) \leq e^{-\theta m} \phi_\theta(w).$$

Proof. Consider $(W, u) \in R_{0+}$, and, for all $s \geq 0$, let $M_s := e^{\theta W_s - 2(\cosh(\theta)-1)s}$. Then $(M_s)_{s\geq 0}$ is a càdlàg martingale. Since T is a stopping time, we have, for all finite $K > 0$, that

$$\mathbb{E}_w(M_{T\wedge K}) = \mathbb{E}_w(M_0) = e^{\theta W_0}. \tag{63}$$

Now we have that $\liminf_{K\to+\infty} M_{T\wedge K} \geq M_T \mathbf{1}(T < +\infty)$, so that, by Fatou's lemma and (63),

$$\mathbb{E}_w(M_T \mathbf{1}(T < +\infty)) \leq e^{\theta W_0}. \tag{64}$$

Now, from our assumptions on T and the fact that $r_0 = 0$, one has that, on $\{T < +\infty\}$, $r_T \geq \alpha T + m$. Using the fact that, by (24), $2(\cosh(\theta) - 1) \leq \alpha\theta$, we deduce that

$$2(\cosh(\theta) - 1)T \leq 2(\cosh(\theta) - 1) \left(\frac{r_T - m}{\alpha} \right) \leq \theta(r_T - m).$$

Writing

$$-\theta r_T + \theta W_T = -\theta r_T + 2(\cosh(\theta) - 1)T - 2(\cosh(\theta) - 1)T + \theta W_T,$$

we finally deduce that, on $\{T < +\infty\}$,

$$-\theta r_T + \theta W_T \leq -\theta m - 2(\cosh(\theta) - 1)T + \theta W_T.$$

In view of (64), we deduce that

$$\mathbb{E}_w(e^{-\theta(r_T - W_T)}\mathbf{1}(T < +\infty)) \leq e^{-\theta m}\mathbb{E}_w(M_T\mathbf{1}(T < +\infty)) \leq e^{-\theta m}e^{\theta W_0}.$$

The result now follows from summing the above inequality over all $(W, u) \in R_{0+}$. □

Lemma 24. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bound:*

$$\mathbb{E}_\nu(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}) \leq c_{30}e^{-\theta L}\mathcal{M}_n + c_{31},$$

where c_{30} is a strictly positive constant depending on $\mathscr{C}$, and c_{31} is a strictly positive constant depending on $\mathscr{C}$ and L.

Proof. Define

$$\mathcal{M}'_{n+1} := \sum_{(W,u)\in R^*_{S'_{n+1}}} \exp\left(-\theta(r_{S'_{n+1}} - W_{S'_{n+1}})\right),$$

where $R^*_{S'_{n+1}}$ is defined as the set $R_{S'_{n+1}}$ from which we remove the particle path that makes the front climb at time S'_{n+1}. By definition we have that $\mathcal{M}'_{n+1} \leq \mathcal{A}^{(1)} + \mathcal{A}^{(2)}$, with

$$\mathcal{A}^{(1)} := \sum_{(W,u)\in R_{S_n}} \exp\left(-\theta(r_{S'_{n+1}} - W_{S'_{n+1}})\right)$$

and

$$\mathcal{A}^{(2)} := \sum_{(W,u)\in B_{S_n}\cap R_{S'_{n+1}}} \exp\left(-\theta(r_{S'_{n+1}} - W_{S'_{n+1}})\right).$$

First, using the fact that for each $(W, u) \in R_{S'_{n+1}}$, one has $W_{S'_{n+1}} \leq r_{S'_{n+1}}$, we have the bound

$$\mathcal{A}^{(2)} \leq \mathcal{L}^{(1)}_n. \tag{65}$$

Now using Lemma 23, we deduce that

$$E_\nu(\mathcal{A}^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}) \leq \mathcal{M}_n + 1, \tag{66}$$

where the $+1$ term comes from the fact that the definition of $\mathcal{M}_n$ involves the particles in $R^*_{S_n}$, not R_{S_n}, so we have to add the contribution to $\mathcal{A}^{(1)}$ of the particle path (W^{*n}, u^{*n}) , which we bound by 1. Now we have that $\mathcal{M}_{n+1} \leq \mathcal{B}^{(1)} + \mathcal{B}^{(2)}$, with

$$\mathcal{B}^{(1)} := \sum_{(W,u)\in R_{S'_{n+1}}} \exp\left(-\theta(r_{S_{n+1}} - W_{S_{n+1}})\right)$$

and

$$\mathcal{B}^{(2)} := \sum_{(W,u)\in B_{S'_{n+1}}\cap R_{S_{n+1}}} \exp\left(-\theta(r_{S_{n+1}} - W_{S_{n+1}})\right).$$

As in (65), we have the bound

$$\mathcal{B}^{(2)} \leq \mathcal{L}_n^{(2)}. \tag{67}$$

On the other hand, using Lemma 23, we deduce that

$$E_\nu(\mathcal{B}^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S'_{n+1}}) \leq e^{-\theta L}\mathcal{M}'_{n+1} + 1. \tag{68}$$

Combining (65), (66), (67), (68), and using the fact that $\mathcal{F}^R_{S_n} \subset \mathcal{F}^R_{S'_{n+1}}$, we deduce that, on $\{S_n < +\infty\}$,

$$\begin{aligned}\mathbb{E}_\nu(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}) &\leq e^{-\theta L}\mathcal{M}_n + e^{-\theta L} + 1 \\ &+ e^{-\theta L}\mathbb{E}_\nu(\mathcal{L}_n^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}) \\ &+ \mathbb{E}_\nu(\mathcal{L}_n^{(2)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}).\end{aligned}$$

The conclusion now follows from Lemmas 20 and 22. □

So far, we have proved results dealing with the behavior of the system during the time-interval $[S_n, S_{n+1}]$, for $n \geq 1$. The case of the interval $[0, S_1]$ is a little bit different since it starts at time $D_0 = 0$, where not all the properties of times S_n, $n \geq 1$ are met. However, the distribution of (R_0, B_0) is exactly known, and, in this case, Proposition 12 directly yields the estimates that we obtained by a combination of Proposition 13 and Corollary 4 in the case $[S_n, S_{n+1}]$. In particular, we have the following results.

Let $\mathcal{N}_0$ denote the number of $(0,\alpha)-$crossings contained in the time-interval $[0, S'_1]$ and associated to an integer k such that $r_0 + k \geq 1$.

Lemma 25. *One has the following bound: for all $n \geq 1$, for all $K > 0$,*

$$\mathbb{P}_\nu(\mathcal{N}_0 \geq K) \leq c_{32}K^{-c_{33}\mathscr{C}}.$$

where c_{15}, c_{33} are strictly positive constants, with c_{32} depending on $\mathscr{C}$.

Proof. We always have $\Upsilon = \emptyset$, so that S'_1 is the smallest positive t such that t is a backward super-α time and $\Xi_t = 1$. As a consequence, we only have to adapt the first part of the argument in the proof of Lemma 19, i.e. the one dealing with $\mathcal{N}_n^{(2)}$, which is straightforward. □

Corollary 8. *One has the following bound: for all $t > 0$,*

$$\mathbb{P}_\nu(S'_1 \geq t) \leq c_{34}t^{-c_{35}\mathscr{C}} \ a.s.,$$

where c_{34}, c_{35} are strictly positive constants, with c_{34} depending on $\mathscr{C}$.

Proof. Similar to the proof of Corollary 5, using Lemma 25 and Proposition 12 with $\mathscr{C} = 0$ instead of invoking Proposition 13 then Corollary 4, and taking care of the fact that we may have $r_0 < 0$. □

Corollary 9. *One has the following bound: for all $t > 0$,*

$$\mathbb{P}_\nu(r_{S'_1} \geq K) \leq c_{36}K^{-c_{37}\mathscr{C}} \ a.s.,$$

where c_{36}, c_{37} are strictly positive constants, with c_{36} depending on $\mathscr{C}$.

Proof. Similar to the proof of Corollary 6, using the inequality $r_{S'_1} \leq \alpha S'_1 + \mathcal{N}_0 + 1$. $\square$

Define $\mathcal{L}_0^{(1)}$ to be the number of particle paths in $B_0 \cap R_{S'_1}$.

Lemma 26. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bound:*

$$\mathbb{E}_\nu(\mathcal{L}_0^{(1)}) \leq c_{38},$$

where c_{38} is a strictly positive constant depending on $\mathscr{C}$.

Proof. Similar to the proof of Lemma 20. $\square$

Lemma 27. *One has the following bound: for all $t > 0$, ,*

$$\mathbb{P}_\nu(S_1 - S'_1 \geq t) \leq c_{39} t^{-c_{40}\mathscr{C}} \ a.s.,$$

where c_{39}, c_{40} are strictly positive constants, with c_{39} depending on $\mathscr{C}$ and L.

Proof. Similar to the proof of Lemma 21. $\square$

Corollary 10. *One has the following bound: for all $K > 0$,*

$$\mathbb{P}_\nu(r_{S_1} \geq K) \leq c_{41} K^{-c_{42}\mathscr{C}} \ a.s.,$$

where c_{41}, c_{42} are strictly positive constants with c_{41} depending on $\mathscr{C}$ and L.

Proof. Similar to the proof of Corollary 7. $\square$

Define $\mathcal{L}_0^{(2)}$ to be the number of particle paths in $B_{S'_1} \cap R_{S_1}$.

Lemma 28. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bound:*

$$\mathbb{E}_\nu(\mathcal{L}_0^{(2)}) \leq c_{43},$$

where c_{43} is a strictly positive constant depending on $\mathscr{C}$ and L.

Proof. Similar to the proof of Lemma 22. $\square$

Lemma 29. *For all large enough $\mathscr{C}$, one has the following bound:*

$$\mathbb{E}_\nu(\mathcal{M}_1) \leq c_{44},$$

where c_{44} is a strictly positive constant depending on $\mathscr{C}$ and L.

Proof. Similar to the proof of Lemma 24. $\square$

Proposition 16. *For all large enough $\mathscr{C}$, and all large enough L (depending on $\mathscr{C}$), there exists $c_{45} < +\infty$, depending on $\mathscr{C}$ and L, such that, for all $n \geq 1$,*

$$\mathbb{E}_\nu(\mathcal{M}_n | D_{n-1} < +\infty) \leq c_{45}.$$

Proof. For $n = 1$, the result is Lemma 29. Consider $n \geq 1$, and write

$$\mathbb{E}_\nu(\mathcal{M}_{n+1} | D_n < +\infty) = \frac{\mathbb{E}_\nu(\mathcal{M}_{n+1} \mathbf{1}(D_n < +\infty) | D_{n-1} < +\infty)}{\mathbb{P}_\nu(D_n < +\infty | D_{n-1} < +\infty)}. \tag{69}$$

Using Lemma 24 and the fact that the event $D_{n-1} < +\infty$ is measurable with respect to $\mathcal{F}^R_{S_n}$, we deduce that

$$\mathbb{E}_\nu(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty)|D_{n-1} < +\infty) \leq c_{30}e^{-\theta L}\mathbb{E}_\nu(\mathcal{M}_n|D_{n-1} < +\infty) + c_{31}.$$

On the other hand, observe that there exists a strictly positive constant d_1 such that

$$\mathbb{P}_\nu(D_n < +\infty|D_{n-1} < +\infty) \geq d_1, \tag{70}$$

considering e.g. the probability for the particle that makes the front climb at time S_n to cross at a time $> S_n$ the half-line of slope α starting at (S_n, r_{S_n}). Combining (69) and (70), we deduce that

$$\mathbb{E}_\nu(\mathcal{M}_{n+1}|D_n < +\infty) \leq d_1^{-1}c_{30}e^{-\theta L}\mathbb{E}_\nu(\mathcal{M}_n|D_{n-1} < +\infty) + d_1^{-1}c_{31}.$$

When L is large enough so that $d_1^{-1}c_{30}e^{-\theta L} < 1$, we deduce, using also Lemma 29, that the sequence $(\mathbb{E}_\nu(\mathcal{M}_n|D_{n-1} < +\infty))_{n\geq 1}$ is bounded. □

We are now ready to prove our main estimate on the regeneration structure, namely, Proposition 10.

Proof of Proposition 10. In this proof we assume that $\mathscr{C}$ and L are large enough so that all the previous results hold.

Define $\mathfrak{K} := \inf\{n \geq 1;\ D_n = +\infty\}$. Our first claim is that, for some strictly positive constant d_1 depending on $\mathscr{C}$ and L, for all $k \geq 1$,

$$\mathbb{P}_\nu(\mathfrak{K} \geq k) \leq d_1^k. \tag{71}$$

From Lemma 13, we have that

$$d_2 := \mathbb{P}_{\nu_{\mathscr{C},+}}(\{0 \text{ is a forward super } \alpha\text{−time for } B_0\} \cap G) > 0.$$

Consider $n \geq 1$. Using Proposition 15 and the fact that

$$G \subset G(Q^{(n)}, (r_{s+S_n} - r_{S_n})_{-S_n \leq s \leq 0}, -r_{S_n}),$$

we deduce that, on $\{D_{n-1} < +\infty\}$, one has that

$$\mathbb{P}_\nu(S_n \text{ is a forward and backward super } \alpha\text{−time for } B_{S_n}\ |\mathcal{G}^R_{S_n}) \geq d_2\ a.s.$$

On the other hand, the event that S_n is a forward sub α−time is measurable with respect to $\mathcal{G}^R_{S_n}$. Call d_3 the probability for a random walk starting at zero to remain at zero during the time-interval $[0, \alpha^{-1}]$ and then to satisfy $W_s \leq \alpha s - 1$ for all $s \geq \alpha^{-1}$, which is > 0 by Lemma 6. Using Lemma 7, we deduce that, for arbitrary $K > 0$, on $\{D_{n-1} < +\infty\}$,

$$\mathbb{P}_\nu(S_n \text{ is a forward sub } \alpha\text{−time}|\mathcal{F}^R_{S_n}) \geq d_3 g(K)\mathbf{1}(\mathcal{M}_n \leq K)\ a.s.$$

We deduce that

$$\mathbb{P}_\nu(D_n < +\infty|\ D_{n-1} < +\infty) \geq d_2 d_3 g(K)\mathbb{P}_\nu(\mathcal{M}_n \leq K|D_{n-1} < +\infty).$$

By Proposition 16, we have that $\mathbb{E}_\nu(\mathcal{M}_n|D_{n-1} < +\infty) \leq c_{45}$, so that, by Markov's inequality, $\mathbb{P}_\nu(\mathcal{M}_n \leq 2c_{45}|D_{n-1} < +\infty) \geq 1/2$. Setting $K := 2c_{45}$ and $d_1 := 1/2(d_2 d_3 g(K))$, we see that (71) is proved.

Now observe that, by definition, $S_{\mathfrak{K}}$ is an $\alpha-$separation time. As a consequence, we have that $\kappa_1 \leq S_{\mathfrak{K}}$. Writing $S_{\mathfrak{K}} := S_1 + \sum_{k=1}^{\mathfrak{K}-1}(S_{k+1} - S_k)$, we deduce that for all t and $n \geq 1$,

$$\begin{aligned} \mathbb{P}_\nu(\kappa_1 \geq t) &\leq \mathbb{P}_\nu(\mathfrak{K} > n) + \mathbb{P}_\nu(S_1 \geq t/n) \\ &+ \sum_{k=1}^{n-1} \mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty). \end{aligned} \tag{72}$$

Let $t' := t/n$. Using Corollary 8 and Lemma 27 and, we deduce that

$$\mathbb{P}_\nu(S_1 \geq t/n) \leq c_{34}(t'/2)^{-c_{35}\mathscr{C}} + c_{39}(t'/2)^{-c_{40}\mathscr{C}}. \tag{73}$$

On the other hand, one has that

$$\mathbb{P}_\nu(S_{k+1}-S_k \geq t/n, D_k < +\infty) \leq \mathbb{P}_\nu(S_{k+1}-S_k \geq t/n, D_k < +\infty | D_{k-1} < +\infty),$$

and, using Corollary 5, Lemma 21 and Proposition 16, we deduce that

$$\begin{aligned} \mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty) &\leq c_{45}e^{-c_{17}(t'/2)} + c_{18}(t'/2)^{-c_{19}\mathscr{C}} \\ &+ c_{25}(t'/2)^{-c_{26}\mathscr{C}}. \end{aligned} \tag{74}$$

Choosing e.g. $n := \lceil t^{1/2} \rceil$, and using (71), (73) and (74) to bound the terms in (72), we deduce that

$$\mathbb{P}_\nu(\kappa_1 \geq t) \leq d_4 t^{-d_5\mathscr{C}+1/2}, \tag{75}$$

where d_4 and d_5 are strictly positive constants, with d_4 depending on $\mathscr{C}$ and L. Choosing $\mathscr{C}$ large enough, this proves the fact that κ_1 has a finite second moment. Now write

$$\mathbb{P}_\nu(r_{\kappa_1} \geq \ell) \leq \mathbb{P}_\nu(\kappa_1 > t) + \mathbb{P}_\nu\left(\sup_{s\in[0,t]} r_s \geq \ell\right).$$

Choosing $t := \ell/C_1'(\rho)$, and using Lemma 17 and (75), we deduce that

$$\mathbb{P}_\nu(r_{\kappa_1} \geq \ell) \leq d_6 \ell^{-d_5\mathscr{C}+1/2}, \tag{76}$$

where d_6 is a strictly positive constant depending on $\mathscr{C}$ and L. Choosing $\mathscr{C}$ large enough, this proves the fact that r_{κ_1} has a finite second moment. □

5. Extension to the case $D_R > D_B$

We now rigorously define the infection dynamics of the remanent infection model for $D_R > D_B$, assuming without loss of generality that $D_B = 2$. To emphasize the similarities, we use as much as possible the same notations that were already used for the single-rate KS infection model.

We use a construction of the dynamics with $D_R > D_B = 2$ that uses random walk trajectories $(\mathcal{W}, u)$ with constant jump rate 2, for which our reference probability space for paths $(\mathcal{W}, u)$ is $(\Omega, \mathcal{F}, \mathbb{P}_w)$. As long as a particle is blue, it follows the corresponding trajectory in the usual way, while, as soon as it is turned into a red particle, it starts following the trajectory with a speed multiplied by a factor $D_R/2$. As a result, the actual path (W, u) followed by a particle is related to the path $(\mathcal{W}, u) \in \Omega$ by a time-change, which we describe below.

Let us first define the trajectory of the front. Since, by definition, the front can only perform upward jumps, it makes sense to start with $r_0 := 0$, which leads to the simplification that $\mathfrak{r}_k := k$ for all $k \geq 0$. We start with $T_0 := 0$, $r_0 := 0$, and define inductively the sequence $(T_k)_{k\geq 0}$ together with the value of $(r_t)_{t\in[0,T_k]}$. Consider $t > T_\ell$. We say that t is upward if there exists $(\mathcal{W}, u) \in \Psi$ such that $\mathcal{W}_s \leq r_s$ for some $s \in [0, t[$ and such that $\mathcal{W}_{v-} = \ell$ and $\mathcal{W}_v = \ell + 1$, where

$$v := \tau + \frac{D_R}{2}(t - \tau), \; \tau := \inf\{s \in [0, t[; \; \mathcal{W}_s \leq r_s\}. \tag{77}$$

Then let

$$T_{\ell+1} := \inf\{t > T_\ell; \quad t \text{ is upward}\},$$

and

$$r_t := \ell \text{ on } [T_\ell, T_{\ell+1}[.$$

The sets R_t and B_t of red and blue particles at time t are then defined exactly as in the single-rate KS infection model, namely

$$B_t := \{(\mathcal{W}, u) \in \Psi; \; \forall s \in [0, t[, \; \mathcal{W}_s > r_s\},$$

$$R_t := \{(\mathcal{W}, u) \in \Psi; \; \exists s \in [0, t[, \; \mathcal{W}_s \leq r_s\}.$$

We now properly define (W, u) as a time-changed version of $(\mathcal{W}, u)$. Using the notations defined in (77), we let $W_t := \mathcal{W}_t$ for $t \in [0, \tau]$ and $W_t := \mathcal{W}_v$ for $t > \tau$.

We now make the following key remark.

Lemma 30. *For all $k \geq 1$, the set B_{T_k} coincides with the set of $(\mathcal{W}, u) \in \Psi$ such that $\mathcal{W}_{T_k} \geq k$, minus the particle that makes the front climb at time T_k.*

Note that the above result is an immediate consequence of the definition when $D_R = D_B$, but not in the present case, due to the time-change.

Proof. One inclusion is immediate: a particle path $(\mathcal{W}, u)$ in B_{T_k} evolves using the jump rate $D_B = 2$ up to at least time T_k, so that $\mathcal{W}_{T_k}$ indeed corresponds to the position W_{T_k} of the corresponding particle at time T_k, and must by definition be $\geq k$. On the other hand, assume that a $(\mathcal{W}, u) \in R_{T_k}$ is such that $\mathcal{W}_{T_k} \geq k$, and hits (or lies below, to include particles in R_{0+}) the front for the first time at a time $\tau < T_k$. Introduce the time $t := \tau + (T_k - \tau)\frac{2}{D_R}$. Since $D_R > D_B = 2$, we have $t < T_k$, and by definition one has $W_t = \mathcal{W}_{T_k} \geq k$, whence the existence of a red particle above k at a time $< T_k$, which contradicts the definition of T_k. □

One now defines the renewal structure exactly as for the single-rate KS infection model, but with the time-changed trajectories W replacing the trajectories $\mathcal{W}$. Similarly, we can define

$$\mathcal{F}_t^R := \sigma((W_s, u); \; s \leq t, \; (\mathcal{W}, u) \in R_t),$$

$$\mathcal{F}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u);\ s \leq t,\ (\mathcal{W}, u) \in R_T),$$
$$\mathcal{G}_t^R := \sigma((W_s, u);\ s \in \mathbb{R},\ (\mathcal{W}, u) \in R_t),$$
$$\mathcal{G}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u);\ s \in \mathbb{R},\ (\mathcal{W}, u) \in R_T).$$

Note that it does not matter whether we define the $\sigma-$algebras $\mathcal{G}_t^R$ using the original or time-changed trajectories, since in both cases the history of the front up to time t is measurable, due to the fact that the $\sigma-$algebra includes the full trajectories (and not just the trajectories up to time t). The same remark is valid for $\mathcal{G}_T^R$, where T is a non-negative random time. With the help of Lemma 30, and of the fact that, for any $(\mathcal{W}, u) \in B_{T_k}$, one has $W_s = \mathcal{W}_s$ for all $s \leq T_k$, it is then possible to re-prove Propositions 8 and 9 in exactly the same way as for the single-rate KS infection model.

The key advantage of introducing remanence in the model is that, when $D_R > D_B = 2$, a comparison holds with the single rate model with jump rate equal to 2.

Lemma 31. *Let ${}^{(1)}r_t$ denote the front of the single-rate KS model with rate 2, and ${}^{(2)}r_t$ denote the front of the remanent KS model. If $D_R > D_B = 2$, one has that ${}^{(1)}r_t \leq {}^{(2)}r_t$ for any t such that ${}^{(1)}r_t$ and ${}^{(2)}r_t$ are distinct from $\dagger$.*

The above Lemma, combined with Proposition 13, yields the key ballisticity estimate needed to reprove the estimates of Section 4 for the remanent KS infection model. The two additional results we need are the following: a version of the strong Markov property restricted to R_T, and an upper bound on the speed exactly similar to Proposition 14. Specifically:

Proposition 17. *The strong Markov property holds for our process: for all $w \in \mathbb{S}_\theta$, all non-negative $(\mathcal{F}_t^R)_{t\geq 0}-$stopping time T, and bounded measurable function F on $\mathcal{D}_+$, one has that, on $\{T < +\infty\}$,*

$$\mathbb{E}_w(F(X(R_T))|\mathcal{F}_T^R) = \mathbb{E}_{X_T(R_T)}(F(X))\ \mathbb{P}_w - a.s., \tag{78}$$

where we use the notation $X := (X_t)_{t\geq 0}$.

Proposition 18. *For the remanent KS infection model, there exist a constant $C_1^\star(\rho) > 0$ and a constant c_{46}, depending on ρ and $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t \geq C_1^\star(\rho)t) \leq c_{46} \exp(-t).$$

It is then possible to reprove all the estimates of Section 4, the only difference being that, at some places, estimates for a random walk with constant jump rate 2 have to be replaced by estimates for a random walk whose jump rate may change from $D_B = 2$ to $D_R > 2$ at some time-point. These estimates are obtained by a simple comparison with a random walk with constant jump rate equal to D_R. One then obtains Proposition 10, leading to the proof of the law of large numbers (Theorem 2), and the central limit theorem (Theorem 3).

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Appendix A: proofs of results in Section 2

Proof of Proposition 1. We do the proof for R_t^K, the argument for Q_t^K being completely similar. For all $(x, i) \in A$ and $t \geq 0$, let $C_{x,i,t} := \exp(\theta Z_t(x, i))$. For $k \geq 0$, let also

$$H_{K,k}(s) := \sum_{(x,i)\in A,\ -K-k\leq x\leq -K} C_{x,i,s}.$$

Now let $\gamma := 2(\cosh\theta - 1)$, and observe that $(C_{x,i,s}\exp(-\gamma s))_{s\geq 0}$ is a càdlàg martingale. As a consequence, so is $(H_{K,k}(s)\exp(-\gamma s))_{s\geq 0}$ for all $k \geq 0$, and we have the following inequality, valid for all $\lambda > 0$:

$$P\left(\sup_{s\in[0,t]} H_{K,k}(s)\exp(-\gamma s) > \lambda\right) \leq \lambda^{-1}E\left(H_{K,k}(0)\right). \tag{79}$$

Since $E\left(H_{K,k}(0)\right) = \sum_{(x,i)\in A;\, -K-k\leq x\leq -K}\exp(\theta x)$, we deduce, replacing λ by $\lambda e^{-\gamma t}$ in (79), that

$$P\left(\sup_{s\in[0,t]} H_{K,k}(s) > \lambda\right) \leq \lambda^{-1}\exp(\gamma t)\sum_{(x,i)\in A,\ -K-k\leq x\leq -K}\exp(\theta x). \tag{80}$$

Now observe that, for every s, the sequence $(H_{K,k}(s))_{k\geq 0}$ is non-decreasing since we are summing non-negative terms. As a consequence,

$$P\left(\sup_{s\in[0,t]} R_s^K > \lambda\right) = P\left(\bigcup_{k=0}^{+\infty}\sup_{s\in[0,t]} H_{K,k}(s) > \lambda\right),$$

and this last probability is the probability of the union of a non-decreasing sequence of events, so it is equal to

$$\lim_{k\to+\infty} P\left(\sup_{s\in[0,t]} H_{K,k}(s) > \lambda\right).$$

As a consequence, by (80),

$$P\left(\sup_{s\in[0,t]} R_s^K > \lambda\right) \leq \lambda^{-1}\exp(\gamma t)\sum_{(x,i)\in A;\, x\leq -K}\exp(\theta x). \tag{81}$$

As a first consequence, this proves that, with probability one, R_s^K is finite for all s. Now observe that, for every s, the sequence $(\sum_{(x,i)\in A;\, x\leq -K} C_{x,i,s})_{K\geq 0}$ is non-increasing, since we are summing non-negative terms. As a consequence, $\lim_{K\to+\infty}\sup_{s\in[0,t]} R_s^K$ exists, and $P\left(\lim_{K\to+\infty}\sup_{s\in[0,t]} R_s^K > \lambda\right)$ equals $P\left(\bigcap_{K\geq 0}\sup_{s\in[0,t]} R_s^K > \lambda\right)$, which is the probability of the intersection of a non-increasing sequence of events, and so is equal to the limit $\lim_{K\to+\infty} P(\sup_{s\in[0,t]} R_s^K > \lambda)$. From Inequality (81) and the assumption that $w \in \mathbb{S}_\theta$, we see that this last expression equals zero. The same argument applies to $\sup_{s\in[-t,0]} R_s^K$. □

Lemma 32. *For all $p \geq 1$, given $a = \{a_1 \geq \ldots \geq a_p\}$, and $b = \{b_1 \geq \ldots \geq b_p\}$, one has that*

$$\sum_{\ell=1}^{p}|b_\ell - a_\ell| = \min\left\{\sum_{\ell=1}^{p}|b_{\sigma(\ell)} - a_\ell|;\ \sigma\in\mathfrak{S}_p\right\}.$$

Lemma 33. *If $\alpha, \beta, \gamma, \delta$ are real numbers satisfying $\alpha \geq \gamma$ and $\beta \geq \delta$, then*

$$|\alpha - \beta| + |\gamma - \delta| \leq |\alpha - \delta| + |\beta - \gamma|.$$

Proof of Lemma 33. Up to exchanging (α, γ) and (β, δ), we may assume in addition that $\alpha \geq \beta$. The inequality we want to prove then amounts to

$$A := \beta - \delta + |\beta - \gamma| - |\delta - \gamma| \geq 0$$

Now consider the three cases:

- if $\gamma \geq \beta \geq \delta$, then $A = 0$;
- if $\beta \geq \gamma \geq \delta$, then $A = 2(\beta - \gamma) \geq 0$;
- if $\beta \geq \delta \geq \gamma$, then $A = 2(\beta - \delta) \geq 0$.

□

Proof of Lemma 32. Consider a permutation $\sigma \in \mathfrak{S}_p$ distinct from the identity permutation, and let $i := \min\{1 \leq \ell \leq p;\ \sigma(\ell) \neq \ell\}$. Define $f(\sigma) := t \circ \sigma$, where t is the transposition that exchanges i and $\sigma(i)$. By construction, one has that $f(\sigma)(\ell) = \ell$ for all $1 \leq \ell \leq i$. We claim that

$$\sum_{\ell=1}^{p} |b_{f(\sigma)(\ell)} - a_\ell| \leq \sum_{\ell=1}^{p} |b_{\sigma(\ell)} - a_\ell|. \tag{82}$$

Assume for the moment that (82) is proved. Then, starting from any $\sigma \in \mathfrak{S}_p$ distinct from the identity permutation, we can consider $\sigma_0 := \sigma, \sigma_1 := f(\sigma_0), \sigma_2 := f(\sigma_1)$, etc., iterating until we obtain a σ_k equal to the identity permutation. Then, for $0 \leq j \leq k$, let $\delta_j := \sum_{\ell=1}^{p} |b_{\sigma_j(\ell)} - a_\ell|$. By (82), we see that the sequence $(\delta_j)_{0 \leq j \leq k}$ is non-increasing, so that $\delta_k \leq \delta_0$. This entails the conclusion of the lemma. We now prove (82). Since $f(\sigma)(\ell) = \sigma(\ell)$ except for $\ell = i$ and $\ell = \sigma^{-1}(i)$, all we have to check is that

$$|b_{\sigma(i)} - a_{\sigma^{-1}(i)}| + |b_i - a_i| \leq |b_i - a_{\sigma^{-1}(i)}| + |b_{\sigma(i)} - a_i|.$$

By definition of i, we must have that $\sigma(i)$ and $\sigma^{-1}(i)$ do not belong to $\{1, \ldots, i-1\}$. As a consequence, one has that $a_i \geq a_{\sigma^{-1}(i)}$ and $b_i \geq b_{\sigma(i)}$. The result is then a consequence of Lemma 33.

□

Proof of Lemma 1. Let $A_K := \{(x, i) \in A;\ |x| \leq K\}$. By definition, for all t and x, $|S_t^K(x)| = \sum_{(y,j) \in A_K} \mathbf{1}(Z_t(y,j) = x)$ and $|S_t(x)| = \sum_{(y,j) \in A} \mathbf{1}(Z_t(y,j) = x)$. As a consequence,

$$|S_t(x)| - |S_t^K(x)| = \sum_{(y,j) \in A \setminus A_K} \mathbf{1}(Z_t(y,j) = x).$$

We deduce that

$$\sum_{x \in \mathbb{Z}} \Big| |S_t(x)| - |S_t^K(x)| \Big| e^{-\theta|x|} \leq Q_t^K + R_t^K.$$

Now one has $S_t^K(x) := \{u(y,j);\ Z_t(y,j) = x,\ (y,j) \in A_K\}$ and $S_t(x) := \{u(y,j);\ Z_t(y,j) = x,\ (y,j) \in A\}$. Using Lemma 32, we see that

$$d(S_t(x), S_t^K(x)) \leq \sum_{(y,j)\in A\setminus A_K} \mathbf{1}(Z_t(y,j) = x),$$

by considering the permutation pairing together particle labels associated with the same (y,j), and bounding above the remaining distance terms by 1. As above, we deduce that

$$\sum_{x\in\mathbb{Z}} d(S_t(x), S_t^K(x))e^{-\theta|x|} \leq Q_t^K + R_t^K.$$

□

Proof of Proposition 2. For $\ell = 1,2$, let $A_\ell := \{(x,i);\ x \in \mathbb{Z},\ 1 \leq i \leq |w_\ell(x)|)\}$. We write $S_\ell(x) = \{u^{(\ell)}(x,1) \geq \cdots \geq u^{(\ell)}(x,|w_\ell(x)|)\}$. Then consider an i.i.d. family of random walks $(Z_t(x,i),\ (x,i) \in A_1 \cup A_2)$ as above, and, for $\ell = 1,2$, let $S_t^{(\ell)}(x) := \{u(y,j);\ Z_t(y,j) = x,\ (y,j) \in A_\ell\}$. Then modify $S^{(1)}$ and $S^{(2)}$ on an event of probability zero N_0, as was done for S. It is clear from the definition that $S^{(1)}$ and $S^{(2)}$ are versions of S starting respectively from w_1 and w_2. Denote by B the symmetric difference of A_1 and A_2, i.e. $B := (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$, and $C := A_1 \cap A_2$. Now introduce

$$D_t := \sum_{(x,i)\in B,\ x\geq 0} \exp(-\theta Z_t(x,i)) + \sum_{(x,i)\in B,\ x<0} \exp(\theta Z_t(x,i)),$$

and

$$\begin{aligned} F_t \quad := \quad & \sum_{\substack{(x,i)\in C,\\ x\geq 0}} |u^{(2)}(x,i) - u^{(1)}(x,i)| \exp(-\theta Z_t(x,i)) \\ + \quad & \sum_{\substack{(x,i)\in C,\\ x<0}} |u^{(2)}(x,i) - u^{(1)}(x,i)| \exp(\theta Z_t(x,i)). \end{aligned}$$

We claim that one has the inequality

$$d(S_t^{(1)}(x), S_t^{(2)}(x)) \leq 2D_t + F_t. \tag{83}$$

Indeed, by definition, for all t and x, $|S_t^{(\ell)}(x)| = \sum_{(y,j)\in A_\ell} \mathbf{1}(Z_t(y,j) = x)$. As a consequence,

$$|S_t^{(2)}(x)| - |S_t^{(1)}(x)| = \sum_{(y,j)\in A_2\setminus A_1} \mathbf{1}(Z_t(y,j) = x) - \sum_{(y,j)\in A_1\setminus A_2} \mathbf{1}(Z_t(y,j) = x),$$

whence the inequality

$$\Big||S_t^{(2)}(x)| - |S_t^{(1)}(x)|\Big| \leq \sum_{(y,j)\in B} \mathbf{1}(Z_t(y,j) = x),$$

from which we deduce that

$$\sum_{x\in\mathbb{Z}} \left| |S_t^{(2)}(x)| - |S_t^{(1)}(x)| \right| e^{-\theta|x|} \leq D_t. \tag{84}$$

Now let us prove that, for all t and x,

$$\begin{aligned} d(S_t^{(1)}(x), S_t^{(2)}(x)) &\leq \sum_{(y,j)\in C} |u^{(2)}(y,j) - u^{(1)}(y,j)| \mathbf{1}(Z_t(y,j) = x) \\ &+ \sum_{(y,j)\in B} \mathbf{1}(Z_t(y,j) = x). \end{aligned}$$

Indeed, this inequality is a consequence of Lemma 32, using a permutation that pairs the particle labels associated with the same (y, j), while bounding above the remaining distance terms by 1. Summing over the values of x, we obtain that

$$\sum_{x\in\mathbb{Z}} d(S_t^{(1)}(x), S_t^{(2)}(x)) e^{-\theta|x|} \leq F_t + D_t. \tag{85}$$

Combining (85) with (84), we deduce the validity of (83). Now observe that, as in the proof of Proposition 1, $(\exp(-\gamma s)(F_s+D_s))_{s\geq 0}$ is the non-decreasing limit of càdlàg martingales involving only a finite number of particles. We deduce that

$$P\left(\sup_{s\in[0,t]} F_s + D_s > \lambda\right) \leq \lambda^{-1} \exp(\gamma t)(F_0 + D_0).$$

Using the fact that $F_0 + D_0 \leq d_\theta(w_1, w_2)$ and (83), we deduce the conclusion of the proposition. □

Proof of Proposition 3. The statement concerning boundedness is immediate, so we focus on the uniform continuity. Consider $w_1, w_2 \in \mathbb{S}_\theta$ and the coupling defined in Proposition 2. Let M be such that $|f_i| \leq M$ for all $1 \leq i \leq m$, and consider $\epsilon > 0$, and let $\delta > 0$ be such that, for all $1 \leq i \leq m$, $|f_i(g_1) - f_i(g_2)| \leq \epsilon$ whenever $d_\theta(g_1, g_2) \leq \delta$. Then let

$$G := \left\{ \max_{1\leq i\leq m} d_\theta(S_{t_i}^{(1)}, S_{t_i}^{(2)}) > \delta \right\}.$$

By Proposition 2, we have that

$$P(G) \leq 2\delta^{-1} \exp(\gamma t_m) d_\theta(w_1, w_2).$$

By coupling, we then have that

$$\left| E(f_1(S_{t_1}^{(1)}) \times \cdots \times f_m(S_{t_m}^{(1)})) - E(f_1(S_{t_1}^{(2)}) \times \cdots \times f_m(S_{t_m}^{(2)})) \right|$$

is bounded above by

$$M^m P(G) + m\epsilon M^{m-1} P(G^c),$$

whence the fact that

$$|\mathbb{E}_{w_1}(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m})) - \mathbb{E}_{w_2}(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m}))|$$

is bounded above by

$$M^m 2\delta^{-1}\exp(\gamma t_m) d_\theta(w_1, w_2) + m\epsilon M^{m-1}.$$

□

Proof of Proposition 4. For functions of the form $F = f_1(X_{t_1})\times\cdots\times f_m(X_{t_m})$, where $m \geq 1$, $f_1, \ldots, f_m$ are uniformly bounded continuous functions from $\mathbb{S}_\theta$ to $\mathbb{R}$, and $t_1 \leq \cdots \leq t_m$, the result is a consequence of Proposition 3. The conclusion for a general F follows by a monotone class argument, using the fact that[14], on a metric space, the Borel $\sigma-$algebra is generated by bounded uniformly continuous functions. □

The following is a reformulation of the approximation of the process by truncated versions that we state for future reference in the proofs. Define X_t^K as X_t, but taking into account only those particle paths whose location at time zero lies within $[-K, K]$.

Corollary 11. *For $w \in \mathbb{S}_\theta$, then, $\mathbb{P}_w-$almost surely, as K goes to infinity, X_t^K converges uniformly to X_t on every bounded interval.*

Proof of Proposition 5. We first prove the result for functions of the form $F = f_1(X_{t_1})\times\cdots\times f_m(X_{t_m})$, where $m \geq 1$, $f_1, \ldots, f_m$ are bounded uniformly continuous functions from $\mathbb{S}_\theta$ to $\mathbb{R}$, and $0 \leq t_1 \leq \cdots \leq t_m$. Note that $(X_t^K)_{t\geq 0}$ is easily seen to be a $(\mathcal{F}_t)_{t\geq 0}-$Markov process. As a consequence,

$$\mathbb{E}_w(F((X_{t+s}^K)_{s\geq 0})|\mathcal{F}_t) = \mathbb{E}_{X_t^K}(F((X_s)_{s\geq 0}))\ \mathbb{P}_w - a.s. \tag{86}$$

Now, using Corollary 11 and the specific form of F, one has that, as K goes to infinity, $F((X_{t+s}^K)_{s\geq 0})$ converges to $F((X_{t+s})_{s\geq 0})$ a.s. By dominated convergence, we deduce that the l.h.s. of (86) converges a.s. to $\mathbb{E}_w(F((X_{t+s})_{s\geq 0})|\mathcal{F}_t)$. Moreover, Proposition 3 shows that the r.h.s. converges a.s. to $\mathbb{E}_{X_t}(F((X_s)_{s\geq 0}))$. The conclusion for a general F follows by a monotone class argument. □

Proof of Proposition 6. For all $k \geq 1$, let $T^{(k)} := 2^{-k}(\lceil 2^k T\rceil)$. By construction, each $T^{(k)}$ is a stopping time satisfying $T^{(k)} \geq T$, and one has that $\lim_{k\to+\infty} T^{(k)} = T$. Since $T^{(k)}$ takes its values in a countable set, the validity of (4) for $T^{(k)}$ stems from the simple Markov property, summing over all the possible values of $T^{(k)}$. Let F be of the form $F = f_1(X_{t_1})\times\cdots\times f_m(X_{t_m})$, where $m \geq 1$, $f_1, \ldots, f_m$ are bounded uniformly continuous functions from $\mathbb{S}_\theta$ to $\mathbb{R}$, and $0 \leq t_1 \leq \cdots \leq t_m$. Introduce the map G defined on $\mathbb{S}_\theta$ by $G(w) := \mathbb{E}_w F((X_t)_{t\geq 0})$. Since (4) holds for $T^{(k)}$, we have that, on $\{T < +\infty\}$,

$$\mathbb{E}_w(F((X_{T^{(k)}+t})_{t\geq 0})|\mathcal{F}_{T^{(k)}}) = \mathbb{E}_{X_{T^{(k)}}}(F((X_t)_{t\geq 0})) = G(X_{T^{(k)}})\ \mathbb{P}_w - a.s.$$

[14] Because, for instance, one can write the indicator function of a closed set as the non-increasing limit of a sequence of bounded uniformly continuous functions.

Taking the conditional expectation in the above identity, and using the fact that $\mathcal{F}_T \subset \mathcal{F}_{T^{(k)}}$ since $T \leq T^{(k)}$, we obtain that

$$\mathbb{E}_w(F((X_{T^{(k)}+t})_{t\geq 0})|\mathcal{F}_T) = \mathbb{E}_w(G(X_{T^{(k)}})|\mathcal{F}_T)\ \mathbb{P}_w - a.s. \tag{87}$$

We now take the limit $k \to +\infty$ on both sides of the identity, working on $\{T < +\infty\}$. Since we work on a space of càdlàg trajectories, $\lim_{k\to+\infty} X_{T^{(k)}} = X_T$. Since G is continuous by Proposition 3, one has that $\lim_{k\to+\infty} G(X_{T^{(k)}}) = G(X_T)$. As a consequence, the r.h.s. of (87) converges, as k goes to infinity, to $\mathbb{E}(G(X_T)|\mathcal{F}_T)$, which is a.s. equal to $G(X_T)$ since X_T is $\mathcal{F}_T-$measurable (using again the fact that trajectories are càdlàg). On the other hand, by continuity of the f_i, one has that $F((X_{T^{(k)}+t})_{t\geq 0})$ converges to $F((X_{T+t})_{t\geq 0})$ a.s. We deduce that (4) holds when F has the specific form we have assumed. The conclusion for a general F follows by a monotone class argument. □

Proof of Proposition 7 (complement). We give a more detailed argument of how we deduce invariance with respect to time-shifts. The conclusion of the proposition is equivalent to the fact that, for any $t_1 < \cdots < t_m$, and for any $t \geq 0$, the distribution of $(X_{t+t_1}, \ldots, X_{t+t_m})$ is the same as the distribution of $(X_{t_1}, \ldots, X_{t_m})$. The computation used in the proof of Proposition 5.3 in [10] shows that, given $0 = t_1 < \cdots < t_m$, and bounded measurable maps $f_1, \ldots, f_m : \mathbb{S}_\theta \to \mathbb{R}$, one has the identity

$$\mathbb{E}_\nu(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m})) = \mathbb{E}_\nu(f_1(X_{t_m-t_1}) \times \cdots \times f_m(X_{t_m-t_m})).$$

Now by definition of the dynamics for negative times, one has that

$$\mathbb{E}_\nu(f_1(X_{t_m-t_1}) \times \cdots \times f_m(X_{t_m-t_m})) = \mathbb{E}_\nu(f_1(X_{t_1-t_m}) \times \cdots \times f_m(X_{t_m-t_m})).$$

(We can neglect the fact that, since the paths $(W_s)_s$ are assumed to be càdlàg, the paths $(W_{-s})_{s\geq 0}$ are in fact càglàd, since we consider a finite number of time indices $t_1, \ldots, t_m$, which are a.s. not jump times of any of the random walk paths $(W, u) \in \Psi$.) We deduce that, for any $s_1 < \cdots < s_m = 0$, we have that

$$\mathbb{E}_\nu(f_1(X_{s_1}) \times \cdots \times f_m(X_{s_m})) = \mathbb{E}_\nu(f_1(X_{s_1-s_1}) \times \cdots \times f_m(X_{s_m-s_1})). \tag{88}$$

Now consider $t_1 < \cdots < t_m$ with $t_\ell = 0$ for some $1 < \ell < m$ (starting from arbitrary $t_1 < \cdots < t_m$, one may always add indices i and functions $f_i \equiv 1$ so that this is the case). Conditioning by $\mathcal{F}_0$, we obtain that

$$\mathbb{E}_\nu(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m})) = \mathbb{E}_\nu(f_1(X_{t_1}) \times \cdots \times f_\ell(X_{t_\ell})G(X_{t_\ell})),$$

where

$$G(w) := \mathbb{E}_w(f_{\ell+1}(X_{t_{\ell+1}}) \times \cdots \times f_m(X_{t_m})).$$

From (88), we deduce that

$$\mathbb{E}_\nu(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m})) = \mathbb{E}_\nu(f_1(X_{t_1-t_1}) \times \cdots \times f_\ell(X_{t_\ell-t_1})G(X_{t_\ell-t_1})),$$

whence, from the definition of G and the Markov property,

$$\mathbb{E}_\nu(f_1(X_{t_1}) \times \cdots \times f_m(X_{t_m})) = \mathbb{E}_\nu(f_1(X_{t_1-t_1}) \times \cdots \times f_m(X_{t_m-t_1})).$$

The r.h.s. of this identity is left unchanged if $t_1, \dots, t_m$ is replaced by $t + t_1, \dots, t + t_m$. □

Appendix B: adaptation of ballisticity results from [6]

The results in [6] showing the ballistic behavior of the front are established for the following kind of initial condition: i.i.d. Poisson numbers of blue particles at each site, plus a deterministic finite and non-zero number of red particles placed arbitrarily. Also, the results are for $\sup_{s\in[0,t]} r_s$ than for r_t. Since our framework is slightly different, we explain here how these results can be adapted to prove Propositions 12 and 14.

We denote by $\tilde{\nu}$ the distribution obtained by adding to the Poisson process of particles defining ν a single particle at the origin[15].

Proof of Proposition 12. The conclusion of the Proposition is established in [6] (Theorem 2) for the random variable $\sup_{s\in[0,t]} r_s$ under $\mathbb{P}_{\tilde{\nu}}$. Considering the particle that first hits level $C_2(\rho)t$ in the case where $\sup_{s\in[0,t]} r_s > C_2(\rho)t$ and its maximum possible deviation over the interval $[s,t]$ as in the proof of Lemma 17, we see that, up to choosing a strictly smaller value for $C_2(\rho)$, the conclusion of the Proposition holds for r_t with respect to $\mathbb{P}_{\tilde{\nu}}$.

Now observe that, with respect to ν, conditional on $r_0 = -k$, the sets of particle labels at sites $x \geq 1$ and $x \leq -k-1$ form i.i.d. Poisson processes on $[0,1]$ with rate ρ, while there are 0 particles at sites $x = -k+1, \dots, 0$, and a Poisson process conditioned on having at least one element at site $x = -k$. On the other hand, with respect to $\tilde{\nu}$ conditioned on the fact that there is only one particle at zero (which must then be the added particle), and that there are no particles at sites $1, 2, \dots, k$, the sets of labelled particles at sites $x \geq k+1$ and $x \leq -1$ are i.i.d. Poisson processes, while there are 0 particles at sites $x = 1, \dots, k$. Denoting by L_k the event that there is a single particle at the origin and no particles at sites $1, 2, \dots, k$, we deduce, using Lemma 14, that the distribution of $r_t - r_0$ with respect to $\mathbb{P}_\nu$ conditioned on $r_0 = -k$ stochastically dominates the distribution of r_t with respect to $\mathbb{P}_{\tilde{\nu}}$ conditioned on L_k. Similarly, when $k_1 \leq k_2$, the distribution of r_t with respect to $\mathbb{P}_{\tilde{\nu}}$ conditioned on L_{k_1} stochastically dominates that of r_t with respect to $\mathbb{P}_{\tilde{\nu}}$ conditioned on L_{k_2}. We thus have, for all $k \geq 1$, that

$$\mathbb{P}_\nu(r_t - k \leq C_2(\rho)t) \leq \mathbb{P}_{\tilde{\nu}}(r_t \leq C_2(\rho)t | L_k)\mathbb{P}_\nu(r_0 \geq -k) + \mathbb{P}_\nu(r_0 < -k). \quad (89)$$

Now we have that $\mathbb{P}_\nu(r_0 < -k) = e^{-\rho k}$, and $\mathbb{P}_{\tilde{\nu}}(L_k) = e^{-\rho(k+1)}$. On the other hand, by Theorem 2 of [6], we have that, for all $K > 0$ there exists a constant $d_1 > 0$ (depending on K) such that, for all $t > 0$, $\mathbb{P}_{\tilde{\nu}}(r_t \leq C_2(\rho)t) \leq d_1 t^{-K}$.

[15] Labels play no real role in the sequel. Still, for the sake of compatibility with the statement of Lemma 14, which involves comparison of sets of labels, not just of particle numbers, one may choose the label of this added particle as an exponential random variable with parameter ρ conditioned upon being ≤ 1, which corresponds to the smallest label in a Poisson process with rate ρ on $[0,1]$ conditioned upon containing at least one point.

Letting $k := \lceil A \log t \rceil$ for some $A > 0$, we deduce from (89) that

$$\mathbb{P}_\nu(r_t - k \leq C_2(\rho)t) \leq d_1 e^\rho t^{-K+A\rho} + t^{-A\rho}.$$

Since we can choose $A > 0$ and $K > 0$ at our convenience, the conclusion follows, up to choosing a strictly smaller value for $C_2(\rho)$. □

Proof of Proposition 14. Observe that $\tilde{\nu}$ dominates ν_+, so that, by Lemma 14, the distribution of r_t with respect to $\mathbb{P}_{\tilde{\nu}}$ dominates the distribution of r_t with respect to $\mathbb{P}_{\nu_+}$. The conclusion of the Proposition is then a consequence of Theorem 1 in [6]. □

Appendix C: miscellaneous lemmas

Proof of Lemma 3. One has that A_1 must be of the form $\{(\zeta^1_{i_1}, \zeta^1_{i_2}, \cdots) \in B_1\}$, where $(i_j)_{j\in\mathbb{N}}$ is a family of elements of I, and where B_1 belongs to the product $\sigma-$algebra $\bigotimes_{j\in\mathbb{N}} \mathcal{S}_j$. The results follows from letting

$$A_2 := \{(\zeta^2_{i_1}, \zeta^2_{i_2}, \cdots) \in B_1\}.$$

□

Lemma 34. *Let T denote a non-negative random variable on $(\Omega, \mathcal{F})$. Then for all $t \geq 0$, B_T is $\mathbb{P}_w-$a.s. equal to a random variable from $(\Omega, \mathcal{F})$ to $(\Omega, \mathcal{F})$.*

Proof. The fact that the labels of particles in B_T are distinct, and that no two particle paths jump at the same time is a consequence of B_T being a subset of Ψ. If $r_T = \dagger$, then B_T is an empty set. If $r_T < +\infty$, Corollary 11 shows that B_T is a.s. equal to B_0 up to a finite number of trajectories, and that a.s. $t \mapsto X_t(B_T)$ is in $\mathcal{D}$. Then one checks that the addition or removal of a finite number of trajectories in ψ does not affect the fact that $t \mapsto X_t(\psi)$ is in $\mathcal{D}$. As for measurability, just note that the random variables of the form $\#(B_t \cap \{(W,u);\ W_s = k,\ u \in [a,b]\})$, where $k \in \mathbb{Z}$, $0 \leq a < b \leq 1$, and $s \in \mathbb{R}$, are $\mathcal{F}-$measurable. □

Lemma 35. *If s is a backward sub-α time and if t is an $(s,\alpha)-$crossing time, then t is also a backward sub-α time.*

Proof. Assume that s is a backward sub-α time and t is an $(s,\alpha)-$crossing time. For $v \in [s,t[$, one has that $r_v < k + \alpha(v-s)$. Combined with the fact that $r_t \geq k+\alpha(t-s)$, this leads to $r_v < r_t - \alpha(t-s) + \alpha(v-s) = r_t - \alpha(t-v)$. For $v \in [0,s[$, we have that $r_v \leq r_s - \alpha(s-v) \leq r_t - \alpha(t-s) - \alpha(s-v) = r_t - \alpha(t-s)$. □

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