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# Brandt’s Fully Private Auction Protocol Revisited

Jannik Dreier\*    Jean-Guillaume Dumas†    Pascal Lafourcade\*

## Abstract

Auctions have a long history, having been recorded as early as 500 B.C. [17]. Nowadays, electronic auctions have been a great success and are increasingly used. Many cryptographic protocols have been proposed to address the various security requirements of these electronic transactions, in particular to ensure privacy. Brandt [4] developed a protocol that computes the winner using homomorphic operations on a distributed ElGamal encryption of the bids. He claimed that it ensures full privacy of the bidders, i.e. no information apart from the winner and the winning price is leaked. We first show that this protocol – when using malleable interactive zero-knowledge proofs – is vulnerable to attacks by dishonest bidders. Such bidders can manipulate the publicly available data in a way that allows the seller to deduce all participants’ bids. Additionally we discuss some issues with verifiability as well as attacks on non-repudiation, fairness and the privacy of individual bidders exploiting authentication problems.

## 1 Introduction

Auctions are a simple method to sell goods and services. Typically a *seller* offers a good or a service, and the *bidders* make offers. Depending on the type of auction, the offers might be sent using sealed envelopes which are opened simultaneously to determine the winner (the “sealed-bid” auction), or an *auctioneer* could announce prices decreasingly until one bidder is willing to pay the announced price (the “dutch auction”). Additionally there might be several rounds, or offers might be announced publicly directly (the “English” or “shout-out” auction). The winner usually is the bidder submitting the highest bid, but in some cases he might only have to pay the second highest offer as a price (the “second-price”- or “Vickrey”-Auction). In general a bidder wants to win the auction at the lowest possible price, and the seller wants to sell his good at the highest possible price. For more information on different auction methods see [17]. To address this huge variety of possible auction settings and to achieve different security and efficiency properties numerous protocols have been developed, e.g. [4, 11, 19, 20, 21, 22, 23] and references therein.

One of the key requirements of electronic auction (e-Auction) protocols is privacy, i.e. the bids of losing bidders remain private. Brandt proposed a first-price sealed-bid auction protocol [4, 3, 2] and claimed that it is fully private, i.e. it leaks no information apart from the winner, the winning bid, and what can be deduced from these two facts (e.g. that the other bids were lower).

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**Our Contributions.** The protocol is based on an algorithm that computes the winner using bids encoded as bit vectors. In this paper we show that the implementation using the homomorphic property of a distributed Elgamal encryption proposed in the original paper suffers from a weakness. In fact, we prove that any two different inputs (i.e. different bids) result in different outcome values, which are only hidden using random values. We show how a dishonest participant can remove this random noise, if malleable interactive zero-knowledge proofs are used. The seller can then efficiently compute the bids of all bidders, hence completely breaking privacy. We also discuss two problems with verifiability, and how the lack of authentication enables attacks on privacy even if the above attack is prevented via non-malleable non-interactive proofs. Additionally we show attacks on non-repudiation and fairness, and propose solutions to all discovered flaws in order to recover a fully resistant protocol.

**Outline.** In the next section, we recall the protocol of Brandt. Then, in the following sections, we present our attacks in several steps. In Section 3, we first study the protocol using interactive zero-knowledge proofs and without noise. Then we show how a dishonest participant can remove the noise, thus mount the attack on the protocol with noise, and discuss countermeasures. Finally, in Section 4, we discuss verifiability and in Section 5 we discuss attacks on fairness, non-repudiation and privacy exploiting the lack of authentication.

## 2 The Protocol

The protocol of Brandt [4] was designed to ensure full privacy in a completely distributed way. It exploits the homomorphic properties of a distributed El-Gamal encryption scheme [12] for a secure multi-party computation of the winner. Then it uses zero-knowledge proofs of knowledge of discrete logarithms to ensure correctness of the bids while preserving privacy. We first give a high level description of the protocol and then present details on its main cryptographic primitives.

### 2.1 Informal Description

The participating  $n$  bidders and the seller communicate essentially using broadcast messages. The latter can for example be implemented using a bulletin board, i.e. an append-only memory accessible to everybody. The bids are encoded as  $k$ -bit-vectors where each entry corresponds to a price. If the bidder  $a$  wants to bid the price  $b_a$ , all entries will be 1, except the entry  $b_a$  which will be  $Y$  (a public constant). Each entry of the vector is then encrypted separately using a  $n$ -out-of- $n$ -encryption scheme set up by all bidders. The bidders use multiplications of the encrypted values to compute values  $v_{a,j}$ , exploiting the homomorphic property of the encryption scheme. Each one of this values is 1 if the bidder  $a$  wins at price  $j$ , and is a random number otherwise. The decryption of the final values takes place in a distributed way to ensure that nobody can access intermediate values.

### 2.2 Mathematical Description (Brandt [4])

Let  $\mathbb{G}_q$  be a multiplicative subgroup of order  $q$ , prime, and  $g$  a generator of the group. We consider that  $i, h \in \{1, \dots, n\}$ ,  $j, bid_a \in \{1, \dots, k\}$  (where  $bid_a$  is the bid chosen

by the bidder with index  $a$ ),  $Y \in \mathbb{G}_q \setminus \{1\}$ . More precisely, the  $n$  bidders execute the following five steps of the protocol [4]:

### 1. Key Generation

Each bidder  $a$ , whose bidding price is  $bid_a$  among  $\{1, \dots, k\}$  does the following:

- chooses a secret  $x_a \in \mathbb{Z}/q\mathbb{Z}$
- chooses randomly  $m_{ij}^a$  and  $r_{aj} \in \mathbb{Z}/q\mathbb{Z}$  for each  $i$  and  $j$ .
- publishes  $y_a = g^{x_a}$  and proves the knowledge of  $y_a$ 's discrete logarithm.
- using the published  $y_i$  then computes  $y = \prod_{i=1}^n y_i$ .

### 2. Bid Encryption

Each bidder  $a$

- sets  $b_{aj} = \begin{cases} Y & \text{if } j = bid_a \\ 1 & \text{otherwise} \end{cases}$
- publishes  $\alpha_{aj} = b_{aj} \cdot y^{r_{aj}}$  and  $\beta_{aj} = g^{r_{aj}}$  for each  $j$ .
- proves that for all  $j$ ,  $\log_g(\beta_{aj})$  equals  $\log_y(\alpha_{aj})$  or  $\log_y\left(\frac{\alpha_{aj}}{Y}\right)$ , and that  $\log_y\left(\frac{\prod_{j=1}^k \alpha_{aj}}{Y}\right) = \log_g\left(\prod_{j=1}^k \beta_{aj}\right)$ .

### 3. Outcome Computation

- Each bidder  $a$  computes and publishes for all  $i$  and  $j$ :

$$\gamma_{ij}^a = \left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \alpha_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \alpha_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \alpha_{hj} \right) \right)^{m_{ij}^a}$$

$$\delta_{ij}^a = \left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \beta_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \beta_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \beta_{hj} \right) \right)^{m_{ij}^a}$$

and proves its correctness.

### 4. Outcome Decryption

- Each bidder  $a$  sends  $\phi_{ij}^a = \left( \prod_{h=1}^n \delta_{ij}^h \right)^{x_a}$  for each  $i$  and  $j$  to the seller and proves its correctness. After having received all values, the seller publishes  $\phi_{ij}^h$  for all  $i, j$ , and  $h \neq i$ .

### 5. Winner determination

- Everybody can now compute  $v_{aj} = \frac{\prod_{i=1}^n \gamma_{aj}^i}{\prod_{i=1}^n \phi_{aj}^i}$  for each  $j$ .
- If  $v_{aw} = 1$  for some  $w$ , then the bidder  $a$  wins the auction at price  $p_w$ .

## 2.3 Malleable proofs of knowledge and discrete logarithms

In the original paper [4] the author suggests using zero-knowledge proofs of knowledge to protect against active adversaries. The basic protocols he proposes are interactive and malleable, but can be converted into non-interactive proofs using the Fiat-Shamir heuristic [13], as advised by the author. We first recall the general idea of such proofs, then we expose the man-in-the-middle attacks on the interactive version, which we will use as part of our first attack.

Let **PDL** denote a *proof of knowledge of a discrete logarithm*. A first scheme for **PDL** was developed in 1986 by Chaum et al. [6]. In the original auction paper [4] Brandt proposes to use a non-interactive variant of **PDL** as developed by Schnorr [24],

which are malleable. Unfortunately, interactive malleable **PDL** are subject to man-in-the-middle attacks [16]. We first recall the classic  $\Sigma$ -protocol on a group with generator  $g$  and order  $q$  [1, 5, 7]. Peggy and Victor know  $v$  and  $g$ , but only Peggy knows  $x$ , so that  $v = g^x$ . She can prove this fact, without revealing  $x$ , by executing the following protocol:

1. Peggy chooses  $r$  at random and sends  $z = g^r$  to Victor.
2. Victor chooses a challenge  $c$  at random and sends it to Peggy.
3. Peggy sends  $s = (r + c \cdot x) \pmod q$  to Victor.
4. Victor checks that  $g^s = z \cdot v^c$ .

### 2.3.1 Man-in-the-middle attacks on interactive **PDL**

Suppose Peggy possesses some secret discrete logarithm  $x$ . We present here the man-in-the-middle attack of [16], where an attacker can pretend to have knowledge of any affine combination of the secret  $x$ , even providing the associated proof of knowledge, without breaking the discrete logarithm. To prove this possession to say Victor, the attacker will start an interactive proof knowledge session with Peggy and another one with Victor. The attacker will transform Peggy's outputs and forward Victor's challenges to her. The idea is to use the proof of possession of Peggy's  $x$ , to prove possession of  $1 - x$  to Victor. Indeed to prove for instance possession of just  $x$  to Victor, an attacker would only have to forward Peggy's messages to Victor and Victor's messages to Peggy. The idea of the attack is similar, except that one needs to modify the messages of Peggy. We show the example of  $1 - x$  in Figure 1 since it is used in Section 3.4 to mount our attack. Upon demand by Victor to prove knowledge of  $1 - x$ , Mallory, the man-in-the-middle, simply starts a proof of knowledge of  $x$  with Peggy. Peggy chooses a random exponent  $r$  and sends the commitment  $z = g^r$  to Mallory. Mallory simply inverts  $z$  and sends  $y = z^{-1}$  to Victor. Then Victor presents a challenge  $c$  that Mallory simply forwards without modification to Peggy. Finally Peggy sends a response  $s$  that Mallory combines with  $c$ , as  $u = c - s$ , to provide a correct answer to Victor. This is summarized in Figure 1.

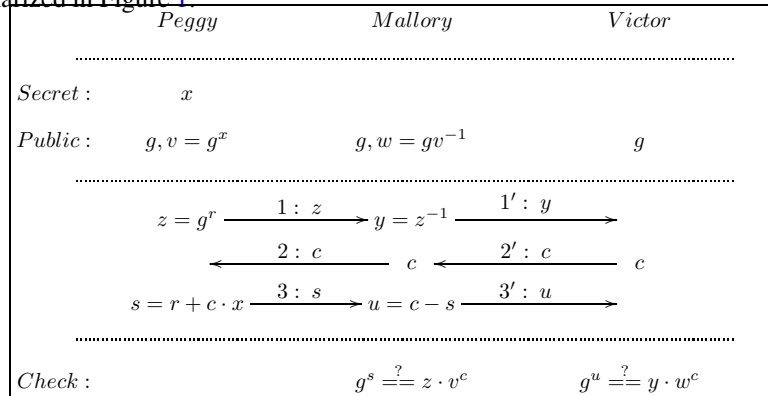


Figure 1: Man-in-the-middle **PDL** of  $1 - x$ , with  $x$  an unknown discrete logarithm.

Actually, the attack works in the generic settings of [5, 18] or of  $\Sigma$ -protocols [10]. We let  $f : \Gamma \rightarrow \Omega$  denote a one way homomorphic function between two commutative groups  $(\Gamma, +)$  and  $(\Omega, \times)$ . We use this generalization to prevent possible countermeasures of our first attack in Section 3.6.

For an integral value  $\alpha$ ,  $\alpha \cdot x \in \Gamma$  (resp.  $y^\alpha \in \Omega$ ) denotes  $\alpha$  applications of the group law  $+$  (resp.  $\times$ ). For a secret  $x \in \Gamma$ , and any  $(h, \alpha, \beta) \in \Gamma \times \mathbb{Z}^2$ , the attacker can build a proof of possession of  $\alpha \cdot h + \beta \cdot x$ . In the setting of the example of Figure 1, we used  $f(x) = g^x$ ,  $h = 1$ ,  $\alpha = 1$  and  $\beta = -1$ .

In the general case also, upon demand of proof by Victor, Mallory starts a proof with Peggy. The secret of Peggy is  $x$ , and the associated witness  $v$  is  $v = f(x)$ . Then Mallory wants to prove that his witness  $w$  corresponds to any combination of  $x$  with a logarithm  $h$  that he knows. With only public knowledge and his chosen  $(h, \alpha, \beta) \in \Gamma \times \mathbb{Z}^2$ , Mallory is able to compute  $w = f(h)^\alpha \cdot v^\beta$ .

For the proof of knowledge, Mallory still modifies the commitment  $z = f(r)$  of Peggy to  $y = z^\beta$ . Mallory forwards the challenge  $c$  of Victor without modification. Finally Mallory transforms the response  $s$  of Peggy, still with only public knowledge and his chosen  $(h, \alpha, \beta) \in \Gamma \times \mathbb{Z}^2$ , as  $u = c \cdot (\alpha \cdot h) + \beta \cdot s$ . We summarize this general attack on Figure 2.

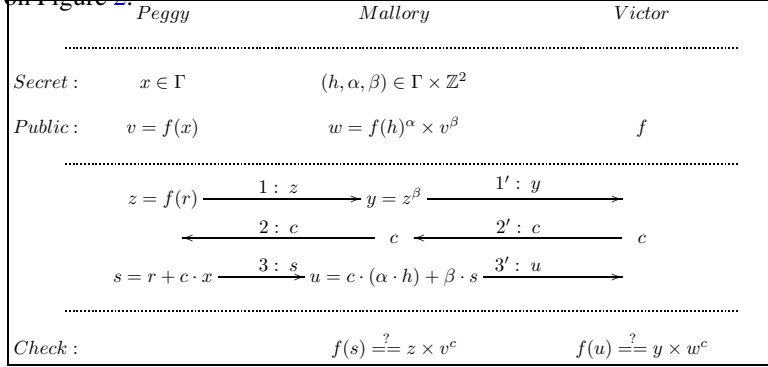


Figure 2: Man-in-the-middle attacks proving knowledge of affine transforms of a secret discrete logarithm in the generic setting.

**Lemma 1.** *In the man-in-the-middle attack of Figure 2 of the interactive proof of knowledge of a discrete logarithm, Victor is convinced by Mallory's proof of knowledge of  $\alpha \cdot h + \beta \cdot x$ .*

*Proof.* Indeed,

$$u = c \cdot (\alpha \cdot h) + \beta \cdot s = c \cdot (\alpha \cdot h) + \beta \cdot (r + c \cdot x) = \beta \cdot r + c \cdot (\alpha \cdot h + \beta \cdot x). \quad (1)$$

Now, since  $z = f(r)$ ,  $y = z^\beta$ ,  $v = f(x)$  and  $f(h)^\alpha \times v^\beta = w$ , the latter Equation (1) proves in turn that

$$f(u) = f(r)^\beta \times f(\alpha \cdot h + \beta \cdot x)^c = z^\beta \times (f(h)^\alpha \times f(x)^\beta)^c = y \times w^c. \quad (2)$$

Now Victor has to verify the commitment-challenge-response  $(y, c, u)$  of Mallory for his witness  $w$ . Then Victor needs to check whether  $f(u)$  corresponds to  $y \times w^c$ , which is the case as shown by the latter Equation (2). □ □

### 2.3.2 Generalizations to equality of discrete logarithms

We let **EQDL** denote a *proof of equality of several discrete logarithms*. Any **PDL** can in general easily be transformed to an **EQDL** by applying it  $k$  times on the same witness. It is often more efficient to combine the application in one as in [8, 9], or more generally as composition of  $\Sigma$ -protocols, here with two logarithms and two generators  $g_1$  and  $g_2$ . Peggy wants to prove that she knows  $x$  such that  $v = g_1^x$  and  $w = g_2^x$ :

1. Peggy chooses  $r$  at random and sends  $\lambda = g_1^r$  and  $\mu = g_2^r$  to Victor.
2. Victor chooses a challenge  $c$  at random and sends it to Peggy.
3. Peggy computes  $s = (r + c \cdot x) \bmod q$  and sends it to Victor.
4. Victor tests if  $g_1^s = \lambda \cdot v^c$  and  $g_2^s = \mu \cdot w^c$ .

This protocol remains malleable, and the previous attacks are still valid since the response remains of the form  $r + c \cdot x$ .

### 2.3.3 Countermeasures

Direct countermeasures to the above attacks are to use non-interactive and/or non-malleable proofs:

- An interactive protocol can be converted into a non-interactive one using the Fiat-Shamir heuristic [13].
- Also the first PDL by [6] uses bit-flipping, and more generally non-malleable protocols like [15] could be used.

We will show in the following that if the proofs proposed in the original paper are not converted into non-interactive proofs, there is an attack on privacy. Note that even if non-interactive non-malleable zero-knowledge proofs are used, a malicious attacker in control of the network can nonetheless recover any bidder's bid as the messages are not authenticated, as we show in Section 5.

## 3 Attacking the fully private computations

The first attack we present uses some algebraic properties of the computations performed during the protocol execution.

### 3.1 Analysis of the outcome computation

The idea is to analyze the computations done in Step 3 of the protocol. Consider the following example with three bidders and three possible prices. Then the first bidder computes

$$\begin{aligned}
\gamma_{11}^1 &= ( (\alpha_{12} \cdot \alpha_{13} \cdot \alpha_{22} \cdot \alpha_{23} \cdot \alpha_{32} \cdot \alpha_{33}) \cdot (1) \cdot (1) )^{m_{11}^1} \\
\gamma_{12}^1 &= ( (\alpha_{13} \cdot \alpha_{23} \cdot \alpha_{33}) \cdot (\alpha_{11}) \cdot (1) )^{m_{12}^1} \\
\gamma_{13}^1 &= ( (1) \cdot (\alpha_{11} \cdot \alpha_{12}) \cdot (1) )^{m_{13}^1} \\
\gamma_{21}^1 &= ( (\alpha_{12} \cdot \alpha_{13} \cdot \alpha_{22} \cdot \alpha_{23} \cdot \alpha_{32} \cdot \alpha_{33}) \cdot (1) \cdot (\alpha_{11}) )^{m_{21}^1} \\
\gamma_{22}^1 &= ( (\alpha_{13} \cdot \alpha_{23} \cdot \alpha_{33}) \cdot (\alpha_{21}) \cdot (\alpha_{12}) )^{m_{22}^1} \\
\gamma_{23}^1 &= ( (1) \cdot (\alpha_{21} \cdot \alpha_{22}) \cdot (\alpha_{13}) )^{m_{23}^1} \\
\gamma_{31}^1 &= ( (\alpha_{12} \cdot \alpha_{13} \cdot \alpha_{22} \cdot \alpha_{23} \cdot \alpha_{32} \cdot \alpha_{33}) \cdot (1) \cdot (\alpha_{11} \cdot \alpha_{21}) )^{m_{31}^1} \\
\gamma_{32}^1 &= ( (\alpha_{13} \cdot \alpha_{23} \cdot \alpha_{33}) \cdot (\alpha_{31}) \cdot (\alpha_{12} \cdot \alpha_{22}) )^{m_{32}^1} \\
\gamma_{33}^1 &= ( (1) \cdot (\alpha_{31} \cdot \alpha_{32}) \cdot (\alpha_{13} \cdot \alpha_{23}) )^{m_{33}^1}
\end{aligned}$$

The second and third bidder do the same computations, but using different random values  $m_{ij}^a$ . Since each  $\alpha_{ij}$  is either the encryption of 1 or  $Y$ , for example the value  $\gamma_{22}^1$  will be an encryption of 1 only if

- nobody submitted a higher bid (the first block) and
- bidder 2 did not bid a lower bid (the second block) and
- no bidder with a lower index submitted the same bid (the third block).

If we ignore the exponentiation by  $m_{ij}^a$ , each  $\gamma_{ij}^a$  is the encryption of the product of several  $b_{ij}$ 's. Each  $b_{ij}$  can be either 1 or  $Y$ , hence  $(\gamma_{ij}^a)^{-m_{ij}^a}$  will be the encryption of a value  $Y^{l_{ij}}$ , where  $0 \leq l_{ij} \leq n$ . The lower bound of  $l_{ij}$  is trivial, the upper bound follows from the observation that each  $\alpha_{ij}$  will be used at most once, and that each bidder will encrypt  $Y$  at most once.

Assume for now that we know all  $l_{ij}$ . We show next that this is sufficient to obtain all bids. Consider the function  $f$  which takes as input the following vector<sup>1</sup>:  $b = \log_Y \left( (b_{11}, \dots, b_{1k}, b_{21}, \dots, b_{2k}, \dots, b_{n1}, \dots, b_{nk})^T \right)$ , and returns the values  $l_{ij}$ . The input vector is thus a vector of all bid-vectors, where 1 is replaced by 0 and  $Y$  by 1. Consider our above example with three bidders and three possible prices, then we have:

$$b = \log_Y \left( (b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33})^T \right).$$

A particular instance where bidder 1 and 3 submit price 1, and bidder 2 submits price 2 would then look as:  $b = (1, 0, 0, 0, 1, 0, 1, 0, 0)^T$ . Hence only the factors  $\alpha_{11}$ ,  $\alpha_{22}$  and  $\alpha_{31}$  are encryptions of  $Y$ , all other  $\alpha$ 's are encryptions of 1. By simply counting how often the factors  $\alpha_{11}$ ,  $\alpha_{22}$  and  $\alpha_{31}$  show up in each equation as described above, we can compute the following result:  $f(b) = (1, 1, 1, 2, 0, 1, 2, 1, 1)^T$ . Note that since we chose the input of  $f$  to be a bit-vector, we have to simply count the ones (which correspond to  $Y$ 's) in particular positions in  $b$ , where the positions are determined by the factors inside  $\gamma_{ij}^a$ . Hence we can express  $f$  as a matrix, i.e.  $f(b) = M \cdot b$  for the following matrix  $M$ :

$$f(b) = M \cdot b = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \mathbf{1} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ \mathbf{0} \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

To see how the matrix  $M$  is constructed, consider for example  $(\gamma_{22}^a)^{-m_{22}^a} = (\alpha_{13} \cdot \alpha_{23} \cdot \alpha_{33}) \cdot (\alpha_{21}) \cdot (\alpha_{12})$  which corresponds to the **second row** in the second vertical block:

- $\alpha_{12}$  and  $\alpha_{13}$ ; hence the two ones at position 2 and 3 in the first horizontal block
- $\alpha_{21}$  and  $\alpha_{23}$ ; hence the two ones at position 1 and 3 in the second horizontal block
- $\alpha_{33}$ ; hence the one at position 3 in the third horizontal block

More generally, we can see that each  $3 \times 3$  block consists of potentially three parts:

<sup>1</sup>By abuse of notation we write  $\log_s(x_1, \dots, x_n)$  for  $(\log_s(x_1), \dots, \log_s(x_n))$ .



- An upper triangular matrix representing all bigger bids.
- On the diagonal we add a lower triangular matrix representing a lower bid by the same bidder,
- In the lower left half we add an identity matrix representing a bid at the current price by a bidder with a lower index.

This corresponds exactly to the structure of the products inside each  $\gamma_{ij}^a$ . It is also equivalent to formula (1) in Section 4.1.1 of the original paper [4] without the random vector  $R_k^*$ . In the following we prove that the function  $f$  is injective. We then discuss how this function can be efficiently inverted (i.e. how to compute the bids when knowing all  $l_{ij}$ 's).

### 3.2 Linear algebra toolbox

Let  $I_k$  be the  $k \times k$  identity matrix;

let  $L_k$  be a lower  $k \times k$  triangular matrix with zeroes on the diagonal, ones in the lower part and zeroes elsewhere; and let  $U_k$  be an upper  $k \times k$  triangular matrix with zeroes on the diagonal, ones in the upper part, and zeroes elsewhere:

$$I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad L_k = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \quad U_k = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

By abuse of notation we use  $I$ ,  $L$  and  $U$  to denote respectively  $I_k$ ,  $L_k$  and  $U_k$ . For a  $k \times k$ -matrix  $M_k$  we define  $(M_k)^r = M \cdots M$  ( $r$  times) and  $(M_k)^0 = I_k$ . Let  $(e_1, \dots, e_k)$  be the canonical basis.

**Lemma 2.** *Matrices  $L_k$  and  $U_k$  have the following properties, for  $0 < j \leq k$  and  $r \geq 0$ :  $(U_k)^r \cdot e_j = \sum_{s=1}^{j-r} e_s$  and  $(L_k)^r \cdot e_j = \sum_{s=j+r}^k e_s$ .*

**Lemma 3.** *Matrices  $L_k$  and  $U_k$  are nilpotent, i.e.  $(U_k)^k = 0$  and  $(L_k)^k = 0$ .*

This follows immediately from Lemma 2 by computing  $(U_k)^k \cdot I_k$  and  $(L_k)^k \cdot I_k$ .

**Lemma 4.** *If  $\sum_{i=1}^k x_i = 1$  then we have  $L_k \cdot x = (1, \dots, 1)^T - (I_k + U_k) \cdot x$ .*

*Proof.* First note that since  $\sum_{i=1}^k x_i = 1$ ,

$$L_k \cdot x = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_1 + x_2 \\ \vdots \\ \sum_{i=1}^{k-1} x_i \end{bmatrix} = \begin{bmatrix} 1 - \sum_{i=1}^k x_i \\ 1 - \sum_{i=2}^k x_i \\ \vdots \\ 1 - x_k \end{bmatrix}$$

On the other hand, if we let  $\mathbf{1} = (1, \dots, 1)^T$ , we have also:

$$\mathbf{1} - (I_k + U_k) \cdot x = \mathbf{1} - \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 1 - \sum_{i=1}^k x_i \\ 1 - \sum_{i=2}^k x_i \\ \vdots \\ 1 - x_k \end{bmatrix}$$

□

**Lemma 5.**  $e_1^T \cdot U^{k-t-1} \cdot z = z_{k-t-1} + e_1^T \cdot U^{k-t} \cdot z$

The proof follows immediately from the fact that  $e_1^T \cdot U^{k-x} = (\underbrace{0, \dots, 0}_{k-x}, \underbrace{1, \dots, 1}_x)$ .

As a direct consequence we obtain the following corollary.

**Corollary 1.**  $e_1^T \cdot U^{k-t} \cdot z = z_{k-t} + e_1^T \cdot U^{k-t+1} \cdot z$

**Lemma 6.** For  $z = e_i - e_j$ , we have that  $(L_k + U_k) \cdot z = -z$ .

*Proof.* If  $i = j$ , then  $z = 0$  and the result is true. Suppose w.l.o.g. that  $i > j$  (otherwise we just prove the result for  $-z$ ). Then  $U_k \cdot (e_i - e_j) = \sum_{s=1}^{i-1} e_s - \sum_{s=1}^{j-1} e_s = \sum_{s=j}^{i-1} e_s$ . Similarly  $L_k \cdot (e_i - e_j) = \sum_{s=i+1}^k e_s - \sum_{s=j+1}^k e_s = \sum_{s=j+1}^i -e_s$ . Therefore  $(L_k + U_k) \cdot (e_i - e_j) = \sum_{s=j}^{i-1} e_s - \sum_{s=j+1}^i e_s = e_j - e_i = -z$ . □

### 3.3 How to recover the bids when knowing the $l_{ij}$ 's

As discussed above, we can represent the function  $f$  as a matrix multiplication. Let  $M$  be the following square matrix of size  $nk \times nk$ :

$$M = \begin{bmatrix} (U+L) & U & \dots & \dots & U \\ (U+I) & (U+L) & U & \dots & U \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (U+I) & \dots & (U+I) & (U+L) & U \\ (U+I) & \dots & \dots & (U+I) & (U+L) \end{bmatrix}. \text{ Then } f(b) = M \cdot b.$$

The function takes as input a vector composed of  $n$  vectors, each of  $k$  bits. It returns the  $nk$  values  $l_{ij}$ ,  $1 \leq i \leq n$  and  $1 \leq j \leq k$ . As explained above, the structure of the matrix is defined by the formula that computes  $\gamma_{ij}^a$ , which consists essentially of three factors: first we multiply all  $\alpha_{ij}$  which encode bigger bids (represented by the matrix  $U$ ), then we multiply all  $\alpha_{ij}$  which encode smaller bids by the same bidder (represented by adding the matrix  $L$  on the diagonal), and finally we multiply by all  $\alpha_{ij}$  which encode the same bid by bidders with a smaller index (represented by adding the matrix  $I$  on the lower triangle of  $M$ ). In our encoding there will be a “1” in the vector for each  $Y$  in the protocol, hence  $f$  will count how many  $Y$ s are multiplied when computing  $\gamma_{ij}^a$ . Using this representation we can prove the following theorem.

**Theorem 1.**  $f$  is injective on valid bid vectors, i.e. for two different correct bid vectors  $u = [u_1, \dots, u_k]^T$  and  $v = [v_1, \dots, v_k]^T$  with  $u \neq v$  we have  $M \cdot u \neq M \cdot v$ .

*Proof.* Let  $u$  and  $v$  be two correct bid vectors such that  $u \neq v$ . We want to prove that  $M \cdot u \neq M \cdot v$ . We make a proof by contradiction, hence we assume that  $M \cdot u = M \cdot v$  or that  $M \cdot (u - v) = 0$ . Because  $u$  and  $v$  are two correct bid vectors, each one of them is an element of the canonical basis  $(e_1, \dots, e_k)$ , i.e.  $u = e_i$  and  $v = e_j$ , as shown in Section 3.1. We denote  $u - v$  by  $z$ , and consequently  $z = e_i - e_j$ . Knowing that  $M \cdot z = 0$ , we prove by induction on  $a$  that for all  $a$  the following property  $P(a)$  holds:

$$P(a) : \forall l, 0 < l \leq a, \text{diag}(U^{k-l}) \cdot z = 0$$

where  $\text{diag}(U^{k-x})$  is a  $nk \times nk$  block diagonal matrix containing only diagonal blocks of the same matrix  $U^{k-x}$ . The validity of  $P(k)$  proves in particular that  $\text{diag}(U^0) \cdot z_l = 0$ , i.e.  $z = 0$  which contradicts our hypothesis.

- Case  $a = 1$ : we also prove this base case by induction, i.e. for all  $b \geq 1$  the property  $Q(b)$  holds, where:

$$Q(b) : \forall m, 0 < m \leq b, U^{k-1} \cdot z_m = 0$$

which gives us that  $U^{k-1} \cdot z = 0$ .

- Base case  $b = 1$ : We start by looking at the multiplication of the first row of  $M$  with  $z$ . We obtain:  $(L + U) \cdot z_1 + U \cdot (z_2 + \dots + z_k) = 0$ . We can multiply each side by  $U^{k-1}$ , and use Lemma 6 to obtain:  $U^{k-1} \cdot [-z_1 + U^k \cdot (z_2 + \dots + z_k)] = 0$ . Since  $U$  is nilpotent, according to Lemma 3 the latter gives  $-U^{k-1} \cdot z_1 = 0$ . Hence we know  $Q(1) : U^{k-1} \cdot z_1 = 0$ , i.e. the last entry of  $z_1$  is 0.
- Inductive step  $b + 1$ : assume  $Q(b)$ . Consider now the multiplication of the  $(b + 1)$ -th row of the matrix  $M$ :  
 $(U + I) \cdot z_1 + \dots + (U + I) \cdot z_b + (L + U) \cdot z_{b+1} + U \cdot (z_{b+2} + \dots + z_k) = 0$ .  
Then by multiplying by  $U^{k-1}$  and using Lemma 6 we obtain:  
 $U^{k-1} \cdot [(U + I) \cdot z_1 + \dots + (U + I) \cdot z_b - z_{b+1} + U \cdot (z_{b+2} + \dots + z_k)] = 0$ .  
Since  $U$  is nilpotent according to Lemma 3 we have  $U^{k-1} \cdot z_1 + \dots + U^{k-1} \cdot z_b - U^{k-1} \cdot z_{b+1} = 0$ . Using the fact that for all  $m < b$  we have  $U^{k-1} \cdot z_m = 0$ , the latter gives  $-U^{k-1} \cdot z_{b+1} = 0$ .

- Inductive step  $a + 1$ : assume  $P(a)$ . By induction on  $b \geq 1$  we will show that  $Q'(b)$  holds, where

$$Q'(b) : \forall m, 0 < m \leq b, U^{k-(a+1)} \cdot z_m = 0$$

which gives us that  $U^{k-(a+1)} \cdot z = 0$ , i.e.  $P(a + 1)$ .

- Base case  $b = 1$ : Consider the multiplication of the first row with  $U^{k-(a+1)}$ :  
 $U^{k-(a+1)} \cdot [(L + U) \cdot z_1 + U \cdot (z_2 + \dots + z_k)] = 0$  which can be rewritten as  $-U^{k-(a+1)} \cdot z_1 + U^{k-a} \cdot (z_2 + \dots + z_k) = 0$ . Using  $U^{k-a} \cdot z_l = 0$  for all  $l$ , we can conclude that  $-U^{k-(a+1)} \cdot z_1 = 0$ , i.e.  $Q'(1)$  holds.
- Inductive step  $b + 1$ : assume  $Q'(b)$ . Consider now the  $(b + 1)$ -th row of the matrix  $M$ :  
 $(U + I) \cdot z_1 + \dots + (U + I) \cdot z_b + (L + U) \cdot z_{b+1} + U \cdot (z_{b+2} + \dots + z_k) = 0$ .  
Then by multiplying by  $U^{k-(a+1)}$  and using Lemma 6 we obtain:  
 $U^{k-(a+1)} \cdot [(U + I) \cdot z_1 + \dots + (U + I) \cdot z_b - z_{b+1} + U \cdot (z_{b+2} + \dots + z_k)] = 0$ .  
Using  $U^{k-a} \cdot z_l = 0$  for all  $l$ , we can conclude that  $U^{k-(a+1)} \cdot z_1 + \dots + U^{k-(a+1)} \cdot z_b - U^{k-(a+1)} \cdot z_{b+1} = 0$ . Now, for all  $m < b$ , we have  $U^{k-(a+1)} \cdot z_m = 0$ , so that  $-U^{k-(a+1)} \cdot z_{b+1} = 0$ ; i.e.  $Q'(b + 1)$  holds.  $\square$

This theorem shows that if there is a constellation of bids that led to certain values  $l_{ij}$ , this constellation is unique. Hence we are able to invert  $f$  on valid outputs. We will now show that this can be efficiently done.

### 3.3.1 An efficient algorithm

Our aim is solve the following linear system:  $M \cdot x = l$ . We will use the same steps we used for the proof of injectivity to solve this system efficiently. First note that

$$M \cdot x = l \Rightarrow \text{diag}(U^{k-t-1}) \cdot M \cdot x = \text{diag}(U^{k-t-1}) \cdot l.$$

Consider the  $r$ -th block of size  $k$  of the latter equality. We have  $x_r = (x_{r,1}, x_{r,2}, \dots, x_{r,k})$ . When multiplying by  $e_1^T$  we obtain the first line of this block. The  $r$ -th block of  $M \cdot x$  is

$$\begin{aligned} (U + I)x_1 + \dots + (U + I)x_{r-1} + (L + U)x_r + Ux_{r+1} + \dots + Ux_k \\ = U(\sum_{i=1}^k x_i) + (\sum_{i=1}^{r-1} x_i) + Lx_r \end{aligned}$$

and the  $r$ -th block of  $l$  is  $l_r$ . Hence:

$$e_1^T \left[ U^{k-t} \left( \sum_{i=1}^k x_i \right) + U^{k-t-1} \left( \sum_{i=1}^{r-1} x_i \right) + U^{k-t-1} Lx_r \right] = e_1^T U^{k-t-1} l_r$$

Using Lemma 4, we can exchange  $L$  in the latter to get:

$$\begin{aligned} e_1^T \left[ U^{k-t} \left( \sum_{i=1}^k x_i \right) + U^{k-t-1} \left( \sum_{i=1}^{r-1} x_i \right) + U^{k-t-1} (\mathbf{1} - (I_n + U_n) x_r) \right] \\ = e_1^T U^{k-t-1} l_r. \end{aligned}$$

We then remark that  $e_1^T U^{k-t-1} \mathbf{1} = t + 1$ , which gives:

$$\begin{aligned} e_1^T \left[ U^{k-t} \left( \sum_{i=1, i \neq r}^k x_i \right) + U^{k-t-1} \left( \sum_{i=1}^{r-1} x_i \right) - U^{k-t-1} x_r \right] \\ = e_1^T U^{k-t-1} l_r - (t + 1). \end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned} e_1^T \left[ U^{k-t} \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right) + U^{k-t-1} \left( \sum_{i=1}^{r-1} x_i \right) \right] + (t + 1) - e_1^T U^{k-t-1} l_r \\ = x_{r, k-t-1} \quad (3) \end{aligned}$$

Using several times Corollary 1 we have:

- $e_1^T U^{k-t} \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right)$   
 $= e_1^T U^{k-t+1} \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right) + e_{k-t}^T \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right)$
- $e_1^T U^{k-t-1} \left( \sum_{i=1}^{r-1} x_i \right) = e_1^T U^{k-t} \left( \sum_{i=1}^{r-1} x_i \right) + e_{k-t-1}^T \left( \sum_{i=1}^{r-1} x_i \right)$
- $e_1^T U^{k-t-1} l_r = e_1^T U^{k-t} l_r + l_{r, k-t-1}$

By changing  $t$  to  $t - 1$  in Equation (3) we get:

$$e_1^T \left[ U^{k-t+1} \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right) + U^{k-t} \left( \sum_{i=1}^{r-1} x_i \right) \right] + t - e_1^T U^{k-t} l_r = x_{r, k-t}.$$

Then regrouping the applications of Corollary 1 and the latter formula within Equation (3), we obtain:

$$x_{r, k-t} + e_{k-t}^T \left( \left( \sum_{i=1}^k x_i \right) - 2x_r \right) + e_{k-t-1}^T \left( \sum_{i=1}^{r-1} x_i \right) + 1 + l_{r, k-t-1} = x_{r, k-t-1} \quad (4)$$

This gives us a formula to compute the values of  $x_{i,j}$ , starting with the last element of the first block  $x_{1,k}$ . Then we can compute the last elements of all other blocks  $x_{2,k}, \dots, x_{n,k}$ , and then the second to last elements  $x_{1, k-1}, \dots, x_{n, k-1}$ , etc.

### 3.3.2 Complexity Analysis.

To obtain all values, we have to apply the above formula for each  $t \leq n$  and  $r \leq k$ , hence we have:

$$\sum_{t=1}^n \sum_{r=1}^k (k+r) = n \left( k^2 + \frac{k(k+1)}{2} \right) = \frac{3}{2}nk^2 + \frac{1}{2}nk \in \mathcal{O}(nk^2)$$

This is efficient enough to be computed on a standard PC for realistic values of  $n$  (the number of bidders) and  $k$  (the number of possible bids). Those could be less than a hundred bidders with a thousand different prices, thus requiring about the order of only a hundred million arithmetic operations. It is anyway the order of magnitude of the number of operations required of each user just to compute her encrypted bids.

### 3.4 Attack on the random noise: how to obtain the $l_{ij}$ 's

In the previous section we showed that knowing the  $l_{ij}$ 's allows us to efficiently break the privacy of all bidders. Here is how to obtain the  $l_{ij}$ 's.

The seller will learn all  $v_{ij} = (Y^{l_{ij}})^{(\sum_{h=1}^n m_{ij}^h)}$  at the end of the protocol. Since the  $m_{ij}^h$  are randomly chosen, this will be a random value if  $l_{ij} \neq 0$ . However a malicious bidder ("Mallory", of index  $a$ ) can cancel out the  $m_{ij}^h$  as follows: in Step 3 of the protocol each bidder will compute his  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ . Mallory waits until all other bidders have published their values (the protocol does not impose any synchronization or special ordering) and then computes his values  $\gamma_{ij}^\omega$  and  $\delta_{ij}^\omega$  as:

$$\begin{aligned}\gamma_{ij}^\omega &= \left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \alpha_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \alpha_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \alpha_{hj} \right) \right) \cdot \left( \prod_{k \neq \omega} \gamma_{ij}^k \right)^{-1} \\ \delta_{ij}^\omega &= \left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \beta_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \beta_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \beta_{hj} \right) \right) \cdot \left( \prod_{k \neq \omega} \delta_{ij}^k \right)^{-1}\end{aligned}$$

The first part is a correct encryption of  $Y^{l_{ij}}$ , with  $m_{ij}^\omega = 1$  for all  $i$  and  $j$ . The second part is the inverse of the product of all the other bidders  $\gamma_{ij}^k$  and  $\delta_{ij}^k$ , and thus it will eliminate the random exponents. Hence after decryption the seller obtains  $v_{ij} = Y^{l_{ij}}$ , where  $l_{ij} < n$  for a small  $n$ . He can compute  $l_{ij}$  by simply (pre-)computing all possible values  $Y^r$  and testing for equality. This allows the seller to obtain the necessary values and then to use the resolution algorithm to obtain each bidder's bid. Note that although we changed the intermediate values, the output still gives the correct result (i.e. winning bid). Therefore, the attack might even be unnoticed by the other participants. Note also that choosing a different  $Y_i$  per bidder does not prevent the attack, since all the  $Y_i$  need to be public in order to prove the correctness of the bid in Step 2 of the protocol.

However the protocol requires Mallory to prove that  $\gamma_{ij}^\omega$  and  $\delta_{ij}^\omega$  have the same exponent. This is obviously the case, but Mallory does not know the exact value of this exponent. Thus it is impossible for him to execute the proposed zero-knowledge protocol directly.

In the original paper [4] the malleable interactive proof of [8], presented in Section 2.3, is used to prove the correctness of  $\gamma_{ij}^a$  and  $\delta_{ij}^a$  in Step 3 of the protocol.

If this proof is not converted into a non-interactive proof, then Mallory is able to fake it as follows.

### 3.5 Proof of equality of the presented outcomes

Note that we can rewrite  $\gamma_{ij}^\omega$  and  $\delta_{ij}^\omega$  as:

$$v = \gamma_{ij}^\omega = \underbrace{\left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \alpha_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \alpha_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \alpha_{hj} \right) \right)}_{g_1}^{1 - (\sum_{k \neq \omega} m_{ij}^k)}$$

$$w = \delta_{ij}^\omega = \underbrace{\left( \left( \prod_{h=1}^n \prod_{d=j+1}^k \beta_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \beta_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \beta_{hj} \right) \right)}_{g_2}^{1 - (\sum_{k \neq \omega} m_{ij}^k)}$$

When Mallory, the bidder  $m$ , is asked by Victor for a proof of correctness of his values, he starts by asking all other bidders for proofs to initialize the man-in-the-middle attack of Figure 1. Each of them answers with values  $\lambda_o = g_1^{z_o}$  and  $\mu_o = g_2^{z_o}$ . Mallory can then answer Victor with values  $\lambda = \prod_o \lambda_o^{-1}$  and  $\mu = \prod_o \mu_o^{-1}$ , where  $o \in ([1, n] \setminus m)$ . Victor then sends a challenge  $c$ , which Mallory simply forwards to the other bidders. They answer with  $r_o = z_o + c \cdot m_{ij}^o$ , and Mallory sends  $r = c - \sum_o r_o$  to Victor, who can check that  $g_1^r = \lambda \cdot v^c$  and  $g_2^r = \mu \cdot w^c$ . If the other bidders did their proofs correctly, then Mallory's proof will appear valid to Victor:

$$\lambda \cdot v^c = \prod_o \lambda_o^{-1} \cdot \left( g_1^{1 - (\sum_o m_{ij}^o)} \right)^c = \prod_o g_1^{-z_o} \cdot g_1^{c - c(\sum_o m_{ij}^o)} = g_1^{c - \sum_o (z_o + c m_{ij}^o)}$$

$$\mu \cdot w^c = \prod_o \mu_o^{-1} \cdot \left( g_2^{1 - (\sum_o m_{ij}^o)} \right)^c = \prod_o g_2^{-z_o} \cdot g_2^{c - c(\sum_o m_{ij}^o)} = g_2^{c - \sum_o (z_o + c m_{ij}^o)}$$

Hence in the case of malleable interactive zero-knowledge proofs Mallory is able to modify the values  $\gamma_{ij}^\omega$  and  $\delta_{ij}^\omega$  as necessary, and even prove the correctness using the bidders. Hence the modifications may stay undetected and the seller will be able to break privacy.

### 3.6 The complete attack and countermeasures

Putting everything together, the attack works as follows:

1. The bidders set up the keys as described in the protocol.
2. They encrypt and publish their bids.
3. They compute  $\gamma_{ij}^h$  and  $\delta_{ij}^h$  and publish them.
4. Mallory, who is a bidder himself, waits until all other bidders have published their values. He then computes his values as defined above, and publishes them.
5. If he is asked for a proof, he can proceed as explained above in Section 3.5.
6. The bidders (including Mallory) jointly decrypt the values.
7. The seller obtains all  $Y^{l_{ij}}$ 's. He can then compute the  $l_{ij}$ 's by testing at most  $n$  possibilities.
8. Once he has all values, he can invert the function  $f$  as explained above.
9. He obtains all bidders bids.

Again, note that for all honest bidders, this execution will look normal, so they might not even notice that an attack took place.

To prevent this attack, one could perform the following actions:

- To counteract the removal of the noise of Section 3.4, the bidders could check whether the product of the  $\gamma_{i,j}^a$  for all bidders  $a$  is equal to the product of the  $\alpha_{hd}$  without any noise (exponent is 1). Unfortunately, the man-in-the-middle attack generalizes to any exponent as shown in Figure 2. Therefore the attacker could use a randomly chosen exponent only known to him.

- As mentioned above, another countermeasure is the use of non-interactive, non-malleable proofs of knowledge. In this case, we will show in Section 5 that it is still possible to attack a targeted bidder’s privacy.

## 4 Attacking verifiability

Brandt claims that the protocol is verifiable as the parties have to provide zero-knowledge proofs for their computations, however there are two problems.

### 4.1 Exceptional values

First, a winning bidder cannot verify if he actually won. To achieve privacy, the protocol hides all outputs of  $v_{aj}$  except for the entry containing “1”<sup>2</sup>. This is done by exponentiation with random values  $m_{ij}^a$  inside all entries  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ , i.e. by computing  $x_{ij}^{\sum_a m_{ij}^a}$  where  $x_{ij}$  is the product of some  $\alpha_{ij}$  as specified in the protocol. If  $x_{ij}$  is one,  $x_{ij}^m$  will still return one for any  $m$ , and in principle something different from one for any other value of  $x_{ij}$ . Now, the random values  $m_{ij}^a$  may add up to zero (mod  $q$ ), hence the returned value will be  $x_{ij}^m = x_{ij}^0 = 1$  and the bidder will conclude that he won, although he actually lost ( $x_{ij} \neq 1$ ). Hence simply verifying the proofs is not sufficient to be convinced that the observed outcome is correct. For the same reason the seller might observe two or more “1”-values, even though all proofs are correct. In such a situation he is unable to decide which bidder actually won since he cannot determine which “1”s correspond to a real bids, and hence which bid is the highest real bid. If two “1”s correspond to real bids, he could even exploit such a situation to his advantage: he can tell both bidders that they won and take money from both, although there is only one good to sell – this is normally prohibited by the protocol’s tie-breaking mechanism. If the bidders do not exchange additional data there is no way for them to discover that something went wrong, since the seller is the only party having access to all values.

A solution to this problem could work as follows: when computing the  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ , the bidders can check if the product

$$x_{ij} = \left( \prod_{h=1}^n \prod_{d=j+1}^k \alpha_{hd} \right) \cdot \left( \prod_{d=1}^{j-1} \alpha_{id} \right) \cdot \left( \prod_{h=1}^{i-1} \alpha_{hj} \right)$$

is equal to one – if yes, they restart the protocol using different keys and random values. If not, they continue, and check if  $\prod_a \gamma_{ij}^a = 1$ . If yes, they choose different random values  $m_{ij}^a$  and re-compute the  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ , otherwise they continue. Since the probability of the random values adding up to zero is low, this will rapidly lead to correct values.

### 4.2 Different private keys

Second, the paper does not precisely specify the proofs that have to be provided in the joint decryption phase. If the bidders only prove that they use the same private key on all decryptions *and not also that it is the one they used to generate their public key*, they may use a wrong one. This will lead to a wrong decryption where with

<sup>2</sup>Note that the protocol contains a mechanism to resolve ties, i.e. there should always be exactly one entry equal to 1, even in the presence of ties.

very high probability no value is “1”, as they will be random. Hence all bidders will think that they lost, thus allowing a malicious bidder to block the whole auction, as no winner is determined. Hence, if we assume that the verification test consists in verifying the proofs, a bidder trying to verify that he lost using the proofs might perform the verification successfully, although the result is incorrect and he actually won – since he would have observed a “1” if the vector had been correctly decrypted.

This problem can be addressed by requiring the bidders to also prove that they used the same private key as in the key generation phase.

## 5 Attacks using the lack of authentication

The protocol as described in the original paper does not include any authentication of the messages. This means that an attacker in control of the network can impersonate any party, which can be exploited in many ways. However, the authors supposed in the original paper a “reliable broadcast channel, i.e. the adversary has no control of communication” [4]. Yet even under this assumption dishonest participants can impersonate other participants by submitting messages on their behalf. Additionally, this assumption is difficult to achieve in asynchronous systems [14]. In the following we consider an attacker in control of the network, however many attacks can also be executed analogously by dishonest parties (which are considered in the original paper) in the reliable broadcast setting.

### 5.1 Another attack on privacy

Our first attack on privacy only works in the case of malleable interactive proofs. If we switch to non-interactive non-malleable proofs, Mallory cannot ask the other bidders for proofs using a challenge of his choice.

However, even with non-interactive non-malleable zero-knowledge proofs, the protocol is still vulnerable to attacks on a targeted bidder’s privacy if an attacker can impersonate any bidder of his choice as well as the seller, which is the case for an attacker controlling the network due to the lack of authentication. In particular, if he wants to know Alice’s bid he can proceed as follows:

1. Mallory impersonates all other bidders. He starts by creating keys on their behalf and publishes the values  $y_i$  and the corresponding proofs for all of them.
2. Alice also creates her secret keyshare and publishes  $y_a$  together with a proof.
3. Alice and Mallory compute the public key  $y$ .
4. Alice encrypts her bid and publishes her  $\alpha_{aj}$  and  $\beta_{bj}$  together with the proofs.
5. Mallory publishes  $\alpha_{ij} = \alpha_{aj}$  and  $\beta_{ij} = \beta_{aj}$  for all other bidders  $i$  and also copies Alice’s proofs.
6. Alice and Mallory execute the computations described in the protocol and publish  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ .
7. They compute  $\phi_{ij}^a$  and send it to the seller.
8. The seller publishes the  $\phi_{ij}^a$  and computes the  $v_{aj}$ .



Since all submitted bids are equal, the seller (which might also be impersonated by Mallory) will obtain Alice's bid as the winning price, hence it is not private any more. This attack essentially simulates a whole instance of the protocol to make Alice indirectly reveal a bid that was intended for another, probably real auction. To counteract this it is not sufficient for Alice to check that the other bids are different: Mallory can produce different  $\alpha_{ij} = \alpha_{aj}y^x$  together with  $\beta_{ij} = \beta_{aj}g^x$  which are still correct encryptions of Alice bids.

Note that the same attack also works if dishonest bidders collude with the seller: they simply re-submit the targeted bidders bid as their own bid.

## 5.2 Attacking fairness, non-repudiation and verifiability

The lack of authentication obviously entails that a winning bidder can claim that he did not submit his bid, hence violating non-repudiation (even in the case of reliable broadcast). Additionally, this also enables an attack on fairness: an attacker in control of the network can impersonate all bidders vis-à-vis the seller, submitting bids of his choice on their behalf and hence completely controlling the winner and winning price. This also causes another problem with verifiability: it is impossible to verify if the bids were submitted by the registered bidders or by somebody else.

## 5.3 Countermeasures

The solution to these problems is simple: all the messages need to be authenticated, e.g. using signatures or Message Authentication Codes (MACs) based on a trust anchor, for example a Public Key Infrastructure (PKI).

# 6 Conclusion

In this paper we analyze the protocol of Brandt [4] from various angles. We show that the underlying computations have a weakness which can be exploited by malicious bidders to break privacy if malleable interactive zero-knowledge proofs are used. We also identified two problems with verifiability and proposed solutions. Finally we showed how the lack of authentication can be used to mount different attacks on privacy, verifiability as well as fairness and non-repudiation. Again we suggested a solution to address the discovered flaws.

So sum up, the following countermeasures have to be implemented:

- Use of non-interactive or non-malleable zero-knowledge proofs.
- All messages have to be authenticated, e.g. using a Public-Key Infrastructure (PKI) and signatures.
- In the outcome computation step: when computing the  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ , the bidders can check if  $x_{ij} = \left(\prod_{h=1}^n \prod_{d=j+1}^k \alpha_{hd}\right) \cdot \left(\prod_{d=1}^{j-1} \alpha_{id}\right) \cdot \left(\prod_{h=1}^{i-1} \alpha_{hj}\right)$  is equal to one – if yes, they restart the protocol using different keys and random values. If not, they continue, and check if  $\prod_a \gamma_{ij}^a = 1$ . If yes, they choose different random values  $m_{ij}^a$  and re-compute the  $\gamma_{ij}^a$  and  $\delta_{ij}^a$ , otherwise they continue.
- In the outcome decryption step: the bidders have to prove that the value  $x_a$  they used to decrypt is the same  $x_a$  they used to generate their public key  $y_a$  in the first step.

The attacks show that properties such as authentication can be necessary to achieve other properties which might appear to be unrelated at first sight, like for instance privacy. It also points out that there is a difference between computing the winner in a fully private way, and ensuring privacy for the bidders: in the second attack we use modified inputs to break privacy even though the computations themselves are secure. Additionally our analysis highlights that the choice of interactive or non-interactive, malleable or non-malleable proofs is an important decision in any protocol design.

As for possible generalizations of our attacks, of course the linear algebra part of our first attack is specific to this protocol. Yet the man-in-the-middle attack on malleable proofs as well as the need of authentication for privacy are applicable to any protocol. Similarly, checking all exceptional cases and ensuring that the same keys are used all along the process are also valid insights for other protocols.

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