Notes on the analytic solution for the Dahl's friction model

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***Abstract*— This document provides the analytical solution of the friction model for different changes in the sign of the vehicle speed, as shown in the paper "Genesis of booster curves in Electric Power Assistance Steering Systems" by the same authors. It is not intended to be self contained; it is only an additional note.**

I. APPENDIX

The load torque τ_a is described by the Dahl's model and is given by the following differential equation:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0 \left(1 - \frac{F}{F_c}\operatorname{sgn}\left(\dot{\theta}\right)\right) \tag{1}$$

Let make the hypothesis to have a positive speed signal during the time interval $t \in [0, t_1]$. At the time $t = 0$, we have the corresponding value for the angle $\theta(0) = \theta_0$. This hypothesis implies that we have $\operatorname{sgn}\left(\dot{\theta}\right) = +1$ in Eq. (1) for $\theta \in [\theta_0, \theta]$. We make also the hypothesis to start from null initial conditions, this implies that we need to solve the following differential equation:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0 \left(1 - \frac{F}{F_c}\right) \tag{2}$$

with the following initial condition:

$$F(\theta_0) = 0 \tag{3}$$

We divide both terms of Eq. (2) by $\left(1 - \frac{F}{F_c}\right)$ and F_c, in order to obtain a suited primitive:

$$\frac{-\frac{1}{F_c}\left(\frac{\mathrm{d}F}{\mathrm{d}\theta}\right)}{\left(1 - \frac{F}{F_c}\right)} = -\frac{\sigma_0}{F_c} \tag{4}$$

To obtain the analytic solution of Eq. (4) for positive speeds, we integrate both terms during the interval $(\theta_0, \theta(t))$:

$$\int_{\theta_0}^{\theta(t)} \frac{-\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1 - \frac{F}{F_c}\right)}\mathrm{d}\theta = -\frac{\sigma_0}{F_c}\int_{\theta_0}^{\theta(t)} \mathrm{d}\theta \tag{5}$$

The result of this integral is:

$$\ln\left|1 - \frac{F(\theta(t))}{F_c}\right| - \ln\left|1 - \frac{F(\theta_0)}{F_c}\right| = -\frac{\sigma_0}{F_c}(\theta(t) - \theta_0) \tag{6}$$

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By substituting the initial condition (3), it is easily to obtain the solution for positive speeds:

$$F(\theta(t)) = F_c\left[1 - e^{-\frac{\sigma_0}{F_c}(\theta(t) - \theta_0)}\right] \tag{7}$$

At the time instant $t = t_1$, the speed changes in sign and becomes negative ($\operatorname{sgn}\left(\dot{\theta}\right) = -1$ in Eq. (1)) during the time interval $t \in [t_1, t_2]$. To guarantee the continuity of the function, we require that the initial condition corresponds to the last values assumed from $F(\theta(t))$ during the interval $t \in [0, t_1]$, i.e. $F(\theta(t_1^-)) = F(\theta(t_1^+))$. Under these hypothesis, the new Cauchy problem to solve is:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0\left(1 + \frac{F}{F_c}\right) \tag{8}$$

$$F(\theta(t_1^-)) = F(\theta(t_1^+)) \tag{9}$$

To solve this problem, we proceed as for Eqs. (4) and (5), then we integrate between $\theta(t_1)$ and $\theta(t)$, as follows:

$$\int_{\theta(t_1)}^{\theta(t)} \frac{\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1 + \frac{F}{F_c}\right)}\mathrm{d}\theta = \int_{\theta(t_1)}^{\theta(t)} \frac{\sigma_0}{F_c}\mathrm{d}\theta \tag{10}$$

The solution of this integral is:

$$\ln\left|1 + \frac{F(\theta(t))}{F_c}\right| - k_1 = \frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1)) \tag{11}$$

Where k_1 is a constant, depending from the initial conditions. By inverting this equation, we obtain:

$$F(\theta(t)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1))} \cdot e^{k_1}\right] \tag{12}$$

To determine the constant k_1, we introduce the initial condition $F(\theta(t_1^-)) = F(\theta(t_1^+))$:

$$F(\theta(t_1^-)) = F_c\left[1 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] \tag{13}$$

$$F(\theta(t_1^+)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t_1) - \theta(t_1))} \cdot e^{k_1}\right] \tag{14}$$

It follows:

$$F_c\left[1 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] = F_c\left[-1 + e^{k_1}\right] \tag{15}$$

By solving for k_1, we obtain:

$$k_1 = \ln\left[2 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] \tag{16}$$

So the solution for negative speeds is:

$$F(\theta(t)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1))} \cdot e^{\ln\left[2 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right]}\right] \tag{17}$$

By rearranging all terms, this expression can be simplified as follows:

$$F(\theta(t)) = F_c\left[-1+2e^{\frac{\sigma_0}{F_c}(\theta(t)-\theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t)-2\theta(t_1))}\right] \quad (18)$$

If we make the hypothesis that the speed changes in sign and becomes positive during the time interval $t \in [t_2, t_f]$, we can find the analytic solution $F(\theta)$ for this last interval, by repeating the same steps as for the previous cases. In this case, the boundary condition is $F(\theta(t_2^-)) = F(\theta(t_2^+))$, so we need to solve:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0\left(1-\frac{F}{F_c}\right) \quad (19)$$

$$F(\theta(t_2^-)) = F(\theta(t_2^+)) \quad (20)$$

We proceed as in the previous cases and we integrate the primitive between $[\theta_2, \theta(t)]$:

$$\int_{\theta_2}^{\theta(t)} \frac{-\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1-\frac{F}{F_c}\right)}\mathrm{d}\theta = -\frac{\sigma_0}{F_c}\int_{\theta_2}^{\theta(t)}\mathrm{d}\theta \quad (21)$$

We obtain the following analytic solution:

$$F(\theta(t)) = F_c\left[1-e^{-\frac{\sigma_0}{F_c}(\theta(t)-\theta(t_2))}\cdot e^{k_2}\right] \quad (22)$$

We can determine the constant k_2 by introducing the boundary condition $F(\theta(t_2^-)) = F(\theta(t_2^+))$:

$$F(\theta(t_2^-)) = F_c\left[-1+2e^{\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \quad (23)$$

$$F(\theta(t_2^+)) = F_c\left[1-e^{-\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_2))}\cdot e^{k_2}\right] \quad (24)$$

It follows:

$$F_c\left[1-e^{k_2}\right] = F_c\left[-1+2e^{\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \quad (25)$$

From Eq. (25) we get the following expression for the constant k_2:

$$k_2 = \ln\left[2-2e^{-\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} + e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \quad (26)$$

If we substitute Eq. (26) in Eq. (22) and we simplify, we get the solution for the friction torque when the speed is positive and the initial condition is different from zero:

$$F(\theta(t)) = F_c\Big[1-2e^{-\frac{\sigma_0}{F_c}(\theta(t)-\theta(t_2))} + 2e^{-\frac{\sigma_0}{F_c}(\theta(t)+\theta(t_1)-2\theta(t_2))} - e^{-\frac{\sigma_0}{F_c}(\theta(t)+2\theta(t_1)-2\theta(t_2))}\Big] \quad (27)$$

□

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