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Abstract

We aim to present mathematical models of smart materials and smart structures. Smart materials are materials which present significant multiphysical couplings. They are integrated in smart structures which take technological advantages of some multiphysical effects. We establish a classification of piezoelectric crystalline materials and propose simplified but accurate models of thin plates or slender rods made of piezoelectric or electromagneto-elastic materials. In the first part algebraic tools from group theory are used, whereas in the second one some tools of variational and functional analysis are suitable to solve some convergence questions for boundary value problems depending on small parameters.

Keywords: Group theory, singular perturbations methods, asymptotic analysis, plates and rods models.

1 Introduction

Smart materials present significant multiphysical couplings, one of the fields of interaction being mechanics. For instance, the application of an electric field on a piezoelectric material generates a deformation and conversely, a deformation generates polarization. Piezoelectric materials are widely used in the design of smart structures. They are integrated in these structures which take technological advantages of the piezoelectric effect. We may distinguish two classes: sensors and actuators. Sensors transform a mechanical loading into an electric signal and the systems they enter convey these signals according to the needs: design of micro weighing machine, airbags, Hi-Fi equipments... The actuators convert an electric signal into a displacement, which can be monitored with precision, of say mirrors, lenses, machine tools... This twofold behavior is known and used for a long time and several derivations of models of smart structures can be found in the engineering literature. Due to the various proceedings, discrepancies and controversies may occur (see [2], [6] and [9] for example). Here we aim to present mathematical models of both smart materials and smart structures.

First we establish a classification of piezoelectric crystalline materials by a suitable description of crystal lattices and a careful study of the symmetries of the piezoelectric tensors which account for the piezoelectric phenomenon. We used mainly algebraic tools from representation theory.

In a second part we intend to propose simplified but accurate models of structures made of piezoelectric or elecromagneto-elastic materials, these structures (thin plates, slender rods) presenting one or two small dimensions. The models are obtained by a rigorous study of the asymptotic behavior of a three dimensional body when some of its dimensions, considered as parameters, tend to zero. We used various tools of variational and functional analysis, the point being to consider boundary value problems depending on small parameters. This study has been carried out in the steady-state and transient cases.

2   To begin with: classifying crystals

From an historical point of view, the word 'crystal' can be seen as a synonym of order and symmetry. And indeed, crystals are the manifestation of order and symmetries taking places inside matter. The first scientific treatises dealing with crystals appeared at the end of the 18th century. They were written by mineralogists who tried to understand the reasons for which certain kinds of mineral species had so regular forms. At this time, the french scientist René-Just Haüy made the hypothesis that the exterior form of these species was the consequence of a *periodical* juxtaposition of microscopic elementary blocks inside theses bodies. That led to the notion of *crystalline state*. Theoretical studies then began and, from the geometric point of view, the architecture of this crystalline state imaginated by Haüy was described well before the development of modern measure devices. In fact, it is only after the discovery of X-ray diffraction in 1912 that the Bragg brothers together with Von Laue showed experimentaly that crystalline state is linked to a periodical ordering of matter at microscopic or molecular scale.

In the last two decades, the use of new mathematical tools coming from calculus of variations gave a new impulse at the mathematical modeling of crystals. An important part of this impulse originates in [1], who showed that mesoscale arrangements of matter can be seen as minimizers of an energy linked to the macroscopic behavior of solids. The need of an up to date and rigourous modeling of microscopic structures then appeared. The history and the results of this modeling can be found in [8].

The very question that arises in mathematical modeling of crystals is the following: which are the tools that allows us to translate the notions of 'order' and 'symmetry' in mathematical terms ? The answer is twofold. First, the need of a *classification* clearly appears. We have to range crystals in different families depending on their 'forms'. Secondly, we have to give a precise meaning to this term of 'form'.

The basic tools then appear. Because we want to classify, we will build *equivalence classes*. And because we want to classify forms, we will pick our tools in the *orthogonal transformations*, that is, transformations that preserve angles (and therefore shapes). This will allow us to say when two bodies are basically two different images of a same form.

We denote by $\mathbf{L}$in the space of second-order tensors, $\mathbf{I}$ the unit element in $\mathbf{L}$in and $\mathbf{L}^t$ the transpose of the element $\mathbf{L} \in \mathbf{L}$in. We recall that the space of orthogonal transformations is $\mathbf{O(3)} = \{\mathbf{L} \in \mathbf{L}\text{in} : \mathbf{L}\,\mathbf{L}^t = \mathbf{L}^t\,\mathbf{L} = \mathbf{I}\}$ and that the subgroup of all elements of $\mathbf{O(3)}$ with determinant equal to 1 (*i.e.* the subgroup of *rotations*) is denoted by $\mathbf{SO(3)}$.

Let $\mathbb{R}^3$ be the three-dimensional Euclidean vector space with a basis $\{\mathbf{e}_a\} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Using the convention of summation over repeated indexes, we define the *lattice* $\mathcal{R}(\{\mathbf{e}_a\})$ by:

$$\mathcal{R}(\{\mathbf{e}_a\}) := \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = M^a \mathbf{e}_a, a = 1, 2, 3, \ \ M^a \in \mathbb{Z}\}. \tag{1}$$

In a way $\mathcal{R}(\{\mathbf{e}_a\})$ is the skeleton of the crystal to model and we say that the 3 linearly independent vectors $\mathbf{e}_a$ $(a = 1, 2, 3)$ are the *basis vectors* of $\mathcal{R}(\{\mathbf{e}_a\})$. The parallelepiped $\mathcal{P}(\{\mathbf{e}_a\})$ spanned by $\mathbf{e}_a$ $(a = 1, 2, 3)$ is called the *unit cell* of $\mathcal{R}(\{\mathbf{e}_a\})$. It is the elementary block that we introduce *supra*.

It is well known that lattices can be ranged in *7 crystal systems* or *14 Bravais lattices*. But it is not so easy to understand that these two types of classification correspond to two types of equivalence classes: "geometrical" equivalence classes and "arithmetic" equivalence classes. These two kinds of equivalence classes arise on the one hand from geometrical actions (namely: elements $\mathbf{Q}$ of $\mathbf{O(3)}$) and on the other hand from arithmetic actions (elements m of $GL(3, \mathbb{Z})$, the space of 3×3 invertible unimodular matrices with integral entries). The reader interested in the details of the process can refer to [8]. Here, let's just say that an element $\mathbf{Q} \in \mathbf{O(3)}$ acts on $\{\mathbf{e}_a\}$ in the following way:

$$\mathbf{Q}\{\mathbf{e}_a\} := (\mathbf{Q}\,\mathbf{e}_1, \mathbf{Q}\,\mathbf{e}_2, \mathbf{Q}\,\mathbf{e}_3) = \{\mathbf{Q}\,\mathbf{e}_a\}. \tag{2}$$

As to arithmetic actions $m = (m_a^b) \in GL(3, \mathbb{Z})$, they are defined by:

$$m\{\mathbf{e}_a\} := (m_1^b\,\mathbf{e}_b, m_2^b\,\mathbf{e}_b, m_3^b\,\mathbf{e}_b) = \{m_a^b\,\mathbf{e}_b\}. \tag{3}$$

These actions are quite different: notice that elements $\mathbf{Q} \in \mathbf{O(3)}$ act on each basis vector independently, while it is not the case of elements m of $GL(3, \mathbb{Z})$.

Up to these two actions, two kinds of equivalence classes arise. We shall not detail their definitions. It is enough to know that crystal lattices can be either geometrically or arithmetically equivalent. The geometrical equivalence classes lead to the 7 *crystal systems* (also called geometrical holoedries), widely known under the following denomination: *triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal* and *cubic*. As to the arithmetical equivalence classes, they lead to the 14 *Bravais lattices*. It is however not possible to justify here the existence of the 32 crystallographic point groups because it would need the introduction of affine symmetries and of multilattices which are far from our purposes.

3 Towards tensors symmetry classes

3.1 A bridge: the Cauchy-Born hypothesis

The Cauchy-Born hypothesis is, roughly speaking, the statement that lattices vectors behave as *material vectors*. It simply means that when a crystal with lattice $\mathcal{R}(\{\mathbf{e}_a^0\})$ in the reference configuration experiences a homogeneous deformation whose gradient is $\mathbf{F}$, we have:

$$\mathbf{e}_a = \mathbf{F}\mathbf{e}_a^0, \tag{4}$$

where $\mathbf{e}_a$ are the lattice vectors of the crystal in the deformed configuration.

This quite simple hypothesis allows us to transfer symmetry properties studied up to now to the macroscopic level, then passing from geometrical to *constitutive* symmetry.

To this aim we first take for granted that the structure of crystals can be described by means of a simple lattice in all their allowed configurations. Second, we assume that the free energy w per unit cell of a deformable crystalline body whose current configuration is a simple lattice $\mathcal{R}(\{\mathbf{e}_a\})$ can be written as:

$$w = w(\mathbf{e}_a). \tag{5}$$

In the piezoelectric case, the energy is a function of the strain tensor $\mathbf{E}$ and of the electric displacement vector $\mathbf{d}$:

$$w = w(\mathbf{E}, \mathbf{d}), \tag{6}$$

while the piezoelectric tensor $\mathbf{P}$ is given by:

$$\mathbf{P} = \frac{\partial^2 w}{\partial \mathbf{E} \partial \mathbf{d}}. \tag{7}$$

In the linear case these expressions lead to:

$$\begin{aligned}
\mathbf{Q}\,\mathbf{P}[\mathbf{d}]\,\mathbf{Q}^t &= \mathbf{P}[\mathbf{Q}\,\mathbf{d}], \\
\mathbf{Q}^t\,\mathbf{P}[\mathbf{E}] &= \mathbf{P}[\mathbf{Q}^t\,\mathbf{E}\,\mathbf{Q}],
\end{aligned} \tag{8}$$

as soon as $\mathbf{Q}$ is an element of the geometrical holoedry of $\mathcal{R}(\{\mathbf{e}_a^0\})$.

This process clearly shows how some of the symmetry elements present in a crystal can be brought across the microscopic level to the macroscopic one, where others equivalence classes can be built. In fact, (8) says that each element of the geometrical holoedry of a crystal is an element of the symmetry group of the piezoelectric tensor of this crystal. This has to be compared to the *Neumann's Principle*, which states that *"the symmetry group of any property of a crystal must include the point group of this last"*. We emphasize on the fact that all hypothesis that have been made since we introduced the Cauchy-Born hypothesis concern the crystal viewed as a *lattice*. The point groups therefore cannot appear in this approach.The important fact, anyway, is how the $\star$ action that will be introduced in the next section, and which is of fundamental use in the tensors symmetries study, naturally arises from the crystal structure via the Cauchy-Born hypothesis.

3.2   The piezoelectric tensors symmetries

We consider a tensor of any order noted $\mathbf{T}$. Its components in a given basis $R = (O; \mathbf{u}, \mathbf{v}, \mathbf{w})$ are denoted $\mathbf{T}_{\dots ijk\dots}$. Let $\mathbf{Q} \in \mathbf{O(3)}$ be an orthogonal transformation. In the new basis $R' = (O; \mathbf{Qu}, \mathbf{Qv}, \mathbf{Qw}) = \mathbf{Q}\,R$, the components of $\mathbf{T}$ are mapped into

$$(\mathbf{Q} \star \mathbf{T})_{\dots ijk\dots} := \cdots Q_{ip} Q_{jq} Q_{kr} \cdots T_{\dots pqr\dots}. \tag{9}$$

Now, let $\mathbf{P}$iez be the space of all third-order tensors, symmetric according to their first two indexes:

$$\mathbf{P}\text{iez} := \{\mathbf{P} : P_{ijk} = P_{jik}\}. \tag{10}$$

A tensor $\mathbf{P} \in \mathbf{P}$iez is called a *piezoelectric* tensor.

In a piezoelectric material, the electric displacement vector d is connected to the stress tensor σ via the relation

$$d_k = P_{ijk}\,\sigma_{ij}, \tag{11}$$

where $\mathbf{P} \in \mathbf{P}$iez.

We define the symmetry group $g(\mathbf{P})$ of an element $\mathbf{P} \in \mathbf{P}$iez by:

$$g(\mathbf{P}) := \{\mathbf{Q} \in \mathbf{O(3)} : \mathbf{Q} \star \mathbf{P} = \mathbf{P}\}, \tag{12}$$

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and we will say that two piezoelectric tensors $\mathbf{P}_1$ and $\mathbf{P}_2$ in **P**iez are *equivalent* if there is an element $\mathbf{Q}$ of $\mathbf{O(3)}$ such that $g(\mathbf{P}_1) = g(\mathbf{Q} \star \mathbf{P}_2)$. In other words, $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent when their symmetry groups are conjugate:

$$\mathbf{P}_1 \overset{\star}{\sim} \mathbf{P}_2 \iff \{\exists\, \mathbf{Q} \in \mathbf{O(3)} : g(\mathbf{P}_1) = \mathbf{Q}\, g(\mathbf{P}_2)\, \mathbf{Q}^t\}. \tag{13}$$

This relation translates the physical intuition that whenever two material bodies can be rigidly rotated so that their symmetry groups become identical, they share the same "symmetry". Therefore, the equivalence classes of **P**iez which result from $\overset{\star}{\sim}$ are called *symmetry classes*. The question we ask now is: how many symmetry classes are there in **P**iez and which are there? Quite recently in [4], it has been introduced a well-suited technique that gives an answer to this question in the elasticity framework. But the tools they used are powerfull enough to be extended to any tensors translating physical properties. The core of the method lies in the fact that it is not so easy to see how orthogonal transformations act on tensors. Therefore, we have to find another space on which the action $\star$ will become elementary. Indeed, this space does exist and is linked to the set of harmonic polynomials. The path from tensors to harmonic polynomials has however to be explained, and we now focus on this point.

3.2.1 Harmonic decomposition

Any second-order tensor $\mathbf{E}$ can be seen as the sum of a symmetric tensor $\mathbf{S} = \frac{1}{2}(\mathbf{E} + \mathbf{E}^t)$ and of an antisymmetric tensor $\mathbf{W} = \frac{1}{2}(\mathbf{E} - \mathbf{E}^t)$ associated with an axial vector $^\epsilon\mathbf{w}$ such that $W_{ij} = \epsilon_{ijk}\, {}^\epsilon w_k$, where ϵ_{ijk} are the components of the alternating tensor. We note tr for the *trace* operator. The relation $\mathbf{S} = \mathbf{S}^D + \frac{1}{3}(\operatorname{tr}\mathbf{S})\mathbf{I} = \mathbf{S}^D + \frac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{I}$, where $\mathbf{S}^D$ is the deviatoric part of $\mathbf{S}$, points out an isomorphism between **L**in and $\mathbf{Dev} \times \mathbb{R}^3 \times \mathbb{R}$ where **D**ev is the space of second-order traceless and symmetric tensors. We therefore write:

$$\mathbf{L}\text{in} \ni \mathbf{E} \approx (\mathbf{S}^D, {}^\epsilon\mathbf{w}, \frac{1}{3}(\operatorname{tr}\,\mathbf{E})) \in \mathbf{Dev} \times \mathbb{R}^3 \times \mathbb{R}. \tag{14}$$

It is easy to check that:

$$\mathbf{Q} \star \mathbf{E} \approx (\mathbf{Q} \star \mathbf{S}^D, \det(\mathbf{Q})\,(\mathbf{Q} \star {}^\epsilon\mathbf{w}), \frac{1}{3}(\operatorname{tr}\,\mathbf{E})), \tag{15}$$

so that the symmetry class of $\mathbf{E}$ verifies:

$$g(\mathbf{E}) = g(\mathbf{S}^D) \cap {}^\epsilon g({}^\epsilon\mathbf{w}), \tag{16}$$

where

$${}^\epsilon g({}^\epsilon\mathbf{w}) = \{\mathbf{Q} \in \mathbf{O(3)} \,:\, \mathbf{Q} \star {}^\epsilon\mathbf{w} = \det(\mathbf{Q})\, {}^\epsilon\mathbf{w}\}. \tag{17}$$

The generalization of the correspondence (14) to tensors of any order (*cf.* [12]) is called *harmonic decomposition*. This decomposition is the bridge between tensors and harmonic polynomials. We say that a tensor is harmonic when it is *totally symmetric* (its components are unchanged under any permutations of indexes) and *traceless* (the trace with respect to any pair of indexes is null). Applied to **P**iez, this decomposition makes it possible to write:

$$\mathbf{P}\text{iez} \ni \mathbf{M} \approx (\mathbf{H}, {}^\epsilon\mathbf{C}, \nu, \mathbf{v}) \in \mathbf{H}\text{rm} \times {}^\epsilon\mathbf{Dev} \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{18}$$

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where **H**rm is the space of third-order harmonic tensors and ${}^{\epsilon}\mathbf{D}$ev the space of harmonic and axial second-order tensors. We then have:

$$\forall\, \mathbf{Q} \in \mathbf{O(3)},\ \mathbf{Q} \star \mathbf{M} \approx (\mathbf{Q} \star \mathbf{H}, \det(\mathbf{Q})\,(\mathbf{Q} \star {}^{\epsilon}\mathbf{C}), \mathbf{Q} \star \nu, \mathbf{Q} \star \mathbf{v}), \tag{19}$$

and thus:

$$g(\mathbf{M}) = g(\mathbf{H}) \cap {}^{\epsilon}g({}^{\epsilon}\mathbf{C}) \cap g(\nu) \cap g(\mathbf{v}),\ \forall\, \mathbf{M} \in \mathbf{P}\text{iez}, \tag{20}$$

where ${}^{\epsilon}g({}^{\epsilon}\mathbf{C})$ follows from (17).

3.2.2   Cartan decomposition

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a vector of $\mathbb{R}^3$ and let $\mathcal{P}_n$ be the space of homogeneous polynomials of degree n in the three variables x, y and z. There is a classical isomorphism ψ between $\mathcal{P}_n$ and $\mathbf{Sym}^n$, the space of totally symmetric tensors of order n (*cf.* [4]):

$$\mathbf{Sym}^n \ni \mathbf{T} \mapsto \psi(\mathbf{T}) := \mathbf{T}[\mathbf{r}, \mathbf{r}, \cdots, \mathbf{r}] = T_{i_1 i_2 \cdots i_n}\, r_{i_1} r_{i_2} \cdots r_{i_n} \in \mathcal{P}_n, \tag{21}$$

where the convention of summation over repeated indexes is understood. Thus, the second-order symmetric tensor $\mathbf{S} = (S_{ij})$ defined in the preceding paragraph is mapped into

$$\psi(\mathbf{S}) = S_{11}\, x^2 + 2\, S_{12}\, xy + 2\, S_{13}\, xz + S_{22}\, y^2 + 2\, S_{23}\, yz + S_{33}\, z^2. \tag{22}$$

A polynomial $h \in \mathcal{P}_n$ is *harmonic* when $\Delta h = 0$, where Δ is the *Laplacian* operator. We write $h \in \mathcal{H}_n$. It is easy to check that the space of harmonic tensors of order n is isomorphic via ψ to $\mathcal{H}_n$. The isomorphism (21) enables us to extend the action $\star$ defined in (9) to $\mathcal{P}_n$: for $p = \psi(\mathbf{T}) \in \mathcal{P}_n$ and $\mathbf{Q} \in \mathbf{O(3)}$, we define

$$(\mathbf{Q} \star p)(x,y,z) := \mathbf{T}[\mathbf{Q}^t \mathbf{r}, \mathbf{Q}^t \mathbf{r}, \cdots, \mathbf{Q}^t \mathbf{r}] = T_{i_1 i_2 \cdots i_n}\, r_{j_1} r_{j_2} \cdots r_{j_n}\, Q_{j_1 i_1} Q_{j_2 i_2} \cdots Q_{j_n i_n}. \tag{23}$$

Moreover, the linear mapping ψ is $\mathbf{O(3)}$-invariant in the following sense

$$\forall\, \mathbf{Q} \in \mathbf{O(3)},\ \mathbf{Q} \star \psi(\mathbf{T}) = \psi(\mathbf{Q} \star \mathbf{T}). \tag{24}$$

The goal is now to map harmonic polynomials (and consequently harmonic tensors) into spaces where the action $\star$ will become elementary. This purpose is achieved by the *Cartan decomposition*, which is an $\mathbf{SO(2)}$-invariant decomposition of $\mathcal{H}_n$. To give just a preliminary idea of it, let us consider a second-order harmonic polynomial f in the three variables x, y and z. We then have $f \in \mathcal{H}_2$. It is clear that f can be expressed on the following "basis":

$$u := z^2 - \frac{1}{3}(x^2 + y^2 + z^2),\ (s_1, t_1) := (xz, yz),\ (s_2, t_2) := (x^2 - y^2, 2xy). \tag{25}$$

This decomposition can also be viewed as the decomposition of second-order deviatoric tensors on the following basis (up to constants):

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},\ \mathbf{S}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \mathbf{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{S}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \mathbf{T}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

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All these tensors are traceless, symmetric and translate very simply the action of orthogonal transformations which operate on them:

$$\mathbf{Q}(\mathbf{k}, \theta) \star s_l = s_l \cos(l\theta) + t_l \sin(l\theta), \ \mathbf{Q}(\mathbf{k}, \theta) \star t_l = -s_l \sin(l\theta) + t_l \cos(l\theta), \quad (26)$$

and

$$\mathbf{Q}(\mathbf{i}, \pi) \star u = (-1)^n u, \ \mathbf{Q}(\mathbf{i}, \pi) \star s_l = (-1)^{n-l} s_l, \ \mathbf{Q}(\mathbf{i}, \pi) \star t_l = (-1)^{n-l+1} t_l. \quad (27)$$

Arguments which amount to the above results and extensions to higher order situations are described in [5] p.109-111.

3.2.3 Symmetry classes of piezoelectric tensors

In the preceding sections, we chosed to focus on tools that allows to understand how symmetry can be modeled at different scales. Obtaining results with the help of these tools is a question of technique, and we prefer to let the technique aside in this paper. Harmonic and Cartan decompositions then lead to the fact that there are 15 equivalence classes of piezoelectric tensors. When we look at a crystallographic table (as the one in [10] for example), we can see that it appears 16 different forms of piezoelectric tensors. It's important to focus on the fact that tools presented here lead to informations that can be considered as deeper as the one we can get with the help of crystallographic tables. Indeed, we can proove that $\overline{6}$ and $\overline{6}m2$ crystal classes materials have the same piezoelectric behavior even if their piezoelectric tensors appear to be different in the crystallographic tables. Invisible informations thus appear when dealing with appropriate mathematical tools.

4 Static behavior of smart plates or rods

As usual we make no difference between the physical space and $\mathbb{R}^3$ and, for all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, we denote (ξ_1, ξ_2) by $\widehat{\xi}$. Greek indices for coordinates take their values in $\{1, 2\}$ whereas latin indices run from 1 to 3.

Let $\mathcal{H} = S^3 \times \mathbb{R}^3$, where S^3 denotes the set of all 3×3 real and symmetric matrices. The set of all linear mappings from a space V into a space W is denoted $\mathcal{L}(V, W)$ and by $\mathcal{L}(V)$ if $V = W$.

In the sequel, for every domain G in $\mathbb{R}^N$, the subset of the Sobolev space $H^1(G)$ whose elements vanish on Γ, included in the boundary ∂G of G, will be denoted by $H^1_\Gamma(G)$.

4.1 Piezoelectric thin plates

Finding the equilibrium of a thin linearly piezoelectric plate can be formulated as follows. The reference configuration of a linearly piezoelectric thin plate is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$ and ε a small positive number. Let $\Gamma^\varepsilon_{\text{lat}} := \partial\omega \times (-\varepsilon, \varepsilon)$, $\Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon\}$ and two suitable partitions of $\partial\Omega^\varepsilon$: $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$ and $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ with Γ^ε_{mD} and Γ^ε_{eD} of strictly positive surface measures. The plate is clamped along Γ^ε_{mD} and at an electrical potential φ^ε_0 on Γ^ε_{eD}. It is subjected to body forces f^ε in Ω^ε and to surface forces g^ε in Γ^ε_{mN}. Furthermore, we will consider an electrical loading d^ε on Γ^ε_{eN}. We note n^ε the outward unit normal to $\partial\Omega^\varepsilon$ and assume that $\Gamma^\varepsilon_{mD} = \gamma_0 \times (-\varepsilon, \varepsilon)$, with $\gamma_0 \subset \partial\omega$. The equations determining the piezoelectric state $s^\varepsilon := (u^\varepsilon, \varphi^\varepsilon)$ at equilibrium are:

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$$\mathcal{P}(\Omega^\varepsilon) \begin{cases} \text{div } \sigma^\varepsilon + f^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \ \sigma^\varepsilon n^\varepsilon = g^\varepsilon \text{ on } \Gamma^\varepsilon_{mN}, \ u^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{mD}, \\ \text{div } D^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \ D^\varepsilon \cdot n^\varepsilon = d^\varepsilon \text{ on } \Gamma^\varepsilon_{eN}, \ \varphi^\varepsilon = \varphi^\varepsilon_0 \text{ on } \Gamma^\varepsilon_{eD}, \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(x)(e(u^\varepsilon), \nabla\varphi^\varepsilon) \text{ in } \Omega^\varepsilon, \end{cases}$$

where u^ε, φ^ε, σ^ε, $e(u^\varepsilon)$ and D^ε respectively stand for the displacement, the electrical potential, the stress tensor, the tensor of small strains (*i.e.* the symmetrized gradient) and the electric induction. The operator M^ε is an element of $\mathcal{L}(\mathcal{H})$ such that:

$$\begin{array}{rcccc} \sigma^\varepsilon & = & \mathsf{a}^\varepsilon\, e(u^\varepsilon) & - & \mathsf{b}^\varepsilon\, \nabla\varphi^\varepsilon \\ D^\varepsilon & = & \mathsf{b}^{\varepsilon^T}\, e(u^\varepsilon) & + & \mathsf{d}^\varepsilon\, \nabla\varphi^\varepsilon \end{array} \tag{28}$$

with b^{ε^T} the transpose of the piezoelectric tensor b^ε, the elastic tensor a^ε and the dielectric one c^ε being symmetric and positive. Note that because of the piezoelectric coupling, M^ε is not symmetric.

It is easy to give a weak (or variational) formulation of the previous linear boundary problem and to conclude to the existence and the uniqueness of a solution in suitable Sobolev spaces through the Stampacchia theorem.

Nevertheless, due to the very low thickness of the plate, this classical model may be difficult to tackle numerically. The essence of our proposal of simplified but accurate modeling is to consider ε as a small *parameter* and to study the asymptotic behavior of s^ε when ε goes to 0. In fact, two different limit behaviors indexed by $p \in \{1, 2\}$ will occur, according to the type of boundary condition in $\mathcal{P}(\Omega^\varepsilon)$.

From the mathematical point of view it is convenient to proceed to a change of coordinates Π^ε and of unknows $s_p(\varepsilon) = S_p(\varepsilon)s^\varepsilon$ in order to consider functional spaces defined on a fixed domain $\Omega = \omega \times (-1, 1)$.

$$\begin{array}{rcl} x = (x_1, x_2, x_3) \in \overline{\Omega} & \mapsto & \Pi^\varepsilon x = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon \\ s_p(\varepsilon) := (u(\varepsilon)(x), \varphi_p(\varepsilon)(x)) & = & ((\varepsilon^{-1}\hat{u}^\varepsilon(\Pi^\varepsilon x), u_3^\varepsilon(\Pi^\varepsilon x)), \varepsilon^{-p}\varphi^\varepsilon(\Pi^\varepsilon x)). \end{array} \tag{29}$$

The formulae defining $S_p(\varepsilon)$ stem from the assumptions on the magnitude of the electromechanical loading and are justified by the convergence results they lead to. If we consider forces and displacements, these hypotheses are the ones of [3] and supply a mathematical justification of the Kirchhoff-Love theory of thin linearly elastic plates. In addition, we assume that φ^ε_0 has an extension into Ω^ε still denoted by φ^ε_0 and that $\varphi_0 \in H^1(\Omega)$ is such that $\varphi^\varepsilon_0(\Pi^\varepsilon x) = \varepsilon^p \varphi_0(x)$ with:

$$\begin{cases} \text{if } p = 1 : \varphi_0 \text{ does not depend on } x_3. \\ \text{if } p = 2 : \text{ the closure of the projection of } \Gamma^\varepsilon_{eD} \text{ on } \omega \text{ coincides with } \overline{\omega}, \\ \qquad \text{moreover, either } d^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{eN} \cap \Gamma^\varepsilon_{\text{lat}} \text{ or } \Gamma^\varepsilon_{eN} \cap \Gamma^\varepsilon_{\text{lat}} = \emptyset. \end{cases} \tag{30}$$

Thus $s(\varepsilon)$ is the solution of the variational problem:

$$\begin{cases} \text{Find } s_p(\varepsilon) \in (0, \varphi_0) + V = \{r = (v, \psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega)\} \text{ such that} \\ \int_\Omega M(x)\, k_p(\varepsilon, s) \cdot k_p(\varepsilon, r)\, dx = L(r), \ \forall r \in V \end{cases}$$

where the linear form L does not depend on ε and

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$$
\begin{cases}
k_p(\varepsilon, r) = k_p(\varepsilon, (v, \psi)) = (e(\varepsilon, v), \nabla_p(\varepsilon, \psi)), \\
e(\varepsilon, v)_{\alpha\beta} = e(v)_{\alpha\beta},\ e(\varepsilon, v)_{\alpha 3} = \varepsilon^{-1} e(v)_{\alpha 3},\ e(\varepsilon, v)_{33} = \varepsilon^{-2} e(v)_{33}, \\
\widehat{\nabla}_p(\varepsilon, \psi) = \varepsilon^{p-1} \widehat{\nabla}\psi,\ \nabla_p(\varepsilon, \psi)_3 = \varepsilon^{p-2} \partial_3 \psi.
\end{cases}
\tag{31}
$$

The signs of the various powers of ε in the components of $k_p(\varepsilon, r)$ induce an orthogonal decomposition of $\mathcal{H}$ in subspaces $\mathcal{H}_p^{\star}$, with $\star \in \{-, 0, +\}$, which is crucial to fully describe plates models in all admissible crystal classes. We denote by $h_p^{\star}$ the projection on $\mathcal{H}_p^{\star}$ of any element h of $\mathcal{H}$ so that M can then be decomposed in nine elements $M_p^{\star\diamond} \in \mathcal{L}(\mathcal{H}_p^{\diamond}, \mathcal{H}_p^{\star})$, with $\star, \diamond \in \{-, 0, +\}$. Because M_p^{00} and M_p^{--} are positive operators on $\mathcal{H}_p^0$ and $\mathcal{H}_p^{-}$, the Schur complement $\widetilde{M}_p := M_p^{00} - M_p^{0-}(M_p^{--})^{-1} M_p^{-0}$ is an element of $\mathcal{L}(\mathcal{H}_p^0)$. The key point of the asymptotic study is to show that if $\overline{k}_p$ is the limit (in a suitable topology) of $k_p(\varepsilon, s_p(\varepsilon))$, then $(M \overline{k}_p)_p^{-} = (\overline{k}_p)_p^{+} = 0$. This will enable us to exhibit $\widetilde{M}_p$ *as the operator governing the limit constitutive equations* due to the fundamental relation:

$$
(M h)_p^{-} = h_p^{+} = 0 \Rightarrow \widetilde{M}_p\, h_p^0 = (M h)_p^0 \text{ and } \widetilde{M}_p\, h_p^0 \cdot h_p^0 = M h \cdot h.
\tag{32}
$$

The limit space of displacements will be the space of Kirchhoff-Love displacements defined by $\mathbf{V}_{KL} := \{v \in H^1_{\Gamma_{mD}}(\Omega)^3; e_{i3}(v) = 0\}$ while the limit electrical spaces will be $\Phi_{e,1} := \{\psi \in H^1_{\Gamma_{eD}}(\Omega); \partial_3 \psi = 0\}$ and $\Phi_{e,2} := \{\psi \in H^1_{\partial_3}(\Omega); \psi_{|\Gamma_{eD} \cap \Gamma^{\pm}} = 0\}$, where $H^1_{\partial_3}(\Omega) := \{\psi \in L^2(\Omega); \partial_3 \psi \in L^2(\Omega)\}$. Finally, we have the following convergence result:

Let $\mathbf{K}_1 := H^1(\Omega)$ and $\mathbf{K}_2 := H^1_{\partial_3}(\Omega)$. When $\varepsilon \to 0$, the family $(s_p(\varepsilon))_{\varepsilon > 0}$ of the unique solutions of $\mathcal{P}(\varepsilon, \Omega)_p$ strongly converges in $\mathbf{X}_p := H^1_{\Gamma_{mD}}(\Omega)^3 \times \mathbf{K}_p$ to the unique solution $\overline{s}_p$ of

$$
\overline{\mathcal{P}}(\Omega)_p
\begin{cases}
\text{Find } s \in (0, \varphi_0) + \mathbf{S}_p \text{ such that} \\
\int_{\Omega} \widetilde{M}_p\, k(s)_p^0 \cdot k(r)_p^0\, dx = L(r),\ \forall\, r \in \mathbf{S}_p := \mathbf{V}_{KL} \times \Phi_{e,p}.
\end{cases}
$$

To get physically meaningful results, we define an electromechanical state $\overline{s}_p^{\varepsilon}$ over the real plate Ω^{ε} by the descaling $\overline{s}_p^{\varepsilon} = S_p(\varepsilon)^{-1} \overline{s}_p$: it is the unique solution of a problem $\overline{\mathcal{P}}(\Omega^{\varepsilon})_p$ posed over Ω^{ε} which is the transportation by Π^{ε} of the (limit scaled) problem $\overline{\mathcal{P}}(\Omega)_p$. This transported problem is our proposal to model the thin linearly piezoelectric plate of thickness 2ε. Our model in fact involves two dimensional problems set on ω, which is very attractive and favourable from the numerical point of view. It is also accurate in the sense that the convergence result on the scaled states implies that s^{ε} is asymptotically equivalent to $\overline{s}_p^{\varepsilon}$.

Thus the main mathematical ingredient of our modeling is a technique of singular perturbation in variational equations in Hilbert spaces. The model involves "reduced" state variables, the sole component k_p^0 of the couple strain/gradient of the electrical potential, and the constitutive equation are supplied by the Schur complement (or the "condensation" of the initial operator M^{ε} with respect to the maintained components. This identification is the keypoint for obtaining some decoupling and symmetry properties very important in practice (see [15] and [16]) by due account of the influence of the crystalline symmetries on the coefficients of $\widetilde{M}_p$. The first model ($p = 1$) with $\varphi_0 = 0$ deals with the physical situation when the plate is used as a sensor, the second model corresponds to an actuator.

4.2   Electromagneto-elastic thin plates

Besides the piezoelectric coupling, some materials are sensitive to magnetic effects, thus in [18] we extended the previous modeling to linearly electromagneto-elastic thin plates. Now the state is described by $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon, \phi^\varepsilon)$ where the additional variable ϕ^ε denotes the magnetic potential and the constitutive equations read as:

$$
\begin{aligned}
\sigma^\varepsilon &= \mathsf{a}^\varepsilon\, e(u^\varepsilon) &-& \mathsf{b}^\varepsilon\, \nabla\varphi^\varepsilon &-& \mathsf{c}^\varepsilon\, \nabla\phi^\varepsilon, \\
D^\varepsilon &= \mathsf{b}^{\varepsilon^T} e(u^\varepsilon) &+& \mathsf{d}^\varepsilon\, \nabla\varphi^\varepsilon &+& \mathsf{e}^\varepsilon\, \nabla\phi^\varepsilon, \\
B^\varepsilon &= \mathsf{c}^{\varepsilon^T} e(u^\varepsilon) &+& \mathsf{e}^{\varepsilon^T}\, \nabla\varphi^\varepsilon &+& \mathsf{f}^\varepsilon\, \nabla\phi^\varepsilon.
\end{aligned}
\tag{33}
$$

In these constitutive equations, c^ε, e^ε and f^ε respectively stand for the piezomagnetic, electromagnetic coupling and magnetic permeability tensors, while B^ε denotes the magnetic induction.

A similar mathematical analysis of the asymptotic behavior of s^ε can be done to derive a simplified but accurate model of thin electromagneto-elastic plate. It involves reduced state variable and constitutive equations supplied by the condensation $\widetilde{M^\varepsilon}$ of M^ε with respect to the maintained components of $(e(u^\varepsilon), \nabla\varphi^\varepsilon, \nabla\phi^\varepsilon)$.

But the novelty here is that *four* limit behaviors may appear according to the type of boundary conditions and the magnitude of the data on the electric *and* magnetic fields. These cases can be described as previously but by a couple of indices $(p, q) \in \{1, 2\}^2$ in place of the sole indice p. The physical situation when the thin plate is used as an electrical (resp. magnetic) sensor corresponds to $p = 1$ (resp. $q = 1$) while the actuator case corresponds to $p, q = 2$. It therefore appears two original mixed behaviors when $p \neq q$. In these situations, the plate is *at the same time* a sensor *and* an actuator excepted for the classes for which the plate is no more electromagneto-elastic (*i.e.* the electromechanical and magnetomechanical coefficients in $\widetilde{M^\varepsilon}$ vanishes). The two cases $p \neq q$ allow the modeling of electrically commanded magnetic devices and of magnetically commanded electric ones, which is of considerable interest in the development of non-volatile magnetic random access memories. We emphasize on the point that this behavior is here fully described for any admissible crystal class.

4.3   Piezoelectric slender rods

From a technological point of view, piezoelectric materials can also be used in wires or slender rods. Now, the reference configuration of the piezoelectric structure is $\Omega^\varepsilon = \varepsilon \times (0, L)$ with L a fixed positive real number. The equations describing the equilibrium of the structure are the same as in the section 4.1 but of course the geometry of the various boundaries is different: we assume that $\Gamma^\varepsilon_{mD} = \varepsilon\omega \times \{0, L\}$.

To get our simplified models, we proceed as in the case of plates. Due to classical assumptions on the mechanical loading (which permits the justification of Bernoulli-Navier theory of elastic slender rods (see [7] and [13]) and on electrical loading:

$$
\begin{cases}
\text{if } p = 1 \text{ the extension of } \varphi^\varepsilon_0 \text{ into } \Omega^\varepsilon \text{ does not depend on } \hat{x} \text{ and } \Gamma^\varepsilon_{eD} \subset \varepsilon\omega \times \{0, L\}, \\
\text{if } p = 2 \text{ there exists } \gamma_e \subset \partial\omega \text{ such that } \Gamma^\varepsilon_{eD} \subset \varepsilon \times (0, L).
\end{cases}
\tag{34}
$$

the scaling is defined by:

$$\begin{cases} x = (\hat{x}, x_3) \in \overline{\Omega} = \omega \times (0, L) \mapsto \Pi^\varepsilon x = (\varepsilon \hat{x}, x_3) \in \overline{\Omega}^\varepsilon \\ s_p(\varepsilon) = S_p(\varepsilon) s^\varepsilon \\ (\hat{u}(\varepsilon)(x), u_3(\varepsilon)(x), \varphi(\varepsilon)(x)) = (\hat{u}^\varepsilon(\Pi^\varepsilon x), \varepsilon^{-1} u_3^\varepsilon(\Pi^\varepsilon x), \varepsilon^{-p} \varphi^\varepsilon(\Pi^\varepsilon x)) \end{cases} \quad (35)$$

so that $s_p(\varepsilon)$ is the unique solution of the variational problem

$$\text{Find } s_p(\varepsilon) \in (0, \varphi_0) + V \text{ such that } \int_\Omega M(x) k_p(\varepsilon, s(\varepsilon)) \cdot k_p(\varepsilon, r) = L(r), \, \forall \, r \in V,$$

with now:

$$k_p(\varepsilon, (v, \psi)) = ((\varepsilon^2 e_{\alpha\beta}(v), \varepsilon e_{\alpha 3}(v), e_{33}(v)), (\varepsilon^{p-2} \hat{\nabla} \psi, \varepsilon^{p-1} \partial_3 \psi)).$$

As in the case of purely elastic slender rods (*cf.* [7]), finding the limit is a little bit more difficult and the limit problems are as follows:

$$\overline{\mathcal{R}}(\Omega)_1 \begin{cases} \text{Find } (\overline{u}, \overline{v}, \overline{w}, \overline{\phi}, \overline{\psi}) \in V_1 \text{ such that} \\ \int_\Omega M(x) \, k_1(\overline{u}, \overline{v}, \overline{w}, \overline{\phi}, \overline{\psi}) \cdot k_1(u', v', w', \phi', \psi') \, dx = L(u'), \, \forall \, (u', v', w', \phi', \psi') \in V_1, \end{cases}$$

with

$$\begin{cases} V_1 = V_{BN}(\Omega) \times R_b(\Omega) \times RD_2^\perp \times \Phi \times \Psi, \\ V_{BN}(\Omega) = \{v \in H^1_{\Gamma_{mD}}(\Omega)^3; e_{\alpha\beta}(v) = e_{\alpha 3}(v) = 0\}, \\ R_b(\Omega) = \{v; \exists c \in H^1_0(0, L); \hat{v}(x) = c(x_3)(-x_2, x_1), v_3 \in L^2(0, L; H^1_m(\omega))\}, \\ H^1_m(\omega) = \{v \in H^1(\omega); \int_\omega \psi(\hat{x}) \, d\hat{x} = 0\}, \\ RD_2^\perp(\Omega) = \{\omega; \hat{w} \in L^2(0, L; H^1_m(\omega)^2), w_3 = 0 \text{ and} \\ \qquad\qquad\qquad \int_\omega (-x_2 w_1(\hat{x}, x_3) + x_1 w_2(\hat{x}, x_3)) \, d\hat{x} = 0, ae \, x_3 \in (0, L)\}, \\ \Phi = \{\phi \in H^1_0(0, L); \phi(x) = \phi(x_3)\}, \\ \Psi = L^2(0, L; H^1_m(\omega)), \\ k_1(u, v, w, \phi, \psi) = (\hat{e}(w), e_{\alpha 3}(v), e_{33}(u), \hat{\nabla} \psi, \frac{d\phi}{dx_3}) \end{cases} \quad (36)$$

and

$$\overline{\mathcal{R}}(\Omega)_2 \begin{cases} \text{Find } (\overline{u}, \overline{v}, \overline{w}, \phi) \in V_2 \text{ such that} \\ \int_\Omega M(x) \, k_2(\overline{u}, \overline{v}, \overline{w}, \phi) \cdot k_2(u', v', w', \phi') \, dx = L(u'), \, \forall \, (u', v', w', \phi') \in V_2, \end{cases}$$

with

$$\begin{cases} V_2 = V_{BN}(\Omega) \times R_b(\Omega) \times RD_2^\perp \times L^2(0, L; H^1_{\gamma_e}(\omega)), \\ k_2(u, v, w, \phi) = (\hat{e}(w), e_{\alpha 3}(v), e_{33}(u), \hat{\nabla} \phi). \end{cases} \quad (37)$$

The space $V_{BN}(\Omega)$ is the Bernoulli-Navier displacements space.

Of course, our proposal of model is obtained by taking the inverse scaling, that is to say a transported problem $\overline{\mathcal{R}}(\Omega^\varepsilon)_p$ posed over Ω^ε. On the contrary to the case of plates, the state variables of the model do not reduce to the couple displacement/electrical potential but involve additional variables: two fields of displacements (easy to interpret

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mechanically) and a scalar field of electrical nature. Nevertheless,the kinematics of the state variables is simpler than the one of the genuine three-dimensional model which is very favourable from a numerical point of view. As in the purely elastic case it is worthwhile to note that for particular classes of monoclinic materials the additional variables v, w and ψ disappear [14]. Anyway, in the case $p = 1$, the additional variables can be eliminated but it leads to non standard equations involving non local terms!

5 Dynamical response of piezoelectric plates

The interest of an efficient modeling of the dynamic response of piezoelectric plates lies in the fact that a major technological application of piezoelectric effects is the control of vibrations of structures through very thin plates or patches. We present two modelings depending on the various extents to which the magnetic effects are taken into account. Actually, because of the large discrepancy between the celerities of the mechanical and electromechanical waves, magnetic effects can be disregarded. That is why first we propose a modeling in the appropriate framework of the quasi-electrostatic approximation which claims that the electrical field still derives from an electrical potential.

5.1 Quasi-electrostatic case

Now a new parameter appears: the density ρ of the plate. In the framework of the realistic quasi-electrostatic approximation, the electrical equilibrium equation remains true but the mechanical equilibrium equation is replaced by

$$\text{div } \sigma^\varepsilon + f^\varepsilon = \rho \ddot{u}^\varepsilon \text{ in } \Omega^\varepsilon$$

where the upper dot denotes the differentiation with respect to time. Under mild assumptions on the initial state and the essential assumption

$$\int_{-1}^{+1} x_3 \widetilde{M}_1(x_1, x_2, x_3)\, dx_3, \; \widetilde{M}_2 \text{ independent of } x_3 \tag{38}$$

it is possible to proceed to the study of the convergence of s_p^ε when ε goes to zero ([15], [17]), the result depends strongly on the relative behaviour of ε and ρ. A unified accurate and simplified modeling is then obtained by simply adding $\int_{\Omega^\varepsilon} \rho \ddot{\overline{u}}_p^\varepsilon\, dx$ to the left hand side of the equation defining the descaled limit problem $\overline{\mathcal{P}}(\Omega^\varepsilon)_p$. Thus the relationship between the *reduced* stress, electric displacement, strain and gradient of electrical potential remains the same as in the static case: $\widetilde{M}_p^\varepsilon$ really describes the constitutive equations of the plate! The displacement fields involved in our simplified modeling being of Kirchoff-Love type, clearly four cases, indexed by q, of relative behaviours of the parameters determine the *essential* nature of the limit response of the plate to the electromechanical loading:

$$\begin{array}{llll} q = 1 & : & \rho \to \overline{\rho} \in (0, +\infty) & , \quad q = 2 \quad : \quad \rho \to 0 \text{ and } \rho/\varepsilon^2 \to \infty \\ q = 3 & : & \rho/\varepsilon^2 \to \overline{\overline{\rho}} \in (0, +\infty) & , \quad q = 4 \quad : \quad \rho = o(\varepsilon^2). \end{array} \tag{39}$$

In the cases $q = 2$ and $q = 4$, the limit response of the plate to the electromechanical loading is essentially quasi-static, while the cases $q = 1$ and $q = 3$ involve the acceleration of the displacement. Moreover, because of the assumption (38), appears a decoupling between the membrane motion and the flexural one. If $q = 1, 2$, the flexion is neglectible and the membrane response is dynamic if $q = 1$, quasi-static if $q = 2$. When $q = 3, 4$, the membrane response is quasi-static whereas the flexural response is dynamic if $q = 3$

and quasi-static if $q = 4$. In these last two cases, the equation giving the flexion does not involve the limit electric potential if $p = 1$. The uncoupled elliptic and hyperbolic involved problems are two-dimensional and set on ω.

The steps of the derivation of our model are the following. Firts we proceed to the same scaling as in section 4.1 and to a decomposition $s(\varepsilon) = s(\varepsilon)_e + s(\varepsilon)_r$, where $s(\varepsilon)_e$ solves a problem like $\mathcal{P}(\varepsilon, \Omega)_p$ and consequently whose asymptotic behavior is provided by section 4.1. Hence $s(\varepsilon)_r = (u(\varepsilon)_r, \varphi(\varepsilon)_r)$ satisfies an homogeneous variational evolution equation. Because the time derivatives do not act on $\varphi(\varepsilon)_r$, it is possible to exhibit a linear evolution equation for $u(\varepsilon)_r$ governed by a maximal monotone operator in a suitable Hilbert space whose norm depends on (ε, ρ). Since the Trotter results of convergence of semi-goups of linear operators acting on variables spaces claim that the study of convergence of the transient problems reduces to the static case, the asymptotic behavior of $u(\varepsilon)_r$, and consequently of $s(\varepsilon)_r$ is easily determined by straightforward variants of the convergence results of the section 4.1.

5.2   The fully dynamic case

In the previous case, the electrical field E^ε was assumed to be curl-free and, consequently, equal to the gradient of the so-called electrical potential φ^ε. If we want to take into account the magnetic effects, the state of the plate is now described by a triplet $z^\varepsilon = (u^\varepsilon, E^\varepsilon, H^\varepsilon)$ where H^ε is the magnetic field and the equations of the problem read as:

$$\begin{cases} \operatorname{div} \sigma^\varepsilon + f^\varepsilon = \rho \ddot{u}^\varepsilon \text{ in } \Omega^\varepsilon \\ \dot{D}^\varepsilon = c \operatorname{curl} H^\varepsilon \text{ in } \Omega^\varepsilon \\ \mu \dot{H}^\varepsilon = -c \operatorname{curl} E^\varepsilon \text{ in } \Omega^\varepsilon \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(e(u^\varepsilon), E^\varepsilon) \text{ in } \Omega^\varepsilon \end{cases}$$

with two kind of boundary conditions intimately linked to those of the previous cases (and, then, still indexed by p!):

$$p = 1 : H^\varepsilon \wedge n^\varepsilon = j^\varepsilon \text{ on } \partial\Omega^\varepsilon, \; p = 2 : E^\varepsilon \wedge n^\varepsilon = E_0^\varepsilon \wedge n^\varepsilon \text{ on } \partial\Omega^\varepsilon$$

Here $c, \mu, j^\varepsilon, E_0^\varepsilon$ stand for the light celerity, the magnetic permeability, the surface current density and the exterior electrical field respectively. We will assume that there exist sufficiently smooth fields E_0, j such that:

$$\begin{cases} E_0^\varepsilon(\Pi^\varepsilon) = \varepsilon^2 E_0(x), \; \forall x \in \partial\Omega, \quad j^\varepsilon(\Pi^\varepsilon) = \varepsilon^2 j(x), \; \forall x \in \Gamma_\pm \\ j_\alpha^\varepsilon(\Pi^\varepsilon x) = \varepsilon j_\alpha(x), \quad j_3^\varepsilon(\Pi^\varepsilon) = \varepsilon^2 j_3(x), \; \forall x \in \Gamma_{lat}. \end{cases} \tag{40}$$

Let

$$\begin{cases} \overline{E}_1^\varepsilon = \{E \in L^2(\Omega^\varepsilon)^3; E_3 = 0, \partial_3 E_\alpha = 0\}, \\ \overline{E}_2^\varepsilon = \{E; E_3 \in L^2(\Omega^\varepsilon); \partial_3(\partial_\alpha E_3 - \partial_3 E_\alpha) = 0, E_\alpha = 0 \text{ on } \Gamma_\pm^\varepsilon\}, \\ \overline{H}_1^\varepsilon = \{H \in L^2(\Omega^\varepsilon)^3; H_\alpha = 0, \partial_3 H_3 = 0\}, \\ \overline{H}_2^\varepsilon = \{H \in L^2(\Omega^\varepsilon)^3; H_3 = 0, \partial_3 H_\alpha = 0\}, \\ \overline{Z}_p^\varepsilon = \overline{V}^\varepsilon \times \overline{E}_p^\varepsilon \times \overline{H}_p^\varepsilon, \; k_1(v, E) = (e_{\alpha\beta}(v), E_\alpha), \; k_2(v, E) = (e_{\alpha\beta}(v), E_3), \; 1 \leqslant \alpha, \beta \leqslant 3. \end{cases} \tag{41}$$

Under (38), (40) and mild assumptions on the smoothness of the initial state, it can be shown that the state z_p^ε is asymptotically equivalent to $\overline{z}_p^\varepsilon = (\overline{u}_p^\varepsilon, \overline{E}_p^\varepsilon, \overline{H}_p^\varepsilon)$ which satisfies:

$$\begin{cases} \int_{\Omega^\varepsilon} \rho \ddot{\overline{u}}_p^\varepsilon \cdot v \, dx + \int_{\Omega^\varepsilon} \widetilde{M}_p^\varepsilon k_p(\overline{u}_p^\varepsilon, \overline{E}_p^\varepsilon) \cdot k_p(v,0) \, dx = L_p^\varepsilon(v,0), \ \forall v \in V_{KL}(\Omega^\varepsilon) \\ (\dot{\overline{D}}_1^\varepsilon)_1(\widehat{x}) = -c \, \partial_2(\overline{H}_1^\varepsilon)_3(\widehat{x}) + j_1^\varepsilon(\widehat{x},\varepsilon) + j_1^\varepsilon(\widehat{x},-\varepsilon), \ \forall \widehat{x} \in \omega \\ (\dot{\overline{D}}_1^\varepsilon)_2(\widehat{x}) = c \, \partial_1(\overline{H}_1^\varepsilon)_3(\widehat{x}) + j_2^\varepsilon(\widehat{x},\varepsilon) + j_2^\varepsilon(\widehat{x},-\varepsilon), \ \forall \widehat{x} \in \omega \\ \dot{\overline{D}}_2^\varepsilon)_2(\widehat{x}) = c \, (\partial_1(\overline{H}_2^\varepsilon)_2 - \partial_2(\overline{H}_2^\varepsilon)_1)(\widehat{x}), \ \forall \widehat{x} \in \omega \\ \mu(\dot{\overline{H}}_1^\varepsilon)_3(\widehat{x}) = -c \, (\partial_1(\overline{E}_1^\varepsilon)_2 - \partial_2(\overline{E}_1^\varepsilon)_1)(\widehat{x}), \ \forall \widehat{x} \in \omega \\ \mu(\dot{\overline{H}}_2^\varepsilon)_1(\widehat{x}) = -c \, (\partial_2(\overline{E}_2^\varepsilon)_3 - \partial_3(\overline{E}_2^\varepsilon)_2)(\widehat{x}), \ \forall \widehat{x} \in \omega \\ \mu(\dot{\overline{H}}_2^\varepsilon)_2(\widehat{x}) = -c \, (\partial_3(\overline{E}_2^\varepsilon)_1 - \partial_1(\overline{E}_2^\varepsilon)_3)(\widehat{x}), \ \forall \widehat{x} \in \omega \\ (\overline{\sigma}_p^\varepsilon, \overline{D}_p^\varepsilon) = \widetilde{M}_p^\varepsilon k_p(\overline{u}_p^\varepsilon, \overline{E}_p^\varepsilon) \end{cases} \tag{42}$$

with the boundary conditions:

$$\overline{H}_1^\varepsilon \wedge n^\varepsilon = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} j^\varepsilon(\cdot, x_3) \, dx_3 \ \text{on} \ \Gamma_{lat}^\varepsilon, \ \overline{E}_2^\varepsilon \wedge n^\varepsilon = \overline{E}_0^\varepsilon \wedge n^\varepsilon \ \text{on} \ \Gamma_\pm^\varepsilon. \tag{43}$$

The structure of the equations of our model is the same that those of the genuine model, but the problems are two-dimensional and with a lesser number of degrees of freedom for the state fields!

Again, the key-point is to formulate a suitable scaling of the problems in terms of an evolution equation governed by a maximal monotone operator in an Hilbert space of possible states with finite scaled energy. By using Trotter theory we only have to consider the limit behavior of a perturbation of the variational equation which defines $\mathcal{P}(\varepsilon, \Omega)_p$. This perturbation taking into account a scaling of the curl operator, the limit behavior is obtained by using weak continuity and integration by parts in the terms involving the curl operator.

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