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Two linearized problems of shallow water flows

Christian Licht

Laboratoire de Mécanique et Génie Civil, UMR CNRS 5508,
Case Courrier 048, Université Montpellier 2, Place Eugène Bataillon,
34095 Montpellier Cedex 05, France
licht@imgc.univ-montp2.fr

Tran Thu Ha

Institute of Mechanics, 264 Doi Can, Hanoi, Vietnam
ttha@imech.ac.vn

Quoc Phong Vu

Department of Mathematics, Ohio University Athens, Ohio 45701, USA

Summary: *A mathematical analysis, based on the theory of semi-groups of operators on Hilbert space, of linearized problems involving Saint-Venant equations governing shallow water flows is carried out. The first problem concerns the classical linearized Saint-Venant problem, while the second deals with a water quality problem: the evolution of a concentration of products in a flow solution of the first problem. The method seems capable to take into account thermal and multi-phase effects*

Key words: *Shallow water flows, Linearized Saint-Venant equations, Semi-group of operators.*

1. Introduction

Many works are devoted to the study of shallow water flows especially by numerical methods. Here, we proceed to a mathematical analysis by the theory of evolution equations in Hilbert spaces of two linearized problems involving the Saint-Venant equations system. The first one concerns the classical linearized Saint-Venant problem. The second deals with a water quality problem: the evolution of a concentration of products in a flow solution of the first problem.

2. Formulation of the linearized Saint-Venant problem

The linearized Saint-Venant equations which model the flow of a viscous fluid in shallow water ([1], [2]) read as:

$$\begin{cases} \frac{\partial \eta}{\partial t} = -h \operatorname{div} V - R \eta \\ \frac{\partial V}{\partial t} = -g \nabla \eta + \mu \Delta V - P V + f & \text{in } \Omega \times (0, T), \\ \eta(x, 0) = \eta^0(x), \quad V(x, 0) = V^0(x) \quad \forall x \in \Omega, \quad \eta^0, V^0 \quad \text{given in } \Omega \end{cases} \quad (2.1)$$

Here, Ω is a bounded domain of \mathbb{R}^2 with a Lipschitz-continuous boundary Γ , T is a positive real number, t denotes the time and η , V stand for the free surface elevation and the velocity field. The positive constants g , μ , h are connected with the gravity acceleration, the viscosity and average height of the fluid, while R is a positive relaxation parameter and P a non negative 2×2 matrix accounting for the Coriolis effect and numerical relaxation. Let σ denote the “stress tensor”:

$$\sigma = -g\eta I_d + \mu \nabla v$$

where I_d is the 2×2 identity matrix.

We assume that Γ is the union of disjoint parts Γ_i , $i=1, \dots, 5$, and, n being the unit normal along Γ outward to Ω , consider the boundary conditions:

$$V = \bar{V} \text{ given on } \Gamma_1 \times [0, T] \quad (2.2)$$

$$\sigma n = \gamma \text{ given on } \Gamma_2 \times [0, T] \quad (2.3)$$

$$V_N = \bar{V}_N, \sigma_T = \gamma_T, V_N, \gamma_T \text{ given on } \Gamma_3 \times [0, T] \quad (2.4)$$

with $V_N := V \cdot n$, $\sigma_T := \sigma n - \sigma_N n$, $\sigma_N := \sigma n \cdot n$,

$$V_T := V - V_N n = \bar{V}_T, \sigma_N = p, \bar{V}_T \text{ and } p \text{ given on } \Gamma_4 \times [0, T] \quad (2.5)$$

$$\sigma n + KV = \delta \text{ given on } \Gamma_5 \times [0, T], \quad (2.6)$$

where K is a given 2×2 non-negative matrix.

The condition on Γ_1 is an adhesion condition whereas the condition on Γ_2 expresses that the stress vector is given. On Γ_3, Γ_4 , the normal and the tangential component of either the velocity or the stress vector are imposed. Last, a friction condition is involved on Γ_5 .

3. A result of existence and uniqueness.

Let $J = \{V \in H^1(\Omega)^2; V|_{\Gamma_1} = 0, V_N|_{\Gamma_3} = 0, V_T|_{\Gamma_4} = 0\}$, the subspace of the Sobolev space $H^1(\Omega)^2$, made of the kinematically admissible velocity fields. Then, introducing the Hilbert space $H := \{u = (\eta, v) \in L^2(\Omega) \times L^2(\Omega)^2\}$ of possible states $u = (\eta, V)$ with finite mechanical energy equipped with the norm and inner product:

$$|u|_H^2 = (u, u)_H = g \int_{\Omega} \eta^2 dx + h \int_{\Omega} |V|^2 dx \quad \forall u = (\eta, V) \in H$$

$$(u, u')_H = g \int_{\Omega} \eta \eta' dx + h \int_{\Omega} V \cdot V' dx \quad \forall u = (\eta, V) \in H, \forall u' = (\eta', V') \in H$$

and the unbounded linear operator with domain $D(A)$ defined by:

$$D(A) = \left\{ \begin{array}{l} u = (\eta, v) \in H; v \in J \text{ and } \exists! w \in L^2(\Omega) \text{ such that:} \\ \int_{\Omega} (g\eta \operatorname{div} \varphi - \mu \nabla v \cdot \nabla \varphi) dx = \int_{\Omega} w \cdot \varphi dx + \int_{\Gamma_5} K v \cdot \varphi ds \quad \forall \varphi \in J \end{array} \right. \quad (3.1)$$

$$Au = (-h \operatorname{div} V - R\eta, w - PV),$$

it is possible to formulate the problem in terms of an evolution in H :

$$\begin{cases} \frac{du_r}{dt} = Au_r + F \\ u_r(0) = u_r^0 \end{cases}$$

where u_r is the difference between u and a suitable lifting u_e of the inhomogeneous Dirichlet boundary conditions, and F stems from the data and u_e . It is easy to show that A is m -dissipative and then generates a continuous semi-group of contractions on H . Hence, if the data are smooth enough the problem has a unique solution. Indeed, the semi group being analytic, the conditions on the data are rather mild.

Remark: It is possible to include a dissipative radiation condition on a part of the boundary which may be useful for numerical approximations by truncated domains when Ω is unbounded. The trick for the formulation of the Saint-Venant problem in terms of an evolution equation consists in introducing a new state variable: the trace of the velocity on the involved part of the boundary!

4. A water quality problem

Many problems dealing with water quality (salinity intrusion, transport of sediments or pollution, etc...) lead to finding a field C of concentration which satisfies:

$$\begin{cases} \frac{\partial C}{\partial t} = K_c \Delta C - V \cdot \nabla C + f_c & \text{in } \Omega \times (0, T) \\ C|_{\partial\Omega} = \bar{C} & \text{and } C(x, 0) = C^0(x) & \text{in } \Omega \end{cases},$$

here K_c is a positive diffusion coefficient and V is the previous velocity field solution of Saint-Venant equations in Ω , which, from now on, is assumed to have a C^2 boundary. It can be established that C_r , the difference between the true concentration and a suitable lifting C_e of \bar{C} , satisfies the evolution equation in $L^2(\Omega)$:

$$\begin{cases} \frac{dC_r}{dt} = -BC_r + \Phi(t)C_r + F_c(t) \\ C_r(0) = C^0 - C_e(0) \end{cases}, \quad (4.1)$$

where B is the realization of $-\Delta$ in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$, $F_c = -\frac{dC_e}{dt} + f_c + V \cdot \nabla C_e$, and $\Phi(t)$ is the operator defined in $D(B^{s/2})$, for all $s \in [1, 2]$, by:

$$\varphi \in D(B^{s/2}) \mapsto \Phi(t)\varphi := -V \cdot \nabla \varphi \in L^2(\Omega)$$

The existence and uniqueness of a mild solution

$$C_r(t) = S_B(t)(C^0 - C_e(0)) + \int_0^t S_B(t-\tau)(F_c(\tau) + \Phi(\tau)C_r(\tau))d\tau$$

of (4.1) is obtained by a fixed point method ([3],[4]) using the analyticity of the semi group $S_B(t)$ generated by $-B$ and the fact that $\Phi(t) \in C^0([0,T], Lin(D(B^{s/2}), L^2(\Omega)))$, a mere consequence of the Sobolev embeddings.

Remark 1: The regularity assumption on Γ may be dropped and various other mixed boundary conditions may be considered in the extent where the induced operator B satisfies $D(B^{s/2}) \subset H^s(\Omega)$.

Remark 2: Because, the water thermal convection equations read as:

$$\begin{cases} \rho \frac{\partial T}{\partial t} = \chi \Delta T - \rho V \cdot \nabla T & \text{in } \Omega \times (0, T) \\ T|_{\partial\Omega} = \bar{T} & \text{and } T(x, 0) = T^0(x) & \text{in } \Omega \end{cases},$$

where T, χ, ρ denote the temperature, the coefficient of heat conductivity and the density of the water, our previous method works for the water thermal convection problem.

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References

1. **S.Guillou**, “Modélisation des écoulements cotiers et estuariens. Etude mathématique et applications en coordonnées généralisées”, Ph. D. thesis, University of Caen, France, 1996.
2. **G. Darblade, R. Baraille, A. Y. Le Roux, X. Carton, D. Pinchon**, “Non reflecting boundary conditions for a two-dimensional linear barotropic shallow water model”, C. R. Acad. Sci. Paris, T. 324, No. 1, pp 485-490, 1997.
3. **Songmu Zheng**, Nonlinear Evolution Equations, Monographs and Surveys in Pure and Applied Mathematics 133, Chapman & Hall, 2004.
4. **Taylor**, Partial Differential Equations III, Applied Mathematical Sciences 117, Springer, 1997.