Two linearized problems of shallow water flows

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Summary: *A mathematical analysis, based on the theory of semi-groups of operators on Hilbert space, of linearized problems involving Saint-Venant equations governing shallow water flows is carried out. The first problem concerns the classical linearized Saint-Venant problem, while the second deals with a water quality problem: the evolution of a concentration of products in a flow solution of the first problem. The method seems capable to take into account thermal and multi-phase effects*

Key words: *Shallow water flows, Linearized Saint-Venant equations, Semi-group of operators.*

1. Introduction

Many works are devoted to the study of shallow water flows especially by numerical methods. Here, we proceed to a mathematical analysis by the theory of evolution equations in Hilbert spaces of two linearized problems involving the Saint-Venant equations system. The first one concerns the classical linearized Saint-Venant problem. The second deals with a water quality problem: the evolution of a concentration of products in a flow solution of the first problem.

2. Formulation of the linearized Saint-Venant problem

The linearized Saint-Venant equations which model the flow of a viscous fluid in shallow water ([1], [2]) read as:

$$\begin{cases} \dfrac{\partial \eta}{\partial t} = -h\,div V - R\eta \\ \dfrac{\partial V}{\partial t} = -g\nabla\eta + \mu\Delta V - PV + f \qquad in \quad \Omega\times(0,T), \\ \eta(x,0) = \eta^0(x), \quad V(x,0) = V^0(x) \quad \forall x\in\Omega, \quad \eta^0, V^0 \quad given \quad in \quad \Omega \end{cases} \tag{2.1}$$

Here, Ω is a bounded domain of R^2 with a Lipschitz-continuous boundary Γ, T is a positive real number, t denotes the time and η, V stand for the free surface elevation and the velocity field. The positive constants g, μ, h are connected with the gravity acceleration, the viscosity and average height of the fluid, while R is a positive relaxation parameter and P a non negative 2×2 matrix accounting for the Coriolis effect and numerical relaxation. Let σ denote the "stress tensor":

$$\sigma = -g\eta I_d + \mu\nabla v$$

where I_d is the 2x2 identity matrix.

We assume that Γ is the union of disjoint parts Γ_i, i=1,…,5, and, n being the unit normal along Γ outward to Ω, consider the boundary conditions:

$$V = \overline{V} \text{ given on } \Gamma_1\times[0,T] \qquad (2.2)$$

$$\sigma n = \gamma \text{ given on } \Gamma_2\times[0,T] \qquad (2.3)$$

$$V_N = \overline{V}_N,\ \sigma_T = \gamma_T,\ V_N, \gamma_T \text{ given on } \Gamma_3\times[0,T] \qquad (2.4)$$

with $V_N := V.n$, $\sigma_T := \sigma n - \sigma_N n$, $\sigma_N := \sigma n \cdot n$,

$$V_T := V - V_N n = \overline{V}_T,\ \sigma_N = p,\ \overline{V}_T \text{ and } p \text{ given on } \Gamma_4\times[0,T] \qquad (2.5)$$

$$\sigma n + KV = \delta \text{ given on } \Gamma_5\times[0,T], \qquad (2.6)$$

where K is a given 2x2 non-negative matrix.

The condition on Γ_1 is an adhesion condition whereas the condition on Γ_2 expresses that the stress vector is given. On Γ_3, Γ_4, the normal and the tangential component of either the velocity or the stress vector are imposed. Last, a friction condition is involved on Γ_5.

3. A result of existence and uniqueness.

Let $J = \left\{V \in H^1(\Omega)^2; V_{|\Gamma_1} = 0, V_{N|\Gamma_3} = 0,\ V_{T|\Gamma_{43}} = 0\right\}$, the subspace of the Sobolev space $H^1(\Omega)^2$, made of the kinematically admissible velocity fields. Then, introducing the Hilbert space $H := \left\{u = (\eta, v) \in L^2(\Omega)\times L^2(\Omega)^2\right\}$ of possible states $u = (\eta, V)$ with finite mechanical energy equipped with the norm and inner product:

$$|u|_H^2 = (u,u)_H = g\int_\Omega \eta^2 dx + h\int_\Omega |V|^2 dx \quad \forall u = (\eta, V) \in H$$

$$(u,u')_H = g\int_\Omega \eta\eta' dx + h\int_\Omega V\cdot V' dx \qquad \forall u = (\eta, V) \in H, \forall u' = (\eta, V') \in H$$

and the unbounded linear operator with domain $D(A)$ defined by:

$$D(A) = \left\{ \begin{array}{l} u = (\eta, \mathrm{v}) \in H;\ \mathrm{v} \in J \text{ and } \exists!\mathrm{w} \in \mathrm{L}^2(\Omega) \text{ such that:} \\ \int_\Omega (g\eta\ div\varphi - \mu\nabla\mathrm{v}\cdot\nabla\varphi)\ dx = \int_\Omega w\cdot\varphi\ dx + \int_{\Gamma_5} K\mathrm{v}.\varphi\ ds \quad \forall\varphi \in \mathrm{J} \end{array} \right\} \qquad (3.1)$$

$$Au = (-h\,divV - R\eta, w - PV),$$

it is possible to formulate the problem in terms of an evolution in H:

$$\begin{cases} \dfrac{du_r}{dt} = Au_r + F \\ \\ u_r(0) = u_r^0 \end{cases}$$

where u_r is the difference between u and a suitable lifting u_e of the inhomogeneous Dirichlet boundary conditions, and F stems from the data and u_e. It is easy to show that A is m-dissipative and then generates a continuous semi-group of contractions on H. Hence, if the data are smooth enough the problem has a unique solution. Indeed, the semi group being analytic, the conditions on the data are rather mild.

Remark: It is possible to include a dissipative radiation condition o a part of the boundary which may be useful for numerical approximations by truncated domains when Ω is unbounded. The trick for the formulation of the Saint-Venant problem in terms of an evolution equation consists in introducing a new state variable: the trace of the velocity on the involved part of the boundary!

4. A water quality problem

Many problems dealing with water quality (salinity intrusion, transport of sediments or pollution, etc…) lead to finding a field C of concentration which satisfies:

$$\begin{cases} \dfrac{\partial C}{\partial t} = K_c \Delta C - V.\nabla C + f_c \qquad in \quad \Omega \times (0,T) \\ C|_{\partial\Omega} = \overline{C} \qquad and \quad C(x,0) = C^0(x) \qquad in \quad \Omega \end{cases},$$

here K_c is a positive diffusion coefficient and V is the previous velocity field solution of Saint-Venant equations in Ω, which, from now on, is assumed to have a C^2 boundary.
It can be established that C_r, the difference between the true concentration and a suitable lifting C_e of $\overline{C}$, satisfies the evolution equation in $L^2(\Omega)$:

$$\begin{cases} \dfrac{dC_r}{dt} = -BC_r + \Phi(t)C_r + F_c(t) \\ C_r(0) = C^0 - C_e(0) \end{cases}, \qquad (4.1)$$

where B is the realization of $-\Delta$ in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$, $F_c = -\dfrac{dC_e}{dt} + f_c + V \cdot \nabla C_e$, and $\Phi(t)$ is the operator defined in $D(B^{s/2})$, for all $s \in [1,2)]$, by:

$$\varphi \in D(B^{s/2}) \mapsto \Phi(t)\varphi := -V.\nabla\varphi \in L^2(\Omega)$$

The existence and uniqueness of a mild solution

$$C_r(t) = S_B(t)(C^0 - C_e(0) + \int_0^t S_B(t-\tau)(F_c(\tau) + \Phi(\tau)C_r(\tau)d\tau$$

of (4.1) is obtained by a fixed point method ([3],[4]) using the analycity of the semi group $S_B(t)$ generated by –*B* and the fact that $\Phi(t) \in C^0([0,T], Lin(D(B^{s/2}), L^2(\Omega))$, a mere consequence of the Sobolev embeddings.

Remark 1: The regularity assumption on Γ may be dropped and various other mixed boundary conditions may be considered in the extent where the induced operator *B* satisfies $D\left(B^{s/2}\right) \subset H^s(\Omega)$.

Remark 2: Because, the water thermal convection equations read as:

$$\begin{cases} \rho \dfrac{\partial T}{\partial t} = \chi \Delta T - \rho V.\nabla T & in \quad \Omega \times (0,T) \\ T\big|_{\partial\Omega} = \overline{T} \qquad and \quad T(x,0) = T^0(x) & in \quad \Omega \end{cases},$$

where T, χ, ρ denote the temperature, the coefficient of heat conductivity and the density of the water, our previous method works for the water thermal convection problem.

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