On the Fattorini Criterion for Approximate Controllability and Stabilizability of Parabolic Systems

Mehdi Badra, Takéo Takahashi

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Mehdi Badra [*†], Takéo Takahashi [‡§]

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Abstract

In this paper, we extend an approximate controllability criterion for infinite dimensional linear systems of type $y' = Ay + Bu$, originally proved by H. O. Fattorini in [11] for bounded input B, to the case where B is unbounded. We also prove that if Fattorini criterion is satisfied and if the set of geometric multiplicities of A is bounded then approximate controllability can be achieved with a finite dimensional control. Thus, we show that Fattorini criterion implies the feedback stabilizability of linear and nonlinear parabolic systems.

When considering systems discribed by partial differential equations such a criterion reduces to a unique continuation theorem for an eigenvalue problem. We then consider flow systems described by coupled Navier-Stokes type equations (such as MHD system or micropolar fluid system) and we sketch a systematic procedure relying on Fattorini criterion for checking stabilizability of such nonlinear system. In particular, we provide local Carleman inequalities for Stokes equations that permit to prove unique continuation theorems related to the stabilizability of coupled Stokes type systems.

Key words. Approximate controllability, stabilizability, parabolic equation, finite dimensional control, coupled - Stokes and MHD system.

AMS subject classifications. 93B05, 93D15, 35Q30, 76D05, 76D07, 76D55, 93B52, 93C20.

1 Introduction

In the present paper we address the question of the stabilizability of the linear system

$$y' = Ay + Bu, \tag{1}$$

where A is the infinitesimal generator of a semigroup (e^{tA}) on a complex Hilbert space H (in particular, $D(A)$ is dense in H and A is a closed linear operator) and where B is a linear and possibly unbounded input operator defined on a complex Hilbert space U. In the particular case where (e^{tA}) is analytic our aim is to show some necessary and sufficient conditions for (1) to be stabilizable by using a frequency approach. Such a stabilizability criterion relies on an approximate controllability criterion originally proved by H. O. Fattorini in [11] in the case of bounded operator B.

More precisely, [11, Corollary 3.3] states that if B is bounded, if the spectrum of A consists in isolated eigenvalues with finite multiplicity and if its family of root vectors is complete, then (1) is approximately controllable if and only if B^* is one-to-one on each eigenspace of A^*, or equivalently, if the following "unique continuation" property holds:

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall \lambda \in \mathbb{C}, \quad A^*\varepsilon = \lambda\varepsilon \quad \text{and} \quad B^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \tag{UC}$$

In above settings, A^* and B^* denote the adjoints of A and B respectively. Condition (UC) is an infinite dimensional version of the classical Hautus test introduced in [20] (see [34] for a stronger version for exact

[*]IMT, UMR CNRS 5219, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France.

[†]LMAP, UMR CNRS 5142, UNIV PAU & PAYS ADOUR, 64013 Pau Cedex, France (mehdi.badra@univ-pau.fr).

[‡]Institut Élie Cartan, UMR 7502, INRIA, Nancy-Université, CNRS, BP239, 54506 Vandœuvre-lès-Nancy Cedex, France

[§]Team-Project CORIDA. INRIA Nancy - Grand Est, 615, rue du Jardin Botanique. 54600 Villers-lès-Nancy. France (takahash@inria.fr).

controllability). Note that (UC) can also be formulated as a linear independence condition related to the family of eigenvectors of A^*: if $\{\lambda_j \mid j \in \mathbb{N}\}$ denotes the set of eigenvalues of A, if $\ell_j = \dim\ker(\lambda_j - A)$ denotes the geometric multiplicity of λ_j and if $\{\varepsilon_k(\bar{\lambda}_j) \mid k = 1, \dots, \ell_j\}$ is a basis of $\ker(\bar{\lambda}_j - A^*)$ then (UC) is true if and only if

$$\text{for all } j \in \mathbb{N} \ \left\{B^*\varepsilon_k(\overline{\lambda}_j) \mid k = 1, \dots, \ell_j\right\} \text{ is a linearly independent family of } U. \tag{2}$$

Here we generalize the Fattorini criterion (UC) to the case of unbounded input operator and we show that it is also a criterion for approximate controllability with finite dimensional control. More precisely, we prove that approximate controllability of (1) by K-dimensional control in $\text{span}_{\mathbb{C}}\{v_l \mid l = 1, \dots, K\} \subset U$ holds if and only if the following rank conditions hold:

$$\forall j \in \mathbb{N}, \quad \text{rank}\left((v_l \mid B^*\varepsilon_k(\bar{\lambda}_j))_U\right)_{1\le k\le \ell_j, 1\le l\le K} = \ell_j. \tag{3}$$

From characterization (2) it is easily seen that (UC) and $K \geq \sup\{\ell_j \mid j \in \mathbb{N}\}$ are necessary conditions for (3). In the present paper, we prove that conditions (UC) and $K \geq \sup\{\ell_j \mid j \in \mathbb{N}\}$ are also sufficient to obtain a family $\{v_l \mid l = 1, \dots, K\}$ satisfying (3). Moreover, we also prove that the set of such families is residual in U^K, which means that approximate controllability of (1) by K-dimensional control is generically true if and only if (UC) and $K \geq \sup\{\ell_j \mid j \in \mathbb{N}\}$ hold (see Theorem 4 below).

In the parabolic case (i.e. if A generates an analytic semigroup on H) and if the spectrum of A consists in isolated eigenvalues with finite multiplicity without any finite cluster point, unique continuation property (UC) is also a stability criterion for (1). Indeed, in such situation only a finite number of the modes of the system have to be controlled. More precisely, approximate controllability of (1) guarantees the stabilizability of the unstable modes with finite dimensional control and then the stabilizability of the whole system is obtained by remarking that the control does not destabilize the stable modes. Here we prove that (1) is generically stabilizable by finite dimensional feedback control for an arbitrary rate of decrease if and only if (UC) is satisfied. More precisely, we prove that (1) is generically stabilizable by K-dimensional feedback control for an exponential rate of decrease $\sigma > 0$ if and only if $K \geq \sup\{\ell_j \mid \Re\lambda_j \geq -\sigma\}$ and the following "unique continuation" property holds (see Corollary 7 below):

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall \lambda \in \mathbb{C},\ \Re\lambda \geq -\sigma \quad A^*\varepsilon = \lambda\varepsilon \quad \text{and} \quad B^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \tag{UC$_\sigma$}$$

Moreover, by using stabilizability results of [6] for a certain class of nonlinear systems for which the linearized system is stabilizable, it follows that (UC) is also a stabilizability criterion (see Theorem 10 and Theorem 11 below).

Finally, let us underline that in many practical examples system (1) is originally defined on a real Hilbert space for control functions with values in a real control space. Thus above approximate controllability and stabilization resuts are also stated in the real case (see Theorem 12 and Corollary 13 below).

The first works dealing with approximate controllability of infinite dimensional linear systems are due to H. O. Fattorini in the pioneer papers [11, 12]. While [12] focus on the particular case of self-adjoint generators, general semigroup generators are considered in [11] where the above mentioned infinite dimensional Hautus test (UC) is proposed. Very surprisingly, the last quoted work has been published three years earlier than M. L. J. Hautus famous paper [20]. It is also suggested in [11] that one can find a finite rank input operator such that approximate controllability holds provided that the rank is greater or equal than the maximum of the geometric multiplicities of A. However, nothing is said about the way of constructing such a finite rank input operator (obtained for instance by simply restricting a given operator B to a finite dimensional subspace of U). A partial answer is given in [37] where a generalization of the Kalman rank condition for approximate controllability of system generated by strongly continuous semigroup in general Banach spaces is proposed. In the particular case where A is a normal operator with compact resolvent, such a generalized Kalman condition reduces to the matrix rank criterion (3) and permits to characterize admissible finite dimensional control subspaces, see [37, eq.(3.5)]. About stabilizability problems, the idea of using finite dimensional rank feedback law stabilizing linear parabolic systems goes back to [36, 38]. Note that in the case of self-adjoint generators with compact resolvent, the rank conditions in (3) for j corresponding to the unstable modes appears in [36, eq.(7.2)] as a stabilizability criterion.

One of the major interest of the Fattorini criterion is that proving (UC) is an easy alternative to obtain the finite cost condition needed to construct stabilizing feedback law from well-posed optimal quadratic problem,

see [3, Remark 2]. For instance, boundary feedback stabilizability of the Navier-Stokes system can be simply reduced to a uniqueness result of type [10] and avoid more sophisticate approachs for the instationary system as in [23, 14] or [15, 16]. Moreover, a systematic generalization of the last quoted works to other analogous more complex systems such as coupled Stokes type systems or fluid-structure systems is not straightforward. Here, with two examples of flow systems described by coupled Stokes type equations, we sketch a systematic procedure relying on (UC) for checking nonlinear system stabilizability. We prove local Carleman estimates that permit to check (UC) for general coupled Navier-Stokes systems and next we deduce feedback and dynamical stabilization of nonlinear MHD and micropolar systems (see Corollary 17, Theorem 23 and Theorem 25). Concerning the use of Fattorini criterion for feedback stabilization of fluid structure system we refer to forthcoming works [5, 4].

The rest of the paper is organized as follows. In Section 2, we state the main assumptions, we recall Fattorini theorem and we enounce our main results. In Section 3, we give two simple examples of application involving the heat equation, Section 4 is devoted to the proofs of the main results and we end the paper with two applications to the stabilization of incompressible coupled Navier-Stokes type systems in Section 5.

2 Assumptions and abstract results

2.1 The Fattorini Theorem

Here we recall the Fattorini Theorem for approximate controllability of (1).

In what follows, H denotes a complex Hilbert space, A an unbounded operator with domain $\mathcal{D}(A)$ which is the infinitesimal generator of a strongly continuous semigroup (e^{At}) on H. We assume that A satisfies the following assumption:

$(\mathcal{H}_1)$ the spectrum of A consists in isolated eigenvalues with finite multiplicity.

More precisely, $(\mathcal{H}_1)$ means that the spectrum of A is composed with complex eigenvalues λ_j for $j \in \mathbb{N}$ and that for all $j \in \mathbb{N}$ the projection operator:

$$R_{-1}(\lambda_j) = \frac{1}{2\pi\imath} \int_{|\lambda-\lambda_j|=\alpha} (\lambda - A)^{-1} \mathrm{d}\lambda, \tag{4}$$

has finite dimensional range. In (4) the value $\alpha > 0$ is chosen small enough so that the circle $\{\lambda \in \mathbb{C} \mid |\lambda-\lambda_j| = \alpha\}$ does not enclose other point of the spectrum than λ_j. It is well-known that there exists some $n \in \mathbb{N}^*$ such that $\mathrm{Ran}(R_{-1}(\lambda_j)) = \ker(\lambda_j - A)^n$ (see [24]) and we denote by $m(\lambda_j) \in \mathbb{N}^*$ the smallest of such n. The finite dimensional subspace $\ker(\lambda_j - A)^{m(\lambda_j)} = \mathrm{Ran}(R_{-1}(\lambda_j))$ is called the *generalized eigenspace* of A associated to λ_j, its dimension $N_j \in \mathbb{N}^*$ is called the *algebraic multiplicity* of λ_j and an element of $\ker(\lambda_j - A)^{m(\lambda_j)}$ is called *a root vector* of A. We also recall that the subspace $\ker(\lambda_j - A)$ is called the *(proper) eigenspace* of A associated to λ_j and its dimension $\ell_j \in \mathbb{N}^*$ is called the *geometric multiplicity* of λ_j (recall that $\ell_j \leq N_j$).

We make the following second assumption on A:

$(\mathcal{H}_2)$ The family of root vectors of A is complete in H, or equivalently:

$$\varepsilon = 0 \quad \Longleftrightarrow \quad \forall j \in \mathbb{N},\ \forall e \in \ker(\lambda_j - A)^{m(\lambda_j)}, \quad (\varepsilon|e)_H = 0.$$

Remark 1. *According to Keldy's Theorem, assumption $(\mathcal{H}_2)$ is satisfied by a class of perturbations of self-adjoint operator: if A_0 is a self-adjoint operator in H with compact resolvent, if A_1 is an operator such that $A_1(-A_0)^{-\alpha}$ is bounded for some $0 \leq \alpha < 1$, and if the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of A_0 satisfy for some $1 \leq p < +\infty$:*

$$\sum_{j\in\mathbb{N}} \frac{1}{|\lambda_j|^p} < +\infty, \tag{5}$$

then $A = A_0 + A_1$ with domain $D(A) = D(A_0)$ is closed and its family of root vectors is complete in H, see [18, Thm.10.1, p.276] or [35, Thm.5.6.1.3, p.394] combined with [35, Thm.5.6.1.1, p.392 and Lem.5.6.1.2, p.395]. Note that Weyl's formula ensures that (5) is satisfied by regular self-adjoint elliptic operator, see [35, p.395].

Next, let denote by U a complex Hilbert space and by $B : U \to H$ a linear mapping. Suppose for the moment that $B : U \to H$ is bounded and for any $\tau > 0$ define the *input map*:

$$\Phi_\tau(u) = \int_0^\tau e^{A(\tau-s)} Bu(s)\ \mathrm{ds}. \tag{6}$$

The boundedness of B clearly implies that $\Phi_\tau : L^2(0,+\infty;U) \to H$ is well-defined and bounded. Moreover, for $t > 0$ and $y_0 \in H$ the solution to (1) and $y(0) = y_0$ is given by $y(t) = e^{At}y_0 + \Phi_t(u)$.

System (1) is said to be *approximately controllable*[1] if $\cup_{\tau>0} \operatorname{Ran} \Phi_\tau$ is dense in H. It can be checked that system (1) is approximately controllable, if and only if, for any $y_1 \in H$ and for any $\varepsilon > 0$, there exists $u \in L^2(0,+\infty;U)$ such that the solution y of (1) with $y(0) = 0$ satisfies $\|y(\tau) - y_1\|_H < \varepsilon$ for some τ, depending in general on y_1 and ε. If τ can be chosen independently on y_1 and ε, or equivalently if $\operatorname{Ran} \Phi_\tau$ is dense in H, then (1) is said to *approximately controllable in time* τ. It can be checked that system (1) is approximately controllable in time τ, if and only if, for any y_0, $y_1 \in H$ and for any $\varepsilon > 0$, there exists $u \in L^2(0,+\infty;U)$ such that the solution y of (1) with $y(0) = y_0$ satisfies $\|y(\tau) - y_1\|_H < \varepsilon$. Finally, (1) is said to be *approximately controllable in any time* if for all $\tau > 0$ it is approximately controllable in time τ.

It is well known that approximate controllability can be characterized in terms of the adjoint of Φ_τ. Let denote by A^* (with domain $\mathcal{D}(A^*)$) the adjoint of A and by $B^* : H \to U$ the adjoint of B. Then the adjoint of Φ_τ is given by:

$$\forall \varepsilon \in H, \qquad (\Phi_\tau^* \varepsilon)(t) = \begin{cases} B^* e^{A^*(\tau-t)}\varepsilon & \text{for } t \in [0,\tau] \\ 0 & \text{for } t > \tau, \end{cases} \tag{7}$$

and a straightforward consequence of the equivalence $\overline{\operatorname{Ran} \Phi_\tau}^H = H \Leftrightarrow \ker \Phi_\tau^* = \{0\}$ is that (1) is approximately controllable in time τ, if and only if the following uniqueness result holds:

$$\forall \varepsilon \in H, \qquad B^* e^{A^* t}\varepsilon = 0 \quad (t \in (0,\tau)) \Longrightarrow \quad \varepsilon = 0. \tag{8}$$

Moreover, from $\overline{\cup_{\tau>0} \operatorname{Ran} \Phi_\tau}^H = H \Leftrightarrow \cap_{\tau>0} \ker \Phi_\tau^* = \{0\}$ we also get that (1) is approximately controllable, if and only if the following uniqueness result holds:

$$\forall \varepsilon \in H, \qquad B^* e^{A^* t}\varepsilon = 0 \quad (t \in (0,+\infty)) \Longrightarrow \quad \varepsilon = 0. \tag{9}$$

We are now in position to enounce the Fattorini Theorem stated in [11, Cor.3.3].

Theorem 2 (Fattorini 1966)**.** *Assume* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *and that* $B : U \to H$ *is bounded. System* (1) *is approximately controllable if and only if* (UC) *is satisfied.*

It is underlined in [12, Rem. 1] that approximate controllability and approximate controllability in any time are equivalent notions under the following additional assumption:

$(\mathcal{H}_3)$ A is the infinitesimal generator of an analytic semigroup on H.

Indeed, this follows from the fact that $(\mathcal{H}_3)$ allows to extend equality $B^* e^{A^* t}\varepsilon = 0$ from $(0,\tau)$ to $(0,+\infty)$ by analytic continuation. As a consequence, the following corollary holds.

Corollary 3. *Assume* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *and that* $B : U \to H$ *is bounded. System* (1) *is approximately controllable in any time if and only if* (UC) *is satisfied.*

2.2 The case of unbounded input operator and of finite dimensional control

Theorem 2 can be extended to the case where B is not necessarily bounded from U into H. In what follows, we fix $\mu_0 > \sup_{j\in\mathbb{N}} \Re\lambda_j$ so that fractional powers of $(\mu_0 - A^*)$ and $(\mu_0 - A)$ are well defined and for $\alpha \in \mathbb{R}$ we introduce the spaces:

$$H_\alpha \overset{\text{def}}{=} \begin{cases} \mathcal{D}((\mu_0 - A)^\alpha) & \text{if } \alpha \geq 0 \\ \mathcal{D}((\mu_0 - A^*)^\alpha)' & \text{if } \alpha < 0, \end{cases} \quad \text{and} \quad H_\alpha^* \overset{\text{def}}{=} \begin{cases} \mathcal{D}((\mu_0 - A^*)^\alpha) & \text{if } \alpha \geq 0 \\ \mathcal{D}((\mu_0 - A)^\alpha)' & \text{if } \alpha < 0, \end{cases} \tag{10}$$

where $\mathcal{D}((\mu_0 - A)^\alpha)'$ [resp. $\mathcal{D}((\mu_0 - A^*)^\alpha)'$] stands for the dual space of $\mathcal{D}((\mu_0 - A)^\alpha)$ [resp. $\mathcal{D}((\mu_0 - A^*)^\alpha)$] with respect to H.

We assume that B satisfies the following assumption:

[1] or *completely controllable* in the terminology of H.O. Fattorini

($\mathcal{H}_4$) (i) $B : U \to H_{-\gamma}$ is bounded for some $\gamma \geq 0$.

(ii) For some $\beta \in [\gamma - 1, \gamma]$ we have:

$$\mathrm{Ran}\Phi_\tau \subset H_{-\beta}. \tag{11}$$

Assumption ($\mathcal{H}_4$)(i) means that $(\mu_0 - A)^{-\gamma} B : U \to H$ is bounded for some $\gamma \geq 0$. For instance, the heat equation with Dirichlet boundary control corresponds to $\gamma = \frac{3}{4} + \epsilon$, $\epsilon > 0$, and the heat equation with Neumann boundary control corresponds to $\gamma = \frac{1}{4} + \epsilon$, $\epsilon > 0$ (see [7, Part IV, par. 2, examples 1.1 and 1.2]). Under ($\mathcal{H}_4$)(i) the linear system (1) can be interpreted in $H_{-\gamma}$ instead of H: Φ_τ defined by (6) is now bounded from $L^2(0, +\infty; U)$ into $H_{-\gamma}$ and then the solution to (1) for $y(0) = y_0 \in H_{-\gamma}$ and $u \in L^2(0, +\infty; U)$ belongs to $C([0, +\infty); H_{-\gamma})$. However, in many situations the trajectory y may be more regular. For instance, when ($\mathcal{H}_3$) is true maximal regularity result for analytic semigroup ensures that $y \in C([0, +\infty); H_{\frac{1}{2}-\gamma})$ which corresponds to $\beta = \gamma - \frac{1}{2}$ in assumption ($\mathcal{H}_4$)(ii). Indeed, if the admissibility condition (11) holds for a specific $\tau > 0$ then (11) holds for all $\tau > 0$ and Φ_τ is bounded from $L^2(0, +\infty; U)$ into $H_{-\beta}$ for all $\tau > 0$, see [40, Prop. 4.2.2.][2]. Then assumption (11) implies that the solution to (1) for $y(0) = y_0 \in H_{-\beta}$ belongs to $C([0, +\infty); H_{-\beta})$.

Of course, previous definitions of approximate controllability must be modified by just replacing H by $H_{-\beta}$: we now say that system (1) is *approximately controllable* if $\cup_{\tau>0} \mathrm{Ran}\, \Phi_\tau$ is dense in $H_{-\beta}$, that system (1) is *approximately controllable in time* τ if $\mathrm{Ran}\, \Phi_\tau$ is dense in $H_{-\beta}$ and that system (1) is *approximately controllable in any time* if $\mathrm{Ran}\, \Phi_\tau$ is dense in $H_{-\beta}$ for all $\tau > 0$.

As before, those definitions can be equivalently stated in terms of uniqueness properties analogue to (8) or (9). However, because the expression of Φ_τ^* given by (7) is valid for $\varepsilon \in H_\gamma^*$ but not necessarily for $\varepsilon \in H_\beta^*$, expressions (8) or (9) must be slightly modified. Following [40, Par. 4.3 and Par. 4.4] we have

$$\forall \varepsilon \in H_\beta^*, \quad (\Phi_\tau^* \varepsilon)(t) = \begin{cases} (\Psi\varepsilon)(\tau - t) & \text{for } t \in [0, \tau] \\ 0 & \text{for } t > \tau, \end{cases} \tag{12}$$

where Ψ denotes the uniquely determined *extended output map* $\Psi : H_\beta^* \to L^2_{\mathrm{loc}}([0, +\infty); U)$ satisfying

$$(\Psi\varepsilon)(t) = B^* e^{A^* t} \varepsilon \qquad \forall \varepsilon \in H_\gamma^*,\ t \geq 0.$$

About existence and uniqueness of such Ψ, see [40, p. 123][3]. Then we verify that (1) is *approximately controllable in time* τ if and only if

$$\forall \varepsilon \in H_\beta^*, \quad (\Psi\varepsilon)(t) = 0 \quad (t \in (0, \tau)) \Longrightarrow \quad \varepsilon = 0, \tag{13}$$

and that (1) is *approximately controllable* if and only if

$$\forall \varepsilon \in H_\beta^*, \quad (\Psi\varepsilon)(t) = 0 \quad (t \in (0, \infty)) \Longrightarrow \quad \varepsilon = 0. \tag{14}$$

Another important extension for Theorem 2, which is of course interesting from applications point of view, consists in replacing U by a strict subspace if possible of finite dimension. In what follows, v denotes an infinite linearly independent family in $\ell^2(\mathbb{N}; U)$ or a finite linearly independent family in U^K for some $K \in \mathbb{N}^*$. Here $\ell^2(\mathbb{N}; U)$ denotes the space of square-summable sequences of U. For such linearly independent family $v = (v_j)_{j \in \mathbb{N}} \in \ell^2(\mathbb{N}; U)$ or $v = (v_j)_{j=1,\ldots,K}$ we set $U_v = \overline{\mathrm{span}_{\mathbb{C}}\{v_j\}}^U$ and we introduce the restriction operator $B|_{U_v} : U_v \to H_{-\gamma}$ defined by $B|_{U_v} u = Bu$ for $u \in U_v$. Then such a linearly independent family v is said to be *admissible* if system

$$y' = Ay + B|_{U_v} u \tag{15}$$

is approximately controllable. Moreover, if there exists an admissible finite family $v = (v_j)_{j=1,\ldots,K}$ (then $\dim U_v = K$) we say that (1) *is approximately controllable by a K-dimensional control.*

As it has already been underlined in the introduction, it is suggested in [11, p.693] that if the geometric multiplicities of A's eigenvalues are bounded above by $K \in \mathbb{N}^*$ then one can find an input operator defined on a K-dimensional control space and such that (1) is approximately controllable. However, for a prescribed input operator approximate controllability by a K-dimensional control is not established i.e. for a given infinite

[2] Although the quoted work only applies for $\gamma = 1$ and $\beta = 0$ the result can be recovered for $\gamma \geq 0$ and $\beta \in [\gamma - 1, \gamma]$ with the change of variable $\widetilde{\Phi}_\tau \stackrel{\mathrm{def}}{=} (\mu_0 - A)^{-\beta} \Phi_\tau$, because for such values γ, β we have $(\mu_0 - A)^{-\beta} B : U \to H_{-1}$ bounded.

[3] Since the quoted work only applies for $\gamma = 1$ and $\beta = 0$ it can be used to justify the existence and the uniqueness of $\widetilde{\Psi} : H \to L^2_{\mathrm{loc}}([0, +\infty); U)$ satisfying $(\widetilde{\Psi}\varepsilon)(t) = B^*(\mu_0 - A^*)^{-\beta} e^{A^* t}\varepsilon$ for all $\varepsilon \in H_1^*$, $t \geq 0$, and we check that $\Psi = \widetilde{\Psi}(\mu_0 - A^*)^\beta$ obeys the desired property.

dimensional input B it is not said if it is possible to choose a K-dimensional subspace $U_v \subset U$ such that (15) is approximately controllable.

Here, we propose a generalization of Theorem 2 to the case of unbounded input which gives a precise characterization of approximate controllability by a K-dimensional control and of admissibility for a family v. For that, we need to introduce some additional notations. We recall that $\ell_j = \ker(\lambda_j - A) = \ker(\overline{\lambda}_j - A^*)$ is the geometric multiplicity of λ_j, we denote by $\varepsilon_k(\overline{\lambda}_j)$, $k = 1, \dots, \ell_j$, a basis of $\ker(\overline{\lambda}_j - A^*)$ and we set

$$\mathcal{E} \stackrel{\text{def}}{=} \overline{\bigoplus_{j\in\mathbb{N}} \ker(\overline{\lambda}_j - A^*)}^H = \overline{\operatorname{span}_{\mathbb{C}} \{\varepsilon_k(\overline{\lambda}_j) \mid j \in \mathbb{N},\ k = 1, \dots, \ell_j\}}^H.$$

Moreover, for a given family $v = (v_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}; U)$ we introduce the bounded linear operator

$$W_j(v) : \mathbb{C}^{\ell_j} \to \ell^2(\mathbb{N}), \quad x \mapsto \left(\sum_{k=1}^{\ell_j} x_k (v_l | B^* \varepsilon_k(\bar{\lambda}_j))_U \right)_{l\in\mathbb{N}}, \tag{16}$$

and for a finite family $v = (v_j)_{j=1,\dots,K}$ of U^K we use the same notation for the following matrix of order $K \times \ell_j$:

$$W_j(v) \stackrel{\text{def}}{=} \left((v_l \mid B^* \varepsilon_k(\bar{\lambda}_j))_U \right)_{1\le l\le K,\, 1\le k\le \ell_j}. \tag{17}$$

The following generalization of Theorem 2 holds.

Theorem 4. *Assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$ and let $K \in \mathbb{N}^*$. Then the following results hold.*

1. *System* (1) *is approximately controllable if and only if* (UC) *is satisfied.*
2. *System* (1) *is approximately controllable by a K-dimensional control if and only if* (UC) *is satisfied and*
$$\sup_{j\in\mathbb{N}} \ell_j \le K. \tag{18}$$
3. *A family v is admissible if and only if*
$$\operatorname{rank} W_j(v) = \ell_j \quad \forall j \in \mathbb{N}. \tag{19}$$
4. *Assume that* (UC) *is true. Then the set of admissible families of $\ell^2(\mathbb{N}, U)$ forms a residual set of $\ell^2(\mathbb{N}, U)$. Moreover, if $v \in \ell^2(\mathbb{N}, U)$ is admissible then its orthogonal projection onto $\ell^2(\mathbb{N}, B^*\mathcal{E})$ is admissible.*
5. *Assume that* (UC) *and* (18) *are true. Then the set of admissible families of U^K forms a residual set of U^K. Moreover, if $v \in U^K$ is admissible then its orthogonal projection onto $(B^*\mathcal{E})^K$ is admissible.*

Recall that a residual set of a topological space $\mathcal{X}$ is the union of countable open and dense subsets of $\mathcal{X}$. In particular, it is a dense subset of $\mathcal{X}$.

Remark 5. *Theorem 4 hold for approximately controllable in time $\tau > 0$ instead of approximately controllable if the following implication holds:*

$$\forall \varepsilon \in H^*_\beta, \qquad \Psi(\varepsilon)(t) = 0 \quad (t \in (0, \tau)) \quad \Longrightarrow \quad \Psi(\varepsilon)(t) = 0 \quad (t \in (0, +\infty)).$$

Moreover, if $(\mathcal{H}_3)$ is satisfied then conclusions of Theorem 4 hold for approximately controllable in any time instead of approximately controllable. Indeed, $B^* e^{A^* t}\varepsilon$ for $\varepsilon \in H^*_\beta$ is then well defined as an analytic function of $t \in (0, +\infty)$ with values in U and the uniqueness of Ψ guarantees that $\Psi(\varepsilon)(t) = B^* e^{A^* t}\varepsilon$ for $t \in (0, +\infty)$ and $\varepsilon \in H^*_\beta$. Then we can extend equality $\Psi(\varepsilon)(t) = 0$ from $(0, \tau)$ to $(0, +\infty)$ by analytic continuation and we get (13) *for any $\tau > 0$. Such a remark is also true in the real case, see Theorem 12 below.*

Remark 6. *If $-A$ is positive we can define its fractional powers $(-A)^\alpha$ with $0 < \alpha < 1$, and we easily verify that the set of eigenvectors of $(-A)^\alpha$ and $(-A^*)^\alpha$ coincide with the set of eigenvectors of A and A^* respectively. Then assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and* (UC) *holds for $-(-A)^\alpha$ when they hold for A. Moreover, if (A, B) satisfies $(\mathcal{H}_4)$ for some $\gamma \ge 0$ then $(-(-A)^\alpha, B)$ satisfies $(\mathcal{H}_4)$ for $\frac{\gamma}{\alpha}$. Then under assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ conclusions of Theorems 4 are also true for $-(-A)^\alpha$ with $0 < \alpha < 1$. This should be compared with the fact that the null controllability of (A, B) does not imply the null controllability of $(-(-A)^\alpha, B)$ for $0 < \alpha < 1/2$, see [30].*

2.3 Stabilizability of linear parabolic systems

Theorem 4 can be used to deduce results of feedback stabilization. In what follows, assume $(\mathcal{H}_3)$ and replace $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$ by the following hypotheses:

$(\mathcal{H}_1')$ the spectrum of A consists in isolated eigenvalues with finite multiplicity without any finite cluster point.

$(\mathcal{H}_4')$ $B : U \to H_{-\gamma}$ is bounded for some $\gamma \in [0, 1)$.

Note that a sufficient condition for $(\mathcal{H}_1')$ is that A has compact resolvent.

To check that (UC) is also a stabilizability criterion for system (1) the point is that $(\mathcal{H}_1')$ and $(\mathcal{H}_3)$ imply that the subspace corresponding to eigenvalues of A with real part greater than zero (i.e the unstable subspace) is of finite dimension. Then approximate controllability of (1) implies that the projection of (1) onto the unstable subspace is controllable, and then stabilizability of (1) can be obtained as in [36] by verifying that stabilizing controls for unstable modes does not destabilize the stables modes of (1). Here, assumption $(\mathcal{H}_4')$ is only required to guarantee that (1) is stabilizable for the topology of H. If we assume $(\mathcal{H}_4)$ instead of $(\mathcal{H}_4')$ then stabilizability of (1) can also be obtained but for the topology of $H_{-\lfloor\gamma\rfloor}$ where $\lfloor\gamma\rfloor$ denotes the integer part of γ (just replace the norm of H by the norm of $H_{-\lfloor\gamma\rfloor}$ in inequality (59) in the proof of Corollary 7).

Before going further, let us give precise definitions. A pair (A, B) is said to be *stabilizable* if there exist a bounded operator $F : H \to U$ and constants $C > 0$, $\epsilon > 0$ such that the solutions of (1) with $u = Fy$ obey $\|y(t)\|_H \leq Ce^{-\epsilon t}\|y(0)\|_H$ for all $t \geq 0$. Note that for a given $\sigma > 0$ it is easily seen that $(A+\sigma, B)$ is stabilizable if and only if there exist a bounded operator $F : U \to H$ and $C > 0$, $\epsilon > 0$ such that the solutions of (1) with $u = Fy$ satisfy

$$\|y(t)\|_H \leq Ce^{-(\sigma+\epsilon)t}\|y(0)\|_H \quad (t \geq 0). \tag{20}$$

Moreover, we say that (A, B) is *stabilizable by K-dimensional control* if there exists a K-dimensional subspace $U_K \subset U$ such that $(A, B|_{U_K})$ is stabilizable. A linearly independent family $(v_j)_{j=1,\dots,K}$ of U^K generating such a U_K (i.e. $U_K = \mathrm{span}_{\mathbb{C}}\{v_j, j = 1, \dots, K\}$) is said to be *admissible for stabilizability of* (A, B). Note that in such case there exists $\widehat{\varepsilon}_j \in H$, $j = 1, \dots, K$ such that the finite dimensional feedback control can be written as follows:

$$Fy(t) = \sum_{j=1}^{K} (y(t), \widehat{\varepsilon}_j)_H v_j. \tag{21}$$

In what follows, for all $\sigma > 0$ we set

$$\mathcal{E}_\sigma \stackrel{\text{def}}{=} \bigoplus_{\Re\lambda_j \geq -\sigma} \ker(\overline{\lambda}_j - A^*) = \mathrm{span}_{\mathbb{C}}\{\varepsilon_k(\overline{\lambda}_j) \mid \Re\lambda_j \geq -\sigma,\ k = 1, \dots, \ell_j\}.$$

The following corollary of Theorem 4 holds.

Corollary 7. *Assume $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ and let $K \in \mathbb{N}^*$ and $\sigma > 0$. Then the following results hold.*

1. *The pair $(A + \sigma, B)$ is stabilizable if and only if (UC_σ) is satisfied.*
2. *The pair $(A + \sigma, B)$ is stabilizable by a K-dimensional control if and only if (UC_σ) is satisfied and*

$$\sup_{\Re\lambda_j \geq -\sigma} \ell_j \leq K. \tag{22}$$

3. *A family $v = (v_j)_{j=1,\dots,K}$ of U^K is admissible for stabilizability of $(A + \sigma, B)$ if and only if*

$$\mathrm{rank}\, W_j(v) = \ell_j \quad \forall \lambda_j,\ \Re\lambda_j \geq -\sigma. \tag{23}$$

4. *Assume that* (UC) *and* (22) *are true. Then the set of admissible families for stabilizability of $(A + \sigma, B)$ forms a residual set of U^K. Moreover, if v is admissible for stabilizability of $(A+\sigma, B)$ then its orthogonal projection onto $(B^*\mathcal{E}_\sigma)^K$ is admissible for stabilizability of $(A + \sigma, B)$.*

Remark 8. *Note that $(\mathcal{H}_2)$ is not required in Corollary 7. This comes from the fact that Corollary 7 is obtained by applying Theorem 4 to the projection of system* (1) *onto the finite dimensional subspace generated by unstable root vectors of A (i.e. the unstable subspace). Then by definition the family of unstable root vectors of A is complete in the unstable subspace and $(\mathcal{H}_2)$ is always satisfied for the projected system.*

Remark 9. *From the applications point of view rank conditions* (23) *are of great interest since they are practical criterions to construct admissible families of stabilizing actuators. For instance, a choice of* v *could be*

$$v = (B^*\varepsilon_k(\bar{\lambda}_j))_{\Re\lambda_j \geq -\sigma, k=1,\dots,\ell_j}.$$

Indeed, for such a v *each matrix* W_j *contains the full rank block* $(B^*\varepsilon_k(\bar{\lambda}_j), B^*\varepsilon_l(\bar{\lambda}_j))_{0\leq k,l\leq \ell_j}$.

Finally, if F is the bounded stabilizing feedback operator given by Corollary 7 we can define the closed loop operator $A_F \stackrel{\text{def}}{=} A+BF$ with domain $\mathcal{D}(A_F)$ and the semigroup generated by A_F is analytic on H (see [6, Prop. 10]). Moreover, if we define:

$$H_{F,\alpha} \stackrel{\text{def}}{=} \mathcal{D}((-A_F)^\alpha) \quad \alpha \geq 0, \tag{24}$$

then we have the following generalization of (20):

$$\|y(t)\|_{H_{F,\alpha}} \leq Ce^{-(\sigma+\epsilon)t}\|y(0)\|_{H_{F,\alpha}} \quad (t \geq 0). \tag{25}$$

2.4 Stabilizability of nonlinear parabolic systems

Here, by following the path sketched in [6] we recall how Corollary 7 can be used to prove the stabilizability of nonlinear systems:

$$y' = Ay + Bu + N(y,u), \tag{26}$$

where $N(\cdot,\cdot)$ is a nonlinear mapping satisfying adequate Lipschitz properties recalled below. For the following F is the bounded stabilizing feedback operator given by Corollary 7 and $A_F \stackrel{\text{def}}{=} A+BF$.

First, we assume that the following interpolation equality is satisfied

$$H_\alpha = [H, \mathcal{D}(A)]_\alpha, \quad \forall \alpha \in [0,1], \tag{27}$$

where $[\cdot,\cdot]_\alpha$ denotes the complex interpolation method (see [35]). Concerning operators satisfying the above interpolation equality we refer to [7, II.1.6] (for instance, it is satisfied if A is dissipative). Equality (27) is a technical assumption that implies $H_{F,\alpha} = [H, \mathcal{D}(A_F)]_\alpha$ for $\alpha \in [0,1]$ and then simplifies the study of the linear nonhomogeneous closed-loop system which is at the basis of the proofs of Theorem 10 and Theorem 11 below, see [6] for details.

Next, we suppose that $N(\cdot,\cdot)$ obeys for $s \in [0,1]$:

$$\begin{aligned} \|N(\xi, F\xi)\|_{H_{\frac{s-1}{2}}} &\leq C\|\xi\|_{H_{F,\frac{s}{2}}}\|\xi\|_{H_{F,\frac{s+1}{2}}} \\ \|N(\xi,F\xi) - N(\zeta,F\zeta)\|_{H_{\frac{s-1}{2}}} &\leq C\big(\|\xi-\zeta\|_{H_{F,\frac{s}{2}}}(\|\xi\|_{H_{F,\frac{s+1}{2}}} + \|\zeta\|_{H_{F,\frac{s+1}{2}}}) \\ &\quad + \|\xi-\zeta\|_{H_{F,\frac{s+1}{2}}}(\|\xi\|_{H_{F,\frac{s}{2}}} + \|\zeta\|_{H_{F,\frac{s}{2}}})\big). \end{aligned} \tag{28}$$

Finally, to state the stabilization theorems for (26) we need to introduce some spaces of Hilbert valued functions of $t \geq 0$. For two Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ we denote by $L^2(0,T;\mathcal{X})$, $L^\infty(0,T;\mathcal{X})$, $H^1(0,T;\mathcal{Y})$ usual Lebesgue and Sobolev spaces, we set $W(0,T;\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} L^2(0,T;\mathcal{X}) \cap H^1(0,T;\mathcal{Y})$ and $W(\mathcal{X},\mathcal{Y}) \stackrel{\text{def}}{=} W(0,+\infty;\mathcal{X},\mathcal{Y})$, we denote by $L^2_{\text{loc}}(\mathcal{X})$, $L^\infty_{\text{loc}}(\mathcal{X})$ the spaces of functions belonging for all $T>0$ to $L^2(0,T;\mathcal{X})$, $L^\infty(0,T;\mathcal{X})$ respectively, and for $\sigma > 0$ we denote by $W_\sigma(\mathcal{X},\mathcal{Y})$ the space of functions y such that $e^{\sigma(\cdot)}y \in W(\mathcal{X},\mathcal{Y})$. Finally, for $s \in [0,1]$ we use the shorter expression:

$$W^s_\sigma \stackrel{\text{def}}{=} W_\sigma(H_{F,\frac{s+1}{2}}, H_{\frac{s-1}{2}}).$$

Then the following theorem can be obtained analogously as [6, Thm. 15].

Theorem 10. *Assume* $(\mathcal{H}'_1)$, $(\mathcal{H}_3)$, $(\mathcal{H}'_4)$ *and* (27), $\sigma > 0$ *and let* F *(and* $K \in \mathbb{N}^*$*) be given by Corollary 7 and* (21). *For* $s \in [0,1]$ *assume also* (28) *and* $y_0 \in H_{F,\frac{s}{2}}$. *There exist* $\rho > 0$ *and* $\mu > 0$ *such that if* $\|y_0\|_{H_{F,\frac{s}{2}}} < \mu$ *then system* (26) *with* $y(0) = y_0$ *admits a solution* $y \in W^s_\sigma$ *such that* $\|y\|_{W^s_\sigma} \leq \rho\|y_0\|_{H_{F,\frac{s}{2}}}$, *which is unique within the class of functions in*

$$L^\infty_{\text{loc}}(H_{F,\frac{s}{2}}) \cap L^2_{\text{loc}}(H_{F,\frac{s+1}{2}}).$$

Moreover, there exists $C > 0$ *such that for all* $t \geq 0$

$$\|y(t)\|_{H_{F,\frac{s}{2}}} \leq Ce^{-\sigma t}\|y_0\|_{H_{F,\frac{s}{2}}}. \tag{29}$$

The main difficulty to apply Theorem 10 in concrete exemples is that one has to identify $H_{F,\alpha}$ for $\alpha \in [0,1]$. Indeed, assumption (28) is usually obtained from boundedness property of $N(\cdot,\cdot)$ in original spaces that are not related to F. Moreover, Theorem 10 is only valid for $y_0 \in H_{F,\frac{s}{2}}$ that can be too restrictive, see [2, 1].

An alternative to avoid such a difficulty and to obtain an exponential decrease in the norm of $H_{\frac{s}{2}}$ is to use a dynamical control. Consider a control function of the form:

$$u(t) = \sum_{j=1}^{K} u_j(t) v_j, \quad \overline{u} = (u_1, \dots, u_K) \in \mathbb{C}^K \tag{30}$$

where (v_j) is an admissible family for (1) and where $\overline{u}$ is solution of the following dynamical system:

$$\overline{u}' = F_1 \overline{u} + F_2 y. \tag{31}$$

Here F_1 and F_2 are adequate coupling operators that are obtained as follows. First, define

$$V\overline{u} \stackrel{\text{def}}{=} \sum_{j=1}^{K} u_j B v_j,$$

so that system (1)-(30) with $\overline{u}' = \overline{g}$ can be rewritten as the extended system:

$$\begin{bmatrix} y \\ \overline{u} \end{bmatrix}' = \mathbb{A} \begin{bmatrix} y \\ \overline{u} \end{bmatrix} + \mathbb{V}\overline{g}, \quad \text{where } \mathbb{A} \stackrel{\text{def}}{=} \begin{bmatrix} A & V \\ 0 & 0 \end{bmatrix}, \quad \mathbb{V} \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ \mathrm{Id} \end{bmatrix} \tag{32}$$

Fuction $\overline{g}$ now plays the role of the control. Next, for $\theta \in [0,1]$ let us introduce the following spaces

$$\mathbb{H}_\theta \stackrel{\text{def}}{=} \{(y, \overline{u}) \in H \times \mathbb{C}^K \mid y + (\mu_0 - A)^{-1} V\overline{u} \in H_\theta\}.$$

We verify that $\mathbb{H}_1$ is the domain of $\mathbb{A}$ and that if (A, V) satisfies $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ then $(\mathbb{A}, \mathbb{V})$ satisfies $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ with $\gamma = 0$. Moreover, it is easily seen that if (A, V) satisfies (UC) then so does $(\mathbb{A}, \mathbb{V})$, which means that if v is admissible for stabilizability of (A, B) then (32) is stabilizable, see [6] for details. Then from Corollary 7 we have the existence of a finite rank operator $(F_1, F_2) : \mathbb{C}^K \times H \to \mathbb{C}^K \times \mathbb{C}^K$ such that the solutions of (1)-(30)-(31) obey

$$\|y(t)\|_H + |\overline{u}(t)| \le C(\|y(0)\|_H + |\overline{u}(0)|)e^{-\sigma t},$$

where $|\cdot|$ denotes the euclidian norm of $\mathbb{C}^K$. Finally, we suppose that for $s \in [0,1]$:

$$\begin{aligned} \|N(\xi,u)\|_{H_{\frac{s-1}{2}}} &\le C\|(\xi,\overline{u})\|_{\mathbb{H}_{\frac{s}{2}}} \|(\xi,\overline{u})\|_{\mathbb{H}_{\frac{s+1}{2}}} \\ \|N(\xi,u) - N(\zeta,w)\|_{H_{\frac{s-1}{2}}} &\le C\big(\|(\xi,\overline{u}) - (\zeta,\overline{w})\|_{\mathbb{H}_{\frac{s}{2}}} (\|(\xi,\overline{u})\|_{\mathbb{H}_{\frac{s+1}{2}}} + \|(\zeta,\overline{w})\|_{\mathbb{H}_{\frac{s+1}{2}}}) \\ &\qquad \|(\xi,\overline{u}) - (\zeta,\overline{w})\|_{\mathbb{H}_{\frac{s+1}{2}}} (\|(\xi,\overline{u})\|_{\mathbb{H}_{\frac{s}{2}}} + \|(\zeta,\overline{w})\|_{\mathbb{H}_{\frac{s}{2}}})\big), \end{aligned} \tag{33}$$

where we have used the notations:

$$u = \sum_{j=1}^{K} u_j v_j, \quad w(t) = \sum_{j=1}^{K} w_j v_j, \quad \overline{u} = (u_1, \dots, u_K), \quad \overline{w} = (w_1, \dots, w_K),$$

and for $s \in [0,1]$ we define

$$\mathbb{W}_\sigma^s = \left\{ (y, \overline{u}) \mid (e^{\sigma(\cdot)} y, e^{\sigma(\cdot)} \overline{u}) \in L^2(\mathbb{H}_{\frac{s+1}{2}}) \cap H^1(H_{\frac{s-1}{2}} \times \mathbb{C}^K) \right\}.$$

Then the following theorem can be obtained analogously as [6, Thm. 18].

Theorem 11. *Assume $(\mathcal{H}_1')$, $(\mathcal{H}_3)$, $(\mathcal{H}_4')$, (27) and let $\sigma > 0$. For $s \in [0,1]$ assume also (33) and $y_0 \in H_{\frac{s}{2}}$. There exist $\rho > 0$ and $\mu > 0$ such that if $\|y_0\|_{H_{\frac{s}{2}}} < \mu$ then system (26)-(30)-(31) with $y(0) = y_0$ and $\overline{u}(0) = 0$ admits a solution $(y, \overline{u}) \in \mathbb{W}_\sigma^s$ such that $\|(y,\overline{u})\|_{\mathbb{W}_\sigma^s} \le \rho \|y_0\|_{H_{\frac{s}{2}}}$, which is unique within the class of functions in $L^\infty_{\mathrm{loc}}(\mathbb{H}_{\frac{s}{2}}) \cap L^2_{\mathrm{loc}}(\mathbb{H}_{\frac{s+1}{2}})$. Moreover, there exists $C > 0$ such that for all $t \ge 0$*

$$\|(y(t), \overline{u}(t))\|_{\mathbb{H}_{\frac{s}{2}}} \le Ce^{-\sigma t} \|y_0\|_{H_{\frac{s}{2}}}.$$

2.5 Approximate controllability and stabilizability of real systems

In many practical examples system (1) is originally defined on a real Hilbert space for control functions with values in a real control space: A is originally defined as an unbounded operator on a real Hilbert space G (i.e. $A : \mathcal{D}(A) \subset G \to G$) and the input B is originally defined as a bounded operator from a real Hilbert space W into the real space $G_{-\gamma}$ (defined as $H_{-\gamma}$ in (10) but now from real operators A, A^*). Then for $y(0) \in G_{-\beta}$ and $u \in L^2(0,+\infty;W)$ the trajectories $t \mapsto y(t)$ are continuous with values in the real Hilbert space $G_{-\beta}$. In such situation, (1) is referred as a *real system* and all the above definitions of approximate controllability and admissibility can be stated for real spaces in the same manner as it has been done for complex spaces. It follows that complex spaces H and U are simply the complexified spaces $H = G + \imath G$ and $U = W + \imath W$ and to recover the above complex framework it suffices to consider extensions of A and B to H and U respectively.

It is clear that the approximate controllability of the complex system implies the approximate controllability of the real system. This follows by remarking that the complex system (1) can be decompose in two uncoupled real systems corresponding to real and imaginary parts of (1). The same argument yields that if $(v_j)_j$ is an admissible family of U for the complex system then $(\Re v_j, \Im v_j)_j$ is an admissible family of W for the real system. In particular, if the complex system is approximately controllable by K-dimensional control then the real system is approximately controllable by $2K$-dimensional control. However, the fact that (18) and (UC) are also sufficient for approximate controllability by K-dimensional control of the real system is not obvious. A slight modification of the proof of Theorem 4 permits to obtain a real version of Theorem 4 stated in Theorem 12 below.

Before stating this result, we need some additional notations. Recall that for each λ_j the (complex) family $\varepsilon_k(\overline{\lambda}_j)$, $k = 1, \dots, \ell_j$, denotes a basis of $\ker(\overline{\lambda}_j - A^*)$ and we denote by $\mathcal{F}$ the subspace of G generated by real and imaginary parts of $\varepsilon_k(\overline{\lambda}_j)$. Note that when $\overline{\lambda}_j = \lambda_j$ is real we can suppose $\varepsilon_k(\overline{\lambda}_j) \in G$. Moreover, since A^* is real its spectrum is symetric with respect to the real line and non real eigenvalues are pairwise conjugate with pairwise conjugate basis of eigenvectors, i.e for all $j \in \mathbb{N}$ the complex value λ_j is also an eigenvalues of A^* with corresponding basis of eigenvector $\{\overline{\varepsilon_k(\overline{\lambda}_j)} \mid k = 1, \dots, \ell_j\}$. Then with

$$\mathcal{J}_0 \stackrel{\text{def}}{=} \{j \in \mathbb{N} \mid \Im\lambda_j = 0\}, \quad \mathcal{J}_+ \stackrel{\text{def}}{=} \{j \in \mathbb{N} \mid \Im\lambda_j > 0\} \quad \text{and} \quad \mathcal{J}_- \stackrel{\text{def}}{=} \{j \in \mathbb{N} \mid \Im\lambda_j < 0\} \tag{34}$$

the subspace $\mathcal{F}$ is also defined by:

$$\mathcal{F} \stackrel{\text{def}}{=} \overline{\text{span}_{\mathbb{R}}\{\chi_k(\overline{\lambda}_j) \mid j \in \mathbb{N},\ k = 1, \dots, \ell_j\}}^{G} \quad \text{where} \quad \chi_k(\overline{\lambda}_j) \stackrel{\text{def}}{=} \begin{cases} \varepsilon_k(\overline{\lambda}_j) & \text{if } j \in \mathcal{J}_0, \\ \Re\varepsilon_k(\overline{\lambda}_j) & \text{if } j \in \mathcal{J}_+, \\ \Im\varepsilon_k(\overline{\lambda}_j) & \text{if } j \in \mathcal{J}_-. \end{cases} \tag{35}$$

The following real version of Theorem 4 holds.

Theorem 12. *Suppose that A and B are real operators defined from respective real Hilbert spaces G and W, assume that the complexified of A and B satisfy $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$ and let $K \in \mathbb{N}^*$. Then the following results hold.*

1. *Real system* (1) *is approximately controllable if and only if* (UC) *is satisfied.*
2. *Real system* (1) *is approximately controllable by a K-dimensional control if and only if* (UC) *and* (18) *are satisfied.*
3. *A family $v \in \ell^2(\mathbb{N}, W)$ or $v \in W^K$ is admissible for stabilizability of $(A + \sigma, B)$ if and only if* (19) *is satisfied.*
4. *Assume that* (UC) *is true. Then the set of admissible families of $\ell^2(\mathbb{N}, W)$ forms a residual set of $\ell^2(\mathbb{N}, W)$. Moreover, if $v \in \ell^2(\mathbb{N}, W)$ is admissible then its orthogonal projection onto $\ell^2(\mathbb{N}, B^*\mathcal{F})$ is admissible.*
5. *Assume that* (UC) *and* (18) *are true. Then the set of admissible families of W^K forms a residual set of W^K. Moreover, if $v \in W^K$ is admissible then its orthogonal projection onto $(B^*\mathcal{F})^K$ is admissible.*

We also have a real version of Corollary 7 stated in Corollary 13 below. If A and B are real operators defined on real Hilbert spaces G and W respectively the pair (A, B) is referred as a *real pair* and the related definitions

of stabilizability introduced in Subsection 2.3 are the same as above by replacing the complex spaces H and U by the real spaces G and W. In such case, a (real) K-dimensional feedback law is of the form

$$Fy(t) = \sum_{j=1}^{K} (y(t), \widehat{\chi}_j)_G v_j, \tag{36}$$

where $\widehat{\chi}_j \in G$, $j = 1, \dots, K$.

To state a real version of Corollary 7, we need to introduce the following subspace of W:

$$\mathcal{F}_\sigma \stackrel{\text{def}}{=} \text{span}_{\mathbb{R}}\{\chi_k(\bar{\lambda}_j) \mid \Re\lambda_j \geq -\sigma,\ k = 1, \dots, \ell_j\},$$

where family $(\chi_k(\bar{\lambda}_j))$ is defined by (35). The following corollary of Theorem 12 holds.

Corollary 13. *Suppose that A and B are real operators defined from respective real Hilbert spaces G and W, assume that the complexified of A and B satisfy $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ and let $K \in \mathbb{N}^*$ and $\sigma > 0$. Then the following results hold.*

1. *The real pair $(A + \sigma, B)$ is stabilizable if and only if (UC_σ) holds.*
2. *The real pair $(A + \sigma, B)$ is stabilizable by a K-dimensional control if and only if (UC_σ) and (22) are satisfied.*
3. *A family $v = (v_j)_{j=1,\dots,K}$ of W^K is admissible for stabilizability of $(A+\sigma, B)$ if and only if (23) is satisfied.*
4. *Assume that (UC) and (22) are true. Then the set of admissible families for stabilizability of $(A + \sigma, B)$ forms a residual set of W^K. Moreover, if v is admissible for stabilizability of $(A+\sigma, B)$ then its orthogonal projection onto $(B^*\mathcal{F}_\sigma)^K$ is admissible for stabilizability of $(A + \sigma, B)$.*

Remark 14. *If F is the bounded stabilizing feedback operator (36) given by Corollary 13 then in a way similar as for (24) we can define $G_{F,\alpha}$. Then the analogue of (29) holds for the real closed-loop linear system and the analogue of Theorem 10 and Theorem 11 hold for the real closed-loop nonlinear system, just replace complex spaces $\mathbb{C}^K$, $H_{F,\alpha}$, $H_{F,\frac{s}{2}}$, $H_{\frac{s}{2}}$ etc by real spaces $\mathbb{R}^K$, $G_{F,\alpha}$, $G_{F,\frac{s}{2}}$, $G_{\frac{s}{2}}$ etc.*

Remark 15. *Note that in [33, 32] the authors choose the whole family of real and imaginary parts of generalized eigenvectors as a stabilizing family. However, according to rank criterion (18) it is sufficient to choose the family of real and imaginary parts of pure eigenvectors. The family v given below is admissible for stabilizability of $(A + \sigma, B)$:*

$$v = (B^*\chi_k(\bar{\lambda}_j))_{\Re\lambda_j \geq -\sigma, k=1,\dots,\ell_j}.$$

Indeed, if $j \in J_0$ the above matrix contains the full rank block $(B^*\varepsilon_k(\bar{\lambda}_j), B^*\varepsilon_l(\bar{\lambda}_j))_{0\leq k,l\leq \ell_j}$, and if $j \notin J_0$ the matrix W_j contains the ℓ_j-rank block $[R_j, I_j]$ where

$$R_j = (B^*\Re\varepsilon_k(\bar{\lambda}_j), B^*\varepsilon_l(\bar{\lambda}_j))_{0\leq k,l\leq \ell_j} \quad \text{and} \quad I_j = (B^*\Im\varepsilon_k(\bar{\lambda}_j), B^*\varepsilon_l(\bar{\lambda}_j))_{0\leq k,l\leq \ell_j}.$$

This last claim comes from the fact that eigenvectors associated to eigenvalues $\{\overline{\lambda}_j \mid \Re\lambda_j \geq -\sigma\}$ are pairwise conjugate.

3 Simple examples with the heat equation

3.1 Example of coupled heat equations

Here we give a simple example of non diagonalizable system which consists in two coupled heat equations:

$$\begin{cases} y_t - \Delta y + z = 0 & \text{in } (0,T)\times\Omega, \\ z_t - \Delta z = \mathbf{1}_\omega h & \text{in } (0,T)\times\Omega, \\ y = z = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 \ \text{ and } \ z(0) = z_0 & \text{in } \Omega. \end{cases} \tag{37}$$

Above, Ω is an open subset of $\mathbb{R}^d$, $d \geq 1$, $(y_0, z_0) \in L^2(\Omega) \times L^2(\Omega)$, ω is a non empty open subset of Ω, $\mathbf{1}_\omega$ is the characteristic function of ω and $h \in L^2((0,T) \times \Omega)$ is the control function. System (37) can be rewritten in the form (1) with

$$A = \begin{pmatrix} \Delta & -I \\ 0 & \Delta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ \mathbf{1}_\omega \end{pmatrix}.$$

It is clear that A with domain $\mathcal{D}(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$ satisfies $(\mathcal{H}_1')$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ with $H = L^2(\Omega) \times L^2(\Omega)$ and that B satisfies $(\mathcal{H}_4)$ with $U = L^2(\Omega) \times L^2(\Omega)$ and $\gamma = 0$ (i.e. B is bounded). Moreover, it is easily seen that

$$A^* = \begin{pmatrix} \Delta & 0 \\ -I & \Delta \end{pmatrix} \quad \text{and} \quad \mathcal{D}(A^*) = \mathcal{D}(A).$$

Let denote by λ_j, $j \in \mathbb{N}$, the Dirichlet Laplacian eigenvalues with geometric multiplicities ℓ_j and related basis of eigenvectors $\{\xi_k(\lambda_j) \mid k = 1, \dots \ell_j\}$. Simple computations show that the eigenvalues of A^* are exactly the λ_j's and that proper eigenspaces and generalized eigenspaces are given by

$$\begin{aligned} \ker(\lambda_j - A^*) &= \operatorname{span}\{\varepsilon_k(\lambda_j) \mid \ k = 1, \dots, \ell_j\} \\ \ker(\lambda_j - A^*)^2 &= \operatorname{span}\{\varepsilon_k(\lambda_j), f_k(\lambda_j) \mid \ k = 1, \dots, \ell_j\}, \end{aligned}$$

where

$$\varepsilon_k(\lambda_j) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ \xi_k(\lambda_j) \end{pmatrix}, \quad f_k(\lambda_j) \stackrel{\text{def}}{=} \begin{pmatrix} \xi_k(\lambda_j) \\ 0 \end{pmatrix}.$$

Thus we are in the situation where $m(\lambda_j) = 2$ and where algebraic multiplicity of λ_j is $N_j = 2\ell_j$ while the geometric multiplicity of λ_j is ℓ_j.

Finally, (UC) reduces to:

$$\left\{ \begin{array}{ll} \Delta y = \lambda y & \text{in } \Omega, \\ \Delta z - y = \lambda z & \text{in } \Omega, \\ y = z = 0 & \text{on } \partial\Omega, \end{array} \right. \quad \text{and} \quad z \equiv 0 \text{ in } \omega \quad \Longrightarrow \quad \left\{ \begin{array}{ll} y \equiv 0 & \text{in } \Omega, \\ z \equiv 0 & \text{in } \Omega. \end{array} \right. \tag{38}$$

To prove (38), suppose that the left part of (38) is satisfied with $\lambda = \lambda_j$ for some j (otherwise the conclusion is obvious). Then ${}^t(y, z) \in \ker(\lambda_j - A^*)$ which implies $y = 0$ and the conclusion follows from Holmgren's uniqueness Theorem, see for instance [21, Thm.8.6.5, p.309]. In conclusion, Theorem 4 applies and (37) is approximately controllable in any time.

3.1.1 Minimal number of actuators for the heat equation in a rectangular domain

Consider the d–dimensional controlled heat equation in a rectangle $\Omega = \prod_{i=1}^d (0, c_i)$ for $d \geq 2$:

$$\left\{ \begin{array}{ll} y_t = \Delta y & \text{in } (0,T) \times \Omega, \\ y = \mathbf{1}_\Gamma h & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{array} \right. \tag{39}$$

In the above setting, $\mathbf{1}_\Gamma$ is the characteristic function of a non empty open subset $\Gamma \subset \{0\} \times \prod_{i=2}^d (0, c_i)$ and $h \in L^2((0,T) \times \partial\Omega)$ is the control function.

The eigenvalues of the Dirichlet Laplacian are given by

$$\lambda_\alpha = -\pi^2 \sum_{i=1}^d \left(\frac{\alpha_i}{c_i} \right)^2, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^{*d},$$

with related eigenvectors

$$\varphi_\alpha(x) = \prod_{i=1}^d \sin\left(\frac{\pi}{c_i} \alpha_i x_i \right), \quad x = (x_1, \dots, x_d) \in \Omega.$$

It is classical that controlled system (39) can be written in the form (1) for linear operator A, B satisfying $(\mathcal{H}_1')$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ with $\gamma > \frac{3}{4}$. Moreover, (UC) reduces to

$$\left\{ \begin{array}{rll} \lambda\varphi - \Delta\varphi &= 0 & \text{in } \Omega, \\ \varphi &= 0 & \text{on } \partial\Omega, \\ \frac{\partial\varphi}{\partial x_1} &= 0 & \text{on } \Gamma, \end{array} \right. \quad \Longrightarrow \quad \varphi = 0 \text{ in } \Omega,$$

which is an easy consequence of Holmgren's uniqueness Theorem with an extension of the domain procedure. As a consequence, (39) is approximately controllable and stabilizable by finite dimensional control. A remaining question is how many actuators are required for approximate controllability or stabilizability?

Suppose that $1/c_i^2$, $i = 1, \dots, d$ are $\mathbb{Q}$-linearly independent. Then a straightforward calculation shows that the mapping $\alpha \mapsto \lambda_\alpha$ is one-to-one and then that the spectrum is simple (note that it is well-known that the spectrum of the Dirichlet-Laplace operator is generically simple with respect to the domain see [29]). It implies that (39) is approximately controllable, as well as stabilizable for all rate $\sigma > 0$, with a one dimensional controller.

Suppose now that $c_i = \pi/c$, $i = 1, \dots, d$ for $c > 0$. Then the mapping $\alpha \mapsto \lambda_\alpha = -(\pi/c)^2|\alpha|^2$, $|\alpha|^2 \stackrel{\text{def}}{=} \sum_{i=1}^d \alpha_i^2$, is no longer one-to-one. This means that (39) is no longer approximately controllable with a one dimensional controller. In fact, the sequence of geometric multiplicities

$$m_\alpha^d = \sharp\{\beta \in \mathbb{N}^{*d} \mid \lambda_\beta = \lambda_\alpha\}, \quad \alpha \in \mathbb{N}^{*d},$$

is unbounded. This follows from the fact that $m_\alpha^d = r_d(|\alpha|^2) \geq r_2(\alpha_1^2 + \alpha_2^2)$ where $r_d(n)$ denotes the total number of the representation of n as a sum of d square of positive integers, and that for instance $r_2(5^{2p}) = p + 1$ for $p \in \mathbb{N}^*$, see [19, Thm. 278]. It means that the geometric multiplicities of the Dirichlet Laplacian in the square are not bounded and that the approximate controllability with finite dimensional controllers is not possible. However, for $\sigma > 0$ one can prove that the maximum of the geometric multicplicities m_α^d corresponding to eigenvalues $\lambda_\alpha \geq -\sigma$ is bounded by $\frac{(\sqrt{\pi})^{d-1}}{(2c)^{d-1}\Gamma(\frac{d+1}{2})}\sigma^{\frac{d-1}{2}}$ and then (39) is stabilizable for a rate $\sigma > 0$ by means of a K^σ-dimensional control with

$$K^\sigma = \left\lfloor \frac{(\sqrt{\pi})^{d-1}}{(2c)^{d-1}\Gamma(\frac{d+1}{2})}\sigma^{\frac{d-1}{2}} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Indeed, by noticing that $r_d(n)$ is also the number of tuples of positive integers which are on the d-sphere of ray $\sqrt{n}$ we deduce first that $r_d(n)$ is bounded above by the number of tuples of positive integers in the $d-1$ ball of ray $\sqrt{n}$ and thus that $r_d(n)$ is bounded above by the volume occupied by the points of the $d-1$ ball of ray $\sqrt{n}$ which have positive coordinates: $r_d(n) \leq \frac{\pi^{\frac{d-1}{2}}}{2^{d-1}\Gamma(\frac{d+1}{2})} n^{\frac{d-1}{2}}$. Then the conclusion follows from:

$$\max_{\lambda_\alpha \geq -\sigma} m_\alpha^d = \max_{|\alpha|^2 \leq \sigma/c^2} r_d(|\alpha|^2).$$

Finally, let us underline that the strategy consisting in choosing as many controllers as the number of modes corresponding to eigenvalues greater than $-\sigma$ (which is the strategy of [33, 32]) would lead to a number of controllers $K_u^\sigma = \lfloor \frac{(\sqrt{\pi})^d}{(2c)^d\Gamma(\frac{d}{2}+1)}\sigma^{\frac{d}{2}} \rfloor$, and that the ratio K_u^σ/K^σ behaves as $\frac{(d+1)\sqrt{\pi}\sqrt{\sigma}}{4c}$ as $\sigma \to \infty$.

4 Proof of the main results

4.1 Proof of the first point of Theorem 4 and of Theorem 12

First, let us prove point 1 of Theorem 4. We assume $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ and we show that system (1) is approximately controllable if and only if (UC) is satisfied.

Recall that system (1) is approximately controllable if and only if (A^*, B^*) is approximately observable in infinite time (i.e. if (14) is true), and let us first prove that (UC) implies (14). For that, assume $\varepsilon \in H_\beta^*$ (defined in (10)) satisfies:

$$(\Psi\varepsilon)(t) = 0 \quad (t \in (0, +\infty)). \tag{40}$$

The Laplace transform of (40) (see [40, Thm. 4.3.7]) first yields:

$$B^*(\lambda - A^*)^{-1}\varepsilon = \int_0^\infty e^{-\lambda t}(\Psi\varepsilon)(t)\mathrm{d}t = 0 \quad \text{for } \Re\lambda \geq \mu_0 > \sup_{j\in\mathbb{N}} \Re\lambda_j$$

and next $B^*(\lambda - A^*)^{-1}\varepsilon = 0$ for all for all $\lambda \in \rho(A^*)$ by analytic continuation. Now define

$$R_{-n}^*(\bar{\lambda}_j) = \frac{1}{2\pi\imath}\int_{|\lambda-\bar{\lambda}_j|=\alpha} (\lambda - \bar{\lambda}_j)^{n-1}(\lambda - A^*)^{-1}\mathrm{d}\lambda, \quad n \in \{1, \dots, m(\lambda_j)\}$$

where $\alpha > 0$ is the same as in (4). Since $B^*(\lambda - A^*)^{-1}\varepsilon = 0$ for all $\lambda \in \rho(A^*)$ we deduce that

$$B^* R^*_{-n}(\bar{\lambda}_j)\varepsilon = 0 \qquad \forall n \in \{1, \dots, m(\lambda_j)\}. \tag{41}$$

Moreover, by easy computations we verify that

$$R^*_{-n-1}(\bar{\lambda}_j) + (\bar{\lambda}_j - A^*)R^*_{-n}(\bar{\lambda}_j) = 0 \qquad \forall n \in \{1, \dots, m(\lambda_j) - 1\}. \tag{42}$$

Finally, let us prove that (UC) with an inductive argument using (41), (42) yields

$$R^*_{-1}(\bar{\lambda}_j)\varepsilon = 0. \tag{43}$$

By (42) we have $R^*_{-m(\lambda_j)}(\bar{\lambda}_j) = (A^* - \bar{\lambda}_j)^{m(\lambda_j)-1} R^*_{-1}(\bar{\lambda}_j)$, and since $\mathrm{Ran}(R_{-1}(\lambda_j)) = \ker(\lambda_j - A)^{m(\lambda_j)}$ we deduce that $R^*_{-m(\lambda_j)}(\bar{\lambda}_j)\varepsilon \in \ker(\bar{\lambda}_j - A^*)$. Then (41) with $n = m(\lambda_j)$ combined with (UC) first gives $R^*_{-m(\lambda_j)}(\bar{\lambda}_j)\varepsilon = 0$. As a consequence, (42) with $n = m(\lambda_j) - 1$ yields $R^*_{-m(\lambda_j)+1}(\bar{\lambda}_j)\varepsilon \in \ker(\bar{\lambda}_j - A^*)$, and (41) with $n = m(\lambda_j) - 1$ combined with (UC) then gives $R^*_{-m(\lambda_j)+1}(\bar{\lambda}_j)\varepsilon = 0$. By reiterating the argument we successively obtain $R^*_{-n}(\bar{\lambda}_j)\varepsilon = 0$ from $n = m(\lambda_j)$ until $n = 1$. Finally, since (43) holds for all $j \in \mathbb{N}$, with assumption $(\mathcal{H}_2)$ we get $\varepsilon = 0$.

Conversely, let us now prove that (14) implies (UC). Suppose that (UC) is false: there exist $j_0 \in \mathbb{N}$ and an eigenvector $\varepsilon(\overline{\lambda}_{j_0}) \neq 0$ (associated with the eigenvalue $\overline{\lambda}_{j_0}$) such that $B^*\varepsilon(\overline{\lambda}_{j_0}) = 0$. Moreover, since we have (see [31, Chapter 2])

$$e^{A^* t}\varepsilon(\overline{\lambda}_{j_0}) = e^{\bar{\lambda}_j t}\varepsilon(\overline{\lambda}_{j_0}),$$

then $(\Psi\varepsilon(\overline{\lambda}_{j_0}))(t) = B^* e^{A^* t}\varepsilon(\overline{\lambda}_{j_0}) = 0$ for all $t \in (0, +\infty)$ and (14) is false.

Next, let us prove point 1 of Theorem 12. We assume $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ and we show that the real system (1) is approximately controllable if and only if (UC) is satisfied. The fact that (UC) implies that real system (1) is approximately controllable follows from the complex case since it reduces to prove

$$\forall \varepsilon \in G^*_\beta, \quad (\Psi\varepsilon)(t) = 0 \quad (t \in (0, \infty)) \Longrightarrow \quad \varepsilon = 0, \tag{44}$$

where G^*_β is the real subspace of H^*_β defined as H^*_β in (10) but now from real domain of fractional powers. Conversely, if (UC) is false then $B^*\varepsilon(\overline{\lambda}_{j_0}) = 0$ for some eigenvector $\varepsilon(\overline{\lambda}_{j_0}) \neq 0$. Then we also have $B^*\Re\varepsilon(\overline{\lambda}_{j_0}) = 0$ and $B^*\Im\varepsilon(\overline{\lambda}_{j_0}) = 0$ and taking real and imaginary part of $e^{A^* t}\varepsilon(\overline{\lambda}_{j_0}) = e^{\bar{\lambda}_j t}\varepsilon(\overline{\lambda}_{j_0})$ (and using the fact that A is real) we get:

$$\begin{aligned} e^{A^* t}\Re\varepsilon(\overline{\lambda}_{j_0}) &= e^{\Re\lambda_j t}(\cos(\Im\lambda_j t)\Re\varepsilon(\overline{\lambda}_{j_0}) + \sin(\Im\lambda_j t)\Im\varepsilon(\overline{\lambda}_{j_0})) \\ e^{A^* t}\Im\varepsilon(\overline{\lambda}_{j_0}) &= e^{\Re\lambda_j t}(\cos(\Im\lambda_j t)\Im\varepsilon(\overline{\lambda}_{j_0}) - \sin(\Im\lambda_j t)\Re\varepsilon(\overline{\lambda}_{j_0})). \end{aligned} \tag{45}$$

It implies $(\Psi\Re\varepsilon(\overline{\lambda}_{j_0}))(t) = B^* e^{A^* t}\Re\varepsilon(\overline{\lambda}_{j_0}) = 0$ and $(\Psi\Im\varepsilon(\overline{\lambda}_{j_0}))(t) = B^* e^{A^* t}\Im\varepsilon(\overline{\lambda}_{j_0}) = 0$ for all $t \in (0, +\infty)$. Then since $\varepsilon(\overline{\lambda}_{j_0}) \neq 0$ we have $\Re\varepsilon(\overline{\lambda}_{j_0}) \neq 0$ or $\Im\varepsilon(\overline{\lambda}_{j_0}) \neq 0$ and it yields the existence of $\varepsilon \in G^*_\beta$ (choose $\varepsilon = \Re\varepsilon(\overline{\lambda}_{j_0})$ or $\varepsilon = \Im\varepsilon(\overline{\lambda}_{j_0})$) satisfying $(\Psi\varepsilon)(t) = 0$ for all $t \in (0, +\infty)$. In conclusion, (14) is false.

4.2 Proof of points 2 to 5 of Theorem 4

In this section, we end the proof of Theorem 4. In the remaining part of this subsection we suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ are satisfied.

Proof of point 3. Let us treat the case $v \in \ell^2(\mathbb{N}, U)$. Consider here the controlled system (1) with

$$u = \sum_{l \in \mathbb{N}} u_l v_l, \tag{46}$$

where $v \overset{\text{def}}{=} (v_j)_{j\in\mathbb{N}}$ is a family to be chosen and where $\bar{u}=(u_j)_{j\in\mathbb{N}}$ is only depending on time and is the new control. Here we are looking for families $v \overset{\text{def}}{=} (v_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}; U)$ such that the corresponding system is still approximately controllable.

Let us introduce for all $v \in \ell^2(\mathbb{N}, U)$ the operator

$$V(v) : \ell^2(\mathbb{N}) \longrightarrow H_{-\gamma}, \quad V(v)\bar{w} \overset{\text{def}}{=} \sum_{l \in \mathbb{N}} w_l B v_l \qquad \text{where} \quad \bar{w} = (w_l)_{l\in\mathbb{N}}. \tag{47}$$

Here $\ell^2(\mathbb{N}) \stackrel{\text{def}}{=} \ell^2(\mathbb{N},\mathbb{C})$ denotes the space of square-summable complex sequences. The system considered here and which is deduced from (1) can be written as

$$y' + Ay = V(v)\bar{u}. \tag{48}$$

In order to control the above system, we will apply point 1 of Theorem 4. Condition (UC) for (48) writes as

$$\forall \varepsilon \in \mathcal{D}(A^*), \quad \forall \lambda \in \mathbb{C}, \quad A^*\varepsilon = \lambda\varepsilon \quad \text{and} \quad V(v)^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \tag{49}$$

It can be checked that the adjoint operator of $V(v)$ is given by

$$V(v)^* : H \longrightarrow \ell^2(\mathbb{N}), \quad V(v)^*\varepsilon = ((v_l|B^*\varepsilon)_U)_{l\in\mathbb{N}}, \tag{50}$$

and it is clear that (49) is equivalent to (19). In conclusion, since the admissiblity of v is equivalent to the approximate controllability of (49), we deduce that v is admissible if an only if (19) is satisfied. Finally, the case $v = (v_j)_{j=1,\dots,K} \in U^K$ can be treated completely similarly. Just note that sums in (46), (47) are now finite and hold over $l = 1,\dots,K$ and that we have to replace $\ell^2(\mathbb{N})$ by $\mathbb{C}^K$ in the definition (47) of $V(v)$.

Proof of point 4. First, let us prove that if (UC) is true then the set of families $v \in \ell^2(\mathbb{N},U)$ such that (19) holds form a residual set of $\ell^2(\mathbb{N},U)$ i.e the set of families $v \in \ell^2(\mathbb{N},U)$ satisfying (19) is the intersection of a countable family of open and dense subsets of $\ell^2(\mathbb{N},U)$.

For all $n \in \mathbb{N}$, let use the notation $\mathcal{L}(\mathbb{C}^n,\ell^2(\mathbb{N}))$ for the space of linear continuous mappings from $\mathbb{C}^n$ into $\ell^2(\mathbb{N})$. We also denote by $\mathcal{R}_n$ the subset of $\mathcal{L}(\mathbb{C}^n,\ell^2(\mathbb{N}))$ composed by the linear mapping of rank n. It is well-known that $\mathcal{R}_n$ is an open and dense subset $\mathcal{L}(\mathbb{C}^n,\ell^2(\mathbb{N}))$.

According to (16) we have

$$W_j : \ell^2(\mathbb{N},U) \to \mathcal{L}(\mathbb{C}^{\ell_j},\ell^2(\mathbb{N})).$$

The set of families $v \in \ell^2(\mathbb{N},U)$ such that (19) holds for all $j \in \mathbb{N}$ can be written as

$$\mathcal{A} = \bigcap_{j\in\mathbb{N}} W_j^{-1}(\mathcal{R}_{\ell_j}).$$

In order to prove that this set is residual, we prove that for all $j \in \mathbb{N}$, $W_j^{-1}(\mathcal{R}_{\ell_j})$ is an open dense subset of $\ell^2(\mathbb{N},U)$. Let us fix $j \in \mathbb{N}$.

First, by using that W_j is a continuous mapping, we deduce that $W_j^{-1}(\mathcal{R}_{\ell_j})$ is an open subset of $\ell^2(\mathbb{N},U)$.

To prove the density, we proceed as in [6]: let us consider for all $z = (z_i^k) \in \mathbb{C}^{\ell_j\times\ell_j}$,

$$v(z) = \left(v(z^1),\dots,v(z^{\ell_j})\right), \quad z^k = (z_1^k,\dots,z_{\ell_j}^k) \tag{51}$$

with

$$v(z^k) = \sum_{i=1}^{\ell_j} z_i^k B^*\varepsilon_i(\bar{\lambda}_j). \tag{52}$$

We consider the following determinant:

$$P_j(z) \stackrel{\text{def}}{=} \det\begin{pmatrix} (v(z^1)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \dots & (v(z^k)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \dots & (v(z^{\ell_j})|B^*\varepsilon_1(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (v(z^1)|B^*\varepsilon_i(\bar{\lambda}_j))_U & \dots & (v(z^k)|B^*\varepsilon_i(\bar{\lambda}_j))_U & \dots & (v(z^{\ell_j})|B^*\varepsilon_i(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (v(z^1)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \dots & (v(z^k)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \dots & (v(z^{\ell_j})|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U \end{pmatrix}.$$

By using the multilinearity of the determinant, we deduce that P_j is polynomial in z and the coefficient of $z_1^1\dots z_{\ell_j}^{\ell_j}$ is equal to

$$\det\begin{pmatrix} (B^*\varepsilon_1(\bar{\lambda}_j)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_k(\bar{\lambda}_j)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_{\ell_j}(\bar{\lambda}_j)|B^*\varepsilon_1(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (B^*\varepsilon_1(\bar{\lambda}_j)|B^*\varepsilon_k(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_k(\bar{\lambda}_j)|B^*\varepsilon_i(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_{\ell_j}(\bar{\lambda}_j)|B^*\varepsilon_i(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (B^*\varepsilon_1(\bar{\lambda}_j)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_k(\bar{\lambda}_j)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \dots & (B^*\varepsilon_{\ell_j}(\bar{\lambda}_j)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U \end{pmatrix}. \tag{53}$$

By using the fact that (UC) is equivalent to (2) for all $j \in \mathbb{N}$, we deduce that the above determinant is not zero and then that the polynomial P_j is not identically zero. As a consequence, the complementary set $\mathbb{O}_j^c$ of its roots $\mathbb{O}_j$ is a dense subset of $\mathbb{C}^{\ell_j \times \ell_j}$. Now we consider the mapping

$$\Psi_j : \left[(B^*\mathcal{E}_j)^\perp\right]^{\ell_j} \times \mathbb{C}^{\ell_j \times \ell_j} \times \ell^2(\mathbb{N}, U) \to \ell^2(\mathbb{N}, U)$$

where $\mathcal{E}_j = \mathrm{span}_{\mathbb{C}}\{\varepsilon_1(\bar{\lambda}_j), \ldots, \varepsilon_{\ell_j}(\bar{\lambda}_j)\}$, defined by

$$\Psi_j(U_\perp, z, w) = \begin{cases} [U_\perp]_k + v(z^k) & \text{if} \quad k \leq \ell_j \\ w_{k-\ell_j-1} & \text{if} \quad k > \ell_j. \end{cases}$$

Since Ψ_j is a linear onto mapping, the subset

$$\Psi_j(\left[(B^*\mathcal{E}_j)^\perp\right]^{\ell_j} \times \mathbb{O}_j^c \times \ell^2(\mathbb{N}, U)) \subset W_j^{-1}(\mathcal{R}_{\ell_j})$$

is dense in $\ell^2(\mathbb{N}, U)$.

Proof of point 5. The proof is completely similar to the proof of point 4. Just, observe that $K \geq \ell_j$ is clearly a necessary condition for (19) with $W_j(v)$ now defined by (17), and that it guarantees $\mathcal{R}_{\ell_j} \neq \emptyset$ and that determinants P_j of size $\ell_j \times \ell_j$ can be extracted from $W_j(v)$.

Proof of point 2. The fact that (UC) and (18) imply that (1) is approximately controllable follows from point 5. To prove the converse implication it suffices to remark that (UC) (or equivalently (2)) and (18) are necessary conditions for (19).

4.3 Proof of points 2 to 5 of Theorem 12

In this section, we end the proof of Theorem 12. In the remaining part of this subsection we suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ are satisfied. It is sufficient to prove the point 4. Indeed, the point 3 is a direct consequence of the point 3 of Theorem 4, the point 2 is a consequence of points 3 and 4, and the proof of the point 5 is completely similar to the proof of the point 4.

Then let us prove that if (UC) is true then the set of families $v \in \ell^2(\mathbb{N}, W)$ such that (19) holds form a residual set of $\ell^2(\mathbb{N}, W)$. The proof of the fact that the set of admissible family $v \in \ell^2(\mathbb{N}, W)$ is the union of open subsets of $\ell^2(\mathbb{N}, W)$ is similar to the proof of Theorem 4, and we skip it. For the density we also follow the same arguments as in the proof of Theorem 4, but instead of (51),(52) and P_j we have to consider for $(x, y) = (x_i^k, y_i^k) \in \mathbb{R}^{\ell_j \times \ell_j} \times \mathbb{R}^{\ell_j \times \ell_j}$:

$$v(x, y) = \left(v(x^1, y^1), \ldots, v(x^{\ell_j}, y^{\ell_j})\right), \quad x^k = (x_1^k, \ldots, x_{\ell_j}^k),\ y^k = (y_1^k, \ldots, y_{\ell_j}^k)$$

with

$$v(x^k, y^k) = \sum_{i=1}^{\ell_j} x_i^k B^* \Re\varepsilon_i(\bar{\lambda}_j) + y_i^k B^* \Im\varepsilon_i(\bar{\lambda}_j),$$

and the following determinants for $j \in \mathbb{N}$:

$$Q_j(x, y) \stackrel{\text{def}}{=} \det \begin{pmatrix} (v(x^1, y^1)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \ldots & (v(x^k, y^k)|B^*\varepsilon_1(\bar{\lambda}_j))_U & \ldots & (v(x^{\ell_j}, y^{\ell_j})|B^*\varepsilon_1(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (v(x^1, y^1)|B^*\varepsilon_i(\bar{\lambda}_j))_U & \ldots & (v(x^k, y^k)|B^*\varepsilon_i(\bar{\lambda}_j))_U & \ldots & (v(x^{\ell_j}, y^{\ell_j})|B^*\varepsilon_i(\bar{\lambda}_j))_U \\ \vdots & & \vdots & & \vdots \\ (v(x^1, y^1)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \ldots & (v(x^k, y^k)|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U & \ldots & (v(x^{\ell_j}, y^{\ell_j})|B^*\varepsilon_{\ell_j}(\bar{\lambda}_j))_U \end{pmatrix}.$$

Since for $z \in \mathbb{C}^{\ell_j \times \ell_j}$ we have $Q_j(z, \imath z) = P_j(z)$ and $P_j \neq 0$ (see subsection 4.2) then $Q_j \neq 0$ and we can conclude in a way completely similar as for the complex case.

4.4 Proof of Corollary 7 and of 13

We only prove Corollary 7. Corollary 13 can be obtained exactly in the same way, just replace complex spaces by real spaces below. For the following we suppose that assumptions $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$ are satisfied and we first rewrite (1) in two equations, one related to the "unstable" modes and the other to the "stable" modes. Let $N \in \mathbb{N}^*$ be such that:

$$\Re\lambda_0 \geq \Re\lambda_1 \geq \cdots \geq \Re\lambda_N \geq -\sigma > \Re\lambda_{N+1} \geq \ldots, \tag{54}$$

and set $\Sigma_N \stackrel{\text{def}}{=} \{\lambda_k \mid k = 1, \ldots, N\}$. Thus, we split (1) in two equations, one related to the "unstable" modes Σ_N and the other to the "stable" modes $\Sigma\backslash\Sigma_N$ (see [24, Par. III.4, Thm. 6.17, p.178]). For that, we introduce the projection operator defined by

$$P_N \stackrel{\text{def}}{=} \frac{1}{2\pi\imath}\int_{\Gamma_N}(\lambda - A)^{-1}\mathrm{d}\lambda, \tag{55}$$

where Γ_N be a contour enclosing Σ_N but no other point of the spectrum of A. Then the space H is the direct sum of the two invariant subspaces $H_N = P_N H$ and $H_N^- = (I - P_N)H$ of A and the solution y of (1), which can be rewritten $y = y_N + y_N^-$ with $y_N = P_N y$ and $y_N^- = (I - P_N)y$, is solution to systems:

$$y_N' = A_N y_N + B_N p_N u \in H_N, \tag{56}$$

$$y_N^{-'} = A_N^- y_N^- + B_N^- p_N^- u \in H_N^-. \tag{57}$$

In the above setting, A_N and A_N^- denote the restriction of A to H_N and H_N^- respectively, and $B_N = P_N B p_N^*$, $B_N = (I - P_N)B p_N^{*-}$ where $p_N : U \to U_N$ and $p_N^- : U \to U_N^-$ are the orthogonal projection operators on

$$U_N \stackrel{\text{def}}{=} \{B^*\varepsilon \mid \varepsilon \in H_N^* \stackrel{\text{def}}{=} P_N^* H\} \quad \text{and} \quad U_N^- \stackrel{\text{def}}{=} \{B^*\varepsilon \mid \varepsilon \in H_N^{*-} \stackrel{\text{def}}{=} (I - P_N^*)H\},$$

respectively. Note that their respective adjoints $p_N^* : U_N \to U$ and $p_N^{-*} : U_N^- \to U$ are the inclusion maps. For a detailed justification of the decomposition (56)-(57), see [6, 33].

Let us prove that $(A + \sigma, B)$ is stabilizable, if and only if, (UC_σ) is satisfied.

The "only if" part is obtained by remarking that if for $j \in \{1, \ldots, N\}$ there is $\varepsilon(\overline{\lambda}_j) \in \ker(A^* - \overline{\lambda}_j)$ obeying $\varepsilon(\overline{\lambda}_j) \neq 0$ and $B^*\varepsilon(\overline{\lambda}_j) = 0$, then multiplying (1) by $\varepsilon(\overline{\lambda}_j)$ yields that every solution y of (1) satisfies:

$$(y(t)|\varepsilon(\overline{\lambda}_j))_H = e^{\lambda_j t}(y(0)|\varepsilon(\overline{\lambda}_j))_H.$$

Since the above equality is independent on u and since $\Re\lambda_j \geq -\sigma$, then (20) is false for any initial datum not orthogonal to $\varepsilon(\overline{\lambda}_j)$.

Suppose now that (UC_σ) is true and let us prove that $(A + \sigma, B)$ is stabilizable. For that, let us first verify that (56) is null controllable. Assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ are obviously satisfied for A_N and the fact that A_N obeys $(\mathcal{H}_2)$ is a direct consequence of the definition of H_N and of A_N (the restriction of A to H_N). Moreover, since the spectrum of A_N is exactly Σ_N, then (UC) for A_N is exactly (UC_σ). Then by Theorem 4 system (56) is approximately controllable, and since null controllability and approximate controllability are equivalent notions for finite dimensional systems we deduce that (56) is null controllable.

Then it follows that $(A_N + \sigma, B_N)$ is stabilizable: there is a $F_N : H_N \to U_N$ such that the solution $\widehat{y}_N$ to (56) with $u = F_N\widehat{y}_N$ satisfies for some $\epsilon > 0$:

$$\|\widehat{y}_N(t)\|_H \leq Ce^{-(\sigma+\epsilon)t}\|P_N y(0)\|_H \quad (t \geq 0). \tag{58}$$

For instance, the finite dimensional feedback law F_N can be constructed with a Riccati operator obtained from a quadratic minimizing problem as in [6]. Moreover the solution $\widehat{y_N^-}$ of (57) with $u = F_N\widehat{y}_N$ is given by:

$$\begin{aligned}\widehat{y_N^-}(t) &= e^{A_N^- t}(I - P_N)y(0) + \int_0^t e^{A_N^-(t-s)}B_N^- p_N^- F_N\widehat{y}_N\mathrm{ds},\\ &= e^{A_N^- t}(I - P_N)y(0) + \int_0^t (\lambda_0 - A)^\gamma e^{A_N^-(t-s)}(I - P_N)(\lambda_0 - A)^{-\gamma}BF_N\widehat{y}_N\mathrm{ds}.\end{aligned}$$

The last above equality follows by remarking that $(\lambda_0 - A_N^-)^{-\gamma} = (\lambda_0 - A)^{-\gamma}(I - P_N)$. Then with $(\mathcal{H}_4')$, with the fact that (54) and $(\mathcal{H}_3)$ guarantee that for $\epsilon' > 0$ such that $-(\sigma + \epsilon') > \lambda_{N+1}$ we have $\|e^{A_N^- t}\|_{H_N^-} \leq Ce^{-(\sigma+\epsilon')t}$ and $\|(\lambda_0 - A_N^-)^\gamma e^{A_N^-(t-s)}\|_{H_N^-} \leq C(t-s)^{-\gamma}e^{-(\sigma+\epsilon')(t-s)}$, and with (58) we deduce:

$$\|\widehat{y_N^-}(t)\|_H \leq C\left(e^{-(\sigma+\epsilon')t} + e^{-(\sigma+\epsilon)t}\int_0^t \frac{e^{(\epsilon-\epsilon')(t-\tau)}}{(t-\tau)^\gamma}d\tau\right)\|y(0)\|_H \quad (t \geq 0). \tag{59}$$

Then if we choose $\epsilon < \epsilon'$ the above inequality means that the feedback control constructed from the unstable part of system (1) does not destabilize the stable part of system (1). Then we have proved that the solution of (1) with $u = F_N P_N y$ obeys (20) which is to say that $(A + \sigma, B)$ is stabilizable.

Finally, points 2, 3 and 4 are a direct consequence of Theorem 4 applied to the projected system (56).

5 Stabilizability of incompressible Navier-Stokes type systems

In the present section, we provide local Carleman inequalities for the Stokes system that are useful to prove Fattorini criterion (UC) corresponding to Stokes and coupled Stokes like systems. Such inequalities will be first used to recover in an easy way a uniqueness theorem for Oseen equations originally obtained by Fabre and Lebeau in [10]. Next, we give two examples of uniqueness theorem for coupled Stokes like system: one for an adjoint MHD system and one for an adjoint micropolar system. Finally, we deduce the stabilizability of nonlinear MHD and micropolar systems.

5.1 Carleman inequalities for the Stokes system

In what follows, $d = 2$ or $d = 3$ and for an open subset $\mathcal{D}$ of $\mathbb{R}^d$ we denote $L^2(\mathcal{D};\mathbb{R})$, $L^2(\mathcal{D};\mathbb{C})$, $H^1(\mathcal{D};\mathbb{R})$, $H^1(\mathcal{D};\mathbb{C})$, etc the usual Lebesgue and Sobolev spaces of functions with values in $\mathbb{R}$ or $\mathbb{C}$. For a scalar function π or a vector field $z = {}^t(z_1, \dots, z_d)$ (^{t} denotes the transpose) we define $\nabla\pi = {}^t(\partial_{x_1}\pi, \dots, \partial_{x_d}\pi)$, $\nabla z = (\partial_{x_j} z_i)_{1\le i,j\le d}$, ${}^t\nabla z = (\partial_{x_i} z_j)_{1\le i,j\le d}$ and we use the notations $D^s z \stackrel{\text{def}}{=} \nabla z + {}^t\nabla z$ and $D^a z \stackrel{\text{def}}{=} \nabla z - {}^t\nabla z$. We recall that the divergence of z is defined by $\operatorname{div} z = \sum_{j=1}^d \partial_{x_j} z_j$ and the curl of z or π is defined by

$$\operatorname{curl} z = \partial_{x_1} z_2 - \partial_{x_2} z_1 \quad \text{and} \quad \operatorname{curl}\pi = \begin{pmatrix} \partial_{x_2}\pi \\ -\partial_{x_1}\pi \end{pmatrix} \quad \text{if } d = 2,$$

and

$$\operatorname{curl} z = \begin{pmatrix} \partial_{x_2} z_3 - \partial_{x_3} z_2 \\ \partial_{x_3} z_1 - \partial_{x_1} z_3 \\ \partial_{x_1} z_2 - \partial_{x_2} z_1 \end{pmatrix} \quad \text{if } d = 3.$$

Note also that for two vector fields y, z the following relation holds:

$$(D^a y)z = S(z)\operatorname{curl} y, \tag{60}$$

where

$$S(z) \stackrel{\text{def}}{=} \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix} \text{ if } d = 2 \quad \text{and} \quad S(z) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix} \quad \text{if } d = 3.$$

In what follows, $\mathcal{O}$ denotes a non empty bounded open subset of $\mathbb{R}^d$ of class C^2 and $\psi : \mathcal{O} \to \mathbb{R}$ is a function satisfying

$$\psi \in C^2(\mathcal{O};\mathbb{R}), \quad \psi > 0 \quad \text{and} \quad |\nabla\psi| > 0 \quad \text{on } \mathcal{O}. \tag{61}$$

Let us recall a well-known Carleman inequality for the Laplace equation.

Theorem 16. *Let $k \in \{0, 1\}$, $F_0 \in L^2(\mathcal{O};\mathbb{C})$ and $F_1 \in (L^2(\mathcal{O};\mathbb{C}))^d$. There exist $C > 0$ and $\widehat{\tau} > 1$ such that for all $\tau \ge \widehat{\tau}$ there exists $\widehat{s}(\tau)$ such that for all $s \ge \widehat{s}(\tau)$ and for all $u \in H_0^1(\mathcal{O};\mathbb{C})$ solution of*

$$-\Delta u = F_0 + \operatorname{div} F_1 \quad \text{in} \quad \mathcal{O}$$

the following inequality holds:

$$\int_{\mathcal{O}} \left(e^{(k-1)\tau\psi} |\nabla u|^2 + s^2\tau^2 e^{(k+1)\tau\psi} |u|^2 \right) e^{2se^{\tau\psi}} \mathrm{dx} \le C \int_{\mathcal{O}} \left(se^{k\tau\psi} |F_1|^2 + s^{-1}\tau^{-2} e^{(k-2)\tau\psi} |F_0|^2 \right) e^{2se^{\tau\psi}} \mathrm{dx}. \tag{62}$$

Inequality (62) for $k = 1$ can be obtained for instance from [22, Thm A.1] and (62) for $k = 0$ is obtained by applying (62) with $k = 1$ to the equation satisfied by $e^{-\frac{\tau}{2}\psi} u$. Note that the proof of (62) proposed in the above quoted work is performed for a $\mathbb{R}$-valued function u, but it is easily checked that it can be done in the same way for a $\mathbb{C}$-valued function u.

From Theorem 16, we deduce the following Carleman inequalities for the Stokes system.

Corollary 17. *Let $\alpha \in \mathbb{R}\backslash\{-1\}$. There exist $C > 0$ and $\widehat{\tau} > 1$ such that for all $\tau \geq \widehat{\tau}$ there exists $\widehat{s}(\tau)$ such that for all $s \geq \widehat{s}(\tau)$ and for all $(z, \pi) \in (H_0^2(\mathcal{O};\mathbb{C}))^d \times H_0^1(\mathcal{O};\mathbb{C})$ the following inequalities hold:*

$$\int_{\mathcal{O}} \left(|\nabla z|^2 + s^2\tau^2 e^{2\tau\psi}|z|^2\right) e^{2se^{\tau\psi}} \mathrm{dx} \leq C \int_{\mathcal{O}} (se^{\tau\psi}|\mathrm{div}\, z + \alpha\pi|^2 + \tau^{-2}|\nabla\pi - \Delta z|^2) e^{2se^{\tau\psi}} \mathrm{dx}, \quad (63)$$

$$\int_{\mathcal{O}} s\tau^2 e^{\tau\psi} \left(|\mathrm{curl}\, z|^2 + |\mathrm{div}\, z - \pi|^2\right) e^{2se^{\tau\psi}} \mathrm{dx} \leq C \int_{\mathcal{O}} |\nabla\pi - \Delta z|^2 e^{2se^{\tau\psi}} \mathrm{dx}. \quad (64)$$

Proof. Set $f \stackrel{\text{def}}{=} -\Delta z + \nabla\pi$ and $g \stackrel{\text{def}}{=} \mathrm{div}\, z$. From $-\Delta z = \mathrm{curl}\,(\mathrm{curl}\, z) - \nabla(\mathrm{div}\, z)$ we get:

$$-\Delta z = \frac{1}{1+\alpha}\left(\mathrm{curl}\,(\mathrm{curl}\, z) - \nabla(g + \alpha\pi) + \alpha f\right) \quad \text{in} \quad \mathcal{O}, \quad (65)$$

$$-\Delta(\mathrm{curl}\, z) = \mathrm{curl}\, f \quad \text{in} \quad \mathcal{O}, \quad (66)$$

$$-\Delta(\pi - g) = -\mathrm{div}\, f \quad \text{in} \quad \mathcal{O}. \quad (67)$$

Then (64) is obtained by applying (62) for $k = 0$ to (66) and to (67). Finally, (63) is obtained by first applying (62) for $k = 1$ to (65) and next using the estimate of $\mathrm{curl}\, z$ given by (64). □

5.2 Uniqueness theorem for the Oseen system

Here we give a first illustration of an application of Corollary 17 that allows to recover in an easy way a uniqueness theorem for the Oseen equations. It also provides a sketch of the method of proof which is used in the next section to obtain uniqueness theorems for coupled Stokes type systems. For $\lambda \in \mathbb{C}$ and $w^S \in (L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}))^d$ consider the following eigenvalue problem:

$$\begin{aligned} \lambda z - \Delta z - (D^s z) w^S + \nabla\pi = 0 \quad &\text{in} \quad \Omega, \\ \mathrm{div}\, z = 0 \quad &\text{in} \quad \Omega. \end{aligned} \quad (68)$$

Let us prove that every $z \in (H^1_{\mathrm{loc}}(\Omega;\mathbb{C}))^d$ solution of (68) which vanishes in a non empty open subset $\omega \subset\subset \Omega$ must be identically zero in Ω. Such a *distributed observability theorem* has been first proved by Fabre and Lebeau in [10] for Stokes equations with bounded potential. When ω has a smooth C^2 boundary and $w^S \in (H^2(\Omega;\mathbb{R}))^d$ the result has been obtained in [39]. Such a uniqueness theorem implies Fattorini criterion (UC) corresponding to the linearized Navier-Stokes equations and permits to construct a feedback or a dynamical control stabilizing the Navier-Stokes equations around a given stationary state w^S, see [6] for details.

More precisely, we prove the following theorem.

Theorem 18. *Let Ω be a connected open subset of $\mathbb{R}^d$ with $d = 2$ or $d = 3$, $\lambda \in \mathbb{C}$ and $w^S \in (L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}))^d$. Suppose that $(z, \pi) \in (H^1_{\mathrm{loc}}(\Omega;\mathbb{C}))^d \times L^2_{\mathrm{loc}}(\Omega;\mathbb{C})$ satisfies (68) and that z vanishes on a non empty open subset of Ω. Then z is identically equal to zero in Ω and the function π is constant.*

Proof. The proof consists in three steps.

Step 1. A local Carleman inequality. Let $\mathcal{O} \subset\subset \Omega$ and ψ satisfying (61). From (63) and (64) with $\alpha = 0$ we get that every $(z, \pi) \in (H_0^2(\mathcal{O};\mathbb{C}))^d \times H_0^1(\mathcal{O};\mathbb{C})$ satisfies:

$$\begin{aligned} &\int_{\mathcal{O}} \left(|\nabla z|^2 + s^2\tau^2 e^{2\tau\psi}|z|^2 + s\tau^2 e^{\tau\psi}|\pi - \mathrm{div} z|^2\right) e^{2se^{\tau\psi}} \mathrm{dx} \leq \\ &\quad C \int_{\mathcal{O}} (se^{\tau\psi}|\mathrm{div}\, z|^2 + |\lambda z - \Delta z - (D^s z) w^S + \nabla\pi|^2) e^{2se^{\tau\psi}} \mathrm{dx}. \end{aligned} \quad (69)$$

Indeed, it suffices to choose τ in (63) large enough so that the first order term $(D^s z) w^S$ is absorbed by the left side of the inequality (since $w^S \in (L^\infty(\mathcal{O};\mathbb{R}))^d$), and thus to choose s large enough so that the zero order term λz is absorbed by the left hand side of the inequality. Thus, we combine the resulting inequality with (64).

Next, suppose that $(z, \pi) \in (H^1_{\mathrm{loc}}(\Omega;\mathbb{C}))^d \times L^2_{\mathrm{loc}}(\Omega;\mathbb{C})$ is a solution of (68). Let $\mathcal{O}_1$ and $\mathcal{O}^*$ be two open subsets of $\mathbb{R}^d$ satisfying

$$\mathcal{O}_1 \subset\subset \mathcal{O} \quad \text{and} \quad \mathcal{O}^* \stackrel{\text{def}}{=} \mathcal{O} \setminus \overline{\mathcal{O}}_1, \quad (70)$$

and let $\chi : \mathcal{O} \to \mathbb{R}$ a cut-off function such that

$$\chi \in C_c^\infty(\mathcal{O};\mathbb{R}), \quad \chi \equiv 1 \ \text{ in } \ \mathcal{O}_1 \subset\subset \mathcal{O} \quad \text{and} \quad 0 \leq \chi \leq 1 \ \text{ in } \ \mathcal{O}^*. \quad (71)$$

From (68) and regularity results for Stokes equations we get that $(\chi z, \chi\pi) \in (H^2_0(\mathcal{O};\mathbb{C}))^d \times H^1_0(\mathcal{O};\mathbb{C})$ satisfies:

$$\begin{array}{c} \lambda(\chi z) - \Delta(\chi z) - (D^s(\chi z))w^S + \nabla(\chi\pi) = -[\Delta + (w^S\cdot\nabla) + (w^S\cdot {}^t\nabla), \chi]z + [\nabla,\chi]\pi \quad \text{in} \quad \Omega \\ \operatorname{div}(\chi z) = [\operatorname{div},\chi]z \quad \text{in} \quad \Omega, \end{array} \tag{72}$$

where $[\cdot,\cdot]$ denote commutators. Then applying (69) to $(\chi z, \chi\pi)$ and using the fact that commutators in (72) are supported in $\mathcal{O}^*$ we deduce that solutions of (68) obey:

$$\int_{\mathcal{O}_1} e^{2se^{\tau\psi}}(|z|^2 + |\pi|^2)\mathrm{dx} \le C_{\chi,\tau,\psi}\int_{\mathcal{O}^*} e^{2se^{\tau\psi}}(|z|^2 + |\nabla z|^2 + |\pi|^2)\mathrm{dx}. \tag{73}$$

Step 2. A local uniqueness theorem. Now let us use inequality (73) to prove that if z vanishes in a relatively compact ball of Ω then z is necessarily zero in a neighborhood of such a ball. More precisely, let $x_0 \in \mathbb{R}^d$ and $R > 0$, denote by $B(x_0, R)$ the open ball centered at x_0 with radius R and suppose that $B(x_0, R) \subset\subset \Omega$ and $z \equiv 0$ in $B(x_0, R)$. Since $\partial B(x_0, R)$ is compact, to prove that z vanishes in an open neighborhood of $\partial B(x_0, R)$ it suffices to prove that for each $x_1 \in \partial B(x_0, R)$ there is a ball $B(x_1, \varepsilon)$, $\varepsilon > 0$, in which z vanishes.

For $x_1 \in \partial B(x_0, R)$ choose $\mathcal{O} = B(x_1, r)$, $\mathcal{O}_1 = B(x_1, r/2)$, $\mathcal{O}^* = \mathcal{O}\backslash\overline{\mathcal{O}}_1$ with $0 < r < R/2$ small enough such that $B(x_0, R) \cup \mathcal{O} \subset \Omega$ and set

$$\psi(x) = C_0 - |x - x_0|^2 - \frac{1}{2}|x - x_1|^2,$$

where $C_0 > 0$ is large enough so that $\psi > 0$ in $\mathcal{O}$. Moreover, we verify that $|\nabla\psi| > 0$ and then (61) is satisfied. Thus, set $\mathcal{V} = \mathcal{O} \setminus B(x_0, R)$, $\mathcal{V}_\varepsilon = B(x_1, \varepsilon) \setminus B(x_0, R)$ for $\varepsilon > 0$ and $\mathcal{V}^* = \mathcal{O}^* \setminus B(x_0, R)$. Since we have $\psi(x) < \psi(x_1)$ for all $x \in \overline{\mathcal{V}^*}$, by continuity we get for $\varepsilon > 0$ small enough:

$$\max_{\mathcal{V}^*}\psi \stackrel{\text{def}}{=} \psi^* < \psi_1 \stackrel{\text{def}}{=} \min_{\mathcal{V}_\varepsilon}\psi. \tag{74}$$

From (68) and $z \equiv 0$ in $B(x_0, R)$ we get that $\nabla\pi$ is zero in $B(x_0, R)$ and then $\pi = \overline{\pi} \in \mathbb{R}$ is constant in $B(x_0, R)$. Then we apply (73) to $(z, \pi - \overline{\pi})$ (which satisfies (68)) and since z and $\pi - \overline{\pi}$ vanish in $B(x_0, R)$ we obtain

$$\int_{\mathcal{V}_\varepsilon}(|z|^2 + |\pi - \overline{\pi}|^2)\mathrm{dx} \le e^{2s(e^{\tau\psi^*} - e^{\tau\psi_1})}C_{\chi,\tau,\psi}\int_{\mathcal{V}^*}(|z|^2 + |\nabla z|^2 + |\pi - \overline{\pi}|^2)\mathrm{dx}.$$

Inequality (74) implies that the right side of the above inequality tends to zero as s goes to infinity, which yields $z \equiv 0$ and $\pi \equiv \overline{\pi}$ in $\mathcal{V}_\varepsilon$.

Step 3. A Connectivity argument. Suppose that z is a non zero solution of (68) which is vanishing in an open subset of Ω, that is to say $\operatorname{Supp}(z) \neq \Omega$ and $\operatorname{Supp}(z) \neq \emptyset$. Because Ω is connected and $\operatorname{Supp}(z)$ is a closed subset of Ω we have that $\operatorname{Supp}(z)$ is not an open subset of Ω. As a consequence, there exists $x^* \in \operatorname{Supp}(z)$ and $R^* > 0$ small enough such that $B(x^*, R^*) \cap (\Omega\backslash\operatorname{Supp}(z)) \neq \emptyset$ and $B(x^*, R^*) \subset \Omega$. Then we can choose $x_0 \in \Omega \setminus \operatorname{Supp}(z)$ and $R_0 > 0$ such that $B(x_0, R_0) \subset B(x^*, R^*)$ and $x^* \in B(x_0, R_0)$ (for instance, for $x_0^* \in B(x^*, R^*) \cap (\Omega\backslash\operatorname{Supp}(z))$ choose $x_0 = x^* + \frac{1}{3}(x_0^* - x^*)$ and $R_0 = \frac{2}{3}|x_0^* - x^*|$). To summerize, we have:

$$x_0 \in \Omega \setminus \operatorname{Supp}(z), \quad B(x_0, R_0) \subset \Omega \quad \text{and} \quad B(x_0, R_0) \cap \operatorname{Supp}(z) \neq \emptyset. \tag{75}$$

Next, because $\Omega\backslash\operatorname{Supp}(z)$ is open in $\mathbb{R}^d$, there exists $r_0 \in (0, R_0)$ such that $B(x_0, r_0) \subset \Omega\backslash\operatorname{Supp}(z)$, and we can introduce the value $\widehat{R} \in [r_0, R_0)$ defined by

$$\widehat{R} \stackrel{\text{def}}{=} \sup\{R > 0 \mid B(x_0, R) \subset \Omega\backslash\operatorname{Supp}(z)\}.$$

Since $\Omega\backslash\operatorname{Supp}(z)$ is open the above infimum is a maximum and $B(x_0, \widehat{R}) \subset \Omega\backslash\operatorname{Supp}(z)$. Moreover, recall that (75) guarantees $B(x_0, \widehat{R}) \subset B(x_0, R_0) \subset \Omega$ and $\widehat{R} < R_0$ which implies $B(x_0, \widehat{R}) \subset\subset \Omega$. Then according to Step 2 above there exists $\varepsilon > 0$ such that $B(x_0, \widehat{R} + \varepsilon) \subset \Omega \setminus \operatorname{Supp}(z)$ which contradicts the definition of $\widehat{R}$. In conclusion, $\operatorname{Supp}(z) = \emptyset$ or $\operatorname{Supp}(z) = \Omega$, which is to say that a non zero solution z of (68) cannot vanish on an open subset of Ω. □

Remark 19. *Note that the key argument in the proof of Theorem 18 is to obtain the local Carleman inequality* (69) *related to Oseen system. Steps 2 and 3 of the proof are standard and have been recalled for readability convenience, see for instance [25] or [40, Appendix IV].*

5.3 Uniqueness theorem for adjoint MHD system

Corollary 17 also permits to prove a *distributed observability theorem* for coupled Oseen systems. Here we prove a uniqueness theorem related to the stabilizability of a MHD system (see Subsection 5.5 below).

For $w^S \in (L^\infty_{\rm loc}(\Omega;\mathbb{R}))^d$ and $\theta^S \in (L^\infty_{\rm loc}(\Omega;\mathbb{R}))^d$ consider the following adjoint linearized MHD system:

$$\left\{\begin{array}{ll} \lambda z - \Delta z - (D^s z) w^S + (D^a \rho)\theta^S + \nabla \pi = 0 & \text{in } \Omega, \\ \lambda \rho - \Delta \rho + (D^s z)\theta^S - (D^a \rho) w^S + \nabla \kappa = 0 & \text{in } \Omega, \\ \qquad\qquad \operatorname{div} z = \operatorname{div} \rho = 0 & \text{in } \Omega. \end{array}\right. \tag{76}$$

Theorem 20. *Let Ω be a connected open subset of $\mathbb{R}^d$ with $d = 2$ or $d = 3$, $\lambda \in \mathbb{C}$, $w^S \in (L^\infty_{\rm loc}(\Omega;\mathbb{R}))^d$ and $\theta^S \in (L^\infty_{\rm loc}(\Omega;\mathbb{R}))^d$. Suppose that $(z,\pi) \in (H^1_{\rm loc}(\Omega;\mathbb{C}))^d \times L^2_{\rm loc}(\Omega;\mathbb{C})$ and $(\rho,\kappa) \in (H^1_{\rm loc}(\Omega;\mathbb{C}))^d \times L^2_{\rm loc}(\Omega;\mathbb{C})$ satisfy (76) and that z and $\operatorname{curl}\rho$ vanish on a non empty open subset $\omega \subset\subset \Omega$. Then z and $\operatorname{curl}\rho$ are identically equal to zero in Ω, the function π is constant and $\nabla\kappa = -\lambda\rho$.*

Proof. The point is to use Corollary 17 to get an inequality for (76) analogue to (73). Then the result will follow from a local uniqueness theorem and a connectivity argument completely analogous to steps 2 and 3 of the proof of Theorem 18.

First, set $\zeta \stackrel{\rm def}{=} \operatorname{curl}\rho$, apply the operator curl to the second equality in (76) and use (60) to get

$$\left\{\begin{array}{ll} \lambda z - \Delta z - (D^s z) w^S + S(\theta^S)\zeta + \nabla \pi = 0 & \text{in } \Omega, \\ \lambda \zeta - \Delta \zeta + \operatorname{curl}\left((D^s z)\theta^S - S(w^S)\zeta\right) = 0 & \text{in } \Omega, \\ \qquad\qquad \operatorname{div} z = 0 & \text{in } \Omega. \end{array}\right. \tag{77}$$

Next, let $\mathcal{O} \subset\subset \Omega$ and $\psi : \mathcal{O} \to \mathbb{R}$ satisfying (61), suppose for the moment that $(z,\zeta,\pi) \in (H^2_0(\mathcal{O};\mathbb{C}))^d \times (H^2_0(\mathcal{O};\mathbb{C}))^{2d-3} \times H^1_0(\mathcal{O};\mathbb{C})$ and let us prove that such (z,ζ,π) obeys

$$\begin{aligned} &\int_{\mathcal{O}} \left(s^2\tau^2 e^{2\tau\psi}|z|^2 + s\tau^2 e^{\tau\psi}(|\pi - \operatorname{div} z|^2 + |\zeta|^2)\right) e^{2se^{\tau\psi}} \mathrm{dx} \\ &\leq C \int_{\mathcal{O}} (se^{\tau\psi}|\operatorname{div} z|^2 + |\mathcal{L}_1(z,\zeta,\pi)|^2 + |\mathcal{L}_2(z,\zeta)|^2) e^{2se^{\tau\psi}} \mathrm{dx}, \end{aligned} \tag{78}$$

where

$$\begin{aligned} \mathcal{L}_1(z,\zeta,\pi) &\stackrel{\rm def}{=} \lambda z - \Delta z - (D^s z) w^S + S(\theta^S)\zeta + \nabla\pi, \\ \mathcal{L}_2(z,\zeta) &\stackrel{\rm def}{=} \lambda\zeta - \Delta\zeta + \operatorname{curl}\left((D^s z)\theta^S - S(w^S)\zeta\right). \end{aligned} \tag{79}$$

For that, first apply (62) to ζ (with $k = 0$) and obtain

$$\int_{\mathcal{O}} s\tau^2 e^{\tau\psi}|\zeta|^2 e^{2se^{\tau\psi}} \mathrm{dx} \leq C \int_{\mathcal{O}} \left(s^{-2}\tau^{-2}|\mathcal{L}_2(z,\zeta) - \lambda\zeta|^2 + |(D^s z)\theta^S - S(w^S)\zeta|^2\right) e^{2se^{\tau\psi}} \mathrm{dx}. \tag{80}$$

Thus, apply (63) (with $\alpha = 0$) and (64) to (z,π), add the resulting inequality to (80) and choose τ and s large enough so that the first order terms $(D^s z)w^S$, $(D^s z)\theta^S$ and the zero order terms $S(\theta^S)\zeta$, $S(w^S)\zeta$, λz, $\lambda\zeta$ are absorbed by the left hand side of the inequality.

Finally, let $\mathcal{O}_1$ and $\mathcal{O}^*$ two open subsets of $\mathbb{R}^d$ satisfying (70) and let $\chi : \mathcal{O} \to \mathbb{R}$ a cut-off function satisfying (71). If $(z,\zeta,\pi) \in (H^1_{\rm loc}(\Omega;\mathbb{C}))^d \times (H^1_{\rm loc}(\Omega;\mathbb{C}))^{2d-3} \times L^2_{\rm loc}(\Omega;\mathbb{C})$ satisfies (77) then by elliptic and Stokes regularity we get that $(\chi z, \chi\zeta, \chi\pi) \in (H^2_0(\mathcal{O};\mathbb{C}))^d \times (H^2_0(\mathcal{O};\mathbb{C}))^{2d-3} \times H^1_0(\mathcal{O};\mathbb{C})$ and (78) applied to $(\chi z, \chi\zeta, \chi\pi)$ yields:

$$\int_{\mathcal{O}_1} e^{2se^{\tau\psi}}(|z|^2 + |\pi|^2 + |\zeta|^2)\mathrm{dx} \leq C_{\chi,\tau,\psi} \int_{\mathcal{O}^*} e^{2se^{\tau\psi}}(|z|^2 + |\nabla z|^2 + |\pi|^2 + |\zeta|^2)\mathrm{dx}. \tag{81}$$

Then we conclude with a local uniqueness theorem and a connectivity argument as for Theorem 18.

Note that the facts that π is constant and that $\nabla\kappa = -\lambda\rho$ are direct consequences of the equations (76) with $z \equiv 0$ and $\operatorname{curl}\rho \equiv 0$ in Ω (since $\operatorname{div}\rho = 0$ implies $-\Delta\rho = \operatorname{curl}\operatorname{curl}\rho$). □

Now assume that Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary, denote by n the unit exterior normal vector field defined on $\partial\Omega$ and introduce the following N-dimensional space:

$$X_N \stackrel{\text{def}}{=} \{y \in (L^2(\Omega))^d \mid \operatorname{div} \mathrm{y} = 0 \text{ in } \Omega, \quad \operatorname{curl} \mathrm{y} = 0 \text{ in } \Omega, \quad y \cdot n = 0 \text{ on } \partial\Omega\}. \tag{82}$$

The fact that the above space is finite dimensional is well-known, see for instance [9, Chap. IX] for a detailed characterization of X_N. We only recall that if Ω is simply-connected we have $N = 0$ and X_N reduces to $\{0\}$ and if Ω is multiply-connected then $N \geq 1$ is the number of cuts required to make Ω simply-connected. Note that if $d = 2$ then $N + 1$ is exactly to the number of connected components of $\partial\Omega$.

The following straightforward consequence of Theorem 20 holds.

Corollary 21. *In addition to the hypotheses of Theorem 20, assume that Ω is a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary and suppose that*

$$\rho \cdot n = 0 \quad \text{on} \quad \partial\Omega. \tag{83}$$

Then the following results hold.

1. *If $\lambda \neq 0$ then z and ρ are identically equal to zero and the functions π and κ are constant in Ω.*
2. *If $\lambda = 0$ then z is identically equal to zero in Ω and $\rho \in X_N$. Moreover, the functions π and κ are constant in Ω. In particular, if Ω is simply-connected then ρ is identically equal to zero in Ω.*

5.4 Uniqueness theorem for adjoint micropolar system

Corollary 17 also provides a *distributed observability theorem* related to stabilizability of a micropolar system (see Subsection 5.6 below). Consider the following adjoint micropolar system:

$$\left\{\begin{array}{rl} \lambda z - \Delta z - (D^s z) w^S - {}^t(\nabla\rho)\theta^S - \operatorname{curl}\rho + \nabla\pi = 0 & \text{in } \Omega, \\ \lambda\rho - \Delta\rho + (D^s z)\theta^S - (\nabla\rho) w^S - \operatorname{curl} z + \nabla\kappa = 0 & \text{in } \Omega, \\ \operatorname{div} z = 0, \quad \operatorname{div}\rho + \kappa = 0 & \text{in } \Omega. \end{array}\right. \tag{84}$$

Theorem 22. *Let Ω be a connected open subset of $\mathbb{R}^d$ with $d = 2$ or $d = 3$, $\lambda \in \mathbb{C}$, $w^S \in (L^\infty_{\text{loc}}(\Omega;\mathbb{R}))^d$ and $\theta^S \in (L^\infty_{\text{loc}}(\Omega;\mathbb{R}))^d$. Suppose that $(z,\pi) \in (H^1_{\text{loc}}(\Omega;\mathbb{C}))^d \times L^2_{\text{loc}}(\Omega;\mathbb{C})$ and $(\rho,\kappa) \in (H^1_{\text{loc}}(\Omega;\mathbb{C}))^d \times L^2_{\text{loc}}(\Omega;\mathbb{C})$ satisfy (84) and that z and ρ vanish on a non empty open subset $\omega \subset\subset \Omega$. Then z and ρ are identically equal to zero in Ω and the functions π and κ are constant.*

Proof. Let us first consider two pairs (z,π) and (ρ,κ) of $(H^2_0(\mathcal{O};\mathbb{C}))^d \times H^1_0(\mathcal{O};\mathbb{C})$. By applying (63), (64) to (z,π) with $\alpha = 0$ and (63), (64) to (ρ,κ) with $\alpha = 1$, and choosing τ and s large enough to absorb the zero and first order terms in z, ρ we get:

$$\begin{aligned} &\int_{\mathcal{O}} \left(s^2\tau^2 e^{2\tau\psi}(|z|^2 + |\rho|^2) + s\tau^2 e^{\tau\psi}(|\pi - \operatorname{div} z|^2 + |\kappa - \operatorname{div}\rho|^2)\right) e^{2se^{\tau\psi}} \mathrm{d}x \\ &\leq C \int_{\mathcal{O}} (se^{\tau\psi}(|\operatorname{div} z|^2 + |\operatorname{div}\rho + \kappa|^2) + (|\mathcal{L}_1(z,\rho,\pi)|^2 + |\mathcal{L}_2(z,\rho,\kappa)|^2)) e^{2se^{\tau\psi}} \mathrm{d}x, \end{aligned} \tag{85}$$

where

$$\begin{aligned} \mathcal{L}_1(z,\rho,\pi) &\stackrel{\text{def}}{=} \lambda z - \Delta z - (D^s z) w^S - {}^t(\nabla\rho)\theta^S - \operatorname{curl}\rho + \nabla\pi, \\ \mathcal{L}_2(z,\rho,\kappa) &\stackrel{\text{def}}{=} \lambda\rho - \Delta\rho + (D^s z)\theta^S - (\nabla\rho) w^S - \operatorname{curl} z + \nabla\kappa. \end{aligned}$$

Thus, let $\mathcal{O}_1$ and $\mathcal{O}^*$ two open subsets of $\mathbb{R}^d$ satisfying (70) and let $\chi : \mathcal{O} \to \mathbb{R}$ a cut-off function satisfying (71). For (z,π,ρ,κ) satisfying (84) we apply inequality (85) to $(\chi z, \chi\pi, \chi\rho, \chi\kappa)$ and obtain

$$\int_{\mathcal{O}_1} e^{2se^{\tau\psi}}(|z|^2 + |\pi|^2 + |\rho|^2 + |\kappa|^2)\mathrm{d}x \leq C_{\chi,\tau,\psi} \int_{\mathcal{O}^*} e^{2se^{\tau\psi}}(|z|^2 + |\nabla z|^2 + |\pi|^2 + |\rho|^2 + |\nabla\rho|^2 + |\kappa|^2)\mathrm{d}x. \tag{86}$$

Then we conclude with a local uniqueness theorem and a connectivity argument as for Theorem 18. □

5.5 Stabilizability of MHD system

Here we show how the use of Theorem 20 permits to check an adequate Fattorini criterion ensuring stabilizability of a nonlinear MHD system.

Let Ω be a bounded open subset of $\mathbb{R}^3$ of class $C^{2,1}$ (in order to freely use H^2 Stokes regularity) and consider a stationary solution (w^S, θ^S, r^S) of the following MHD system:

$$\left\{\begin{array}{rl} -\Delta w^S + (w^S\cdot\nabla)w^S - (\operatorname{curl}\theta^S)\times\theta^S + \nabla r^S = f^S & \text{in } \Omega,\\ \operatorname{curl}(\operatorname{curl}\theta^S) - \operatorname{curl}(w^S\times\theta^S) = 0 & \text{in } \Omega,\\ \operatorname{div} w^S = \operatorname{div}\theta^S = 0 & \text{in } \Omega. \end{array}\right. \tag{87}$$

Here $w^S(x)\in\mathbb{R}^3$ represents the velocity of the fluid, $p^S(x)\in\mathbb{R}$ is the pressure, $\theta^S(x)$ is the magnetic field, $f^S(x)$ is a given stationary body force, n denotes the unit exterior normal vector field defined on $\partial\Omega$ and $\times$ denotes the vector product. Let also underline that in (87), usual non-dimensional constants that characterize the flow (Hartmann number, interaction parameter and magnetic Reynolds number) are supposed to be equal to one for simplicity. Concerning the well-posedness of system (87) we refer to [28]. In the following, w^S and θ^S are supposed to be smooth enough: $w^S\in(H^2(\Omega;\mathbb{R}))^3$ and $\theta^S\in(H^2(\Omega;\mathbb{R}))^3$.

Our aim is to stabilize a given solution of (87) by means of a distributed control localized in an open subset $\omega\subset\subset\Omega$. More precisely, for $(y_0,\vartheta_0)\in(L^2(\Omega;\mathbb{R}))^3\times(L^2(\Omega;\mathbb{R}))^3$ satisfying

$$\operatorname{div} y_0 = \operatorname{div}\vartheta_0 = 0 \ \text{ in } \Omega \quad \text{ and } \quad y_0\cdot n = \vartheta_0\cdot n = 0 \ \text{ on } \partial\Omega, \tag{88}$$

consider the following instationary MHD system

$$\left\{\begin{array}{rl} w_t - \Delta w + (w\cdot\nabla)w - (\operatorname{curl}\theta)\times\theta + \nabla r = f^S + \mathbf{1}_\omega u^1 & \text{in } Q,\\ \theta_t + \operatorname{curl}(\operatorname{curl}\theta) - \operatorname{curl}(w\times\theta) = \mathbf{1}_\omega P_\omega u^2 & \text{in } Q,\\ \operatorname{div} w = \operatorname{div}\theta = 0 & \text{in } Q,\\ w = w^S, \quad \theta\cdot n = \theta^S\cdot n & \text{on } \Sigma,\\ (\operatorname{curl}\theta - w\times\theta)\times n = (\operatorname{curl}\theta^S - w^S\times\theta^S)\times n & \text{on } \Sigma,\\ w(0) = w^S + y_0, \quad \theta(0) = \theta^S + \vartheta_0 & \text{in } \Omega, \end{array}\right. \tag{89}$$

where $Q \stackrel{\text{def}}{=} (0,+\infty)\times\Omega$ and $\Sigma \stackrel{\text{def}}{=} (0,+\infty)\times\partial\Omega$. Here $\mathbf{1}_\omega$ is the extension operator defined on $(L^2(\omega))^3$ by $\mathbf{1}_\omega(y)(x) = y(x)$ if $x\in\omega$ and $\mathbf{1}_\omega(y)(x) = 0$ else, $u = (u^1,u^2)$ is a control function in $(L^2((0,T)\times\omega;\mathbb{R}))^3\times(L^2((0,T)\times\omega;\mathbb{R}))^3$ and P_ω is the classical Helmholtz projector related to ω (i.e the orthogonal projection operator from $(L^2(\omega;\mathbb{R}))^3$ onto the completion of $\{v\in(C_0^\infty(\omega;\mathbb{R}))^3 \mid \operatorname{div} v = 0 \text{ in } \omega\}$ for the norm of $(L^2(\omega;\mathbb{R}))^3$). Note that the particular form of the second control operator guarantees that the right hand side of the magnetic field equation is divergence free and then permits to deduce the boundary stabilization from internal stabilization with a classical extension of the domain procedure. Indeed, because the orthogonal complement of the range of P_ω is composed with $\nabla p\in(L^2(\omega;\mathbb{R}))^3$ for $p\in(H^1_{\text{loc}}(\omega;\mathbb{R}))^3$ (see [17, Chap. III]), we have

$$\forall p\in(H_0^1(\Omega))^3 \quad \langle \operatorname{div}\mathbf{1}_\omega P_\omega u^2\,,\,p\rangle_{(H^{-1}(\Omega))^3,(H_0^1(\Omega))^3} = \int_\Omega \mathbf{1}_\omega P_\omega u^2\cdot\nabla p\,\mathrm{d}x = \int_\omega P_\omega u^2\cdot\nabla p\,\mathrm{d}x = 0.$$

The idea of using such a type of control for the magnetic field equation is due to [26] for the two dimensional version of system (89). If Ω is supposed to be only bounded in directions x_1, x_2 and invariant in direction x_3, if all functions only depend on x_1, x_2 and if each vector field has its third component equal to zero (and then is identified to its two first components i.e $w(x) = {}^t(w_1(x_1,x_2),w_2(x_1,x_2),0) \equiv {}^t(w_1(x_1,x_2),w_2(x_1,x_2))$ etc), then we can identify Ω to its bounded two dimensional section and (89) reduces to:

$$\left\{\begin{array}{rl} w_t - \Delta w + (w\cdot\nabla)w + (\operatorname{curl}\theta)\theta^\perp + \nabla r = f^S + \mathbf{1}_\omega u^1 & \text{in } Q,\\ \theta_t + \operatorname{curl}(\operatorname{curl}\theta) - \operatorname{curl}(w\cdot\theta^\perp) = \mathbf{1}_\omega P_\omega u^2 & \text{in } Q,\\ \operatorname{div} w = \operatorname{div}\theta = 0 & \text{in } Q,\\ w = w^S,\ \theta\cdot n = \theta^S\cdot n & \text{on } \Sigma\\ \operatorname{curl}\theta - w\cdot\theta^\perp = \operatorname{curl}\theta^S - w^S\cdot\theta^{S\perp} & \text{on } \Sigma\\ w(0) = w^S + y_0, \quad \theta(0) = \theta^S + \vartheta_0 & \text{in } \Omega, \end{array}\right. \tag{90}$$

where $(y_0, \vartheta_0) \in (L^2(\Omega;\mathbb{R}))^2 \times (L^2(\Omega;\mathbb{R}))^2$ satisfies (88) and u^1, u^2 both belong to $(L^2((0,T)\times\omega;\mathbb{R}))^2$. We have used the notation ${}^t(a_1,a_2)^\perp = {}^t(a_2,-a_1)$. In [26] the stabilizability of (90) is obtained for a stationary pair $(w^S,\theta^S) \in (W^{2,\infty}(\Omega))^2 \times (W^{2,\infty}(\Omega))^2$ satisfying homogeneous boundary conditions and for Ω simply-connected. The main step of the proof is the checking of the approximate controllability criterion (8) related to the linear instationary adjoint system associated to (90)

Here we extend such a stabilizability result to the three dimensional case and for stationary state only H^2. More precisely, we show the existence of finite dimensional control functions u, v in feedback form, such that for all (y_0,ϑ_0) sufficiently small the solution (w,θ) of (89) (or of (90)) satisfies $(w(t),\theta(t)) \to (w^S,\theta^S)$ as $t\to\infty$, see Theorem 23 below. Note that our proof relies on a uniqueness theorem for a stationary system which is simpler to handle (i.e we prove (UC) instead of (8)). Moreover, we will see that assuming Ω to be simply-connected is essential for the stabilizability of the linear system obtained from (89) (or (90)) by linearizing around (w^S,θ^S), see Remark 24 below.

Our strategy consists in first rewriting (89) in the abstract form (26) with A, B, satisfying $(\mathcal{H}_1')$, $(\mathcal{H}_3)$, $(\mathcal{H}_4')$; second proving Fattorini criterion (UC) by using Corollary 21 with the fact that Ω is simply-connected; third applying Theorem 10. Note that is the strategy sketched in [6] for the Navier-Stokes system and for which (UC) is obtained from the uniqueness result for the Oseen system recalled in Theorem 18. For sake of clarity, we only detail the calculations in the three dimensional case. Adaptation to the 2D case is straightforward and is left to the reader.

Step 1. Abstract reformulation. With the following formulas of vectorial analysis:

$$(\operatorname{curl} a)\times a = (a\cdot\nabla a) - \nabla(|a|^2/2) \quad \text{and} \quad \operatorname{curl}(a\times b) = (b\cdot\nabla)a - (a\cdot\nabla)b + (\operatorname{div} b)a - (\operatorname{div} a)b$$

we first deduce that $(y,\vartheta,p) = (w-w^S, \theta-\theta^S, r-r^S + (|\theta|^2-|\theta^S|^2)/2)$ obeys

$$\begin{cases} y_t - \Delta y + (w^S\cdot\nabla)y + (y\cdot\nabla)w^S - (\theta^S\cdot\nabla\vartheta) - (\vartheta\cdot\nabla\theta^S) + \nabla p = (\vartheta\cdot\nabla\vartheta) - (y\cdot\nabla y) + \mathbf{1}_\omega u^1 & \text{in } Q,\\ \vartheta_t + \operatorname{curl}(\operatorname{curl}\vartheta) - \operatorname{curl}(y\times\theta^S) - \operatorname{curl}(w^S\times\vartheta) = (\vartheta\cdot\nabla)y - (y\cdot\nabla)\vartheta + \mathbf{1}_\omega P_\omega\, u^2 & \text{in } Q,\\ \operatorname{div} y = \operatorname{div}\vartheta = 0 & \text{in } Q,\\ y(0) = y_0, \quad \vartheta(0) = \vartheta_0 & \text{in } \Omega, \end{cases} \tag{91}$$

with boundary conditions

$$y = 0, \quad \vartheta\cdot n = 0, \quad (\operatorname{curl}\vartheta - w^S\times\vartheta)\times n = 0 \quad \text{on } \Sigma.$$

Note that with $(w^S\times\vartheta)\times n = (w^S\cdot n)\vartheta - (\vartheta\cdot n)w^S$ and $\vartheta\cdot n = 0$, the above boundary conditions become

$$y = 0, \quad \vartheta\cdot n = 0, \quad (\operatorname{curl}\vartheta)\times n - (w^S\cdot n)\vartheta = 0 \quad \text{on } \Sigma. \tag{92}$$

Next, we introduce the (real) spaces

$$G \overset{\text{def}}{=} \mathbf{V}_n^0(\Omega)\times\mathbf{V}_n^0(\Omega) \quad \text{where} \quad \mathbf{V}_n^0(\Omega) \overset{\text{def}}{=} \{y\in(L^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} y = 0 \text{ in } \Omega,\ y\cdot n = 0 \text{ on } \partial\Omega\} \tag{93}$$

we denote by P the Helmholtz projection operator related to Ω (i.e the orthogonal projection operator from $(L^2(\Omega;\mathbb{R}))^3$ onto $\mathbf{V}_n^0(\Omega)$) and we define the following linear operator $A : \mathcal{D}(A)\subset G\to G$ by

$$\begin{aligned} A\begin{bmatrix} y\\ \vartheta\end{bmatrix} &= \begin{bmatrix} P(\Delta y - (w^S\cdot\nabla)y - (y\cdot\nabla)w^S + (\theta^S\cdot\nabla)\vartheta + (\vartheta\cdot\nabla)\theta^S)\\ -\operatorname{curl}(\operatorname{curl}\vartheta) + \operatorname{curl}(w^S\times\vartheta) + \operatorname{curl}(y\times\theta^S)\end{bmatrix}\\ \mathcal{D}(A) &= \big\{(y,\vartheta)\in(H^2(\Omega;\mathbb{R}))^3\times(H^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} y = \operatorname{div}\vartheta = 0 \text{ in } \Omega,\\ &\qquad y = 0,\ \ \vartheta\cdot n = 0 \ \text{ and } \ (\operatorname{curl}\vartheta\times n - (w^S\cdot n)\vartheta) = 0 \text{ on } \partial\Omega\big\}. \end{aligned}$$

Note that P does not appear in the second component of $A\,{}^t[y,\vartheta]$ because $\operatorname{curl}(\operatorname{curl}\vartheta - w^S\times\vartheta - y\times\theta^S)$ belongs to $\mathbf{V}_n^0(\Omega)$. In particular, the fact that its normal component is zero on the boundary comes from $y = (\operatorname{curl}\vartheta - w^S\times\vartheta)\times n = 0$ on $\partial\Omega$ with the following calculations for all $q\in H^1(\Omega)$:

$$\begin{aligned} \int_{\partial\Omega}\operatorname{curl}(\operatorname{curl}\vartheta - w^S\times\vartheta - y\times\theta^S)\cdot n q\,\mathrm{d}\sigma &= \int_\Omega \operatorname{curl}(\operatorname{curl}\vartheta - w^S\times\vartheta - y\times\theta^S)\cdot\nabla q\,\mathrm{dx}\\ &= \int_{\partial\Omega}(\operatorname{curl}\vartheta - w^S\times\vartheta - y\times\theta^S)\times n\cdot\nabla q\,\mathrm{d}\sigma = 0. \end{aligned}$$

Since Ω and (w^S, θ^S) are regular enough, with analogous arguments as in [6, 3] we verify that A generates an analytic semigroup and that it has compact resolvent. It implies that $(\mathcal{H}_3)$ and $(\mathcal{H}'_1)$ are satisfied. Moreover, we can verify that the adjoint of A is given by:

$$\begin{aligned} A^* \begin{bmatrix} z \\ \rho \end{bmatrix} &= \begin{bmatrix} P(\Delta z + (D^s z) w^S - (D^a \rho)\theta^S) \\ P(\Delta \rho - (D^s z)\theta^S + (D^a \rho) w^S) \end{bmatrix} \\ \mathcal{D}(A^*) &= \big\{ (z,\rho) \in (H^2(\Omega;\mathbb{R}))^3 \times (H^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} z = \operatorname{div} \rho = 0 \ \text{ in } \Omega, \\ &\qquad\qquad z = 0 \text{ on } \partial\Omega, \rho \cdot n = (\operatorname{curl} \rho) \times n = 0 \text{ on } \partial\Omega \big\}. \end{aligned} \tag{94}$$

Moreover, if we define $B : W \to G$ as follows

$$W \stackrel{\text{def}}{=} (L^2(\omega;\mathbb{R}))^3 \times (L^2(\omega;\mathbb{R}))^3, \quad B \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} P\mathbf{1}_\omega u^1 \\ \mathbf{1}_\omega P_\omega \, u^2 \end{bmatrix}, \tag{95}$$

then $(\mathcal{H}'_4)$ (for $\gamma = 0$) is satisfied. Finally, (91)-(92) can be rewritten in the form (26) with:

$$N\left(\begin{bmatrix} y \\ \vartheta \end{bmatrix}, \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \right) \stackrel{\text{def}}{=} \begin{bmatrix} P((\vartheta \cdot \nabla \vartheta) - (y \cdot \nabla y)) \\ (\vartheta \cdot \nabla) y - (y \cdot \nabla)\vartheta \end{bmatrix}. \tag{96}$$

Step 2. Verification of Fattorini criterion. Since A and B satisfy $(\mathcal{H}'_1)$, $(\mathcal{H}_3)$ and $(\mathcal{H}'_4)$, Corollary 13 applies and stabilizability of $(A+\sigma, B)$ for all $\sigma > 0$ is reduced to (UC), which is to say that for all $\lambda \in \mathbb{C}$, every (z, π, ρ, κ) that satisfies:

$$\begin{cases} \lambda z - \Delta z - (D^s z) w^S + (D^a \rho)\theta^S + \nabla \pi = 0 & \text{in } \Omega, \\ \lambda \rho - \Delta \rho + (D^s z)\theta^S - (D^a \rho) w^S + \nabla \kappa = 0 & \text{in } \Omega, \\ \operatorname{div} z = \operatorname{div} \rho = 0 & \text{in } \Omega, \end{cases} \tag{97}$$

with boundary data

$$z = 0, \quad \rho \cdot n = 0, \quad (\operatorname{curl} \rho) \times n = 0 \quad \text{on } \partial\Omega, \tag{98}$$

and such that

$$z = 0, \quad P_\omega \rho = 0 \quad \text{in } \omega, \tag{99}$$

must be identically equal to zero in Ω. Note that (99) is equivalent to

$$z = 0, \quad \rho = \nabla p \quad \text{in } \omega \quad \text{ for some } p \in (H^1_{\text{loc}}(\omega;\mathbb{R}))^3. \tag{100}$$

Then (100) implies $z = \operatorname{curl} \rho = 0$ in ω and from Corollary 21, the fact that Ω is simply-connected guarantees that the above uniqueness result is true.

Step 3. Stabilizing control for linear and nonlinear systems. From Corollary 13, there are families

$$((v_j^1, v_j^2))_{j=1,\dots,K} \in (B^* \mathcal{F}_\sigma)^K \subset W^K \quad \text{ and } \quad ((\widehat{z}_j, \widehat{\rho}_j))_{j=1,\dots,K} \in G^K \tag{101}$$

and a finite rank feedback operator $F : G \to W$ defined by

$$F \begin{bmatrix} \xi \\ \zeta \end{bmatrix} = \sum_{j=1}^K \begin{bmatrix} v_j^1 \\ v_j^2 \end{bmatrix} \int_\Omega (\xi \cdot \widehat{z}_j + \zeta \cdot \widehat{\rho}_j) \, \mathrm{dx}, \tag{102}$$

such that $A + BF$ with domain $\mathcal{D}(A + BF) = \mathcal{D}(A)$ (since B is bounded) is the infinitesimal generator af an exponentially stable semigroup on G. Moreover, as it happens for the Navier-Stokes system treated in [6, 2], Sobolev embeddings guarantee that nonlinearity (96) satisfies (28) only for $s \in [\frac{d-2}{2}, 1]$. This yield the following stabilization theorem for system (89) with feedback control

$$\begin{aligned} u^1(t) &= \sum_{j=1}^K v_j^1 \int_\Omega \big((w(t) - w^S) \cdot \widehat{z}_j + (\theta(t) - \theta^S) \cdot \widehat{\rho}_j \big) \, \mathrm{dx}, \\ u^2(t) &= \sum_{j=1}^K v_j^2 \int_\Omega \big((w(t) - w^S) \cdot \widehat{z}_j + (\theta(t) - \theta^S) \cdot \widehat{\rho}_j \big) \, \mathrm{dx}. \end{aligned} \tag{103}$$

Theorem 23. *Let $s \in [\frac{d-2}{2}, 1]$, let Ω be a bounded and simply-connected open subset of $\mathbb{R}^d$ of class $C^{2,1}$, let $(w^S, \theta^S) \in (H^2(\Omega;\mathbb{R}))^d \times (H^2(\Omega;\mathbb{R}))^d$ satisfy (87) and let $(y_0, \vartheta_0) \in (H_0^s(\Omega;\mathbb{R}))^d \times (H_0^s(\Omega;\mathbb{R}))^d$ satisfy (88).*

For all $\sigma > 0$ there exists $\mu > 0$ such that if

$$\|y_0\|_{(H^s(\Omega;\mathbb{R}))^d} + \|\vartheta_0\|_{(H^s(\Omega;\mathbb{R}))^d} \leq \mu,$$

then system (89)-(103) if $d = 3$ or system (90)-(103) if $d = 2$ admits a solution (w, r, θ) in

$$\{(w^S, r^S, \theta^S)\} + W_\sigma((H^{s+1}(\Omega;\mathbb{R}))^d, (H^{s-1}(\Omega;\mathbb{R}))^d) \times H^{\frac{s-1}{2}}(L^2(\Omega;\mathbb{R})/\mathbb{R}) \times W_\sigma((H^{s+1}(\Omega;\mathbb{R}))^d, (H^{s-1}(\Omega;\mathbb{R}))^d)$$

which is unique within the class of function in

$$\{(w^S, r^S, \theta^S)\} + W_{\mathrm{loc}}((H^{s+1}(\Omega;\mathbb{R}))^d, (H^{s-1}(\Omega;\mathbb{R}))^d) \times H^{\frac{s-1}{2}}_{\mathrm{loc}}(L^2(\Omega;\mathbb{R})/\mathbb{R}) \times W_{\mathrm{loc}}((H^{s+1}(\Omega;\mathbb{R}))^d, (H^{s-1}(\Omega;\mathbb{R}))^d).$$

Moreover, there exists $C > 0$ such that for all $t \geq 0$ the following estimate holds:

$$\|w(t) - w^S\|_{(H^s(\Omega;\mathbb{R}))^d} + \|\theta(t) - \theta^S\|_{(H^s(\Omega;\mathbb{R}))^d} \leq Ce^{-\sigma t}(\|w_0 - w^S\|_{(H^s(\Omega;\mathbb{R}))^d} + \|\theta_0 - \theta^S\|_{(H^s(\Omega;\mathbb{R}))^d}).$$

Remark 24. *According to Corollary 21 the uniqueness result stated in step 2 is true if and only if Ω is simply-connected. When Ω is multiply-connected and $\lambda = 0$, for each nonzero $\widetilde{\rho} \in X_N$ (defined by (82)) we have that $(z, \pi, \rho, \kappa) \stackrel{def}{=} (0, 0, \rho_N, 0)$ satisfies (97)-(98)-(99). Then there is a N-dimensional subspace of eigenvectors related to the zero eigenvalue which is uncontrollable and the linear system obtained from (89) by linearizing around (w^S, θ^S) is not stabilizable. Of course, this does not imply that the full nonlinear system is not stabilizable (see [8]).*

5.6 Stabilizability of micropolar system

The methodology described in the last section can also be used to stabilize the following 3D micropolar system:

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla)w - \operatorname{curl}\theta + \nabla r = f^S & \text{in } Q, \\ \theta_t - \Delta\theta + (w \cdot \nabla)\theta - \operatorname{curl} w - \nabla(\operatorname{div}\theta) = g^S & \text{in } Q, \\ \operatorname{div} w = 0 & \text{in } Q, \\ w(0) = w^S + y_0, \quad \theta(0) = \theta^S + \vartheta_0 & \text{in } \Omega. \end{cases} \tag{104}$$

About micropolar systems and related control problems see for instance [27, 13] and references therein. Here Ω is a bounded open subset of $\mathbb{R}^3$ of class $C^{2,1}$, $(w^S, \theta^S) \in (H^2(\Omega;\mathbb{R}))^3 \times (H^2(\Omega;\mathbb{R}))^3$ verifies

$$\begin{cases} -\Delta w^S + (w^S \cdot \nabla)w^S - \operatorname{curl}\theta^S + \nabla r^S = f^S & \text{in } \Omega, \\ -\Delta\theta^S + (w^S \cdot \nabla)\theta^S - \operatorname{curl} w^S - \nabla(\operatorname{div}\theta^S) = g^S & \text{in } \Omega, \\ \operatorname{div} w^S = 0 & \text{in } \Omega, \end{cases} \tag{105}$$

and $(y_0, \vartheta_0) \in (L^2(\Omega;\mathbb{R}))^3 \times (L^2(\Omega;\mathbb{R}))^3$ satisfies

$$\operatorname{div} y_0 = 0 \ \text{ in } \Omega \quad \text{ and } \quad y_0 \cdot n = 0 \ \text{ on } \partial\Omega. \tag{106}$$

We want to find a control function $u = (u^1, u^2)$ such that for all (y_0, ϑ_0) sufficiently small the solution (w, θ) of (104) with nonhomegeneous boundary data of the form

$$w = w^S + mu^1 \quad \text{ and } \quad \theta = \theta^S + mu^2 \quad \text{ on } \Sigma \tag{107}$$

are such that $(w(t), \theta(t)) \to (w^S, \theta^S)$ as $t \to \infty$. Here, $m \in C^2(\partial\Omega;\mathbb{R})$ is a non zero cut-off function that allows to localize the action of the control (u^1, u^2) on a subset of $\partial\Omega$ and for all $t \geq 0$:

$$u^1(t) \in \mathbf{V}_m^0(\partial\Omega) \stackrel{\text{def}}{=} \left\{ v \in (L^2(\partial\Omega;\mathbb{R}))^3 \mid \int_{\partial\Omega} mv \cdot n \mathrm{d}\sigma = 0 \right\}, \quad u^2(t) \in (L^2(\partial\Omega;\mathbb{R}))^3. \tag{108}$$

For that, we follow the same strategy as in Subsection 5.5: first we rewrite (104) in the abstract form (26) for A, B, satisfying $(\mathcal{H}_1')$, $(\mathcal{H}_3)$, $(\mathcal{H}_4')$; second we prove Fattorini criterion (UC) by using Theorem 22; third we apply Theorem 11.

Step 1. Abstract reformulation. We first werify that $(y, \vartheta, p) = (w - w^S, \theta - \theta^S, r - r^S)$ obeys

$$\begin{cases} y_t - \Delta y + (w^S \cdot \nabla) y + (y \cdot \nabla) w^S - \operatorname{curl} \vartheta + \nabla p = -(y \cdot \nabla) y & \text{in } Q, \\ \vartheta_t - \Delta \vartheta + (w^S \cdot \nabla)\vartheta + (y \cdot \nabla)\theta^S - \operatorname{curl} y - \nabla(\operatorname{div} \vartheta) = -(y \cdot \nabla)\vartheta & \text{in } Q, \\ \operatorname{div} y = 0 & \text{in } Q, \\ y(0) = y_0, \quad \vartheta(0) = \vartheta_0 & \text{in } \Omega, \end{cases} \tag{109}$$

with boundary conditions

$$y = mu^1 \quad \text{and} \quad \vartheta = mu^2 \text{ on } \Sigma.$$

Next, we introduce the (real) spaces

$$G \stackrel{\text{def}}{=} \mathbf{V}_n^0(\Omega) \times (L^2(\Omega;\mathbb{R}))^3 \quad \text{where} \quad \mathbf{V}_n^0(\Omega) \stackrel{\text{def}}{=} \{y \in (L^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\} \tag{110}$$

and we define the following linear operator $A : \mathcal{D}(A) \subset G \to G$ by

$$\begin{aligned} A \begin{bmatrix} y \\ \vartheta \end{bmatrix} &= \begin{bmatrix} P(\Delta y - (w^S \cdot \nabla) y - (y \cdot \nabla) w^S + \operatorname{curl} \vartheta) \\ \Delta \vartheta - (w^S \cdot \nabla)\vartheta - (y \cdot \nabla)\theta^S + \operatorname{curl} y + \nabla(\operatorname{div} \vartheta) \end{bmatrix} \\ \mathcal{D}(A) &= \big\{(y, \vartheta) \in (H^2(\Omega;\mathbb{R}))^3 \times (H^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} y = \operatorname{div} \vartheta = 0 \text{ in } \Omega,\ y = \vartheta = 0 \text{ on } \partial\Omega\big\}. \end{aligned}$$

Recall that P denotes the orthogonal projection operator from $(L^2(\Omega;\mathbb{R}))^3$ onto $\mathbf{V}_n^0(\Omega)$. Since Ω and (w^S, θ^S) are regular enough, with analogous arguments as in [6, 3] we verify that A generates an analytic semigroup and that it has compact resolvent. It implies that $(\mathcal{H}_3)$ and $(\mathcal{H}_1')$ are satisfied. Moreover, we can verify that the adjoint of A is given by:

$$\begin{aligned} A^* \begin{bmatrix} z \\ \rho \end{bmatrix} &= \begin{bmatrix} P(\Delta z + (D^s z) w^S + {}^t(\nabla \rho)\theta^S + \operatorname{curl} \rho) \\ \Delta \rho - (D^s z)\theta^S + (\nabla \rho) w^S + \operatorname{curl} z + \nabla(\operatorname{div} \rho) \end{bmatrix} \\ \mathcal{D}(A^*) &= \big\{(z, \rho) \in (H^2(\Omega;\mathbb{R}))^3 \times (H^2(\Omega;\mathbb{R}))^3 \mid \operatorname{div} z = \operatorname{div} \rho = 0 \text{ in } \Omega, z = \rho = 0 \text{ on } \partial\Omega\big\}. \end{aligned} \tag{111}$$

By multiplying the two first equations of (109) by $(z, \rho) \in \mathcal{D}(A^*)$ and integrating by parts we get:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \left(\begin{bmatrix} Py \\ \vartheta \end{bmatrix}, \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G &= \left(\begin{bmatrix} y \\ \vartheta \end{bmatrix}, \begin{bmatrix} \Delta z + (D^s z) w^S + {}^t(\nabla \rho)\theta^S + \operatorname{curl} \rho \\ \Delta \rho - (D^s z)\theta^S + (\nabla \rho) w^S + \operatorname{curl} z + \nabla(\operatorname{div} \rho) \end{bmatrix} \right)_G - \left(\begin{bmatrix} (y \cdot \nabla) y \\ (y \cdot \nabla)\vartheta \end{bmatrix}, \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G \\ &\quad - \int_{\partial\Omega} (mu^2) \left(\frac{\partial \rho}{\partial n} + (\operatorname{div} \rho) n \right) \mathrm{d}\sigma - \int_{\partial\Omega} (mu^1) \frac{\partial z}{\partial n} \, \mathrm{d}\sigma. \end{aligned} \tag{112}$$

Thus, to rewrite (112) in the abstract form (26) let us first write

$$\begin{bmatrix} \Delta z + (D^s z) w^S + {}^t(\nabla \rho)\theta^S + \operatorname{curl} \rho \\ \Delta \rho - (D^s z)\theta^S + (\nabla \rho) w^S + \operatorname{curl} z + \nabla(\operatorname{div} \rho) \end{bmatrix} = A^* \begin{bmatrix} z \\ \rho \end{bmatrix} + \begin{bmatrix} \nabla \pi \\ 0 \end{bmatrix}$$

where the pressure function π is uniquely determined by

$$\nabla \pi = (I - P)(\Delta z + (D^s z) w^S + {}^t(\nabla \rho)\theta^S + \operatorname{curl} \rho) \quad \int_{\partial\Omega} m^2 \pi \mathrm{d}\sigma = 0. \tag{113}$$

Then an integration by parts gives:

$$\left(\begin{bmatrix} y \\ \vartheta \end{bmatrix}, \begin{bmatrix} \Delta z + (D^s z) w^S + {}^t(\nabla \rho)\theta^S + \operatorname{curl} \rho \\ \Delta \rho - (D^s z)\theta^S + (\nabla \rho) w^S + \operatorname{curl} z + \nabla(\operatorname{div} \rho) \end{bmatrix} \right)_G = \left(\begin{bmatrix} Py \\ \vartheta \end{bmatrix}, A^* \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G + \int_{\partial\Omega} m(u^1 \cdot n)\pi \, \mathrm{d}\sigma, \tag{114}$$

and (112) becomes:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \left(\begin{bmatrix} Py \\ P\vartheta \end{bmatrix}, \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G &= \left(\begin{bmatrix} Py \\ \vartheta \end{bmatrix}, A^* \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G - \int_{\partial\Omega} m \left(u^2 \cdot \left(\frac{\partial \rho}{\partial n} + (\operatorname{div} \rho) n \right) + u^1 \cdot \left(\frac{\partial z}{\partial n} - \pi n \right) \right) \mathrm{d}\sigma \\ &\quad - \left(\begin{bmatrix} (y \cdot \nabla) y \\ (y \cdot \nabla)\vartheta \end{bmatrix}, \begin{bmatrix} z \\ \rho \end{bmatrix} \right)_G, \end{aligned} \tag{115}$$

which suggests to define the control operator $B : W \to [\mathcal{D}(A^*)]'$ as follows

$$W \stackrel{\text{def}}{=} \mathbf{V}_m^0(\partial\Omega) \times (L^2(\partial\Omega;\mathbb{R}))^3, \quad B^* \begin{bmatrix} z \\ \rho \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} m\left(\pi n - \frac{\partial z}{\partial n}\right) \\ m\left(-(\operatorname{div}\rho)n - \frac{\partial \rho}{\partial n}\right) \end{bmatrix}. \tag{116}$$

Boundedness properties of trace operators ensure that B satisfy $(\mathcal{H}_4')$ with $\gamma \in (\frac{3}{4}, 1)$. Note that normalization conditions in (113) guarantee that the range of B^* is included in W.

Next, define the lifting mapping $Dv = \xi$ where ξ is solution of

$$\begin{cases} -\Delta\xi + \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div}\xi = 0 & \text{in } \Omega, \\ \xi = mv & \text{on } \partial\Omega. \end{cases} \tag{117}$$

Then from $y - Du^1 \in \mathbf{V}_n^0(\Omega)$ we get $(I-P)y = (I-P)Du^1$ and

$$\begin{bmatrix} y \\ \vartheta \end{bmatrix} = \begin{bmatrix} Py \\ \vartheta \end{bmatrix} + \begin{bmatrix} (I-P)Du^1 \\ 0 \end{bmatrix},$$

and the nonlinearity in (115) can be rewritten in terms of Py, ϑ, u as follows:

$$N\left(\begin{bmatrix} Py \\ \vartheta \end{bmatrix}, \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}\right) \stackrel{\text{def}}{=} -\begin{bmatrix} ((Py + (I-P)Du^1)\cdot\nabla)(Py + (I-P)Du^1) \\ ((Py + (I-P)Du^1)\cdot\nabla)\vartheta \end{bmatrix}. \tag{118}$$

Finally, (115) is reduced to the following abstract formulation of type (26):

$$\begin{bmatrix} Py \\ \vartheta \end{bmatrix}' = A\begin{bmatrix} Py \\ \vartheta \end{bmatrix} + B\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} + N\left(\begin{bmatrix} Py \\ \vartheta \end{bmatrix}, \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}\right) \quad \text{in } [\mathcal{D}(A^*)]'. \tag{119}$$

Step 2. Verification of Fattorini criterion. Since A and B satisfy $(\mathcal{H}_1')$, $(\mathcal{H}_3)$ and $(\mathcal{H}_4')$, Corollary 13 applies and stabilizability of $(A+\sigma, B)$ for all $\sigma > 0$ is reduced to (UC), which is to say (see (111) and (116)) that for all $\lambda \in \mathbb{C}$, every (z, π, ρ, κ) satisfying (84) with boundary conditions

$$z = \rho = 0 \quad \text{on } \partial\Omega$$

and overspecified conditions

$$\frac{\partial z}{\partial n} - \pi n = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial n} - \kappa n = 0 \quad \text{on } \Gamma \stackrel{\text{def}}{=} \operatorname{Supp}(m), \tag{120}$$

must be identically equal to zero in Ω.

With a classical extension of the domain procedure, such above uniqueness theorem can be obtained from Theorem 22 applied in an extended domain. Let us briefly recall the argument. From $z = 0$ and $\operatorname{div} z = 0$ on Γ we get that $\frac{\partial z}{\partial n}$ is tangential on Γ. Then $\frac{\partial z}{\partial n} - \pi n = 0$ yields $\pi = 0$, $\frac{\partial z}{\partial n} = 0$ and finally $\nabla z = 0$ on Γ. Moreover, $\rho = 0$ on Γ implies $\frac{\partial \rho}{\partial n} \cdot n = \operatorname{div}\rho = -\kappa$ on Γ. Since from the second equality in (120) we also have $\frac{\partial \rho}{\partial n} \cdot n = \kappa$ we deduce that $\kappa = 0$ and then $\frac{\partial \rho}{\partial n} = 0$ on Γ, and since $\rho = 0$ on Γ we get $\nabla\rho = 0$ on Γ. Finally, we have proved that ∇z, $\nabla\rho$, π, κ vanish on Γ and we can smoothly extend (z, π, ρ, κ) by zero in a larger domain containing an outside open set ω. Then it suffices to invoke Theorem 22 for this larger domain.

Step 3. Stabilizing control for linear and nonlinear systems. From Corollary 13 we get the existence of families of type (101) (where W and G are given by (110), (116)) and a finite rank feedback operator $F : G \to W$ of the form (102) such that $A + BF$ with domain $\mathcal{D}(A+BF)$ is the infinitesimal generator of an exponentially stable semigroup on G. However, because B is now unbounded, $\mathcal{D}(A+BF)$ is not equal to $\mathcal{D}(A)$ as it was the case in Subsection 5.5. More precisely, by following the method considered in [6] it can be proved that $\mathcal{D}(A+BF)$ is composed of $(P\xi, \zeta)$ where $(\xi, \zeta) \in (H^2(\Omega))^3 \times (H^2(\Omega))^3$ obey

$$\begin{bmatrix} \xi \\ \zeta \end{bmatrix} = m\sum_{j=1}^{K} \begin{bmatrix} v_j^1 \\ v_j^2 \end{bmatrix} \int_\Omega (\xi\cdot\widehat{z}_j + \zeta\cdot\widehat{\rho}_j)\,\mathrm{dx} \quad \text{on } \partial\Omega, \tag{121}$$

and that for each $s \in (0,1)$ the space $\mathcal{D}((-A_F)^{\frac{s}{2}})$ is a closed subspace of $(H^s(\Omega;\mathbb{R}))^3 \times (H^s(\Omega;\mathbb{R}))^3$ composed with elements satisfying trace conditions on $\partial\Omega$ related to the feedback law F when $s \geq 1/2$. Moreover, as it happens for the Navier-Stokes system treated in [6, 2], Sobolev embeddings guarantee that the nonlinearity defined in (118) satisfies (28) only for $s \in [\frac{d-2}{2}, 1]$ where d is the space dimension. Then feedback stabilization of 3D micropolar system (109) cannot be obtained unless very specific initial trace compatibility conditions related to F are satisfied. However, since when $d = 2$ inequality (28) is satisfied for $s \in [0, 1/2)$, a relevant stabilization theorem could be obtained from Theorem 10 for the two dimensional version of system (104)-(107) with feedback control (103).

To obtain a 3D stabilization result we have to consider a dynamical control. From (UC) we have the existence of linear mappings $F_1 : \mathbb{R}^K \to \mathbb{R}^K$, $F_i : (L^2(\Omega;\mathbb{R}))^3 \to \mathbb{R}^K)$, $i = 2, 3$ such that dynamical controls:

$$\begin{aligned} \begin{bmatrix} u^1(t) \\ u^2(t) \end{bmatrix} &= \sum_{j=1}^{K} u_j(t) \begin{bmatrix} v_j^1 \\ v_j^2 \end{bmatrix}, \quad \overline{u} \stackrel{\text{def}}{=} (u_1, \ldots, u_K) \\ \overline{u}' &= F_1\overline{u} + F_2(w - w^S) + F_3(\theta - \theta^S) \\ \overline{u}(0) &= 0 \end{aligned} \tag{122}$$

stabilize the linear system obtained from (119) by linearizing around zero. The following stabilization theorem can be deduced from from Theorem 11 with $s = 1$.

Theorem 25. *Let Ω be a bounded and open subset of $\mathbb{R}^d$ of class $C^{2,1}$, let $(w^S, \theta^S) \in (H^2(\Omega;\mathbb{R}))^d \times (H^2(\Omega;\mathbb{R}))^d$ satisfy* (105) *and let $(y_0, \vartheta_0) \in (H_0^1(\Omega;\mathbb{R}))^d \times (H_0^1(\Omega;\mathbb{R}))^d$ satisfy* (106)*. For all $\sigma > 0$ there exists $\mu > 0$ such that if*

$$\|y_0\|_{(H^1(\Omega;\mathbb{R}))^3} + \|\vartheta_0\|_{(H^1(\Omega;\mathbb{R}))^3} \leq \mu,$$

then system (104)-(107)-(122) *admits a solution $(w, r, \theta, \overline{u})$ in*

$$\begin{aligned} \{(w^S, r^S, \theta^S, 0)\} + W_\sigma((H^2(\Omega;\mathbb{R}))^3, (L^2(\Omega;\mathbb{R}))^3) \times L^2(H^1(\Omega;\mathbb{R})/\mathbb{R}) \\ \times W_\sigma((H^2(\Omega;\mathbb{R}))^3, (L^2(\Omega;\mathbb{R}))^3) \times H^1(\mathbb{R}^+;\mathbb{R}^K), \end{aligned}$$

which is unique within the class of function in

$$\begin{aligned} \{(w^S, r^S, \theta^S, 0)\} + W_{\text{loc}}((H^2(\Omega;\mathbb{R}))^3, (L^2(\Omega;\mathbb{R}))^3) \times L^2_{\text{loc}}(H^1(\Omega;\mathbb{R})/\mathbb{R}) \\ \times W_{\text{loc}}((H^2(\Omega;\mathbb{R}))^3, (L^2(\Omega;\mathbb{R}))^3) \times H^1_{\text{loc}}(\mathbb{R}^+;\mathbb{R}^K), \end{aligned}$$

Moreover, there exists $C > 0$ such that for all $t \geq 0$ the following estimate holds:

$$\begin{aligned} \|w(t) - w^S\|_{(H^1(\Omega;\mathbb{R}))^3} + \|\theta(t) - \theta^S\|_{(H^1(\Omega;\mathbb{R}))^3} + \|\overline{u}(t)\|_{\mathbb{R}^K} \\ \leq Ce^{-\sigma t}(\|w_0 - w^S\|_{(H^1(\Omega;\mathbb{R}))^3} + \|\theta_0 - \theta^S\|_{(H^1(\Omega;\mathbb{R}))^3}). \end{aligned}$$

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