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# SPECTRUM OF HYPERSURFACES WITH SMALL EXTRINSIC RADIUS OR LARGE $\lambda_1$ IN EUCLIDEAN SPACES

ERWANN AUBRY, JEAN-FRANÇOIS GROSJEAN

ABSTRACT. The Reilly and Hasanis-Koutroufiotis inequalities give sharp bounds on  $\lambda_1$  and on the extrinsic radius of Euclidean hypersurfaces in term of the  $L^2$  norm of their mean curvature. The equality case of these inequalities characterizes the Euclidean spheres. In this paper, we study the spectral properties of the almost extremal hypersurfaces. We prove that the spectrum of the limit sphere asymptotically appears in the spectrum of almost extremal hypersurfaces for these inequalities. We also construct some examples of extremizing sequences that prove that the limit spectrum can be essentially any closed subset of  $\mathbb{R}^+$  that contains the spectrum of the limit sphere. We also provide natural sharp condition to recover exactly the spectrum of the unit sphere.

## 1. INTRODUCTION

Throughout the paper,  $X: M^n \rightarrow \mathbb{R}^{n+1}$  is a closed, connected, immersed Euclidean hypersurface (with  $n \geq 2$ ). We let  $v_M$  be its volume,  $B_M$  its second fundamental form,  $H_M = \frac{1}{n} \operatorname{tr} B_M$  its mean curvature,  $r_M$  its extrinsic radius (i.e. the least radius of the Euclidean balls containing  $M$ ),  $0 = \lambda_0^M < \lambda_1^M \leq \lambda_2^M \leq \dots$  the non-decreasing sequence of its eigenvalues labelled with multiplicities,  $Sp(M) = (\lambda_i^M)_{i \in \mathbb{N}}$  and its center of mass. For any function  $f: M \rightarrow \mathbb{R}$ , we set  $\|f\|_\alpha = \left( \frac{1}{v_M} \int_M |f|^\alpha dv \right)^{\frac{1}{\alpha}}$ .

The Hasanis-Koutroufiotis inequality ([8], see also section 3 of this paper) and the Reilly inequality ([12], see also section 3 of this paper) assert respectively that

$$(1.1) \quad \{1 \leq r_M \|H_M\|_2\} \text{ and } \{\lambda_1^M \leq n \|H_M\|_2^2\},$$

with equality in one of these inequalities if and only if  $M$  is a Euclidean sphere (which is then uniquely determined).

Our aim is to study the spectral properties of the hypersurfaces that are almost extremal for at least one of the inequalities (1.1). In the sequel, for any immersed hypersurface  $M \hookrightarrow \mathbb{R}^{n+1}$ , we let  $S_M$  be the sphere of radius  $\frac{1}{\|H\|_2}$  and center  $\bar{X} := \frac{1}{v_M} \int_M X dv$ . It follows from the above-mentioned results of Hasanis-Koutroufiotis and Reilly that equality holds in one of the inequalities in (1.1) if and only if  $M = S_M$ .

For any  $k \geq 0$ , we let  $\mu_k^{S_M} := k(n+k-1) \|H\|_2^2$  be the  $k$ -th eigenvalue of  $S_M$  (labelled without multiplicities) and  $m_k$  be its multiplicity.

Our first result is the following

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**Theorem 1.1.** *We fix  $n \geq 2$  and  $\tau > 0$ . Then, there exists  $\varepsilon_0(n, k, \tau) > 0$  depending only on  $n$ ,  $\tau$  and  $k$  such that for any  $\varepsilon < \varepsilon_0(n, k, \tau)$  and any immersed hypersurface  $M \hookrightarrow \mathbb{R}^{n+1}$  satisfying*

$$(1.2) \quad \text{either } \{1 \leq r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon\} \text{ or } \{\lambda_1^M \leq n \|\mathbf{H}\|_2^2 \leq (1 + \varepsilon) \lambda_1^M\}$$

*then the interval  $[(1 - \tau)\mu_k^{S^M}, (1 + \tau)\mu_k^{S^M}]$  contains at least  $m_k$  eigenvalues of  $M$  counted with multiplicities.*

We will see in the proof that  $\varepsilon_0(n, k, \tau)$  tends to 0 when  $k \rightarrow \infty$  or  $\tau \rightarrow 0$ . Note that almost extremal hypersurfaces for the Reilly inequality must have at least  $n + 1$  eigenvalues close to  $\lambda_1^{S^M} = n \|\mathbf{H}\|_2^2$ . However, they can have the topology of any immersed hypersurface of  $\mathbb{R}^{n+1}$  (see below) and can be as close as wanted of any closed, connected subset of  $\mathbb{R}^{n+1}$  that contains  $\mathbb{S}^n$  (see [3]). So almost extremal hypersurfaces for the Reilly inequality are very different from almost extremal manifolds for the Lichnerowicz Inequality in positive Ricci curvature (see for instance [1]).

Now for any sequence  $(M_k)_{k \in \mathbb{N}}$ , let us define

$$\text{LimSet}_{k \rightarrow \infty} \text{Sp}(M_k) := \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{\ell \geq k} \text{Sp}(M_\ell)}.$$

This is the union of the limit-sets of all the sequences  $(\mu_k)_{k \in \mathbb{N}}$  with  $\mu_k \in \text{Sp}(M_k)$  for any  $k \in \mathbb{N}$ . Obviously if  $(\text{Sp}(M_k))_k$  converges to a set  $F$  for the Attouch-Wetts-Hausdorff distance (see section 2 below for the definition), then  $\text{LimSet}_{k \rightarrow \infty} \text{Sp}(M_k) = F$ . As a consequence of Theorem 1.1, we have

**Corollary 1.2.** *Let  $(M_k)_{k \in \mathbb{N}}$  be a sequence of immersed hypersurfaces of  $\mathbb{R}^{n+1}$  normalized by  $\|\mathbf{H}_{M_k}\|_2 = 1$  and such that*

$$(1.3) \quad \text{either } \left\{ \lim_{k \rightarrow \infty} r_{M_k} = 1 \right\} \text{ or } \left\{ \lim_{k \rightarrow \infty} \lambda_1^{M_k} = n \right\}.$$

*Then we have*

$$\text{LimSet}_{k \rightarrow \infty} \text{Sp}(M_k) \supset \text{Sp}(\mathbb{S}^n).$$

*In other words,  $F \supset \text{Sp}(\mathbb{S}^n)$  for any limit-point  $F$  of the sequence  $(\text{Sp}(M_k))_{k \in \mathbb{N}}$  for the Attouch-Wetts-Hausdorff distance.*

Conversely, our result is optimal in the sense that any closed set containing the spectrum of a Euclidean sphere can be achieved as the spectrum of an "almost extremal" manifold. This is the object of our second result:

**Theorem 1.3.** *Let  $M \hookrightarrow \mathbb{R}^{n+1}$  be any immersed hypersurface. Let  $F$  be any closed subset such that  $\text{Sp}(\mathbb{S}^n) \subset F \subset [0, +\infty[$ . Then there exists a sequence  $(i_k)_k$  of immersions  $i_k : M \hookrightarrow \mathbb{R}^{n+1}$  such that, denoting  $M_k := i_k(M)$ , it satisfies*

$$\lim_{k \rightarrow +\infty} r_{M_k} \|\mathbf{H}_{M_k}\|_2 = 1 \text{ and } \lim_{k \rightarrow \infty} \text{Sp}(M_k) = F$$

*for the Attouch-Wetts-Hausdorff distance. If moreover we have  $F \subset \{0\} \cup [n, +\infty[$  then we can obtain  $\lim_{k \rightarrow +\infty} \frac{\lambda_1^{M_k}}{n \|\mathbf{H}_{M_k}\|_2} = 1$ . The sequence of immersions  $i_k : M \hookrightarrow \mathbb{R}^{n+1}$  is such that*

$\|H_{M_k}\|_2 = 1$  and  $\lim_{k \rightarrow \infty} v_{M_k} = v_{\mathbb{S}^n}$ . In addition, we have the following curvature properties

$$(1.4) \quad \lim_{k \rightarrow \infty} \int_{M_k} |B_{M_k}|^\alpha dv = \int_{\mathbb{S}^n} |B_{\mathbb{S}^n}|^\alpha dv$$

for any  $1 \leq \alpha < n$ .

Theorem 1.3 is a special case ( $M_1 = \mathbb{S}^n$  and  $M_2 = M$ ) of the more general Theorem 2.1 of Section 3.

**Remark 1.4.** In the case  $\alpha = n$ , we are only able to get a weak version of Theorem 1.3 with  $F = \text{Sp}(M_1) \cup G$ , where  $G$  is a finite set whose elements are known up to an error term and such that  $\lim \int_{M_k} |B|^n dv$  is bounded above by a constant that depends on  $M$ , on the cardinal of  $G$ , on the distance between  $G$  and  $\text{Sp}(M_1)$ , and on the error term.

We now investigate a natural condition on "almost extremal" manifolds to rule out the formation of a non-spherical spectrum. As proved by the authors in [3], any "almost extremal" hypersurface is arbitrary Hausdorff-close to its spherical model provided an  $L^\alpha$ -control ( $\alpha > n$ ) on the second fundamental form. This result combined with the  $C^{1,\beta}$  pre-compactness theorem of [9] (or a Moser iteration as in [2]) implies the following stability in Lipschitz distance  $d_L$ :

**Theorem 1.5.** We fix  $\alpha \in (n, +\infty]$ ,  $A > 0$  and  $\tau > 0$ . Then there exists  $\varepsilon_0(n, \alpha, A, \tau) > 0$  depending only on  $n, \alpha, A$  and  $\tau$  such that for any  $\varepsilon < \varepsilon_0$  and any immersed hypersurface  $M \hookrightarrow \mathbb{R}^{n+1}$  satisfying

$$\text{either } \{1 \leq r_M \|H\|_2 \leq 1 + \varepsilon\} \text{ or } \{\lambda_1^M \leq n \|H\|_2^2 \leq (1 + \varepsilon) \lambda_1^M\}$$

and

$$v_M \|B\|_\alpha^n \leq A,$$

then  $M$  is diffeomorphic to  $S_M$  and satisfies  $d_L(M, S_M) < \tau$ .

Moreover there exists  $\varepsilon_1(n, k, \alpha, A, \tau) > 0$  depending only on  $n, k, \alpha, A$  and  $\tau$  such that if  $\varepsilon < \varepsilon_1$  then  $|\lambda_k^M - \lambda_k^{S_M}| \leq \tau$ .

Therefore, Theorem 1.3 is optimal in the sense that it is enough to improve slightly (1.4) to get convergence to the spectrum of the sphere.

In the following theorem proved in [2], we construct almost extremal hypersurfaces for the Hasanis-Koutroufiotis inequality, not diffeomorphic to  $\mathbb{S}_M$ , Gromov-Hausdorff close to  $S_M$ , with  $\|H\|_\infty$  bounded, where the limit spectrum is that of  $\mathbb{S}^n$ . But the number of eigenvalues of  $M$  close to each eigenvalue  $\mu_k$  of  $\mathbb{S}^n$  is a multiple of the multiplicity  $m_k$ .

**Theorem 1.6.** For any integers  $l, p$  there exists sequence of embedded hypersurfaces  $(M_j)_j$  of  $\mathbb{R}^{n+1}$  diffeomorphic to  $p$  spheres  $\mathbb{S}^n$  glued by connected sum along  $l$  points, such that  $\|H_j\|_\infty \leq C(n)$ ,  $\|B_j\|_n \leq C(n)$ ,  $r_{M_j} \rightarrow 1$ ,  $\|H_i\|_2 \rightarrow 1$ , and for any  $\sigma \in \mathbb{N}$  we have

$$\lambda_\sigma(M_j) \rightarrow \lambda_{E(\frac{\sigma}{p})}(\mathbb{S}^n).$$

In particular, the  $M_j$  have at least  $pm_k$  eigenvalues close to  $\mu_k$ .

The structure of the paper is as follows: in section 2 we state the theorem 2.1 which is a general construction which gives the theorem 1.3. After a preliminary section 3, where we give short proofs of the Reilly and Hasanis-Koutroufiotis inequalities, we prove in section 4

some concentration properties for the volume, the mean curvature and the position vector  $X$  for almost extremal hypersurfaces. Section 5 is devoted to estimates on the restriction to hypersurfaces of the homogeneous, harmonic polynomials of  $\mathbb{R}^{n+1}$ . These estimates are used in Section 6 to prove Theorem 1.1. We end the paper in section 7 by the proof of the constructions of Theorem 2.1. The results and estimates of this paper are used in [3] to study the metric shape of the almost extremal hypersurfaces.

**Notations:** Note that throughout the paper we adopt the notation that  $C(n, k, p, \dots)$  is function greater than 1 which depends on  $p, q, n, \dots$ . It eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though  $C$  might change from line to line in a calculation it still maintains these basic features. Note that all these constants are computable. For convenience, we will often write  $B = B_M$ ,  $H = H_M$ , and more generally we will drop the index  $M$  in the geometric quantities.

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## 2. MISCELLANEOUS ON THEOREMS 1.1 AND 1.3

We will prove the general construction Theorem 2.1 below. As mentioned in the introduction, Theorem 1.3 is an immediate consequence of Theorem 2.1.

**Theorem 2.1.** *Let  $M_1, M_2 \hookrightarrow \mathbb{R}^{n+1}$  be two immersed compact submanifolds of dimension  $m \geq 3$ ,  $M_1 \# M_2$  be their connected sum and  $F$  be any closed subset of  $(0, +\infty)$  containing  $\text{Sp}(M_1)$ . Then there exists a sequence of immersions  $i_k : M_1 \# M_2 \hookrightarrow \mathbb{R}^{n+1}$  with induced metric  $g_k$  and volume  $v_k$  such that*

- (1)  $i_k(M_1 \# M_2)$  converges to  $M_1$  in Hausdorff topology,
- (2) the curvatures of  $g_k$  satisfy

$$\left\{ \begin{array}{l} \lim_{k \rightarrow \infty} \frac{1}{v_k} \int_{i_k(M_1 \# M_2)} |B|^\alpha dv = \frac{1}{v_{M_1}} \int_{M_1} |B|^\alpha dv \quad \text{for any } 1 \leq \alpha < m \\ \lim_{k \rightarrow \infty} \frac{1}{v_k} \int_{i_k(M_1 \# M_2)} |H|^\alpha dv = \frac{1}{v_{M_1}} \int_{M_1} |H|^\alpha dv \quad \text{for any } 1 \leq \alpha < m \end{array} \right.$$

- (3)  $\lim_{k \rightarrow \infty} \text{Sp}(i_k(M_1 \# M_2)) = F$ , for the Attouch-Wetts-Hausdorff distance,
- (4)  $\lim_{k \rightarrow \infty} v_k = v_{M_1}$ .

**Remark 2.2.** *In the case  $\alpha = m$ , we are only able to get a weak version of Theorem 2.1 with  $F = \text{Sp}(M_1) \cup G$ , where  $G$  is a finite set whose elements are known up to an error term and where the point (2) is replaced by  $\int_{i_k(M_1 \# M_2)} |B|^m dv$  is bounded by a constant that depend on  $M_1, M_2$ , on the cardinal of  $G$ , on the distance between  $G$  and  $\text{Sp}(M_1)$  and on the error term.*

Now we recall the definition of the Attouch-Wetts-Hausdorff distance for the sets of  $\mathbb{R}$ . If  $d_A : \mathbb{R} \rightarrow \mathbb{R}$  denotes the distance function to the subset  $A$ , we have  $d_H(A, B) = \|d_A - d_B\|_\infty$  and so the Hausdorff topology on compact subsets of  $\mathbb{R}$  coincides with the topology of

the uniform convergence on  $\mathbb{R}$  of the associated distance functions. Seemingly, on the set of closed subset of  $\mathbb{R}$  we consider the Attouch-Wetts topology, that is the topology of the uniform convergence on compact subsets of the distance functions. It is a complete, metrizable topology induced by the distance

$$d_{AW}(A, B) = \sum_{N \in \mathbb{N}^*} 2^{-N} \inf(1, \sup_{x \in [0, N]} |d_A(x) - d_B(x)|)$$

We have  $\lim_k d_{AW}(A_k, B) = 0$  if and only if  $\lim_k d_N(A_k, B) = 0$  for any  $N \in \mathbb{N}$  large enough, where

$$d_N(A, B) = \inf\{\varepsilon > 0 \mid A \cap [0, N] \subset B_\varepsilon \text{ et } B \cap [0, N] \subset A_\varepsilon\}$$

and  $A_\varepsilon := \{x \in \mathbb{R} \mid d(x, A) \leq \varepsilon\}$  (see the proof of Proposition 3.1.6 in [5]).

In the proof of theorem of 2.1, we will need of the following construction. If  $F$  is a closed subset of  $\mathbb{R}$ , there exists an increasing sequence of finite sets  $F_N := \{x_1, \dots, x_{k_N}\}$  such that

$$F_N \subset [0, N] \cap F \subset \bigcup_{i=1}^{k_N} \left(x_i - \frac{1}{N}, x_i + \frac{1}{N}\right) = F_{N, 1/N}. \text{ In this case we can easily prove that } \lim_{N \rightarrow \infty} d_{AW}(F_N, F) = 0 \text{ and } F = \text{LimSet}_{N \rightarrow \infty} F_N.$$

### 3. SOME GEOMETRIC OPTIMAL INEQUALITIES

Any function  $F$  on  $\mathbb{R}^{n+1}$  gives rise to a function  $F \circ X$  on  $M$  which, for more convenience, will be also denoted  $F$  subsequently. If  $\Delta$  denotes the Laplace operator of  $(M, g)$ , then we have

$$(3.1) \quad \Delta F = n \text{H} dF(\nu) + \Delta^0 F + \nabla^0 dF(\nu, \nu),$$

where  $\nu$  denotes a local normal vector field of  $M$  in  $\mathbb{R}^{n+1}$ ,  $\nabla^0$  is the Euclidean connection and  $\Delta^0$  is the Laplace operator of  $\mathbb{R}^{n+1}$ . Applied to  $F(x) = \langle x - \bar{X}, x - \bar{X} \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the canonical product on  $\mathbb{R}^{n+1}$ , Formula 3.1 gives the Hsiung formulae,

$$(3.2) \quad \frac{1}{2} \Delta |X - \bar{X}|^2 = n \text{H} \langle \nu, X - \bar{X} \rangle - n, \quad \int_M \text{H} \langle \nu, X - \bar{X} \rangle dv = v_M$$

**3.1. A rough geometrical bound.** The integrated Hsiung formula (3.2) and the Cauchy-Schwarz inequality give the following

$$(3.3) \quad 1 = \int_M \frac{\text{H} \langle \nu, X - \bar{X} \rangle dv}{v_M} \leq \| \text{H} \|_2 \| X - \bar{X} \|_2$$

This inequality  $\| \text{H} \|_2 \| X - \bar{X} \|_2 \geq 1$  is optimal since  $M$  satisfies  $\| \text{H} \|_2 \| X - \bar{X} \|_2 = 1$  if and only if  $M$  is a sphere of radius  $\frac{1}{\| \text{H} \|_2}$  and center  $\bar{X}$ . Indeed, in this case  $X - \bar{X}$  and  $\nu$  are collinear on  $M \setminus \{H = 0\}$ , hence  $|X - \bar{X}|^2$  (and so  $\text{H}$ ) is locally constant on  $M \setminus \{H = 0\}$ . By connectedness and compactness of  $M$ , this implies that  $\text{H}$  is constant and non zero on  $M$ .  $\{H = 0\} = \emptyset$  and that  $X$  is an isometric-cover of  $M$  on the sphere  $S$  of center  $\bar{X}$  and radius  $\| X - \bar{X} \|_2 = \frac{1}{\| \text{H} \|_2}$ , hence an isometry.

**3.2. Hasanis-Koutroufiotis inequality on extrinsic radius.** We set  $R$  the extrinsic Radius of  $M$ , i.e. the least radius of the balls of  $\mathbb{R}^{n+1}$  which contain  $M$ . Then Inequality (3.3) gives  $\| \text{H} \|_2 r_M = \| \text{H} \|_2 \inf_{u \in \mathbb{R}^{n+1}} \| X - u \|_\infty \geq \| \text{H} \|_2 \inf_{u \in \mathbb{R}^{n+1}} \| X - u \|_2 = \| \text{H} \|_2 \| X - \bar{X} \|_2 \geq 1$  and  $r_M = \frac{1}{\| \text{H} \|_2}$  if and only if we have equality in (3.3).

**3.3. Reilly inequality on  $\lambda_1^M$ .** Since we have  $\frac{1}{v_M} \int_M (X_i - \bar{X}_i) dv = 0$  for any component function of  $X - \bar{X}$ , by the min-max principle and Inequality (3.3), we have  $\lambda_1^M \frac{1}{\|\mathbf{H}\|_2^2} \leq \lambda_1^M \|X - \bar{X}\|_2^2 = \lambda_1^M \sum_i \|X_i - \bar{X}_i\|_2^2 \leq \sum_i \|\nabla X_i\|_2^2 = n$  where  $\lambda_1^M$  is the first non-zero eigenvalue of  $M$  and where the last equality comes from the fact that  $\sum_i |\nabla X_i|^2$  is the trace with respect to the canonical scalar product of the quadratic form  $Q(u) = |p(u)|^2$ , where  $p$  is the orthogonal projector from  $\mathbb{R}^{n+1}$  to  $T_x M$ . This gives the Reilly inequality in (1.1).

Here also, equality in the Reilly inequality gives equality in 3.3 and so it characterizes the sphere of radius  $\frac{1}{\|\mathbf{H}\|_2} = \|X\|_2 = \sqrt{\frac{n}{\lambda_1^M}}$ .

#### 4. CONCENTRATION ESTIMATES

In the section, we prove that almost extremal hypersurfaces are close to a sphere and have almost constant mean curvature in  $L^2$ -norm.

We say that  $M$  satisfies the pinching  $(P_{p,\varepsilon})$  when  $\|\mathbf{H}\|_p \|X - \bar{X}\|_2 \leq 1 + \varepsilon$ . From the proofs of Inequalities (1.1) above, it appears that pinchings  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$  or  $n \|\mathbf{H}\|_2^2 / \lambda_1 \leq 1 + \varepsilon$  imply the pinching  $(P_{2,\varepsilon})$ . In all the results of this section, we have  $0 < \varepsilon < 1$ .

From now on, we assume, without loss of generality, that  $\bar{X} = 0$ . Let  $X^T(x)$  denote the orthogonal projection of  $X(x)$  on the tangent space  $T_x M$ . In the following lemma, we see that the position vector  $X$  almost satisfies, in  $L^2$ -norm, characteristics properties of the Euclidean spheres ( $X^T = 0$ ).

**Lemma 4.1.** *If  $(P_{2,\varepsilon})$  holds, then we have  $\|X^T\|_2 \leq \sqrt{3\varepsilon} \|X\|_2$  and  $\|X - \frac{\mathbf{H}}{\|\mathbf{H}\|_2^2} \nu\|_2 \leq \sqrt{3\varepsilon} \|X\|_2$ .*

*Proof.* Since we have  $1 = \frac{1}{v_M} \int_M \mathbf{H} \langle X, \nu \rangle dv \leq \|\mathbf{H}\|_2 \|\langle X, \nu \rangle\|_2$ , Inequality  $(P_{2,\varepsilon})$  gives us  $\|X\|_2 \leq (1 + \varepsilon) \|\langle X, \nu \rangle\|_2$  and  $1 \leq \|\mathbf{H}\|_2 \|X\|_2 \leq 1 + \varepsilon$ . Hence  $\|X - \langle X, \nu \rangle \nu\|_2 \leq \sqrt{3\varepsilon} \|X\|_2$  and  $\|X - \frac{\mathbf{H} \nu}{\|\mathbf{H}\|_2^2}\|_2^2 = \|X\|_2^2 - \|\mathbf{H}\|_2^{-2} \leq 3\varepsilon \|X\|_2^2$ .  $\square$

In the lemma below, we see that in  $L^2$ -norm,  $M$  is close to a sphere and has  $L^2$ -almost constant mean curvature. In particular, the volume of  $M$  is concentrated in a tubular neighborhood  $A_\eta$  of the sphere  $S_M$  where  $A_\eta := B_0(\frac{1+\eta}{\|\mathbf{H}\|_2}) \setminus B_0(\frac{1-\eta}{\|\mathbf{H}\|_2})$  for some  $\eta$ .

**Lemma 4.2.** *If  $(P_{p,\varepsilon})$  (for  $p > 2$ ), or  $n \|\mathbf{H}\|_2^2 / \lambda_1^M \leq 1 + \varepsilon$ , or  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$  holds (with  $\varepsilon \leq \frac{1}{100}$ ), then we have  $\| |X| - \frac{1}{\|\mathbf{H}\|_2} \|_2 \leq \frac{C}{\|\mathbf{H}\|_2} \sqrt[p]{\varepsilon}$ ,  $\| |\mathbf{H}| - \|\mathbf{H}\|_2 \|_2 \leq C \sqrt[p]{\varepsilon} \|\mathbf{H}\|_2$  and  $\text{Vol}(M \setminus A_{\sqrt[p]{\varepsilon}}) \leq C \sqrt[p]{\varepsilon} v_M$ , where  $C = 6 \times 2^{\frac{2p}{p-2}}$  in the case  $(P_{p,\varepsilon})$  and  $C = 100$  in the other cases.*

*Proof.* When  $(P_{p,\varepsilon})$  holds, we have

$$\|\mathbf{H}\|_p \|X\|_2 \leq (1 + \varepsilon) \leq (1 + \varepsilon) \|\mathbf{H}\|_p \|X\|_{\frac{-p}{p-1}} \leq (1 + \varepsilon) \|\mathbf{H}\|_p \|X\|_1^{1-\frac{2}{p}} \|X\|_2^{\frac{2}{p}},$$

hence we get  $\left\| |X| - \frac{1}{\|\mathbf{H}\|_2} \right\|_2^2 = \|X\|_2^2 - 2\frac{\|\mathbf{H}\|_1}{\|\mathbf{H}\|_2} + \frac{1}{\|\mathbf{H}\|_2^2} \leq 2^{\frac{2p}{p-2}} \frac{1}{\|\mathbf{H}\|_2^2} \varepsilon$ . Combined with the second inequality of Lemma 4.1, it gives

$$\left\| |\mathbf{H}| - \|\mathbf{H}\|_2 \right\|_2 \leq \|\mathbf{H}\|_2^2 \left\| |X| - \frac{|\mathbf{H}|}{\|\mathbf{H}\|_2^2} \right\|_2 + \|\mathbf{H}\|_2^2 \left\| |X| - \frac{1}{\|\mathbf{H}\|_2} \right\|_2 \leq C\sqrt[4]{\varepsilon} \|\mathbf{H}\|_2$$

Now, by the Chebyshev inequality and Lemma 4.1, we get

$$\begin{aligned} \text{Vol}(M \setminus A_{\sqrt[4]{\varepsilon}}) &= \text{Vol} \left\{ x \in M / \left| |X(x)| - \frac{1}{\|\mathbf{H}\|_2} \right| \geq \frac{\sqrt[4]{\varepsilon}}{\|\mathbf{H}\|_2} \right\} \\ &\leq \frac{\|\mathbf{H}\|_2^2}{\sqrt{\varepsilon}} \int_M \left| |X| - \frac{1}{\|\mathbf{H}\|_2} \right|^2 dv \leq C(p)\sqrt{\varepsilon} v_M \end{aligned}$$

When  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$  holds. We set  $X_0$  the center of the circumsphere to  $M$  of radius  $r_M$ . We have  $\|X - X_0\|_2^2 = \|X\|_2^2 + |X_0|^2 = r_M^2 \leq \frac{(1+\varepsilon)^2}{\|\mathbf{H}\|_2^2}$  and then we have  $|X_0| \leq \frac{\sqrt{3\varepsilon}}{\|\mathbf{H}\|_2}$  and  $|X| \leq |X_0| + r_M \leq \frac{1+3\sqrt{\varepsilon}}{\|\mathbf{H}\|_2}$ . So we have  $\frac{1}{\|\mathbf{H}\|_2^2} - |X|^2 \in [\frac{\sqrt[4]{\varepsilon}}{\|\mathbf{H}\|_2^2}, \frac{1}{\|\mathbf{H}\|_2^2}]$  on  $M \setminus A_{\sqrt[4]{\varepsilon}}$ . Chebyshev inequality and (3.3) give us

$$\begin{aligned} \frac{\text{Vol}(M \setminus A_{\sqrt[4]{\varepsilon}})}{v_M} \frac{\sqrt[4]{\varepsilon}}{\|\mathbf{H}\|_2^2} &\leq \frac{1}{v_M} \int_{M \setminus A_{\sqrt[4]{\varepsilon}}} \left( \frac{1}{\|\mathbf{H}\|_2^2} - |X|^2 \right) dv \\ &\leq \frac{1}{v_M} \int_{M \cap A_{\sqrt[4]{\varepsilon}}} \left( |X|^2 - \frac{1}{\|\mathbf{H}\|_2^2} \right) dv \leq \frac{9\sqrt{\varepsilon}}{\|\mathbf{H}\|_2^2} \end{aligned}$$

where in the last inequality we have used  $|X| \leq \frac{1+3\sqrt{\varepsilon}}{\|\mathbf{H}\|_2}$  and, so we get

$$\begin{aligned} \left\| |X| - \frac{1}{\|\mathbf{H}\|_2} \right\|_2^2 &= \frac{1}{v_M} \int_{M \cap A_{\sqrt[4]{\varepsilon}}} \left| |X| - \frac{1}{\|\mathbf{H}\|_2} \right|^2 dv + \frac{1}{v_M} \int_{M \setminus A_{\sqrt[4]{\varepsilon}}} \left| |X| - \frac{1}{\|\mathbf{H}\|_2} \right|^2 dv \\ &\leq \frac{\sqrt{\varepsilon}}{\|\mathbf{H}\|_2^2} + \frac{\text{Vol}(M \setminus A_{\sqrt[4]{\varepsilon}})}{v_M} \frac{1}{\|\mathbf{H}\|_2^2} \leq \frac{10\sqrt[4]{\varepsilon}}{\|\mathbf{H}\|_2^2} \end{aligned}$$

Combined with the second inequality of Lemma 4.1, we get  $\left\| \frac{1}{\|\mathbf{H}\|_2} - \frac{|\mathbf{H}|}{\|\mathbf{H}\|_2^2} \right\|_2 \leq \frac{C\sqrt[8]{\varepsilon}}{\|\mathbf{H}\|_2}$ .

When  $n\|\mathbf{H}\|_2^2/\lambda_1^M \leq 1 + \varepsilon$  holds, we have  $\int_M (|X|^2 - \|X\|_2^2) dv = 0$  and so by the Poincare inequality we get  $\left\| |X|^2 - \|X\|_2^2 \right\|_2^2 \leq \frac{4\|X^T\|_2^2}{\lambda_1^M} \leq \frac{12(1+\varepsilon)^2 \varepsilon \|X\|_2^2}{n\|\mathbf{H}\|_2^2} \leq \frac{200\varepsilon}{n\|\mathbf{H}\|_2^4}$ , which gives  $\frac{1}{\|\mathbf{H}\|_2} \left\| |X| - \frac{1}{\|\mathbf{H}\|_2} \right\|_2 \leq \left\| |X|^2 - \frac{1}{\|\mathbf{H}\|_2^2} \right\|_2 \leq \left\| |X|^2 - \|X\|_2^2 \right\|_2 + \left\| \|X\|_2^2 - \frac{1}{\|\mathbf{H}\|_2^2} \right\|_2 \leq \frac{12\sqrt{\varepsilon}}{\|\mathbf{H}\|_2^2}$  and then we get the estimate on the volume of  $A_{\sqrt[4]{\varepsilon}}$  by the same Chebyshev procedure as for  $P_{p,\varepsilon}$  and the estimate on the mean curvature by the same procedure as for  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$ .  $\square$

For our last estimates, we need some notations. Let  $\psi: [0, \infty) \rightarrow [0, 1]$  be a smooth function with  $\psi=0$  outside  $\left[ \frac{(1-2\sqrt[16]{\varepsilon})^2}{\|\mathbf{H}\|_2^2}, \frac{(1+2\sqrt[16]{\varepsilon})^2}{\|\mathbf{H}\|_2^2} \right]$  and  $\psi=1$  on  $\left[ \frac{(1-\sqrt[16]{\varepsilon})^2}{\|\mathbf{H}\|_2^2}, \frac{(1+\sqrt[16]{\varepsilon})^2}{\|\mathbf{H}\|_2^2} \right]$ . Let us consider the function  $\varphi$  on  $M$  defined by  $\varphi(x) = \psi(|X_x|^2)$  and the vector field  $Z$  on  $M$  defined by  $Z = \nu - \mathbf{H}X$ . For any sphere  $RS^n$ ,  $Z$  is vanishing. The previous estimates then imply the following lemma and we see that in  $L^2$ -norm,  $Z$  is small.

**Lemma 4.3.** ( $P_{p,\varepsilon}$ ) (for  $p > 2$ ) or  $n\|\mathbf{H}\|_2^2/\lambda_1 \leq 1 + \varepsilon$  or  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$  implies  $\|\mathbf{H}^2 - \|\mathbf{H}\|_2^2\|_1 \leq C\sqrt[8]{\varepsilon} \|\mathbf{H}\|_2^2$ ,  $\|Z\|_2 \leq C\varepsilon^{\frac{3}{32}}$  and  $\|\varphi\|_2^2 - 1 \leq C\sqrt[8]{\varepsilon}$ , where  $C$  is a constant which depends on  $p$  in the case  $(P_{p,\varepsilon})$ .

*Proof.* We have  $\|\mathbf{H}^2 - \|\mathbf{H}\|_2^2\|_1 \leq 2\|\|\mathbf{H}\|_2 - \|\mathbf{H}\|_2\|_2\|\mathbf{H}\|_2 \leq C\sqrt[8]{\varepsilon}\|\mathbf{H}\|_2^2$  and

$$\begin{aligned} \|Z\|_2^2 &= \frac{1}{v_M} \int_M |Z|^2 dv = \frac{1}{v_M} \int_M 1 - 2\mathbf{H}\langle \nu, X \rangle + \mathbf{H}^2 |X|^2 dv \\ &= \frac{\|\mathbf{H}\|_2^2}{v_M} \int_M |X - \frac{\mathbf{H}}{\|\mathbf{H}\|_2^2} \nu|^2 dv + \frac{1}{\|\mathbf{H}\|_2^2 v_M} \int_M (\|\mathbf{H}\|_2^2 - \mathbf{H}^2)(1 - |X|^2 \|\mathbf{H}\|_2^2) dv \\ &\leq \|\mathbf{H}\|_2^2 \|X - \frac{\mathbf{H}}{\|\mathbf{H}\|_2^2} \nu\|_2^2 + 8\sqrt[16]{\varepsilon} \frac{\|\mathbf{H}^2 - \|\mathbf{H}\|_2^2\|_1}{\|\mathbf{H}\|_2^2}, \end{aligned}$$

which gives the result by Lemma 4.1. Finally, we have  $1 - \frac{\text{Vol}(M \setminus A_{\sqrt[8]{\varepsilon}})}{v_M} \leq \frac{\text{Vol}(A_{\sqrt[8]{\varepsilon}} \cap M)}{v_M} \leq \|\varphi\|_2^2$  and  $\|\varphi\|_2^2 \leq 1$ .  $\square$

## 5. HOMOGENEOUS, HARMONIC POLYNOMIALS OF DEGREE $k$

The eigenfunctions of  $\mathbb{S}^n$  are restrictions to  $\mathbb{S}^n$  of homogeneous, harmonic polynomials of the ambient space  $\mathbb{R}^{n+1}$ . To prove Theorem 1.1, we will use restrictions to  $M$  of homogeneous, harmonic polynomials as quasi-modes. In that purpose, we prove in this section, some estimates on harmonic homogeneous polynomials and their restrictions to Euclidean hypersurfaces.

**5.1. General estimates.** Let  $\mathcal{H}^k(\mathbb{R}^{n+1})$  be the space of homogeneous, harmonic polynomials of degree  $k$  on  $\mathbb{R}^{n+1}$ . Note that  $\mathcal{H}^k(\mathbb{R}^{n+1})$  induces on  $\mathbb{S}^n$  the spaces of eigenfunctions of  $\Delta^{\mathbb{S}^n}$  associated to the eigenvalues  $\mu_k := k(n+k-1)$  with multiplicity  $m_k := \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}$ .

On the space  $\mathcal{H}^k(\mathbb{R}^{n+1})$ , we set  $(P, Q)_{\mathbb{S}^n} := \frac{1}{v_{\mathbb{S}^n}} \int_{\mathbb{S}^n} PQ dv_{\text{can}}$ , where  $dv_{\text{can}}$  denotes the element volume of the sphere with its standard metric.

Remind that for any  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$  and any  $X, Y \in \mathbb{R}^{n+1}$ , we have the Euler identities

$$(5.1) \quad d_X P(X) = kP(X) \text{ and } \nabla^0 d_X P(X, Y) = (k-1)d_X P(Y).$$

**Lemma 5.1.** *For any  $x \in \mathbb{R}^{n+1}$  and  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ , we have  $|P(x)|^2 \leq \|P\|_{\mathbb{S}^n}^2 m_k |x|^{2k}$ .*

*Proof.* Let  $(P_i)_{1 \leq i \leq m_k}$  be an orthonormal basis of  $\mathcal{H}^k(\mathbb{R}^{n+1})$ . For any  $x \in \mathbb{S}^n$ ,  $Q_x(P) = P^2(x)$  is a quadratic form on  $\mathcal{H}^k(\mathbb{R}^{n+1})$  whose trace is given by  $\sum_{i=1}^{m_k} P_i^2(x)$ . Since for any  $x' \in \mathbb{S}^n$  and any  $O \in O_{n+1}$  such that  $x' = Ox$  we have  $Q_{x'}(P) = Q_x(P \circ O)$  and since  $P \mapsto P \circ O$  is an isometry of  $\mathcal{H}^k(\mathbb{R}^{n+1})$ , we have  $\sum_{i=1}^{m_k} P_i^2(x) = \text{tr}(Q_x) = \sum_{i=1}^{m_k} P_i^2(x') = \text{tr}(Q_{x'})$ . We infer that  $\sum_{i=1}^{m_k} \frac{1}{v_{\mathbb{S}^n}} \int_{\mathbb{S}^n} P_i^2(x) dv_{\text{can}} = m_k = \frac{1}{v_{\mathbb{S}^n}} \int_{\mathbb{S}^n} \left( \sum_{i=1}^{m_k} P_i^2(x) \right) dv_{\text{can}}$  and so

$$(5.2) \quad \sum_{i=1}^{m_k} P_i^2(x) = m_k |x|^{2k},$$

and by the Cauchy-Schwarz inequality applied to  $P(x) = \sum_i (P, P_i)_{\mathbb{S}^n} P_i(x)$ , we get the result.  $\square$

As an immediate consequence, we have the following lemma.

**Lemma 5.2.** *For any  $x, u \in \mathbb{R}^{n+1}$  and  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ , we have*

$$|d_x P(u)|^2 \leq \|P\|_{\mathbb{S}^n}^2 m_k \left( \frac{\mu_k}{n} |x|^{2(k-1)} |u|^2 + \left( k^2 - \frac{\mu_k}{n} \right) \langle u, x \rangle^2 |x|^{2(k-2)} \right).$$

*Proof.* Let  $x \in \mathbb{S}^n$  and  $u \in \mathbb{S}^n$  so that  $\langle u, x \rangle = 0$ . Once again the quadratic forms  $Q_{x,u}(P) = (d_x P(u))^2$  are conjugate (since  $O_{n+1}$  acts transitively on orthonormal couples) and so  $\sum_{i=1}^{m_k} (d_x P_i(u))^2$  does not depend on  $u \in x^\perp$  nor on  $x \in \mathbb{S}^n$ . By choosing an orthonormal basis  $(u_j)_{1 \leq j \leq n}$  of  $x^\perp$ , we obtain that

$$\begin{aligned} \sum_{i=1}^{m_k} (d_x P_i(u))^2 &= \frac{1}{n} \sum_{i=1}^{m_k} \sum_{j=1}^n (d_x P_i(u_j))^2 = \frac{1}{n \nu_{\mathbb{S}^n}} \int_{\mathbb{S}^n} \sum_{i=1}^{m_k} |\nabla^{\mathbb{S}^n} P_i|^2 d\nu_{\text{can}} \\ &= \frac{1}{n \nu_{\mathbb{S}^n}} \int_{\mathbb{S}^n} \sum_{i=1}^{m_k} P_i \Delta^{\mathbb{S}^n} P_i d\nu_{\text{can}} = \frac{m_k \mu_k}{n} \end{aligned}$$

Now suppose that  $u \in \mathbb{R}^{n+1}$ . Then  $u = v + \langle u, x \rangle x$ , where  $v = u - \langle u, x \rangle x$ , and we have

$$\begin{aligned} \sum_{i=1}^{m_k} (d_x P_i(u))^2 &= \sum_{i=1}^{m_k} (d_x P_i(v) + k \langle u, x \rangle P_i(x))^2 \\ &= \sum_{i=1}^{m_k} (d_x P_i(v))^2 + 2k \langle u, x \rangle \sum_{i=1}^{m_k} d_x P_i(v) P_i(x) + m_k \langle u, x \rangle^2 k^2 \\ &= \frac{m_k \mu_k}{n} |v|^2 + m_k \langle u, x \rangle^2 k^2 = m_k \left( \frac{\mu_k}{n} |u|^2 + \left( k^2 - \frac{\mu_k}{n} \right) \langle u, x \rangle^2 \right), \end{aligned}$$

where we have taken the derivative the equality (5.2) to compute  $\sum_{i=1}^{m_k} d_x P_i(v) P_i(x)$ . By

homogeneity of  $P_i$  we get  $\sum_{i=1}^{m_k} (d_x P_i(u))^2 = m_k \left( \frac{\mu_k}{n} |x|^{2(k-1)} |u|^2 + \left( k^2 - \frac{\mu_k}{n} \right) \langle u, x \rangle^2 |x|^{2(k-2)} \right)$  and conclude once again by the Cauchy-Schwarz inequality.  $\square$

**Lemma 5.3.** *For any  $x \in \mathbb{R}^{n+1}$  and  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ , we have*

$$|\nabla^0 dP(x)|^2 \leq \|P\|_{\mathbb{S}^n}^2 m_k \alpha_{n,k} |x|^{2(k-2)},$$

where  $\alpha_{n,k} = (k-1)(k^2 + \mu_k)(n + 2k - 3) \leq C(n)k^4$ .

*Proof.* The Bochner equality gives

$$\begin{aligned} \sum_{i=1}^{m_k} |\nabla^0 dP_i(x)|^2 &= \sum_{i=1}^{m_k} \left( \langle d\Delta^0 P_i, dP_i \rangle - \frac{1}{2} \Delta^0 |dP_i|^2 \right) \\ (5.3) \qquad \qquad \qquad &= -\frac{1}{2} m_k (k^2 + \mu_k) \Delta^0 |X|^{2k-2} = m_k \alpha_{n,k} |X|^{2k-4} \end{aligned}$$

$\square$

**5.2. Estimates on hypersurfaces.** The main result of this section is the Lemma 5.6 which is fundamental in the proof of theorem 1.1. It controls the defect of the localized restriction map  $P \in \mathcal{H}^k(\mathbb{R}^{n+1}) \mapsto \|\mathbf{H}\|_2^k \varphi P \circ X \in L^2(M)$  to be an isometry. Note that it applies to any Euclidean hypersurface. In the case of almost extremal hypersurface, it will prove that the localized restriction map is a quasi isometry (see Lemma 6.1).

Let  $\mathcal{H}^k(M) = \{P \circ X, P \in \mathcal{H}^k(\mathbb{R}^{n+1})\}$  be the space of functions induced on  $M$  by  $\mathcal{H}^k(\mathbb{R}^{n+1})$ . We will identify  $P$  and  $P \circ X$  subsequently. There is no ambiguity since we have

**Lemma 5.4.** *Let  $M^n$  be a compact manifold immersed by  $X$  in  $\mathbb{R}^{n+1}$  and let  $(P_1, \dots, P_m)$  be a linearly independent set of homogeneous polynomials of degree  $k$  on  $\mathbb{R}^{n+1}$ . Then the set  $(P_1 \circ X, \dots, P_m \circ X)$  is also linearly independent.*

*Proof.* Any homogeneous polynomial  $P$  which is zero on  $M$  is zero on the cone  $\mathbb{R}^+ \cdot M$ . Since  $M$  is compact there exists a point  $x \in M$  so that  $X_x \notin T_x M$  and so  $\mathbb{R}^+ \cdot M$  has non empty interior. Hence  $P \circ X = 0$  implies  $P = 0$ .  $\square$

We first need to precise the localisation functions  $\varphi$  for which Lemma 5.6 applies. Let  $0 < \eta < 1$  be fixed. We still denote  $\psi : [0, \infty) \rightarrow [0, 1]$  a smooth function which is 0 outside  $\left[\frac{(1-\eta)^2}{\|\mathbf{H}\|_2^2}, \frac{(1+\eta)^2}{\|\mathbf{H}\|_2^2}\right]$ , is 1 on  $\left[\frac{(1-\eta/2)^2}{\|\mathbf{H}\|_2^2}, \frac{(1+\eta/2)^2}{\|\mathbf{H}\|_2^2}\right]$  and satisfies the upper bounds  $|\psi'| \leq \frac{4\|\mathbf{H}\|_2^2}{\eta}$  and  $|\psi''| \leq \frac{8\|\mathbf{H}\|_2^4}{\eta^2}$ . We set  $\varphi(x) = \psi(|X_x|^2)$  on  $M$ .

**Lemma 5.5.** *With the above restrictions on  $\psi$  we have*

$$\|\Delta\varphi^2\|_1 \leq \frac{192\|\mathbf{H}\|_2^4}{\eta^2} \|X^T\|_2^2 + \frac{16n\|\mathbf{H}\|_2^2}{\eta} \|Z\|_1$$

*Proof.* An easy computation yields that

$$\begin{aligned} \Delta(\varphi^2) &= -(\psi^2)''(|X|^2)|d|X|^2|^2 + (\psi^2)'(|X|^2)\Delta|X|^2 \\ &= -4(\psi^2)''(|X|^2)|X^T|^2 - 2n(\psi^2)'(|X|^2)\langle \nu, Z \rangle \end{aligned}$$

But the bound on the derivatives of  $\psi$  gives us  $|(\psi^2)'| \leq \frac{8\|\mathbf{H}\|_2^2}{\eta}\psi$  and  $|(\psi^2)''| \leq \frac{48\|\mathbf{H}\|_2^4}{\eta^2}$ . Hence we get  $\|\Delta\varphi^2\|_1 \leq \frac{192\|\mathbf{H}\|_2^4}{\eta^2} \|X^T\|_2^2 + \frac{16n\|\mathbf{H}\|_2^2}{\eta} \|\varphi Z\|_1$ .  $\square$

**Lemma 5.6.** *Let  $\varphi : M \rightarrow [0, 1]$  be as above. There exists a constant  $C = C(n)$  such that for any isometrically immersed hypersurface  $M$  of  $\mathbb{R}^{n+1}$  and any  $P \in \mathcal{H}^k(M)$ , we have*

$$\left| \|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2 \right| \leq DC(n) \|P\|_{\mathbb{S}^n}^2 \sum_{i=1}^k m_i (1+\eta)^{2i}$$

where  $D = \|Z\|_2 + \|Z\|_2^2 + \frac{200\|\mathbf{H}\|_2^2}{\eta^2} \|X^\perp\|_2^2 + \frac{16n}{\eta} \|Z\|_1 + \frac{\|\mathbf{H}^2 - \|\mathbf{H}\|_2^2\|_1}{\|\mathbf{H}\|_2^2}$  and  $Z = \nu - XH$ .

*Proof.* For any  $P \in \mathcal{H}^k(M)$ , the Euler identities (5.1) give us

$$\begin{aligned} \|\varphi \nabla^0 P\|_2^2 &= \|\varphi dP(\nu)\|_2^2 + \|\varphi \nabla^M P\|_2^2 \\ &= \|\varphi dP(Z)\|_2^2 + \|\varphi dP(HX)\|_2^2 \\ &\quad + \frac{1}{v_M} \int_M (2\varphi^2 H dP(Z) dP(X) + \langle \nabla^M \varphi^2 P, \nabla^M P \rangle - \frac{1}{2} \langle \nabla^M \varphi^2, \nabla^M P^2 \rangle) dv \\ &= \|\varphi dP(Z)\|_2^2 + k^2 \|\varphi HP\|_2^2 + \frac{1}{v_M} \int_M (2k H dP(\varphi Z) \varphi P + \varphi^2 P \Delta P - \frac{1}{2} P^2 \Delta(\varphi^2)) dv \end{aligned}$$

Now, Formula (3.1) applied to  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$  gives

$$(5.4) \quad \Delta P = \mu_k \mathbf{H}^2 P + (n + 2k - 2) \mathbf{H} dP(Z) + \nabla^0 dP(Z, Z)$$

hence, we get

$$\begin{aligned} \|\varphi \nabla^0 P\|_2^2 &= \|dP(\varphi Z)\|_2^2 + (\mu_k + k^2) \|\mathbf{H} \varphi P\|_2^2 \\ &\quad + \frac{1}{v_M} \int_M (\varphi^2 P \nabla^0 dP(Z, Z) + (n + 4k - 2) \varphi \mathbf{H} dP(\varphi Z) P - \frac{1}{2} P^2 \Delta(\varphi^2)) dv \\ &= \frac{1}{v_M} \int_M \left( (\mu_k + k^2) (\mathbf{H}^2 - \|\mathbf{H}\|_2^2) \varphi^2 P^2 + (n + 4k - 2) \mathbf{H} dP(\varphi Z) \varphi P \right) dv \\ &\quad + \frac{1}{v_M} \int_M (P \nabla^0 dP(\varphi Z, \varphi Z) - \frac{1}{2} P^2 \Delta(\varphi^2)) dv \\ &\quad + (\mu_k + k^2) \|\mathbf{H}\|_2^2 \|\varphi P\|_2^2 + \|dP(\varphi Z)\|_2^2 \end{aligned}$$

Now we have

$$(5.5) \quad \|\nabla^0 P\|_{\mathbb{S}^n}^2 = \|\nabla^{\mathbb{S}^n} P\|_{\mathbb{S}^n}^2 + k^2 \|P\|_{\mathbb{S}^n}^2 = (\mu_k + k^2) \|P\|_{\mathbb{S}^n}^2$$

Hence

$$\begin{aligned} \|\mathbf{H}\|_2^{2k-2} \|\varphi \nabla^0 P\|_2^2 - \|\varphi\|_2^2 \|\nabla^0 P\|_{\mathbb{S}^n}^2 &= (\mu_k + k^2) (\|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2) + \|\mathbf{H}\|_2^{2k-2} \|dP(\varphi Z)\|_2^2 \\ &\quad + \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M \varphi^2 P \left( (\mu_k + k^2) (\mathbf{H}^2 - \|\mathbf{H}\|_2^2) P + \mathbf{H} (n + 4k - 2) dP(Z) + \nabla^0 dP(Z, Z) \right) dv \\ &\quad - \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M \frac{1}{2} P^2 \Delta(\varphi^2) dv \end{aligned}$$

Which gives

$$(5.6) \quad \begin{aligned} &\left| \|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2 \right| \\ &\leq \frac{1}{\mu_k + k^2} \left| \|\mathbf{H}\|_2^{2k-2} \|\varphi \nabla^0 P\|_2^2 - \|\varphi\|_2^2 \|\nabla^0 P\|_{\mathbb{S}^n}^2 \right| \\ &\quad + \frac{\|\mathbf{H}\|_2^{2k-2}}{\mu_k + k^2} \int_M \left( (n + 4k - 2) |\mathbf{H} \varphi P| |dP(\varphi Z)| + |dP(\varphi Z)|^2 + |P| |\nabla^0 dP| |\varphi Z|^2 \right) \\ &\quad + \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M (\varphi^2 |\mathbf{H}^2 - \|\mathbf{H}\|_2^2| P^2 + \frac{P^2}{2} |\Delta(\varphi^2)|) dv \end{aligned}$$

By Lemma 5.1, we have

$$\begin{aligned} \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |\mathbf{H}^2 - \|\mathbf{H}\|_2^2| (\varphi P)^2 dv &\leq \frac{m_k \|P\|_{\mathbb{S}^n}^2 \|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |\varphi^2 (\mathbf{H}^2 - \|\mathbf{H}\|_2^2)| |X|^{2k} dv \\ &\leq \|P\|_{\mathbb{S}^n}^2 m_k (1 + \eta)^{2k} \frac{\|\varphi^2 (\mathbf{H}^2 - \|\mathbf{H}\|_2^2)\|_1}{\|\mathbf{H}\|_2^2} \end{aligned}$$

In the same way, we have

$$\frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M \frac{P^2}{2} |\Delta(\varphi^2)| dv \leq \|P\|_{\mathbb{S}^n}^2 m_k (1 + \eta)^{2k} \frac{\|\Delta \varphi^2\|_1}{\|\mathbf{H}\|_2^2}$$

and using Lemma 5.2, we get

$$\frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M \varphi^2 |PdP(Z)\mathbf{H}| dv \leq \|P\|_{\mathbb{S}^n}^2 m_k k (1+\eta)^{2k} \|\varphi^2 Z\|_2$$

and

$$\begin{aligned} \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |dP(\varphi Z)|^2 &\leq \|P\|_{\mathbb{S}^n}^2 m_k k^2 \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |\varphi Z|^2 |X|^{2(k-1)} dv \\ &\leq \|P\|_{\mathbb{S}^n}^2 m_k k^2 (1+\eta)^{2k} \|\varphi Z\|_2^2 \end{aligned}$$

Finally, using Lemma 5.3, we get

$$\begin{aligned} \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |P|\nabla^0 dP|\varphi Z|^2 &\leq \|P\|_{\mathbb{S}^n}^2 m_k \sqrt{\alpha_{n,k}} \frac{\|\mathbf{H}\|_2^{2k-2}}{v_M} \int_M |X|^{2(k-1)} |\varphi Z|^2 dv \\ &\leq \|P\|_{\mathbb{S}^n}^2 m_k \sqrt{\alpha_{n,k}} (1+\eta)^{2k} \|\varphi Z\|_2^2 \end{aligned}$$

which, combined with (5.6) and equation (5.5), gives

$$\begin{aligned} \frac{|\|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2|}{\|P\|_{\mathbb{S}^n}^2} &\leq \frac{|\|\mathbf{H}\|_2^{2k-2} \|\varphi \nabla^0 P\|_2^2 - \|\varphi\|_2^2 \|\nabla^0 P\|_{\mathbb{S}^n}^2|}{\|\nabla^0 P\|_{\mathbb{S}^n}^2} \\ &+ C(n) m_k (1+\eta)^{2k} \left( \|\varphi^2 Z\|_2 + \|\varphi Z\|_2^2 + \frac{\|\Delta(\varphi^2)\|_1}{\|\mathbf{H}\|_2^2} + \frac{\|\varphi^2(\mathbf{H}^2 - \|\mathbf{H}\|_2^2)\|_1}{\|\mathbf{H}\|_2^2} \right) \\ &\leq \frac{|\|\mathbf{H}\|_2^{2k-2} \|\varphi \nabla^0 P\|_2^2 - \|\varphi\|_2^2 \|\nabla^0 P\|_{\mathbb{S}^n}^2|}{\|\nabla^0 P\|_{\mathbb{S}^n}^2} + C(n) m_k (1+\eta)^{2k} D \end{aligned}$$

where we have used the previous lemma. Since in case  $k=1$ ,  $|\nabla^0 P|$  is constant we get

$$|\|\mathbf{H}\|_2^2 \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2| \leq C(n) m_1 (1+\eta)^2 D \|P\|_{\mathbb{S}^n}^2$$

Now, let  $B_k = \sup \left\{ \frac{|\|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|\varphi\|_2^2 \|P\|_{\mathbb{S}^n}^2|}{\|P\|_{\mathbb{S}^n}^2} \mid P \in \mathcal{H}^k(\mathbb{R}^{n+1}) \setminus \{0\} \right\}$ . Then using that for any  $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ , we have  $|\nabla^0 P|^2 = \sum_i |\partial_i P|^2$  with  $\partial_i P \in \mathcal{H}^{k-1}(\mathbb{R}^{n+1})$ , we get

$$\begin{aligned} |\|\mathbf{H}\|_2^{2k-2} \|\varphi \nabla^0 P\|_2^2 - \|\varphi\|_2^2 \|\nabla^0 P\|_{\mathbb{S}^n}^2| &\leq \sum_i |\|\mathbf{H}\|_2^{2k-2} \|\varphi \partial_i P\|_2^2 - \|\varphi\|_2^2 \|\partial_i P\|_{\mathbb{S}^n}^2| \\ &\leq B_{k-1} \sum_i \|\partial_i P\|_{\mathbb{S}^n}^2 = B_{k-1} \|\nabla^0 P\|_{\mathbb{S}^n}^2 \end{aligned}$$

and by (5.5), it gives  $B_k \leq B_{k-1} + C(n) m_k (1+\eta)^{2k} D \leq C(n) D \sum_{i=1}^k m_i (1+\eta)^{2i}$ .  $\square$

## 6. PROOF OF THEOREM 1.1

To prove theorem 1.1 we will show (Lemma 6.3) that for extremal hypersurfaces  $M$ , the functions  $\varphi P$  are almost eigenfunctions of  $M$  in  $L^2$  sense.

The estimates of Lemmas 6.3 and 6.1 need to be compared to the fact proved in [3] that the limit set for the Hausdorff distance of an extremizing sequence of hypersurfaces can contain any closed, connected subset of  $\mathbb{R}^{n+1}$  that contains  $\mathbb{S}^n$ .

Under the assumption of Theorem 1.1, we can use Lemma 4.1, Lemma 4.3 and the pinching  $P_{2,\varepsilon}$  to improve the estimate in Lemma 5.6 in the case  $\eta = 2^{\frac{1}{\sqrt[4]{\varepsilon}}}$ .

**Lemma 6.1.** *For any isometrically immersed hypersurface  $M \hookrightarrow \mathbb{R}^{n+1}$  with  $r_M \|\mathbf{H}\|_2 \leq 1 + \varepsilon$  (or  $\lambda_1(1 + \varepsilon)^2 \geq n \|\mathbf{H}\|_2^2$  or  $(P_{p,\varepsilon})$  for  $p > 2$ ) and for any  $P \in \mathcal{H}^k(M)$ , we have*

$$\left| \|\mathbf{H}\|_2^{2k} \|\varphi P\|_2^2 - \|P\|_{\mathbb{S}^n}^2 \right| \leq C \sqrt[32]{\varepsilon} \|P\|_{\mathbb{S}^n}^2,$$

where  $C = C(n, k)$  in the first two cases and  $C = C(p, k, n)$  in the latter case. Note that  $C$  tends to infinity when  $k$  tends to infinity.

As a consequence, the map  $P \mapsto \varphi P$  is injective on  $\mathcal{H}^k(M)$  for  $\varepsilon$  small enough and is a quasi-isometry.

**Lemma 6.2.** *Under the assumption of Lemma 6.1, if  $\varepsilon \leq \frac{1}{(2C)^{32}}$  then  $\dim(\varphi \mathcal{H}^k(M)) = m_k$ .*

Lemma 6.1 allows us to prove the following estimate on  $\Delta P$ , which says that for extremal hypersurfaces,  $\varphi P$  is in  $L^2$ -norm an almost eigenfunction on  $M$ .

**Lemma 6.3.** *Under the assumptions of Lemma 6.1, if  $\varepsilon \leq \frac{1}{(2C)^{32}}$ , then for any  $P \in \mathcal{H}^k(M)$ , we have  $\|\Delta(\varphi P) - \mu_k^{S_M} \varphi P\|_2 \leq C \sqrt[16]{\varepsilon} \mu_k^{S_M} \|\varphi P\|_2$  where  $C = C(n, k)$  ( $C = C(n, k, p)$  under the pinching  $(P_{p,\varepsilon})$ ).*

*Proof.* Let  $P \in \mathcal{H}^k(M)$ . Using (3.1) we have

$$\begin{aligned} \Delta(\varphi P) &= P\Delta\varphi - 2\langle dP, d\varphi \rangle + \varphi\Delta P = P\Delta\varphi - 2\langle dP, d\varphi \rangle + \varphi n \text{Hd}P(\nu) + \varphi \nabla^0 dP(\nu, \nu) \\ &= P\Delta\varphi - 2\langle dP, d\varphi \rangle + \varphi \mu_k |\mathbf{H}| \|\mathbf{H}\|_2 P + \varphi(n+k-1) \frac{\mathbf{H}}{|\mathbf{H}|} \|\mathbf{H}\|_2 dP(Z) \\ &\quad + \varphi(n+k-1) \frac{\mathbf{H}}{|\mathbf{H}|} (|\mathbf{H}| - \|\mathbf{H}\|_2) dP(\nu) + \varphi \nabla^0 dP(\nu, Z) \end{aligned}$$

hence, we get

$$\begin{aligned} \|\Delta(\varphi P) - \mu_k \|\mathbf{H}\|_2^2 \varphi P\|_2 &\leq \|(\Delta\varphi)P\|_2 + 2\| \langle d\varphi, dP \rangle \|_2 + \mu_k \|(|\mathbf{H}| - \|\mathbf{H}\|_2) \varphi P\|_2 \|\mathbf{H}\|_2 \\ (6.1) \quad &+ (n+k-1) \|\mathbf{H}\|_2 \|\varphi dP\|_2 + (n+k-1) \|\varphi (|\mathbf{H}| - \|\mathbf{H}\|_2) dP(\nu)\|_2 + \|\varphi \nabla^0 dP\|_2 \end{aligned}$$

Let us estimate  $\|(\Delta\varphi)P\|_2$ .

$$\begin{aligned} \|(\Delta\varphi)P\|_2^2 &\leq \frac{1}{v_M} \int_M (4|\psi''(|X|^2)| |X^T|^2 + 2n|\psi'(|X|^2)| |Z|)^2 P^2 dv \\ &\leq \frac{m_k}{v_M} \left( \int_M |X|^{2k} (4|\psi''(|X|^2)| |X^T|^2 + 2n|\psi'(|X|^2)| |Z|)^2 dv \right) \|P\|_{\mathbb{S}^n}^2 \\ &\leq \frac{m_k}{v_M} \frac{(1 + 2\sqrt[16]{\varepsilon})^{2k}}{\|\mathbf{H}\|_2^{2k}} \left( \int_{A_{2\sqrt[16]{\varepsilon}}} \left( \frac{8\|\mathbf{H}\|_2^4}{\sqrt[8]{\varepsilon}} |X^T|^2 + 2n \frac{2\|\mathbf{H}\|_2^2}{\sqrt[16]{\varepsilon}} |Z|^2 \right) dv \right) \|P\|_{\mathbb{S}^n}^2 \\ &\leq \frac{m_k}{v_M} \frac{(1 + 2\sqrt[16]{\varepsilon})^{2k}}{\|\mathbf{H}\|_2^{2k}} \left( \int_{A_{2\sqrt[16]{\varepsilon}}} \left( \frac{128\|\mathbf{H}\|_2^8}{\sqrt[4]{\varepsilon}} |X^T|^4 + 32n^2 \frac{\|\mathbf{H}\|_2^4}{\sqrt[8]{\varepsilon}} |Z|^2 \right) dv \right) \|P\|_{\mathbb{S}^n}^2 \end{aligned}$$

Since we have  $|X^T| \leq |X|$  and since Lemma 4.3 is valid with  $\|\varphi Z\|_2^2$  replaced by  $\frac{1}{v_M} \int_{A_2 \sqrt[16]{\varepsilon}} |Z|^2 dv$ , we get

$$\begin{aligned} \|(\Delta\varphi)P\|_2^2 &\leq \frac{C(n,k)\mu_k}{v_M} \frac{\|P\|_{\mathbb{S}^n}^2}{\|H\|_2^{2k}} \int_{A_2 \sqrt[16]{\varepsilon}} \left( \frac{\|H\|_2^6}{\sqrt[4]{\varepsilon}} |X^T|^2 + \frac{\|H\|_2^4}{\sqrt[8]{\varepsilon}} |Z|^2 \right) dv \\ &\leq \frac{C(n,k)\mu_k}{\|H\|_2^{2k}} \|H\|_2^4 \sqrt[16]{\varepsilon} \|P\|_{\mathbb{S}^n}^2 \end{aligned}$$

From the lemma 6.1,  $\varepsilon \leq \frac{1}{(2C)^{32}}$  implies that

$$(6.2) \quad \|P\|_{\mathbb{S}^n}^2 \leq 2\|H\|_2^{2k} \|\varphi P\|_2^2$$

which gives

$$(6.3) \quad \|(\Delta\varphi)P\|_2^2 \leq C(n,k)\mu_k \|H\|_2^4 \sqrt[16]{\varepsilon} \|\varphi P\|_2^2$$

Now

$$\begin{aligned} \|\langle d\varphi, dP \rangle\|_2^2 &\leq 4\|\psi'(|X|^2)|X^T\| \|dP\|_2^2 \leq \frac{16\|H\|_2^4}{\sqrt[16]{\varepsilon}v_M} \int_{A_2 \sqrt[16]{\varepsilon}} |X^T|^2 |dP|^2 dv \\ &\leq \frac{16\|H\|_2^4}{\sqrt[16]{\varepsilon}v_M} \|P\|_{\mathbb{S}^n}^2 \int_{A_2 \sqrt[16]{\varepsilon}} |X^T|^2 m_k n k^2 |X|^{2(k-1)} dv \\ (6.4) \quad &\leq C(n,k)\mu_k \sqrt[16]{\varepsilon} \|H\|_2^{4-2k} \|P\|_{\mathbb{S}^n}^2 \leq C(n,k)\|H\|_2^4 \sqrt[16]{\varepsilon} \|\varphi P\|_2^2 \end{aligned}$$

By the same way, we get

$$(6.5) \quad \|\varphi|dP|Z\|_2^2 \leq C(n,k)\mu_k \|H\|_2^2 \sqrt[16]{\varepsilon} \|\varphi P\|_2^2$$

Now, by Lemma 4.2, we have

$$\begin{aligned} \|(|H| - \|H\|_2)\varphi P\|_2^2 &\leq \frac{m_k}{v_M} \|P\|_{\mathbb{S}^n}^2 \int_M ||H| - \|H\|_2|^2 |X|^{2k} \varphi^2 dv \\ &\leq \frac{C(n,k)}{\|H\|_2^{2k}} \|P\|_{\mathbb{S}^n}^2 \|\varphi(|H| - \|H\|_2)\|_2^2 \\ (6.6) \quad &\leq C(n,k)\mu_k \|H\|_2^2 \sqrt[16]{\varepsilon} \|\varphi P\|_2^2 \end{aligned}$$

By the same way, we get

$$(6.7) \quad \|\varphi(|H| - \|H\|_2)dP(v)\|_2^2 \leq C(n,k)\mu_k \sqrt[16]{\varepsilon} \|H\|_2^4 \|\varphi P\|_2^2$$

Now let us estimate the last terms of (6.1)

$$\begin{aligned} \|\varphi|\nabla^0 dP\|Z\|_2^2 &\leq \frac{C(n,k)\mu_k}{v_M} \|P\|_{\mathbb{S}^n}^2 \int_M \varphi^2 |X|^{2k-4} |Z|^2 dv \\ (6.8) \quad &\leq C(n,k)\mu_k \|H\|_2^4 \sqrt[16]{\varepsilon} \|\varphi P\|_2^2 \end{aligned}$$

Reporting (6.3), (6.4), (6.5), (6.6), (6.7) and (6.8) in (6.1) we get

$$\|\Delta(\varphi P) - \mu_k \|H\|_2^2 \varphi P\|_2 \leq C(n,k) \sqrt[16]{\varepsilon} \mu_k \|H\|_2^2 \|\varphi P\|_2$$

□

Let  $E_k^\varepsilon$  be the space spanned by the eigenfunctions of  $M$  associated to an eigenvalue in the interval  $[(1 - \sqrt[16]{\varepsilon}2C(n, k))\mu_k^{S_M}, (1 + \sqrt[16]{\varepsilon}2C(n, k))\mu_k^{S_M}]$ . If  $\dim E_k^\varepsilon < m_k$ , then there exists  $\varphi P \in (\varphi\mathcal{H}^k(M)) \setminus \{0\}$  which is  $L^2$ -orthogonal to  $E_k^\varepsilon$ . Let  $\varphi P = \sum_i f_i$  be the decomposition of  $\varphi P$  in the Hilbert basis given by the eigenfunctions  $f_i$  of  $M$  associated respectively to  $\lambda_i$ . Putting  $N := \{i / f_i \notin E_k^\varepsilon\}$ , by assumption on  $P$  we have

$$\begin{aligned} 4C(n, k)^2 \sqrt[8]{\varepsilon} (\mu_k^{S_M})^2 \|\varphi P\|_2^2 &\leq \sum_{i \in N} (\lambda_i - \mu_k^{S_M})^2 \|f_i\|_2^2 = \|\Delta(\varphi P) - \mu_k^{S_M} \varphi P\|_2^2 \\ &\leq (\mu_k^{S_M})^2 C(n, k)^2 \sqrt[8]{\varepsilon} \|\varphi P\|_2^2 \end{aligned}$$

which gives a contradiction. We then have  $\dim E_k^\varepsilon \geq m_k$ . This complete the proof of theorem 1.1.

## 7. PROOF OF THEOREM 2.1

We adapt the constructions made in [4, 13, 3]. There will be two steps. We first consider submanifolds obtained by connected sum of a small submanifold  $\varepsilon M_2$  with a fixed submanifold  $M_1$  along a small, adequately pinched cylinder  $\varepsilon T'_\varepsilon$ . Note that contrary to the constructions in [4, 13], this is a 2 scales collapsing sequence of submanifolds. It will first give Theorem 2.1 in the case where  $F \setminus \text{Sp}(M_1)$  is a singleton. We will then get the general case by iterating the construction (i.e. by glueing several such cylinders).

Subsequently, for any subset  $A$  of  $\mathbb{R}^{n+1}$ , we denote by  $\lambda A$  the set obtained by applying an homothety of factor  $\lambda$  to  $A$ .

### 7.1. Case $F = \text{Sp}(M_1) \cup \{\lambda\}$ .

**7.1.1. Flattening of submanifolds.** For any submanifold  $M$  of  $\mathbb{R}^{n+1}$  and  $\varepsilon > 0$  small enough, we set  $\tilde{M}^\varepsilon$  a submanifold of  $\mathbb{R}^{n+1}$  obtained by smooth deformation of  $M$  at the neighbourhood of a point  $x_0 \in M$  such that  $B_{x_0}(4\varepsilon)$  is flat in  $\tilde{M}^\varepsilon$  and  $\tilde{M}^\varepsilon \setminus B_{x_0}(10\varepsilon)$  is a subset of  $M$ . We also set  $M^\varepsilon = \tilde{M}^\varepsilon \setminus B_{x_0}(3\varepsilon)$  whose boundary has a neighbourhood isometric to the flat annulus  $B_0(4\varepsilon) \setminus B_0(3\varepsilon)$  in  $\mathbb{R}^m$ . We describe precisely how to construct such flattening  $\tilde{M}^\varepsilon$  in [3] so that it also satisfies the following curvature estimates for any  $\alpha \geq 1$ .

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\tilde{M}^\varepsilon} |\mathbf{H}_\varepsilon|^\alpha dv &= \lim_{\varepsilon \rightarrow 0} \int_{M^\varepsilon} |\mathbf{H}_\varepsilon|^\alpha dv = \int_M |\mathbf{H}|^\alpha dv \\ \lim_{\varepsilon \rightarrow 0} \int_{\tilde{M}^\varepsilon} |\mathbf{B}_\varepsilon|^\alpha dv &= \lim_{\varepsilon \rightarrow 0} \int_{M^\varepsilon} |\mathbf{B}_\varepsilon|^\alpha dv = \int_M |\mathbf{B}|^\alpha dv \end{aligned}$$

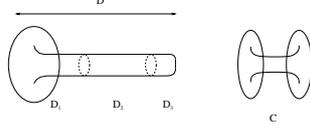
Note that  $H^1(\tilde{M}^\varepsilon)$  tends to isometric to  $H^1(M)$  as  $\varepsilon$  tends to 0.

For more convenience in this section the norms in the different spaces will not be normalized by the volume.

**7.1.2. A small manifold with a prescribed eigenvalue.** Let  $M_1, M_2$  be 2 manifolds of dimension  $m$  isometrically immersed in  $\mathbb{R}^{n+1}$  and  $\lambda, L$  be some positive real numbers with  $\lambda \notin \text{Sp}(M_1)$  and  $L > \max(1, \frac{C(M_1, D_1)(1+\lambda)^2}{d^2})$ , where  $d$  is the distance  $\lambda$  to  $\text{Sp}(M_1)$  and  $C$  is a constant that will be fixed later.

Let  $0 < \eta < 1$  small enough such that the flattening  $\tilde{M}_2^\eta$  of  $M_2$  around the point  $x_2$  exists. Let  $D$  be a smooth hypersurface of revolution in  $\mathbb{R}^{m+1}$ , composed of three parts,  $D_1, D_2, D_3$ , where  $D_1$  is a cylinder of revolution isometric to  $B_0(3) \setminus B_0(2) \subset \mathbb{R}^{m+1}$  at the neighbourhood of one of its boundary component and isometric to  $[0, 1] \times \mathbb{S}^{m-1}$  at the

neighbourhood of its other boundary component, where  $D_2 = [0, L] \times \mathbb{S}^{m-1}$  and where  $D_3$  is a disc of revolution with pole  $x_3$  and isometric to  $[0, 1] \times \mathbb{S}^{m-1}$  at its boundary and to a flat disc at the neighbourhood of  $x_3$ . Let  $C$  be a cylinder of revolution of dimension  $m$  isometric to  $B_0(2) \setminus B_0(1) \subset \mathbb{R}^m$  at the neighbourhood of its 2 boundary components.

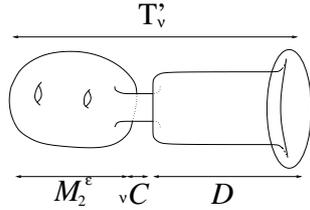


For any  $\nu < \eta/4$  small enough, the gluing of  $\tilde{M}_2^\eta \setminus B_{x_2}(2\nu)$ , of  $\nu C$  and of  $D \setminus B_{x_3}(2\nu)$  along their isometric boundary components exists and is a smoothly immersed submanifold  $T'_\nu$  of dimension  $m$ .

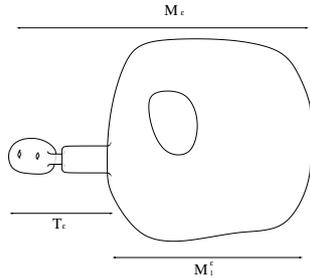
By now classical arguments (see for instance [4]), when  $\nu$  tends to 0, the Dirichlet spectrum of  $T'_\nu$  converges to the disjoint union of the Dirichlet spectrum of  $D$  and of the spectrum of  $\tilde{M}_2^\eta$ . In particular, the limit spectrum has 0 as isolated eigenvalue with multiplicity one. Moreover, since  $\lambda_1^D(T'_\nu)$  has multiplicity one, it depends continuously on  $\nu$ .

We infer that for any  $\varepsilon < \varepsilon_0(\lambda, \tilde{M}_2^\eta, L, D_1, D_3, C)$  there exists  $\nu_\varepsilon < \nu_0(\lambda, \tilde{M}_2^\eta, L, D_1, D_3, C)$  such that  $\lambda_1^D(T'_{\nu_\varepsilon}) = \varepsilon^2 \lambda$  and  $\lambda_2^D(T'_{\nu_\varepsilon}) \geq \Lambda_2(\lambda, \tilde{M}_2^\eta, L, D_1, D_3, C) > 0$ . We set  $T_\varepsilon = \varepsilon T'_{\nu_\varepsilon}$ . Note that for any  $\varepsilon \leq \varepsilon_0$ , we have

$$(7.1) \quad \lambda_1^D(T_\varepsilon) = \lambda \quad \text{and} \quad \lambda_2^D(T_\varepsilon) \geq \frac{\Lambda_2}{\varepsilon^2}$$



**7.1.3. Gluing and control of its curvature.** Now let  $x_1 \in M_1$  and  $\zeta > 0$  fixed. We first assume that  $B_{x_1}(4\zeta) \cup M_1$  is flat. For any  $\varepsilon < \zeta$ , we set  $M_1^\varepsilon = \tilde{M}_1^{4\zeta} \setminus B_{x_1}(3\varepsilon)$ . So we set  $M_\varepsilon$  the  $m$ -submanifold of  $\mathbb{R}^{n+1}$  obtained by gluing  $M_1^\varepsilon$  and  $T_\varepsilon$  along their boundaries in a fixed direction  $\nu \in N_{x_1}M_1$ . Note that  $M_\varepsilon$  is a smooth immersion  $i_\varepsilon$  of  $M_1 \# M_2$  (resp. an embedding when  $M_1$  and  $M_2$  are embedded).



By the computations above, the sequence  $i_\varepsilon(M_1 \# M_2) = M_\varepsilon$  converges to  $M_1$  in Hausdorff distance and we have

$$(7.2) \quad \int_{M_\varepsilon} |\mathbf{H}_\varepsilon|^\alpha dv \leq \varepsilon^{m-\alpha} \left( \int_{M_2^\eta \cup \nu_\varepsilon C \cup D_1 \cup D_3} |\mathbf{H}_\varepsilon|^\alpha dv + C(m, \alpha)L + \int_{M_1} |\mathbf{H}_\varepsilon|^\alpha dv \right)$$

$$(7.3) \quad \int_{M_\varepsilon} |\mathbf{B}_\varepsilon|^\alpha dv \leq \varepsilon^{m-\alpha} \left( \int_{M_2^\eta \cup \nu_\varepsilon C \cup D_1 \cup D_3} |\mathbf{B}_\varepsilon|^\alpha dv + C(m, \alpha)L + \int_{M_1} |\mathbf{B}_\varepsilon|^\alpha dv \right)$$

7.1.4. *Computation of the spectrum of  $M_\varepsilon$ .* We will prove that there exists a sequence  $(\varepsilon_p)_{p \in \mathbb{N}}$  such that  $\varepsilon_p \rightarrow 0$  and the spectrum of  $M_{\varepsilon_p}$  converges to the disjoint union of  $\text{Sp}(M_1)$  and of  $\{\tilde{\lambda}\}$ , where  $\tilde{\lambda}$  satisfies  $\lambda - \frac{C(M_1, D_1)(1+\lambda)}{\sqrt{L}} \leq \tilde{\lambda} \leq \lambda$ . The collapsing of  $M_\varepsilon$  is multiscale, after rescaling of  $T_\varepsilon$ , we get another collapsing sequence of submanifolds with no uniform control of the trace and Sobolev Inequalities, so the cutting and rescaling technique of [4, 13] does not work directly in our case and need to be adapted.

We denote by  $(\lambda_k)_{k \in \mathbb{N}}$  the union with multiplicities of the spectrum of  $M_1$  and of  $\{\lambda\}$ , by  $(\lambda_k^\varepsilon)_{k \in \mathbb{N}}$  the spectrum of  $M_\varepsilon$  and by  $(\mu_k^\varepsilon)_{k \in \mathbb{N}}$  the Dirichlet spectrum of the disjoint union  $T_\varepsilon \cup M_1^\varepsilon$ . By the Dirichlet principle, we have  $\lambda_k^\varepsilon \leq \mu_k^\varepsilon$  for any  $k \in \mathbb{N}$ . It is well known (see for instance [7]) that the Dirichlet spectrum of  $M_1^\varepsilon$  converges to the spectrum of  $M_1$ . We infer that  $\mu_k^\varepsilon \rightarrow \lambda_k$  as  $\varepsilon \rightarrow 0$  and so  $\limsup_{\varepsilon \rightarrow 0} \lambda_k^\varepsilon \leq \lambda_k$  for any  $k \in \mathbb{N}$ .

We set  $\alpha_k = \liminf_{\varepsilon \rightarrow 0} \lambda_k^\varepsilon$ . To get some lower bound on the  $\alpha_k$ , we need some local trace inequalities at the neighbourhood of  $\partial M_1^\varepsilon$ .

#### Local trace Inequalities.

We set  $S_t = \{x \in T_\varepsilon \mid d(x, \partial T_\varepsilon) = -t\}$  for any  $t \leq 0$  and  $S_t = \{x \in M_1^\varepsilon \mid d(x, \partial M_1^\varepsilon) = t\}$  for any  $t \geq 0$ . Obviously we have  $\partial T_\varepsilon = S_0 = \partial M_1^\varepsilon$ . Let  $\varepsilon l$  be the distance in  $M_\varepsilon$  between  $M_1^\varepsilon$  and  $\varepsilon D_2$  (i.e.  $l$  is the distance between the two boundary components of  $D_1$  in  $D_1$ ). Let  $\eta : [-(1+L+l)\varepsilon, \zeta] \rightarrow [0, 1]$  be a smooth function such that  $\eta(t) = 1$  for any  $t \leq \frac{\zeta}{2}$ ,  $\eta(\zeta) = 0$  and  $|\eta'| \leq \frac{4}{\zeta}$ . For any  $r \in [-(1+L+l)\varepsilon, \zeta]$  and any  $f \in H^1(M_\varepsilon)$ , we have :

$$\begin{aligned} \int_{S_r} f^2 d\sigma_r &= \int_{S_0} f(r, u)^2 \theta_\varepsilon(r, u) du \\ &= \int_{S_0} \left( \int_r^\zeta \frac{\partial}{\partial s} [\eta(\cdot) f(\cdot, u)] ds \right)^2 \theta_\varepsilon(r, u) du \\ &= \int_{S_0} \left( \int_r^\zeta \left( \frac{\partial}{\partial s} [\eta(\cdot) f(\cdot, u)] \sqrt{\theta_\varepsilon(r, u)} \right) \frac{ds}{\sqrt{\theta_\varepsilon(r, u)}} \right)^2 \theta_\varepsilon(r, u) du \\ &\leq \int_{S_0} \left( \int_r^\zeta \left( \frac{\partial}{\partial s} [\eta(\cdot) f(\cdot, u)] \right)^2 \theta_\varepsilon(s, u) ds \right) \left( \int_r^\zeta \frac{\theta_\varepsilon(r, u)}{\theta_\varepsilon(s, u)} ds \right) du \end{aligned}$$

For  $r \in [-\varepsilon, \zeta]$  and  $m \geq 3$  :

$$\begin{aligned} \int_r^\zeta \frac{\theta_\varepsilon(r, u)}{\theta_\varepsilon(s, u)} ds &= \int_r^\zeta \frac{\left(1 + \frac{r}{3\varepsilon}\right)^{m-1}}{\left(1 + \frac{s}{3\varepsilon}\right)^{m-1}} ds \\ &= 3\varepsilon \left(1 + \frac{r}{3\varepsilon}\right)^{m-1} \int_{r/(3\varepsilon)}^{\zeta/(3\varepsilon)} \frac{dt}{(1+t)^{m-1}} \\ &= \left(1 + \frac{r}{3\varepsilon}\right)^{m-1} \frac{3\varepsilon}{m-2} \left[ -\frac{1}{(1+t)^{m-2}} \right]_{r/(3\varepsilon)}^{\zeta/(3\varepsilon)} \\ &\leq C(m)(\varepsilon + |r|) \end{aligned}$$

And if  $r \in [-(1+L+l)\varepsilon, -\varepsilon]$  using the fact that  $\theta_\varepsilon(s, u)$  is increasing in  $s$  we have :

$$\int_r^\zeta \frac{\theta_\varepsilon(r, u)}{\theta_\varepsilon(s, u)} ds \leq \int_r^{-\varepsilon} \frac{\theta_\varepsilon(r, u)}{\theta_\varepsilon(s, u)} ds + \int_{-\varepsilon}^\zeta \frac{\theta_\varepsilon(-\varepsilon, u)}{\theta_\varepsilon(s, u)} ds \leq C(m)(\varepsilon + |r|)$$

which gives

$$\begin{aligned} \int_{S_r} f^2 d\sigma_r &\leq C(m)(\varepsilon + |r|) \left[ \left( \int_{S_0} \int_r^\zeta \frac{16}{\zeta^2} f^2(s, u) \theta_\varepsilon(s, u) ds du \right)^{1/2} \right. \\ &\quad \left. + \left( \int_{S_0} \int_r^\zeta \left( \frac{\partial}{\partial s} f(\cdot, u) \right)^2 \theta_\varepsilon(s, u) ds du \right)^{1/2} \right]^2 \\ (7.4) \quad &\leq c(\zeta)(\varepsilon + |r|) \|f\|_{H^1(M_\varepsilon)}^2 \end{aligned}$$

### First estimates on eigenfunctions.

We now use this local trace inequality to get some estimates on the eigenfunctions of  $M_\varepsilon$ . We set  $\varphi : M_\varepsilon \rightarrow [0, 1]$  be a smooth function equal to 1 on  $M^\varepsilon \setminus \bigcup_{-(l+L)\varepsilon \leq s \leq \varepsilon} S_s$ ,

equal to 0 on  $\bigcup_{-(l+L-\sqrt{L})\varepsilon \leq s \leq \varepsilon/2} S_s$  and such that  $|d\varphi| \leq \frac{4}{\varepsilon}$  on  $\bigcup_{\varepsilon/2 \leq s \leq \varepsilon} S_s$  and  $|d\varphi| \leq \frac{2}{\varepsilon\sqrt{L}}$  on  $\bigcup_{-(l+L)\varepsilon \leq s \leq -(l+L-\sqrt{L})\varepsilon} S_s$ . For any  $f_1, f_2 \in H^1(M_\varepsilon)$ , integration of Inequality (7.4) gives us

$$\begin{aligned} \left| \int_{M_\varepsilon} f_1 f_2 dv - \int_{M_\varepsilon} \varphi f_1 \varphi f_2 dv \right| &\leq \int_{M_\varepsilon} |\varphi^2 - 1| |f_1| |f_2| dv \\ &\leq \int_{-(l+L)\varepsilon}^\varepsilon \left( \int_{S_s} |f_1|^2 d\sigma_s \right)^{1/2} \left( \int_{S_s} |f_2|^2 d\sigma_s \right)^{1/2} ds \\ &\leq c(\zeta) \|f_1\|_{H^1(M^\varepsilon)} \|f_2\|_{H^1(M^\varepsilon)} \int_{-(l+L)\varepsilon}^\varepsilon (\varepsilon + |s|) ds \\ (7.5) \quad &\leq c(\zeta, l, L) \varepsilon^2 \|f_1\|_{H^1(M_\varepsilon)} \|f_2\|_{H^1(M_\varepsilon)} \end{aligned}$$

and putting  $I_\varepsilon = [-(l+L)\varepsilon, -(l+L-\sqrt{L})\varepsilon] \cup [\varepsilon/2, \varepsilon]$  we have

$$\begin{aligned} \int_{M^\varepsilon} |d(\varphi f_1)|^2 dv &\leq \int_{M_\varepsilon} (|d\varphi|^2 f_1^2 + 2\varphi f_1 (df_1, d\varphi) + \varphi^2 |df_1|^2) dv \\ &\leq \frac{16}{\varepsilon^2} \int_{I_\varepsilon} \left( \int_{S_s} f_1^2 d\sigma_s \right) ds \\ &\quad + \frac{8}{\varepsilon} \left( \int_{I_\varepsilon} \left( \int_{S_s} f_1^2 d\sigma_s \right) ds \right)^{1/2} \left( \int_{M^\varepsilon} |df_1|^2 dv \right)^{1/2} + \int_{M_\varepsilon} |df_1|^2 dv \\ (7.6) \quad &\leq c(\zeta, l, L) \|f_1\|_{H^1(M_\varepsilon)}^2 \end{aligned}$$

Let  $(f_k^\varepsilon)$  be a  $L^2$ -orthonormal, complete set of eigenfunctions of  $M_\varepsilon$ . For any  $k$ , we set  $\tilde{f}_k^\varepsilon$  the function on  $M_1$  equal to  $\varphi f_k^\varepsilon$  on  $M_1^\varepsilon$  and extended by 0. By Inequality (7.6), we have  $\|\tilde{f}_k^\varepsilon\|_{H^1(M_1)}^2 \leq c(M_1)(1 + \lambda_k)$  for  $\varepsilon$  small enough. We infer by diagonal extraction that there exists some sequences  $(\varepsilon_p)_{p \in \mathbb{N}}$  and  $(h_k)_{k \in \mathbb{N}} \in H^1(M_1)^\mathbb{N}$  such that  $\lambda_k^{\varepsilon_p} \rightarrow \alpha_k$  and  $(\tilde{f}_k^{\varepsilon_p})_p$  converges weakly in  $H^1(M_1)$  and strongly in  $L^2(M_1)$  to  $h_k$ , for any  $k$ . It is easy to prove

that  $h_k$  is a weak solution of  $\Delta h_k = \alpha_k h_k$  on  $H^1(M_1 \setminus \{x_1\}) = H^1(M_1)$  (see for instance [13]). By elliptic regularity, either  $h_k = 0$  or  $\alpha_k$  is an eigenvalue of  $M_1$ .

Estimate (7.4) will not be good enough to control the eigenvalues  $\lambda_k^\varepsilon$  whose eigenfunctions tends to concentrate on  $T_\varepsilon$  so we need to improve it.

**Improved estimate on eigenfunctions.**

Let  $k_0 \in \mathbb{N}$  such that  $\lambda_{k_0} = \lambda$ . Since  $D_2$  isometric to  $[0, L] \times \mathbb{S}^{m-1}$ , any  $f_k^{\varepsilon_p}$  can be seen as a function on  $[0, \varepsilon_p L] \times \varepsilon_p \mathbb{S}^{m-1}$ . For any  $f = \sum_{i \leq k_0} \beta_i f_i^{\varepsilon_p} \in \text{Vect}\{f_i^{\varepsilon_p} \mid i \leq k_0\}$ , we define the rescaling  $F_p$  on  $c = [0, 1] \times \mathbb{S}^{m-1}$  by  $F_p(t, x) = \varepsilon_p^{\frac{m}{2}-1} L^{-\frac{1}{2}} f(\varepsilon_p L t, \varepsilon_p x)$ . By Inequality (7.4), we have

$$\begin{aligned} \int_c F_p^2 dv &= \frac{1}{\varepsilon_p^2 L^2} \int_{\varepsilon_p D_2} f^2 dv = \frac{1}{\varepsilon_p^2 L^2} \int_{-\varepsilon(L+l)}^{-\varepsilon l} \left( \int_{S_r} f^2 d\sigma_r \right) dr \\ &\leq \frac{c(M_1)}{\varepsilon_p^2 L^2} \left( \int_{-\varepsilon(L+l)}^{-\varepsilon l} (\varepsilon + |r|) dr \right) \|f\|_{H^1(M_\varepsilon)}^2 \\ &= c(\zeta) \left( \frac{l+1}{L} + \frac{1}{2} \right) (1 + \lambda) \|f\|_2^2 \end{aligned}$$

$$\begin{aligned} \int_{\{0\} \times \mathbb{S}^{m-1}} F_p^2 dv &= \frac{1}{L \varepsilon_p} \int_{\varepsilon_p(D_1 \cap D_2)} f^2 dv = \frac{1}{L \varepsilon_p} \int_{S_{-d\varepsilon_p}} f^2 d\sigma_{-d\varepsilon_p} \\ (7.7) \quad &\leq \frac{c(\zeta)}{L} (1+l)(1+\lambda) \|f\|_2^2 = \frac{c(M_1, D_1)}{L} (1+\lambda) \|f\|_2^2 \end{aligned}$$

and

$$\begin{aligned} \int_{\{1\} \times \mathbb{S}^{m-1}} F_p^2 &= \frac{1}{L \varepsilon_p} \int_{\varepsilon_p(D_3 \cap D_2)} f^2 = \frac{1}{L \varepsilon_p} \int_{S_{-(l+L)\varepsilon_p}} f^2 d\sigma_{-(l+L)\varepsilon_p} \\ (7.8) \quad &\leq c(\zeta) \left( 1 + \frac{1+l}{L} \right) (1+\lambda) \|f\|_2^2 \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_c |dF_p|^2 dv &= \int_{[0,1] \times \mathbb{S}^{m-1}} |dF_p|^2 dt dx = \int_{[0,1] \times \mathbb{S}^{m-1}} \varepsilon_p^m L \left( \frac{\partial f}{\partial s} \right)^2 (\varepsilon_p L t, \varepsilon_p x) dt dx \\ &\quad + \int_{[0,1] \times \mathbb{S}^{m-1}} \varepsilon_p^m L^{-1} |d^{\varepsilon_p \mathbb{S}^{m-1}} f|^2 (\varepsilon_p L t, \varepsilon_p x) dt dx \\ &= \int_{\varepsilon_p D_2} \left( \frac{\partial f}{\partial s} \right)^2 dv + \frac{1}{L^2} \int_{\varepsilon_p D_2} |d^{\varepsilon_p \mathbb{S}^{m-1}} f|^2 dv \\ &\leq \int_{M_\varepsilon} |df|^2 dv \leq \lambda \|f\|_2^2 \end{aligned}$$

Note that we have used the fact that  $L > 1$ . So we can assume that there exists  $F_\infty \in H^1(c)$  such that the sequence  $(F_p)$  converges to  $F_\infty$  weakly in  $H^1(c)$  and strongly in  $L^2(c)$ . We set

$$\begin{aligned} j_p(t) &= \int_{\mathbb{S}^{m-1}} F_p(t, x) dx \text{ and } j_\infty(t) = \int_{\mathbb{S}^{m-1}} F_\infty(t, x) dx, \text{ we have } j_p, j_\infty \in H^1([0, 1]) \text{ (with} \\ j_p'(t) &= \int_{\mathbb{S}^{m-1}} \frac{\partial F_p}{\partial t}(t, x) dx), j_p \rightarrow j_\infty \text{ strongly in } L^2([0, 1]) \text{ and weakly in } H^1([0, 1]). \text{ By the} \end{aligned}$$

estimates 7.7 and 7.8 and the compactness of the trace operator on  $c$ , we have

$$|j_\infty(0)| \leq \frac{c(\zeta)\sqrt{(1+l)(1+\lambda)}\|f\|_2}{\sqrt{L}}$$

and

$$|j_\infty(1)| \leq c(\zeta)\sqrt{1 + \frac{1+l}{L}}\sqrt{1+\lambda}\|f\|_2$$

Hence  $\ell(t) = j_\infty(t) - (j_\infty(0) + (j_\infty(1) - j_\infty(0))t)$  is in  $H_0^1([0, 1])$ . For any  $\psi \in C_c^\infty([0, 1])$ , we set  $\psi_p(t, x) = \varepsilon_p L \psi(\frac{t}{\varepsilon_p L})$  seen as a function in  $H_0^1(\varepsilon_p D_2)$ . We have

$$\begin{aligned} \int_0^1 \ell' \psi' dt &= \int_0^1 j_\infty' \psi' dt \\ &= \lim_p \int_0^1 j_p'(t) \psi'(t) dt = \lim_p \int_c \frac{\partial F_p}{\partial t} \psi' dv = \lim_p \frac{1}{\varepsilon_p^{\frac{m}{2}} \sqrt{L}} \int_{\varepsilon_p D_2} \langle df, d\psi_p \rangle dt dx \\ &= \lim_p \sum_i \frac{\beta_i \lambda_i^{\varepsilon_p}}{\varepsilon_p^{\frac{m}{2}} \sqrt{L}} \int_{\varepsilon_p D_2} f_i^{\varepsilon_p} \psi_p dt dx = \sum_i \alpha_i \beta_i L^2 \lim_p \varepsilon_p^2 \int_c F_{i,p} \psi dt dx = 0, \end{aligned}$$

where  $F_{i,p}(t, x) = \varepsilon_p^{\frac{m}{2}-1} L^{-\frac{1}{2}} f_i^{\varepsilon_p}(\varepsilon_p L t, \varepsilon_p x)$ . We infer  $\ell$  is harmonic and in  $H_0^1([0, 1])$ , i.e.  $\ell = 0$  and  $j_\infty(t) = j_\infty(0) + (j_\infty(1) - j_\infty(0))t$  on  $[0, 1]$ . Since the Poincaré inequality on  $\mathbb{S}^{m-1}$  gives us

$$\begin{aligned} \int_{\mathbb{S}^{m-1}} F_p(t, x)^2 dx &\leq \frac{1}{\text{Vol } \mathbb{S}^{m-1}} \left( \int_{\mathbb{S}^{m-1}} F_p(t, x) dx \right)^2 + \frac{1}{m-1} \int_{\mathbb{S}^{m-1}} |d_{\mathbb{S}^{m-1}} F_p|^2 dx \\ &\leq \frac{1}{\text{Vol } \mathbb{S}^{m-1}} j_p^2(t) + \frac{\varepsilon_p}{(m-1)L} \int_{\varepsilon_p \mathbb{S}^{m-1}} |d_{\varepsilon_p \mathbb{S}^{m-1}} f|^2(\varepsilon_p L t, x) dx, \end{aligned}$$

we get that

$$\begin{aligned} \frac{1}{L \varepsilon_p^2} \int_{[0, \varepsilon_p \sqrt{L}] \times \varepsilon_p \mathbb{S}^{m-1}} f^2 dv &= L \int_{[0, \frac{1}{\sqrt{L}}] \times \mathbb{S}^{m-1}} F_p^2 dt dx \\ &\leq \frac{L}{\text{Vol } \mathbb{S}^{m-1}} \int_0^{\frac{1}{\sqrt{L}}} j_p^2(t) dt + \frac{1}{(m-1)L} \int_{[0, \varepsilon_p \sqrt{L}] \times \varepsilon_p \mathbb{S}^{m-1}} |d_{\varepsilon_p \mathbb{S}^{m-1}} f|^2 dv \\ &\leq \frac{L}{\text{Vol } \mathbb{S}^{m-1}} \int_0^{\frac{1}{\sqrt{L}}} j_p^2(t) dt + \frac{\lambda}{(m-1)L} \|f\|_2^2 \\ &\rightarrow \frac{L}{\text{Vol } \mathbb{S}^{m-1}} \int_0^{\frac{1}{\sqrt{L}}} j_\infty(t)^2 dt + \frac{\lambda}{(m-1)L} \|f\|_2^2 \end{aligned}$$

Now a straightforward computation shows that  $\int_0^{\frac{1}{\sqrt{L}}} j_\infty(t)^2 dt \leq \frac{c(\zeta)(1+l)(1+\lambda)\|f\|_2^2}{L^{3/2}}$

and for  $p$  great enough

$$(7.9) \quad \frac{1}{L \varepsilon_p^2} \int_{[0, \varepsilon_p \sqrt{L}] \times \varepsilon_p \mathbb{S}^{m-1}} f^2 dv \leq \frac{c(\zeta)(1+\lambda)(1+l)\|f\|_2^2}{\sqrt{L}}$$

Note that this estimate is better than which could be deduced from (7.4).

**Control of the limit spectra.**

If the family  $(h_i)_{i < k_0}$  is not free in  $L^2(M_1)$ , then either one  $h_i$  is null or they are all eigenfunctions of  $M_1$ . Since the eigenspaces are in direct sum, we infer that there exists a not free subfamily of  $(h_i)_{i < k_0}$  included in a given eigenspace of  $M_1$ . In other words there exists  $\mu \leq \lambda_{k_0-1}$  and  $(\beta_i)_{i < k_0} \in \mathbb{R}^{k_0} \setminus \{0\}$  such that  $\sum_{i=0}^{k_0-1} \beta_i^2 = 1$ ,  $\sum_{i=0}^{k_0-1} \beta_i h_i = 0$  and  $\alpha_i = \mu$  for any  $i$  such that  $\beta_i \neq 0$  (we recall that  $\lambda_k^{\varepsilon_p} \rightarrow \alpha_k$  for any  $k \in \mathbb{N}$  and  $\lambda_{k_0} = \lambda$ ). Setting  $u_{\varepsilon_p} = \sum_{i=0}^{k_0-1} \beta_i f_i^{\varepsilon_p}$  we then have

$$\begin{aligned} & \left| \int_{T_{\varepsilon_p}} |d(\varphi u_{\varepsilon_p})|^2 dv - \mu \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \right| = \left| \int_{T_{\varepsilon_p}} (|d\varphi|^2 u_{\varepsilon_p}^2 + \varphi^2 u_{\varepsilon_p} \Delta u_{\varepsilon_p}) dv - \mu \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \right| \\ & \leq \frac{4}{L \varepsilon_p^2} \int_{\varepsilon_p([-l+L, -(l+L-\sqrt{L})] \times \mathbb{S}^{m-1})} u_{\varepsilon_p}^2 dv + \int_{T_{\varepsilon_p}} \sum_{i,j} (\lambda_i^{\varepsilon_p} - \mu) \beta_i \varphi f_i^{\varepsilon_p} \beta_j \varphi f_j^{\varepsilon_p} dv \end{aligned}$$

Since  $\varepsilon_p([-l+L, -(l+L-\sqrt{L})] \times \mathbb{S}^{m-1})$  is isometric to  $[0, \varepsilon_p \sqrt{L}] \times \varepsilon_p \mathbb{S}^{m-1}$  we deduce from 7.9 that

$$(7.10) \quad \begin{aligned} & \left| \int_{T_{\varepsilon_p}} |d(\varphi u_{\varepsilon_p})|^2 dv - \mu \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \right| \\ & \leq \frac{c(\zeta)(1+l)(1+\lambda)}{\sqrt{L}} \|u_{\varepsilon_p}\|_2^2 + \int_{T_{\varepsilon_p}} \sum_{i,j} (\lambda_i^{\varepsilon_p} - \mu) \beta_i \varphi f_i^{\varepsilon_p} \beta_j \varphi f_j^{\varepsilon_p} dv \end{aligned}$$

We recall that  $\varphi f_k^{\varepsilon_p}|_{M_1}$  converges strongly in  $L^2(M_1)$  and then  $\varphi u_{\varepsilon_p}$  converges strongly in  $L^2(M_1)$  to  $\sum_{i=0}^{k_0-1} \beta_i h_i = 0$ . Moreover from (7.5) we have  $\left| \int_{M_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv - 1 \right| \rightarrow 0$ . Then we deduce that  $\int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \rightarrow 1$ . Since  $\varphi u_{\varepsilon_p}|_{T_{\varepsilon_p}} \in H_0^1(T_{\varepsilon_p})$  and since by construction of  $T_{\varepsilon_p}$ , we have  $\lambda_1^D(T_{\varepsilon_p}) = \lambda = \lambda_{k_0}$ , we then have  $\int_{T_{\varepsilon_p}} |d(\varphi u_{\varepsilon_p})|^2 dv \geq \lambda \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv$ . Then for  $p$  large enough

$$\begin{aligned} 0 & < (\lambda - \lambda_{k_0-1}) \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \leq (\lambda - \mu) \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \\ & \leq \int_{T_{\varepsilon_p}} |d(\varphi u_{\varepsilon_p})|^2 dv - \mu \int_{T_{\varepsilon_p}} (\varphi u_{\varepsilon_p})^2 dv \end{aligned}$$

From now we assume that  $C(M_1, D_1) > \sqrt{c(\zeta)}(1+l)$ . Letting  $p$  tend to  $\infty$  in (7.10) we get that  $d \leq \lambda - \lambda_{k_0-1} \leq \frac{C(M_1, D_1)(1+\lambda)}{\sqrt{L}}$  which contradicts the choice made on  $L$ .

We infer that  $(h_i)_{i < k_0}$  is free in  $L^2(M_1)$ . This implies that  $\alpha_i$  is an eigenvalue of  $M_1$  and  $h_i$  is an eigenfunction of  $M_1$  for any  $i < k_0$ . Since  $\alpha_i = \lim \lambda_i^{\varepsilon_p} \leq \lambda_i = \lambda_i(M_1)$  for any  $i < k_0$ , we infer that  $\alpha_i = \lambda_i$  for any  $i < k_0$  and that the  $(h_i)_{i < k_0}$  is a basis of the eigenspaces of  $M_1$  associated to the first  $k_0$  eigenvalues. By the same way, if  $h_{k_0} \neq 0$ , then  $\alpha_{k_0} = \lambda_{k_0-1}$  (since it is an eigenvalue of  $M_1$  less than  $\lambda$ ) and so the family  $(h_i)_{i \leq k_0}$  is not free. The same argument as above gives a contradiction. So we have that  $h_{k_0} = 0$ .

Assume that there exists another index  $k_1 \neq k_0$  such that  $h_{k_1} = 0$ . Then, Inequality (7.5) gives that  $\int_{T_{\varepsilon_p}} \varphi f_{k_0}^{\varepsilon_p} \varphi f_{k_1}^{\varepsilon_p} dv \rightarrow 0$ ,  $\int_{T_{\varepsilon_p}} (\varphi f_{k_0}^{\varepsilon_p})^2 dv \rightarrow 1$  and  $\int_{T_{\varepsilon_p}} (\varphi f_{k_1}^{\varepsilon_p})^2 dv \rightarrow 1$  and Inequality (7.6) gives that  $\int_{T_{\varepsilon_p}} |d(\varphi f_{k_0}^{\varepsilon_p})|^2 dv$  and  $\int_{T_{\varepsilon_p}} |d(\varphi f_{k_1}^{\varepsilon_p})|^2 dv$  remain bounded as  $\varepsilon_p \rightarrow 0$ . We set  $g_p$  a unitary eigenfunction of  $T_{\varepsilon_p}$  for the Dirichlet problem associated to the eigenvalue  $\lambda$ . If we set  $(\varphi f_{k_0}^{\varepsilon_p})|_{T_{\varepsilon_p}} = \beta_{k_0}^p g_p + \gamma_{k_0}^p$  and  $(\varphi f_{k_1}^{\varepsilon_p})|_{T_{\varepsilon_p}} = \beta_{k_1}^p g_p + \gamma_{k_1}^p$ , with  $\beta_{k_0}^p, \beta_{k_1}^p \in \mathbb{R}$  and  $\gamma_{k_0}^p, \gamma_{k_1}^p$  orthogonal to  $g_p$  in  $H_0^1(T_{\varepsilon_p})$ . The previous relations and the lower bound on  $\lambda_2^D(T_{\varepsilon_p})$  imply that

$$\int_{T_{\varepsilon_p}} |d(\varphi f_{k_0}^{\varepsilon_p})|^2 dv \geq \lambda(\beta_{k_0}^p)^2 + \lambda_2^D(T_{\varepsilon_p}) \|\gamma_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2 \geq (\beta_{k_0}^p)^2 \lambda + \frac{\Lambda_2}{\varepsilon_p^2} \|\gamma_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2.$$

By the same way,  $(\beta_{k_1}^p)^2 \lambda + \frac{\Lambda_2}{\varepsilon_p^2} \|\gamma_{k_1}^p\|_{L^2(T_{\varepsilon_p})}^2$  is bounded, and so  $\|\gamma_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2$  and  $\|\gamma_{k_1}^p\|_{L^2(T_{\varepsilon_p})}^2$  tend to 0 with  $\varepsilon_p$ . Now, we have  $(\beta_{k_0}^p)^2 + \|\gamma_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2 \rightarrow 1$  and so  $|\beta_{k_0}^p| \rightarrow 1$ . Up to change of sign of  $f_{k_0}^{\varepsilon_p}$ , we can assume that  $\beta_{k_0}^p \rightarrow 1$ . By the same way, we have  $|\beta_{k_1}^p| \rightarrow 1$ , which contradicts the fact that  $\int_{T_{\varepsilon_p}} \varphi f_{k_0}^{\varepsilon_p} \varphi f_{k_1}^{\varepsilon_p} dv \rightarrow 0$ . We infer that for any  $k \in \mathbb{N} \setminus \{k_0\}$  we have that  $\alpha_k$  is an eigenvalue of  $M_1$ . Moreover, if we decompose  $(\varphi f_k^{\varepsilon_p})|_{T_{\varepsilon_p}} = \beta_k^p g_p + \gamma_k^p$  as above, Inequality (7.6) implies that  $(\beta_k^p)^2 + \frac{\Lambda_2}{\varepsilon_p^2} \|\gamma_k^p\|_{L^2(T_{\varepsilon_p})}^2$  remains bounded and so we have  $\lim_{p \rightarrow \infty} \|\gamma_k^p\|_{L^2(T_{\varepsilon_p})}^2 = 0$  and Inequality (7.5) gives

$$0 = \lim_{p \rightarrow \infty} \int_{M_\varepsilon} f_{k_0}^{\varepsilon_p} f_k^{\varepsilon_p} dv = \lim_{p \rightarrow \infty} \beta_k^p \beta_{k_0}^p = \lim_{p \rightarrow \infty} \beta_k^p$$

and so  $(\varphi f_k^{\varepsilon_p})|_{T_{\varepsilon_p}} \rightarrow 0$  in  $L^2(T_{\varepsilon_p})$  for any  $k \neq k_0$ . Once again, Inequality (7.5) gives us that for any  $k, k' \in \mathbb{N} \setminus \{k_0\}$ , we have

$$\begin{aligned} \int_{M_1} h_k h_{k'} dv &= \lim_{p \rightarrow \infty} \int_{M_1} \varphi f_{k_0}^{\varepsilon_p} \varphi f_{k'}^{\varepsilon_p} dv = \lim_{p \rightarrow \infty} \int_{M_1^{\varepsilon_p}} \varphi f_{k_0}^{\varepsilon_p} \varphi f_{k'}^{\varepsilon_p} dv \\ &= \lim_{p \rightarrow \infty} \int_{M_{\varepsilon_p}} \varphi f_{k_0}^{\varepsilon_p} \varphi f_{k'}^{\varepsilon_p} dv = \delta_{kk'} \end{aligned}$$

From the min-max principle, it gives that we have  $\alpha_k \geq \lambda_k$  for any  $k \neq k_0$ . Since we have  $\alpha_k \leq \lambda_k$  for any  $k \in \mathbb{N}$ , we infer that for any  $k \in \mathbb{N} \setminus \{k_0\}$  we have  $\alpha_k = \lambda_k$ . Moreover we have  $\alpha_{k_0} \leq \lambda_{k_0} = \lambda$ . Finally, since  $\varphi f_{k_0}^{\varepsilon_p}|_{T_{\varepsilon_p}} \in H_0^1(T_{\varepsilon_p})$ , Inequality (7.10), applied to  $f = f_{k_0}^{\varepsilon_p}$  and  $\mu = \alpha_{k_0}$  gives that

$$\begin{aligned} (\lambda - \alpha_{k_0}) \int_{T_{\varepsilon_p}} (\varphi f_{k_0}^{\varepsilon_p})^2 dv &\leq \int_{T_{\varepsilon_p}} |d(\varphi f_{k_0}^{\varepsilon_p})|^2 dv - \alpha_{k_0} \int_{T_{\varepsilon_p}} (\varphi f_{k_0}^{\varepsilon_p})^2 dv \\ &\leq \frac{c(\zeta)(1+l)(1+\lambda)}{\sqrt{L}} + \int_{T_{\varepsilon_p}} (\lambda_{k_0}^{\varepsilon_p} - \alpha_{k_0})(\varphi f_{k_0}^{\varepsilon_p})^2 dv \end{aligned}$$

Now we have seen that  $\tilde{f}_{k_0}^{\varepsilon_p}$  tends to  $h_{k_0} = 0$  in  $L^2(M_1)$ . It follows that from (7.5),  $\lim_{p \rightarrow \infty} \int_{T_{\varepsilon_p}} (\varphi f_{k_0}^{\varepsilon_p})^2 dv = 1$  and we deduce that  $\alpha_{k_0} \in \left[ \lambda - \frac{c(\zeta)(1+l)(1+\lambda)}{\sqrt{L}}, \lambda \right]$ .

At this stage of the proof, we get that for any sequence  $(\varepsilon_k)$  such that  $\lim_k \varepsilon_k = 0$ , the sequence  $(M_{\varepsilon_k})_{k \in \mathbb{N}}$  of immersions of  $M_1 \# M_2$  satisfies the point (1), (2) and (4) of Theorem 2.1 and we have  $\lim_k \text{Sp}(M_{\varepsilon_k}) \subset \text{Sp}(M_1) \cup \left[ \lambda - \frac{c(\zeta)(1+\lambda)(1+l)}{\sqrt{L}}, \lambda \right]$ .

By an easy diagonal extraction taking  $L = i$ , there exists a subsequence  $(\varepsilon_{p(i)})$ , such that  $\int_{M_{\varepsilon_{p(i)}}} |H|^\alpha dv \rightarrow \int_{M_1} |B|^\alpha dv$  and  $\int_{M_{\varepsilon_{p(i)}}} |B|^\alpha dv \rightarrow \int_{M_1} |B|^\alpha dv$  for any  $\alpha < m$  (see (7.2) and (7.3)) and we get Theorem 2.1 for  $F = \text{Sp}(M_1) \cup \{\lambda\}$  when  $B_{x_1}(4\zeta)$  is flat in  $M_1$ .

Now if we assume that  $B_{x_1}(4\zeta)$  is not flat, we use the fact that  $\text{Sp}(\tilde{M}_1^\zeta)$  converges to  $\text{Sp}(M_1)$  and by a new diagonal extraction we get the desired result.

**7.2. End of the proof of Theorem 2.1.** Let  $F$  be a closed subset containing  $\text{Sp}(M_1)$ . As explained in section 2, there exists an increasing sequence of finite sets  $F_N$  such that  $F_N \subset [0, N] \cap F \subset F_{N,1/N}$ . We can assume that  $\text{Sp}(M_1) \cap [0, N]$  is contained in  $F_N$ . Thus  $F_N = G_N \cup (\text{Sp}(M_1) \cap [0, N])$  where  $G_N$  and  $\text{Sp}(M_1) \cap [0, N]$  are disjoint and  $G_N$  is finite. First we have  $F = \text{LimSet}_{N \rightarrow \infty} F_N$ ,  $F_N$  converges to  $F$  for the distance of Attouch-Wetts-Hausdorff as well as  $G_N \cup \text{Sp}(M_1)$  converges to  $F$ .

Now, iterating the construction (with  $M_2^\eta$  replaced by  $\mathbb{S}^m$  for any supplementary gluing) we obtain a sequence  $M_{N,\varepsilon_p}$  such that  $\text{Sp}(M_{N,\varepsilon_p})$  converges to  $G_N \cup \text{Sp}(M_1)$  when  $p$  tends to infinity. Since  $G_N \cup \text{Sp}(M_1)$  converges to  $F$  when  $N$  tends to infinity, by diagonal extraction there exists subsequences  $(N_k)_k$  and  $(\varepsilon_k)_k$  such that  $\text{LimSet}_{k \rightarrow \infty} \text{Sp}(M_{N_k, \varepsilon_k}) = F$  and the point (2) of the theorem 2.1 on the curvatures is true.

In the case  $\alpha = m$ , the limit  $\int_{M_\varepsilon} |B|^m dv$  depends on  $L$  and so we are only able to get a weak version of Theorem 2.1 with  $F = \text{Sp}(M_1) \cup G$ , where  $G$  is a finite set whose elements are known up to an error term and where the point (2) is replaced by  $\int_{i_k(M_1 \# M_2)} |B|^m dv$  is bounded by a constant that depend on  $M_1, M_2, D_1, D_3, G$  and on the error term.

## REFERENCES

- [1] E. AUBRY, *Pincement sur le spectre et le volume en courbure de Ricci positive*, Ann. Sci. École Norm. Sup. (4) **38** (2005), n3, p. 387–405.
- [2] E. AUBRY, J.-F. GROSJEAN, J. ROTH, *Hypersurfaces with small extrinsic radius or large  $\lambda_1$  in Euclidean spaces*, preprint (2010) arXiv:1009.2010v1.
- [3] E. AUBRY, J.-F. GROSJEAN, *Metric shape of hypersurfaces with small extrinsic radius or large  $\lambda_1$* , preprint (2012).
- [4] C. ANNÉ, *Spectre du Laplacien et écrasement d'anses*, Ann. Sci. École Norm. Sup. (4) **20** (1987), p. 271–280.
- [5] G. BEER, *Topologies on closed and closed convex sets*, Mathematics and its Applications, **268**. Kluwer Academic Publishers Group, Dordrecht, 1993.
- [6] B. COLBOIS, J.-F. GROSJEAN, *A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space*, Comment. Math. Helv. **82**, (2007), p. 175–195.
- [7] G. COURTOIS, *Spectrum of manifolds with holes*, J. Funct. Anal. **134** (1995), no.1, p. 194–221.
- [8] T. HASANIS, D. KOUTROFIOTIS, *Immersion of bounded mean curvature*, Arc. Math. **33**, (1979), p. 170–171.
- [9] S. DELLADIO *On Hypersurfaces in  $\mathbb{R}^{n+1}$  with Integral Bounds on Curvature*, J. Geom. Anal., 11: 17741, 2000.

- [10] J.-F. GROSJEAN, J. ROTH *Eigenvalue pinching and application to the stability and the almost umbilicity of hypersurfaces*, to appear in Math. Z.
- [11] J. H. MICHAEL, L. M. SIMON, *Sobolev and mean-value inequalities on generalized submanifolds of  $R^n$* , Comm. Pure Appl. Math. **26** (1973), p. 361–379.
- [12] R.C. REILLY, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52**, (1977), p. 525–533.
- [13] J. TAKAHASHI, *Collapsing of connected sums and the eigenvalues of the Laplacian*, J. Geom. Phys. **40** (2002), p. 201–208.

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