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Jacques Duparc[1], Olivier Finkel[2], and Jean-Pierre Ressayre[2]

[1] Faculty of Business and Economics, University of Lausanne, CH-1015 - Lausanne
[2] Equipe de Logique Mathématique, Institut de Mathématiques de Jussieu, CNRS et Université Paris 7, U.F.R. de Mathématiques, case 7012, site Chevaleret 75205 Paris Cedex 13, France.
jacques.duparc@unil.ch, finkel@math.univ-paris-diderot.fr, ressayre@math.univ-paris-diderot.fr

Dedicated to Victor Selivanov on the occasion of his sixtieth birthday.

Abstract. We describe the Wadge hierarchy of the ω-languages recognized by deterministic Petri nets. This is an extension of the celebrated Wagner hierarchy which turned out to be the Wadge hierarchy of the ω-regular languages. Petri nets are an improvement of automata. They may be defined as partially blind multi-counter automata. We show that the whole hierarchy has height ω^{ω^2}, and give a description of the restrictions of this hierarchy to some fixed number of partially blind counters.

1 Introduction

The languages of infinite words, also called ω-languages, accepted by finite automata were first studied by Büchi to prove the decidability of the monadic second order theory of one successor over the integers. Since then regular ω-languages have been much studied and many applications have been found for specification and verification of non-terminating systems, see [44, 43, 30] for many results and references. The acceptance of infinite words by other finite machines, like pushdown automata, multicounter automata, Petri nets, Turing machines, have also been considered, see [43, 9, 4, 20].

Since the set Σ^ω of infinite words over a finite alphabet Σ is naturally equipped with the Cantor topology, a way to study the complexity of languages of infinite words accepted by finite machines is to study their topological complexity and firstly to locate them with regard to the Borel and the projective hierarchies, and next to the Wadge hierarchy which is a great refinement of the Borel hierarchy. This work was analysed in [41, 42, 44, 40, 9, 26, 43, 38, 36].

It is well known that every ω-language accepted by a deterministic Büchi automaton is a $\mathbf{\Pi}^0_2$-set, and that an ω-language accepted by a non-deterministic Büchi (or Muller) automaton is a $\mathbf{\Delta}^0_3$-set. The Borel hierarchy of regular ω-languages is then determined. Moreover Landweber proved that one can effectively determine the Borel complexity of a regular ω-language accepted by a given Muller or Büchi automaton, see [25, 44, 43, 30]. The Wadge hierarchy obtained on ω-regular languages is called the Wagner hierarchy. It was completely described by Klaus Wagner [47]; its length is the ordinal ω^ω. Wagner gave an

automaton-like characterization of this hierarchy, based on the notions of chain and superchain, together with an algorithm to compute the Wadge (Wagner) degree of any given ω-regular language. Later, Wilke and Yoo proved that the Wadge degree of an ω-regular language may be computed in polynomial time [48]. This hierarchy was thouroughly studied by Carton and Perrin in [2, 3], and by Victor Selivanov in [32, 33, 37].

Since there are various classes of finite machines recognizing ω-languages, each of them yields a countable sub-hierarchy of the Wadge hierarchy. Since the 1980's it has been an endeavor to describe these sub-hierarchies. It started with the work of Klaus Wagner on the ω-regular languages – although Wagner was unaware at the time of the connections between the Wadge hierarchy and his own work. The Wadge hierarchy of deterministic context-free ω-languages was determined, together with its length: $\omega^{(\omega^2)}$ [8, 7]. The problem whether this hierarchy is decidable remains open. The Wadge hierarchy induced by the subclass of deterministic one blind counter automata was determined in an effective way [11], and other partial decidability results have been obtained [12]. It was then proved that the Wadge hierarchy of context-free ω-languages is the same as the one of effective analytic sets [3], [15, 20]. Intriguingly, the only Wadge class for which one can decide whether a given context-free ω-language belongs to it or not, is the rudimentary $\{\emptyset\}$ [12–14]. In particular, one cannot decide whether a non-deterministic pushdown automaton is universal or not. This latter decision problem is actually Π^1_2-complete, hence located at the second level of the analytical hierarchy and "highly undecidable", [18]. Moreover the second author has shown that the topological complexity of a context-free ω-language may depend on the models of set theory, [17]. And some similar results hold for ω-languages accepted by 2-tape Büchi automata, [16, 17]. The Wadge hierarchy of ω-languages of deterministic Turing machines was determined by Victor Selivanov, [34, 35].

Among the many accepting devices that are more powerful than the finite automata to recognize ω-languages, are the Petri nets which are used for the description of distributed systems. A Petri net is a directed bipartite graph, in which the nodes represent transitions and places, and any distribution of tokens over the places defines a configuration of the net. Petri nets are an improvement of automata, because they may be defined as *partially blind multicounter automata* [21]. Petri nets have been extensively examined, particularly in concurrency theory (see for instance [10] [31]). The infinite behavior of Petri nets was first studied by Valk [45], and of deterministic Petri nets, by Carstensen [1].

In this paper, we first consider deterministic blind multicounter automata (corresponding to deterministic Petri nets) and the ω-languages that they accept when they are equipped with a Muller acceptance condition. This forms the class of deterministic Petri net ω-languages denoted $\mathcal{L}^3_{\omega dt}$ in [1].

We describe the Wadge hierarchy of the ω-languages recognized by deterministic Petri nets. This is an extension of the celebrated Wagner hierarchy of

[3] The effective analytic sets (denoted Σ^1_1) is the class of all the ω-languages recognized by (non-deterministic) Turing machines.

the ω-regular languages. We show that the whole hierarchy has height ω^{ω^2}, and give a description of the restrictions of this hierarchy to some fixed number of partially blind counters.

2 Recalls on ω-languages, automata and Petri nets

We assume the reader to be familiar with the theories of formal languages and ω-regular languages (see [22, 44, 30]). We recall some of the definitions and results involving the ω-regular languages.

Through along the paper, we assume Σ to be any finite set, called the alphabet. A finite word (string) over Σ is any sequence of the form $u = a_1 \dots a_k$, where $k \in \mathbb{N}$ and $a_i \in \Sigma$ holds for each $i \leq k$. Notice that when $k = 0$, u is the empty word denoted by ε. We denote by $|u|$ the length of the word u (here $|u| = k$). We write $u(i) = a_i$ and $u[i] = u(1) \dots u(i)$ for $i \leq k$ and $u[0] = \varepsilon$. The set of all finite words over Σ is denoted Σ^*.

An infinite word over Σ is some sequence of the form $x = a_1 a_2 \dots a_n \dots$ where $a_i \in \Sigma$ holds for all non-zero integer i. These infinite words are called ω-words for their length corresponds to ω: the first infinite ordinal. An infinite word x over Σ can be viewed as a mapping $x : \mathbb{N} \longrightarrow \Sigma$, so we write $x = x(1)x(2)\dots$ and $x[n] = x(1)x(2)\dots x(n)$ for its prefix of length n[4]. We write Σ^ω for the set of all ω-words over the alphabet Σ, so that an ω-language over the alphabet Σ is nothing but a subset of Σ^ω.

As usual, the concatenation of two finite words u and v is denoted uv. It naturally extends to the concatenation of a finite word u and an ω-word x to give the ω-words $y = ux$ defined by: $y(k) = u(k)$ if $k \leq |u|$, and $y(k) = x(k-|u|)$ if $k > |u|$. Given any finite word u, and any finite or infinite word x, u is a prefix of x (denoted $u \sqsubseteq x$) if $u(i) = x(i)$ holds for every non-zero integer $i \leq |u|$.

For $V \subseteq \Sigma^*$, $V^\omega = \{\sigma = u_1 \dots u_n \dots \in \Sigma^\omega \mid u_i \in V, \forall i \geq 1\}$

Definition 1. *A* finite state machine (FSM) *is a quadruple $M = (Q, \Sigma, \delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state and δ is a mapping from $Q \times \Sigma$ into 2^Q . It is* deterministic (DFSM) *if $\delta : Q \times \Sigma \longrightarrow Q$.*

Given an infinite word x, the infinite sequence of states $\rho = q_1 q_2 q_3 \dots$ is called an (infinite) run of M on x starting in state p, if both $q_1 = p$ and $q_{i+1} \in \delta(q_i, a_i)$ ($\forall i \geq 1$) hold. In case p is the initial state of M ($p = q_0$), then ρ is simply called an infinite run of M on x.

By $In(\rho)$, we denote the set of states that appear infinitely often in ρ:

$$In(\rho) = \{q \in Q \mid \forall m \ \exists n > m \quad q_n = q\}.$$

Equipped with an acceptance condition F, a finite state machine becomes a finite state automaton $M = (Q, \Sigma, \delta, q_0, F)$. It is a Büchi automaton (BA) *when*

[4] note that the enumeration $x = x(1)x(2)\dots$ does not start at 0 so that we recover the empty word as $x[0]$.

$F \subseteq Q$, and a Muller automaton (MA) *when $F \subseteq 2^Q$. A Büchi automaton (respectively a Muller automaton) accepts x if for some infinite run of M on x, $In(\rho) \cap F$ is not empty (respectively $In(\rho) \in F$ holds). The ω-language accepted by an automaton is the set of all the infinite words it accepts.*

The classical result of R. Mc Naughton [29] establishes that the following automata recognize the exact same ω-languages:

(a) deterministic Muller automata (DMA),
(b) non deterministic Büchi automata (NDBA), and
(c) non deterministic Muller automata (NDMA).

These ω-languages have also a characterization by means of the "ω-Kleene closure".

Definition 2. *Given any family L of ω-languages over the alphabet Σ, the ω-Kleene closure of L, is:*

$$\omega - KC(L) = \{ \bigcup_{1 \leq i \leq n} U_i.V_i^\omega \mid U_i, V_i \in L \}.$$

We denote by REG the class of all the (finitary) regular languages.

Theorem 1. *If L is an ω-language, the following are equivalent:*

(a) L belongs to $\omega - KC(REG)$,
(b) L is accepted by some DMA,
(c) L is accepted by some MA,
(d) L is accepted by some BA.

An ω-language that satisfies any of the above conditions is called an ω-regular language (or regular ω-language). We denote by REG_ω the class of all ω-regular languages.

We now consider partially blind multicounter automata.

A partially blind multicounter automaton is a finite automaton equipped with a finite number (k) of partially blind counters. The content of any such counter is a non-negative integer. A counter is said to be partially blind when the multicounter automaton cannot test whether the content of the counter is zero. This means that if a transition of the machine is enabled when the content of a counter is zero then the same transition is also enabled when the content of the same counter is a non-zero integer. In order to get a partially blind multicounter automaton – simply called a *blind multicounter automaton* – which accepts the same language as a given Petri net, one can distinguish between the places of a Petri net by dividing them into the bounded ones (the number of tokens in such a place at any time is uniformly bounded) and the unbounded ones. Then each unbounded place may be seen as a blind counter, and the tokens in the bounded places determine the state of the blind multicounter automaton. The transitions of the Petri net may then be seen as the finite control of the blind multicounter automaton and the labels of these transitions are then the input symbols.

Contrary to what happens with non-deterministic Petri nets, allowing ε-transitions does not increase the expressive power of deterministic Petri nets which read ω-words [1]. For this reason, we restrict ourselves to the sole real time – *i.e.*, ε-transition free – blind multicounter automata. Also, without loss of generality we may assume that every transition, for every counter, either increases or decreases its content by 1 or leave it untouched.

Definition 3. *Let k be any non-zero integer. A (real time) deterministic k-blind-counter machine (k-BCM) is a 4-tuple*

$$\mathcal{M} = (Q, \Sigma, \delta, q_0)$$

where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state, and the transition relation δ is a partial mapping from $Q \times \Sigma \times \{0,1\}^k$ into $Q \times \{0,1,-1\}^k$.

If the machine $\mathcal{M}$ is in state q, and for each i, $c_i \in \mathbb{N}$ is the content of the counter $\mathcal{C}_i$, then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

Given any $a \in \Sigma$, any $q, q' \in Q$, and any $(c_1, \ldots, c_k) \in \mathbb{N}^k$. If both $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$, and $j_l \in E = \{l \in \{1, \ldots, k\} \mid c_l = 0\} \Rightarrow j_l \in \{0, 1\}$ hold, then we write:

$$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$$

We write $\mapsto^\star_{\mathcal{M}}$ (or even $\mapsto^\star$) for the transitive and reflexive closure of $\mapsto_{\mathcal{M}}$.

Thus we see that the transition relation must verify: if $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$, and $i_m = 0$ holds for some $m \in \{1, \ldots, k\}$, then we must have $j_m = 0$ or $j_m = 1$ (but $j_m = -1$ is prohibited).

Moreover the k counters of $\mathcal{M}$ are blind, i.e., *if $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$ holds, and $i_m = 0$ for $m \in E \subseteq \{1, \ldots, k\}$, then $\delta(q, a, i'_1, \ldots, i'_k) = (q', j_1, \ldots, j_k)$ holds also whenever $i_m = i'_m$ for $m \notin E$, and $i'_m = 0$ or $i'_m = 1$ for $m \in E$.*

Let $u = a_1 a_2 \ldots a_n$ be any finite word over Σ. A sequence of configurations $\rho = (q_i, c^i_1, \ldots c^i_k)_{1 \leq i \leq n+1}$ is called a run of $\mathcal{M}$ on u, starting in configuration $(p, c_1, \ldots, c_k)$ if and only if

- $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$, *and*
- $a_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$ *(for all $1 \leq i \leq n$).*

Let $x = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $\rho = (q_i, c^i_1, \ldots c^i_k)_{i \geq 1}$ is called a complete run of $\mathcal{M}$ on x, starting in configuration $(p, c_1, \ldots, c_k)$ if and only if

- $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$, *and*

- $a_i : (q_i, c_1^i, \dots c_k^i) \mapsto_{\mathcal{M}} (q_{i+1}, c_1^{i+1}, \dots c_k^{i+1})$ *(for all* $1 \leq i$*).*

A complete run ρ of $\mathcal{M}$ on x , starting in configuration $(q_0, 0, \dots, 0)$, is simply called "a run of $\mathcal{M}$ on x". We write $In(\rho)$ for the set of all the states visited infinitely often during the complete run ρ.

Definition 4. *A Büchi (resp. Muller) deterministic k-blind-counter automaton is some k-BCM $\mathcal{M}' = (Q, \Sigma, \delta, q_0)$, equipped with an acceptance condition F:*

$$\mathcal{M} = (Q, \Sigma, \delta, q_0, F).$$

It is a Büchi (resp. Muller [5]) k-blind-counter automaton when $F \subseteq Q$ (resp. $F \subseteq 2^Q$), and it accepts x if the infinite run of $\mathcal{M}'$ on x verifies $In(\rho) \cap F \neq \emptyset$ (respectively $In(\rho) \in F$).

We write $L(\mathcal{M})$ for the ω-language accepted by $\mathcal{M}$. We also write $\mathbf{BC}(\mathbf{k})$ for the class of ω-languages accepted by Muller deterministic k-blind-counter automata.

3 Borel and Wadge hierarchies

We assume the reader to be familiar with basic notions of topology that may be found in [26, 24], and of ordinals (in particular the operations of multiplication and exponentiation) that may be found in [39].

For any given finite alphabet X – that contains at least two letters – we consider X^ω as the topological space equipped with the Cantor topology [6]. The open sets of X^ω are those of the form WX^ω, for some $W \subseteq X^*$. The closed sets are the complements of the open sets. The class that contains both the open sets and the closed sets, and is closed under countable union and intersection is the class of Borel sets. It is nicely set up in a hierarchy but counting how many times these latter operations are needed.

This defines the Borel Hierarchy: $\mathbf{\Sigma}_1^0$ is the class of open sets , and $\mathbf{\Pi}_1^0$ is the class of closed sets. For any non-zero integer n, $\mathbf{\Sigma}_{n+1}^0$ is the class of countable unions of sets inside $\mathbf{\Pi}_n^0$, while $\mathbf{\Pi}_{n+1}^0$ is the class of countable intersections of sets inside $\mathbf{\Sigma}_n^0$. More generally, for any non-zero countable ordinal α, $\mathbf{\Sigma}_\alpha^0$ is the class of countable unions of sets in $\cup_{\gamma<\alpha}\mathbf{\Pi}_\gamma^0$, and $\mathbf{\Pi}_\alpha^0$ is the class of countable unions of sets in $\cup_{\gamma<\alpha}\mathbf{\Sigma}_\gamma^0$.

We state a few basic results about the Borel classes, whose proofs may be found in [28, 23].

Proposition 1. *(a)* $\mathbf{\Sigma}_\alpha^0 \cup \mathbf{\Pi}_\alpha^0 \subsetneq \mathbf{\Sigma}_{\alpha+1}^0 \cap \mathbf{\Pi}_{\alpha+1}^0$*, holds for each countable ordinal* $\alpha \geq 1$.

(b) $\cup_{\gamma<\alpha}\mathbf{\Sigma}_\gamma^0 = \cup_{\gamma<\alpha}\mathbf{\Pi}_\gamma^0 \subsetneq \mathbf{\Sigma}_\alpha^0 \cap \mathbf{\Pi}_\alpha^0$*, holds for each countable limit ordinal* α.

[5] The Muller acceptance condition was denoted 3-acceptance in [25, 1], and $(inf, =)$ in [43].

[6] The product topology of the discrete topology on X.

(c) A set $W \subseteq X^\omega$ *belongs to the class* $\mathbf{\Sigma}^0_\alpha$ *iff* $X^\omega \setminus W$ *belongs to* $\mathbf{\Pi}^0_\alpha$.

(d) $\mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha \neq \emptyset$ *and* $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha \neq \emptyset$ *holds for every countable ordinal* $\alpha \geq 1$.

The Borel rank of a subset A of X^ω is the least ordinal $\alpha \geq 1$ such that A belongs to $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$.

By ways of continuous pre-image, the Borel hierarchy turns into the very refined Wadge Hierarchy:

Definition 5 ($\leq_w, \equiv_w, <_w$)**.** *We let* X, Y *be two finite alphabets, and* $A \subseteq X^\omega, B \subseteq Y^\omega$, *A is said Wadge reducible to B (denoted* $A \leq_W B$*) if and only if there exists some continuous function* $f : X^\omega \longrightarrow Y^\omega$ *that satisfies*

$$\forall x \in X \quad (x \in A \Leftrightarrow f(x) \in B).$$

We write $A \equiv_w B$ *when both* $A \leq_w B$ *and* $B \leq_w A$ *hold, and* $A <_w B$ *when both* $A \leq_w B$ *and* $B \not\leq_w A$ *hold.*

A set $A \subseteq X^\omega$ is called *self dual* if $A \equiv_W A^{\complement}$ is verified. It is called *non-self dual* otherwise [7].

It is easy to verify that the relation $\leq_w$ is both reflexive and transitive, and that $\equiv_W$ is an equivalence relation. Now a topological class is a class that is closed under continuous pre-images. Given any set A, the class of all its continuous pre-images forms a topological class $\mathbf{\Gamma}$ called a Wadge class, and A is said to be $\mathbf{\Gamma}$-*complete* since it both belongs to $\mathbf{\Gamma}$, and (Wadge) reduces every element in it. It follows that two sets are complete for the same topological class if and only if they are Wadge equivalent. We write $[A]_W$ for the Wadge class generated by A. So, formally

$$[A]_W = \{B \subseteq X^\omega \mid B \leq_w A\}.$$

From Proposition 1, we derive that $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) is a Wadge class and any set in $\mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$) is $\mathbf{\Sigma}^0_\alpha$-complete (resp. $\mathbf{\Pi}^0_\alpha$-complete). Both $\mathbf{\Sigma}^0_n$-complete and $\mathbf{\Pi}^0_n$-complete sets (any $0 < n < \omega$) are examined in [41].

The Wadge reducibility is intricately related to game theory for continuous functions may be regarded as strategies for a player in a two-player game of perfect information and infinite length:

Definition 6. *Given any mapping* $f : X^\omega \longrightarrow Y^\omega$, *the game* $\mathbf{G}(f)$ *is the two-player game where players take turn picking letters in X for I and Y for II, player I starting the game, and player II being allowed in addition to pass her turn, while player I is not.*

After ω*-many moves, player I and player II have respectively constructed* $x \in X^\omega$ *and* $y \in Y^* \cup Y^\omega$. *Player II wins the game if* $y = f(x)$, *otherwise player I wins.*

[7] Non-self dual sets are precisely those that verify $A \not\leq_w A^{\complement}$.

So, in the game $\mathbf{G}(f)$, a strategy for player I is a mapping $\sigma : (Y \cup \{s\})^\star \longrightarrow X$, where s is a new letter not in Y that stands for II's moves when she passes her turn [8]. A strategy for player II is a mapping $f : X^+ \longrightarrow Y \cup \{s\}$. A strategy is called winning if it ensures a win whatever the opponent does.

This game was designed to characterize the continuous functions.

Theorem 2 (Wadge). *Let $f : X^\omega \longrightarrow Y^\omega$, the following are equivalent:*

$$f \text{ is continuous} \iff II \text{ has a winning strategy in } \mathbf{G}(f).$$

Proof. This is an easy exercise (see [28, 23]).

Definition 7. *For $A \subseteq X^\omega$ and $B \subseteq Y^\omega$, the Wadge game $\mathbf{W}(A, B)$ is the same game as $\mathbf{G}(f)$, except for the winning condition: II wins if and only if both $y \in Y^\omega$ and $(x \in A \iff y \in B)$ hold.*

One sees immediately that a winning strategy for II in $\mathbf{W}(A, B)$ yields a continuous mapping $f : X^\omega \longrightarrow Y^\omega$ that guaranties that $A \leq_w B$ holds, whereas any continuous function f that witnesses the reduction relation $A \leq_w B$ gives rise to some winning strategy for II in $\mathbf{G}(f)$ which is also winning for II in $\mathbf{W}(A, B)$.

Theorem 3 (Wadge). *For $A \subseteq X^\omega$ and $B \subseteq Y^\omega$,*

$$A \leq_w B \iff II \text{ has a winning strategy in } \mathbf{W}(A, B).$$

In 1975, Martin proved a statement called Borel determinacy [27, 23] whose consequence is that for all Wadge game $\mathbf{W}(A, B)$, either player I or II has a winning strategy as long as both A and B are Borel.

An immediate consequence is the following lemma.

Lemma 1 (Wadge). *For X any finite set, and $A, B \subseteq X^\omega$ both Borel,*

$$A \not\leq_w B \text{ and } B \not\leq_w A \Longrightarrow A \equiv_w B^{\complement}.$$

Proof. By determinacy, from $A \not\leq_w B$, it follows that I has a w.s. in $\mathbf{W}(A, B)$ which gives a w.s. for II in $\mathbf{W}\left(B^{\complement}, A\right)$ which shows that $B^{\complement} \leq_w A$. By the same argument, from $B \not\leq_w A$, one derives $A \leq_w B^{\complement}$.

Another immediate consequence of determinacy is Wadge's Lemma.

[8] "s" stands for "skips".

Lemma 2 (Wadge). *For X any finite set,*

there are no three $\leq_w$-incomparable Borel sets.

Proof. Assume $A, B, C \subseteq X^\omega$ are all Borel $\leq_w$-incomparable sets. By Lemma 1, both $A \equiv_w B^{\complement}$ and $C \equiv_w B^{\complement}$ hold, which leads to $A \equiv_w C$.

We recall that a set S is well ordered by the binary relation $<$ on S if and only if $<$ is a linear order on S such that there is no strictly infinite $<$-decreasing sequence of elements from S.

Theorem 4 (Martin-Monk). *For X any finite set, there is no sequence $(A_i)_{i\in\omega}$ of Borel subsets of X^ω such that*

$$A_0 >_w A_1 >_w A_2 >_w \ldots A_n >_w A_{n+1} >_w \ldots$$

Proof. See [23, 46].

It follows that up to complementation and $\equiv_W$, the class of Borel subsets of X^ω, is well-ordered by $<_w$. Therefore, there is a unique ordinal $|WH|$ isomorphic to this well-ordering, together with a mapping d_W^0 from the Borel subsets of X^ω onto $|WH|$, such that for all Borel subsets A, B:

- $d_W^0 A < d_W^0 B \Leftrightarrow A <_w B$, and
- $d_W^0 A = d_W^0 B \Leftrightarrow (A \equiv_w B$ or $A \equiv_w B^{\complement})$.

This well-ordering restricted to the Borel sets of finite ranks [9] has length the first ordinal that is a fixpoint of the operation $\alpha \longrightarrow {\omega_1}^\alpha$ [46, 6], where ω_1 is the first uncountable ordinal.

In order to study the Wadge hierarchy of the class $\mathbf{BC}(\mathbf{k})$ of ω-languages accepted by Muller deterministic k-blind-counter automata, we concentrate on the non-self dual sets as in [6], and slightly modify the definition of the Wadge degree:

Definition 8. *For $A \subseteq X^\omega$, such that $A >_w \emptyset$, we set*

- $d_w(\emptyset) = d_w(\emptyset^{\complement}) = 1$,
- $d_w(A) = sup\{d_w(B) + 1 \mid B$ *non-self dual and* $B <_W A\}$.

Every ω-language which is accepted by a deterministic Petri net – more generally by a deterministic $\mathbf{X}$-automaton in the sense of [9] or by a deterministic Turing machine – is a boolean combination of $\mathbf{\Sigma_2^0}$-sets thus its Wadge degree inside the whole Wadge hierarchy of Borel sets is located below ω_1^ω.

Moreover, every non-zero ordinal $\alpha < \omega_1^\omega$ admits a unique Cantor normal form of base ω_1 [39], *i.e.*, it can be written as

$$\alpha = \omega_1^{n_j}.\delta_j + \omega_1^{n_{j-1}}.\delta_{j-1} + \ldots + \omega_1^{n_1}.\delta_1$$

[9] The Borel sets of finite ranks are those in $\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}_n^0 = \bigcup_{n\in\mathbb{N}} \mathbf{\Pi}_n^0$.

where $0 < j < \omega$, $\omega > n_j > n_{j-1} > \ldots > n_1 \geq 0$, and $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are non-zero countable ordinals.

It is well known, from Wagner's study, that such an ordinal is the Wadge degree of an ω-regular language if and only if all the multiplicative coefficients $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are integers.

It is also known that such an ordinal is the Wadge degree of a deterministic context-free ω-language if and only if these multiplicative coefficients are all below ω^ω [7].

We will add to this picture the following results that will exhibit the Wadge hierarchy of **BC(k)**:

(a) for every non-null ordinal α whose Cantor normal form of base ω_1 is

$$\alpha = \omega_1^{n_j}.\delta_j + \omega_1^{n_{j-1}}.\delta_{j-1} + \ldots + \omega_1^{n_1}.\delta_1$$

where the multiplicative coefficients $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are (non-null) ordinals $< \omega^{k+1}$, for some integer $k \geq 1$, there exists some ω-language $L \in \mathbf{BC(k)}$ whose Wadge degree is precisely α.

(b) Every non-self dual ω-language in **BC(k)** has a Wadge degree of the above form.

In the next section we introduce several operations over sets of ω-words, that will be useful in the first step of the proof.

4 Operations over sets of ω-words

4.1 The sum

Definition 9. *For $\{X_+, X_-\}$ a partition in non-empty sets of $X_B \setminus X_A$ with $X_A \subseteq X_B$, $A \subseteq X_A^\omega$, and $B \subseteq X_B^\omega$,*

$$B + A = A \ \cup \ X_A^* X_+ B \ \cup \ X_A^* X_- B^{\complement}.$$

A player in charge of $B + A$ in a Wadge game is like a player who would begin the play in charge of A, and at any moment may also decide to start anew but being in charge this time of either B or of $B^{\complement}$. (The first letter in $X_B \setminus X_A$ that is played decides the choice of B or $B^{\complement}$).

Notice that given any finite alphabets X, Y which contain at least two letters, and any $B \subseteq X^\omega$, there exists $B' \subseteq Y^\omega$ such that $B \equiv_w B'$. Moreover, if for some integer $k \geq 0$ we have $B \in \mathbf{BC}(\mathbf{k})$, then B' can be taken in $\mathbf{BC}(\mathbf{k})$. So that we may write $B + A$ whatever space B is a subset of, simply meaning $B' + A$ where B' is any set that satisfies both $B' \equiv_w B$ and $B' \subseteq X^\omega$ for some X that contains the alphabet from which A is taken from, and strictly extends if with at least two new letters.

Proposition 2 (Wadge). *For non-self dual Borel sets A and B,*

$$d_w(B + A) = d_w(B) + d_w(A).$$

Notice that this operation is associative since $A + (B + C) \equiv_W (A + B) + C$ holds for non-self dual Borel sets A, B, C. Notice also that $(B + A)^{\complement} \equiv_w B + A^{\complement}$ always holds.

Although the class $\mathbf{BC}(\mathbf{k})$ is not closed under complementation, and $B + A$ was defined as $A \cup X_A^* X_+ B \cup X_A^* X_- B^{\complement}$, we may however make use of the formulation $B + A \in \mathbf{BC}(\mathbf{k})$ for $A, B \in \mathbf{BC}(\mathbf{k})$ as long as there exists some $C \in \mathbf{BC}(\mathbf{k})$ that verifies $C \equiv_w B^{\complement}$.

4.2 The countable multiplication

We first need to define the supremum of a countable family of sets:

Definition 10. *For any bijection $f : \mathbb{N} \longrightarrow I$, any family $(A_i)_{i \in I}$ of non-self dual Borel subsets of X^ω, we fix some letter $e \in X$ to define*

$$\sup_{i \in I} A_i = \bigcup_{n \in \mathbb{N}} (X \smallsetminus \{e\})^n e A_{f(n)}.$$

Proposition 3. *For $(A_i)_{i \in I}$ any countable family of non-self dual Borel subsets of X^ω such that $\forall i \in I \quad \exists j \in I \quad A_i \leq_w A_j$, then*

(a) $\sup_{i \in I} A_i$ *is a non-self dual Borel subset of X^ω, and*

(b) $d_w(\sup_{i \in I} A_i) = \sup\{d_w(A_i) \mid i \in I\}$.

Proof. See [6, 7].

By combining the two operations of sum and supremum, we define the multiplication by a countable ordinal.

Definition 11. *For $A \subseteq X^\omega$, and $0 < \alpha < \omega_1$, $A \bullet \alpha$ is inductively defined by:*
$A \bullet 1 = A$
$A \bullet (\nu + 1) = (A \bullet \nu) + A$
$A \bullet \beta = \sup_{\delta \in \beta} A \bullet \delta$ *when β is a limit ordinal.*

By Propositions 2 and 3, this operation verifies the following:

Proposition 4. *Let $A \subseteq X^\omega$ be some non-self dual Borel set, and $0 < \alpha < \omega_1$,*

$$d_w(A \bullet \alpha) = d_w(A) \cdot \alpha.$$

For a player in charge of $A \bullet \alpha$ in a Wadge game, everything goes as if (s)he could switch again and again between being in charge of A or $A^{\complement}$ – starting anew every time (s)he does so – but restrained from doing so infinitely often by having to construct a decreasing sequence of ordinals $< \alpha$ on the side every time (s)he switches.

Notice that the operation $A \longrightarrow A \bullet \alpha$ was denoted $A \longrightarrow A\hat{}\alpha$ in [8].

4.3 The multiplication by ω_1

Definition 12. *For $A \subseteq X^\omega$, and $a, b \notin X$ two different letters, $Y = X \cup \{a, b\}$, $A \bullet \omega_1 \subseteq (X \cup \{a, b\})^\omega$ is defined* [10] *by*

$$A \bullet \omega_1 = A \ \cup \ Y^* aA \ \cup \ Y^* bA^{\complement}.$$

Inside a Wadge game, a player in charge of $A \bullet \omega_1$ may switch indefinitely between being in charge of A or its complement, deleting what (s)he has already played each time (s)he switches.

Proposition 5. *For any non-self dual Borel $A \subseteq X^\omega$,*

(a) $A \bullet \omega_1$ is non-self dual Borel, and

(b) $d_w(A \bullet \omega_1) = d_w(A) \cdot \omega_1$.

Proof. See [6].

The following property will be very useful.

Proposition 6. *If $A \subseteq X^\omega$ is regular, then $A \bullet \omega_1$ is also regular.*

Proof. It follows directly from the definition of $A \bullet \omega_1$ and the closure of the class REG_ω under finite union, complementation, and left concatenation by finitary regular languages [8].

4.4 Canonical non-self dual sets

The empty set, considered as an ω-language over a finite alphabet is a Borel set of Wadge degree 1, *i.e.*, $d_w(\emptyset) = 1$. It is a non-self dual set and its complement has the same Wadge degree[11].

On the basis of the emptyset or its complement, the operations defined above provide non-self dual Borel sets for every Wadge degree $< \omega_1^\omega$. For notational purposes, given any $A \subseteq X^\omega$ we define $A \bullet \omega_1^n$ by induction on $n \in \mathbb{N}$:

- $A \bullet \omega_1^0 = A$,
- $A \bullet \omega_1^{n+1} = (A \bullet \omega_1^n) \bullet \omega_1$.

Clearly, by Proposition 5, $d_w(A \bullet \omega_1^n) = d_w(A) \cdot \omega_1^n$ holds for every non-self dual Borel $A \subseteq X^\omega$. It follows that the ω-language $\emptyset \bullet \omega_1^n$ is a non-self dual Borel set whose Wadge degree is precisely ω_1^n.

Every non-null ordinal $\alpha < \omega_1^\omega$ admits a unique Cantor normal form of base ω_1:

[10] This operation was denoted A^+ in [5], $A \longrightarrow A\hat{}\infty$ in [8], and $A \longrightarrow A^\natural$ in [7].
[11] *i.e.*, $d_w(\emptyset) = d_w(X^\omega) = 1$.

$$\alpha = \omega_1^{n_j} \cdot \delta_j + \omega_1^{n_{j-1}} \cdot \delta_{j-1} + \ldots + \omega_1^{n_1} \cdot \delta_1.$$

where $\omega > j > 0$, $\omega > n_j > n_{j-1} > \ldots > n_1 \geq 0$, and $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are non-zero countable ordinals [39].

By mean of the set theoretical sum and countable multiplication we define the following as in [6, 7]:

$$\Omega(\alpha) = (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \ldots + (\emptyset \bullet \omega_1^{n_1}) \bullet \delta_1.$$

By Propositions 2, 4, and 5 $d_w(\Omega(\alpha)) = \alpha$ holds.

5 A hierarchy of $BC(k)$

From now on, we restrain ourselves to the sole ordinals $\alpha < \omega_1^\omega$ whose Cantor normal form of base ω_1 contains only multiplicative coefficients strictly below ω^{k+1}, and we construct for every such α some Muller deterministic k-blind-counter automata $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^-$ such that both $L(\mathcal{M}_\alpha) \equiv_W \Omega(\alpha)$ and $L(\mathcal{M}_\alpha^-) \equiv_W \Omega(\alpha)^{\complement}$ hold.

To start with, notice that for every integer n since $\emptyset \bullet \omega^n \in REG_\omega$ is verified, there exist deterministic Muller automata $\mathcal{O}_n = (Q_n, X_n, \delta_n, q_n^0, \mathcal{F}_n)$, where $\mathcal{F}_n \subseteq 2^{Q_n}$ is the collection of designated state sets, such that $L(\mathcal{O}_n) = \emptyset \bullet \omega^n$.

Proposition 7. *For any ω-regular language A, any integer $j \geq 1$ there exist ω-languages $B, C \in \mathbf{BC}(\mathbf{j})$ such that*

$$B \equiv_W (A \bullet \omega^j) \text{ and } C \equiv_W (A \bullet \omega^j)^{\complement}.$$

Proof. Recall that being in charge of $A \bullet \omega^j$ in a Wadge game is like being able to swap A and $A^{\complement}$, re-starting each time together with building a strictly decreasing sequence $< \omega^j$ on the side. Now any ordinal $\alpha < \omega^j$ has a unique Cantor normal form of base ω:

$$\alpha = \omega^{j-1} \cdot n_{j-1} + \omega^{j-2} \cdot n_{j-2} + \ldots + \omega^1 \cdot n_1 + \omega^0 \cdot n_0.$$

where $n_{j-1}, n_{j-2}, \ldots, n_1, n_0$ are all (possibly null) integers . Then we see that j-many counters suffice to keep track of the ordinal α: the first counter value is n_0, the second counter value is n_1, ..., the j^{th} counter value is n_{j-1}.

Let then A be an ω-regular language accepted by a deterministic Muller automaton

We let $\mathcal{A}, \mathcal{A}^-$ be deterministic Muller automata such that $L(\mathcal{A})^{\complement} = L(\mathcal{A}^-)$. For any non-zero integer j we describe deterministic k-blind counter automata $\mathcal{A}_{\omega^j}$ and $\mathcal{A}_{\omega^j}^-$ such that both $L(\mathcal{A}_{\omega^j}) \equiv_w L(\mathcal{A}) \bullet \omega^j$ and $L(\mathcal{A}_{\omega^j}^-) \equiv_w \left(L(\mathcal{A}) \bullet \omega^j\right)^{\complement}$ hold.

We first add the following new letters to the alphabet of $\mathcal{A}$ (and also $\mathcal{A}^-$):

$$\uparrow_0, \uparrow_1, \ldots, \uparrow_{j-1}, \downarrow_0, \downarrow_1, \ldots, \downarrow_{j-1}, a, a^-.$$

We describe $\mathcal{A}_{\omega^j}$ (respectively $\mathcal{A}^-_{\omega^j}$) by its behavior while reading an ω-word. The reader may easily verify that the conditions below may be carried away by a k-blind counter automaton.

The machine starts reading "up arrow" letters, i.e., those in $\{\uparrow_0, \uparrow_1, \ldots, \uparrow_{j-1}\}$ (otherwise it simply rejects). As long as it does so, it increases the content of its counters – $\uparrow_i$ stands for an increase by 1 of counter i – but if it does so infinitely often – that is the infinite word x that it reads belongs to $\{\uparrow_0, \uparrow_1, \ldots, \uparrow_{j-1}\}^\omega$ – $\mathcal{A}_{\omega^j}$ rejects while $\mathcal{A}^-_{\omega^j}$ accepts.

Then, after the machine has read the whole prefix of "up arrow" letters, it should encounter either the letter a_+ or the letter a^- (otherwise it rejects). If it reads a_+ (resp. a^-) it starts working just like $\mathcal{A}$ (resp. $\mathcal{A}^-$). This goes on until eventually the machine reads a letter inside $\{\uparrow_0, \uparrow_1, \ldots, \uparrow_{j-1}, \downarrow_0, \downarrow_1, \ldots, \downarrow_{j-1}, a, a^-.\}$ If it is the case, then this letter has to be a "down arrow" letter of the form $\downarrow_i$ followed by some (possibly zero) "up arrow" letters of the form $\uparrow_p$ for some $p < i$, followed by either a_+ or a^- (otherwise $\mathcal{A} \bullet \omega^j$ (resp. $\mathcal{A}^- \bullet \omega^j$) rejects). When this word is of the form $\downarrow_i (\uparrow_p)^l a$ (resp. $\downarrow_i (\uparrow_p)^l a^-$), it decreases counter i by one and increases counter p by l, then it behaves like $\mathcal{A}$ (resp. $\mathcal{A}^-$).

By playing the underlying wadge games, the reader may easily verify that both $L(\mathcal{A}_{\omega^j}) \equiv_w L(\mathcal{A}) \bullet \omega^j$ and $L(\mathcal{A}^-_{\omega^j}) \equiv_w (L(\mathcal{A}_{\omega^j}))^{\mathtt{C}}$ hold.

Proposition 8. *For any ω-regular language A, any integer k, and any ordinal $\omega^k \leq \alpha < \omega^{k+1}$, there exist ω-languages $B, C \in \mathbf{BC}(\mathbf{k})$ such that*

$$B \equiv_W (A \bullet \alpha) \text{ and } C \equiv_W (A \bullet \alpha)^{\mathtt{C}}.$$

Proof.

(a) If $0 < \alpha < \omega$, then from the very definition of the sum, it is immediate to see that if A belongs to REG_ω, then both $A \bullet \alpha$ and its complement belong to REG_ω.

(b) If $\omega \leq \alpha < \omega^{k+1}$, we distinguish between two cases:

  (i) If $\alpha = \omega^k$, this was done in Proposition 7.

  (ii) If $\omega^k < \alpha < \omega^{k+1}$, then we consider the Cantor normal form of base ω of α:

$$\alpha = \omega^{m_j} \cdot n_j + \omega^{m_{j-1}} \cdot n_{j-1} + \ldots + \omega^{m_0} \cdot n_0.$$

  where $k = m_j > m_{j-1} > \ldots > m_0$ holds, and $n_j, n_{j-1}, \ldots, n_0$ are all non-null integers.

  By Proposition 7 and case (a), for each m_i $(0 \leq i \leq j)$ there are m_i-blind counter automata $\mathcal{A}_{\omega^{m_i}}$ and $\mathcal{A}^-_{\omega^{m_i}}$ such that $L(\mathcal{A}_{\omega^{m_i}}) \equiv_w L(\mathcal{A}) \bullet \omega^{m_i}$

and $L(\mathcal{A}^-_{\omega^i}) \equiv_w (L(\mathcal{A}_{\omega^i}))^{\complement}$ hold [12]. Without loss of generality, we may assume that

- for each $i \leq j$, $\mathcal{A}_{\omega^{m_i}}$ and $\mathcal{A}^-_{\omega^{m_i}}$ are k-blind counter automata whose transition function only deals with counters c_l for $1 \leq l \leq m_i$ leaving untouched the counters c_l for $m_i < l \leq k$.
- The alphabets of the different machines do not overlap. Which means that for $0 \leq h < i \leq j$, if Σ_h, Σ_i denote the respective alphabets of $\mathcal{A}_{\omega^{m_h}}, \mathcal{A}_{\omega^{m_i}}$ (and equivalently of $\mathcal{A}^-{}_{\omega^{m_h}}, \mathcal{A}^-{}_{\omega^{m_i}}$), then $\Sigma_h \cap \Sigma_i = \emptyset$ holds.

Moreover we set $\Sigma = \bigcup_{i \leq j} \Sigma_i$, and $\check{\mathcal{A}}_{\omega^{m_i}}$ (resp. $\check{\mathcal{A}}^-_{\omega^{m_i}}$) to be the machine that works on Σ – rather than Σ_i – just like $\mathcal{A}^-{}_{\omega^{m_h}}$ (resp. $\mathcal{A}^-{}_{\omega^{m_i}}$) as long as it reads letters in Σ_i, and rejects as soon as it reads a letter in $\Sigma \smallsetminus \Sigma_i$.

We consider two new letters t_+, t_- for transitions from a machine to another, and we build $\mathcal{A}_\alpha, \mathcal{A}^-_\alpha$ as follows.

(A) $\mathcal{A}_\alpha$ starts working like $\check{\mathcal{A}}_{\omega^{m_0}}$ (resp. $\mathcal{A}^-_\alpha$ starts like $\check{\mathcal{A}}^-_{\omega^{m_0}}$) as long as no letter t_+ nor t_- is encountered.

(B) If a letter among $\{t_+, t_-\}$ is encountered for the n^{th} time for some $n < n_0 + \ldots + n_j$, we let i be the integer that verifies

$$n_0 + \ldots + n_{i-1} < n + 1 \leq n_0 + \ldots + n_i$$

if $n > n_0$, and $i = 0$ otherwise. Then right after this letter is read and until another transition letter (either t_+ or t_-) is eventually read:

- if this letter is t_+, $\mathcal{A}_\alpha$ (resp. $\mathcal{A}^-_\alpha$) works like $\check{\mathcal{A}}_{\omega^{m_i}}$, and
- if this letter is t_-, $\mathcal{A}_\alpha$ (resp. $\mathcal{A}^-_\alpha$) works like $\check{\mathcal{A}}^-_{\omega^{m_i}}$.

(C) If a letter among $\{t_+, t_-\}$ is encountered for the $n_0 + \ldots + n_j^{\text{th}}$ time, then $\mathcal{A}_\alpha$ (resp. $\mathcal{A}^-{}_\alpha$) rejects.

The reader should notice that in case (b)((ii))(B) when a machine hits a t_+ or t_- letter, some of its counters may already have a non-zero content. But this is not a problem since the initial process of $\check{\mathcal{A}}_{\omega^{m_i}}$ ($\check{\mathcal{A}}^-_{\omega^{m_i}}$ as well) consists in filling up these same counters plus eventually some extra ones. From there, it is tedious but straightforward to verify that both $\mathcal{A}_\alpha \equiv_w A \bullet \alpha$ and $\mathcal{A}^-_\alpha \equiv_w A \bullet \alpha$ hold.

Theorem 5. *Let $\alpha < \omega_1^\omega$ be any ordinal whose Cantor normal form of base ω_1 only has multiplicative coefficients $< \omega^\omega$:*

$$\alpha = \omega_1^{n_j} \cdot \delta_j + \omega_1^{n_{j-1}} \cdot \delta_{j-1} + \ldots + \omega_1^{n_0} \cdot \delta_0$$

where $\omega > j \geq 0$, $\omega > n_j > n_{j-1} > \ldots > n_0 \geq 0$, and $\omega^\omega > \delta_j, \delta_{j-1}, \ldots, \delta_0 > 0$.

[12] The case $m_i = 0$ corresponds to 0-blind counter automata $\mathcal{A}_{\omega^0} = \mathcal{A}$ and $\mathcal{A}^-_{\omega^0}$ its dual.

Let k be the least integer such that $\forall i \leq j \quad \delta_i < \omega^{k+1}$. Then there exist ω-languages $B, C \in \mathbf{BC}(\mathbf{k})$ such that

$$B \equiv_w \Omega(\alpha) \text{ and } C \equiv_w \Omega(\alpha)^{\complement}.$$

We recall that $\Omega(\alpha)$ is defined by

$$\Omega(\alpha) = (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \ldots + (\emptyset \bullet \omega_1^{n_0}) \bullet \delta_0.$$

Proof. For every $i \leq j$ the set $\emptyset \bullet \omega_1^{n_j}$ is ω-regular, so that for each integer $i \leq j$ we have machines $\mathcal{A}^i$ and $\mathcal{A}^{i^-}$ such that $L(\mathcal{A}^i) \equiv_w \emptyset \bullet \omega_1^{n_i}$ and $L(\mathcal{A}^{i^-}) \equiv_w (\emptyset \bullet \omega_1^{n_i})^{\complement}$ hold for every $i \leq j$. The case $j = 0$ was already proved in Proposition 8, so that we may assume that $j > 0$ holds.

Now we consider also for each $i \leq j$, the "exact" k_i-blind-counter automata $\mathcal{A}^i_{\delta_i}$ and $\mathcal{A}^{i^-}_{\delta_i}$ that were designed in the proof of Proposition 8. (Notice that k_i was defined as the least integer such that $\delta_i < \omega^{k_i+1}$.) We then form for each $i \leq j$, the k-blind-counter automata $\hat{\mathcal{A}}^i_{\delta_i}$ and $\hat{\mathcal{A}}^{i^-}_{\delta_i}$ that work exactly as $\mathcal{A}^i_{\delta_i}$ and $\mathcal{A}^{i^-}_{\delta_i}$ on the first k_i counters, leaving untouched the last $k - k_i$ ones .

For simplicity – and without loss of generality – we may assume that both $\Sigma_h \cap \Sigma_i = \emptyset$ and $\Sigma_i = \Sigma_i^-$ hold for every $0 \leq h < i \leq j$, where Σ_i, Σ_i^- stand for the respective alphabets of $\hat{\mathcal{A}}^i_{\delta_i}, \hat{\mathcal{A}}^{i^-}_{\delta_i}$.

Then we form, for each $i \leq j$, the k-blind-counter automata $\check{\mathcal{A}}^i_{\delta_i}$ and $\check{\mathcal{A}}^{i^-}_{\delta_i}$ that work on the alphabet $\Sigma = \bigcup_{i \leq j} \Sigma_i$. The machine $\check{\mathcal{A}}^i_{\delta_i}$ (resp. $\check{\mathcal{A}}^{i^-}_{\delta_i}$) works like $\hat{\mathcal{A}}^i_{\delta_i}$ (resp. $\hat{\mathcal{A}}^{i^-}_{\delta_i}$) on the alphabet Σ_i, and both of them reject as soon as they read a letter not in Σ_i.

At last we take two new letters t_+, t_- for transitions from a machine to another, and we build $\mathcal{A}_\alpha, \mathcal{A}_\alpha^-$ as follows.

(a) As long as neither the letter t_+ nor t_- is encountered, $\mathcal{A}_\alpha$ (resp. $\mathcal{A}_\alpha^-$) works as $\check{\mathcal{A}}^0_{\delta_0}$ (resp. $\check{\mathcal{A}}^{0^-}_{\delta_0}$).

(b) If a letter among $\{t_+, t_-\}$ is encountered for the i^{th} time for some $i < j$, then right after this letter is read and until another transition letter t_+ or t_- is eventually read:
   - if this letter is t_+, $\mathcal{A}_\alpha$ (resp. $\mathcal{A}_\alpha^-$) works like $\check{\mathcal{A}}^i_{\delta_i}$, and
   - if this letter is t_-, $\mathcal{A}_\alpha$ (resp. $\mathcal{A}_\alpha^-$) works like $\check{\mathcal{A}}^{i^-}_{\delta_i}$.

(c) If a letter among $\{t_+, t_-\}$ is encountered for the j^{th}, then both $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha^-$ reject right away.

We leave to the reader to verify the tedious details of

$$L(\mathcal{A}_\alpha) \equiv_w \Omega(\alpha) \text{ and } L(\mathcal{A}_\alpha^-) \equiv_w \Omega(\alpha)^{\complement}.$$

6 Localisation of $BC(k)$

This section is dedicated to proving that there is no other Wadge class generated by some non-self dual ω-language in $BC(k)$ than the ones described in Theorem 5. Prior to this we need a technical result about the Wadge hierarchy together with a few others on ordinal combinatorics, and notations.

For some $A \subseteq X^\omega$ and some $u \in X^*$, we write $u^{-1}A$ for the set $\{x \in X^\omega \mid ux \in A\}$, and we say that A is *initializable* if the second player has a w.s. in the Wadge game $\mathbf{W}(A, A)$ even though she is restricted to positions $u \in X^*$ that verify $u^{-1}A \equiv_w A$.

Lemma 3. *For $A \subseteq X^\omega$ any initializable set, $B \subseteq Y^\omega$, and δ, θ any countable ordinals,*

$$A \bullet (\theta+1) \leq_w B \leq_w A \bullet \delta \implies \exists u \in Y^* \begin{cases} u^{-1}B \equiv_w A \bullet (\theta+1) \\ \text{or} \\ u^{-1}B \equiv_w (A \bullet (\theta+1))^{\complement}. \end{cases}$$

Proof. The case $\theta + 1 = \delta$ is obvious since the empty word works for u. So in the sequel we assume $\theta + 1 < \delta$. The proof goes by induction on δ.

Assume δ is limit there are two different cases.

(a) If $B \equiv_w A \bullet \delta$, then clearly the set

$$\{d_W^0 u^{-1}B : u \in Y^* \text{ and } u^{-1}B <_w B\}$$

is unbounded in $d_W^0(A \bullet \delta) = d_W^0(A) \cdot \delta$. Hence there exists some ordinal $\xi < \delta$ and some $v \in Y^*$ that both satisfy

$$A \bullet (\theta+1) \leq_w v^{-1}B \leq_w (A \bullet \xi) <_w A \bullet \delta$$

Then by induction hypothesis one gets some $u' \in Y^*$ such that

$$\begin{cases} v^{-1}u'^{-1}B \equiv_w A \bullet (\theta+1) \\ \text{or} \\ v^{-1}u'^{-1}B \equiv_w (A \bullet (\theta+1))^{\complement} \end{cases}$$

Hence $u = vu'$ works.

(b) If $B <_w A \bullet \delta$, then $d_W^0(B) < d_W^0(A \bullet \delta) = d_W^0(A) \cdot \delta$. Hence, for some $\xi < \delta$ we have

$$A \bullet (\theta+1) \leq_w B \leq_w (A \bullet \xi)$$

which gives the result using the induction hypothesis on ξ.

Assume δ is successor.

(a) **Assume $\delta = \zeta + 2$.**

(i) **Assume $(A \bullet (\zeta+1))^{\complement} \leq_w B \leq_w A \bullet (\zeta+2)$.**

We consider the following combination of Wadge games with 3 players : *I*, *II* and *III*:

- I is in charge of $((A \bullet \zeta) + A^{\complement})$ – which is Wadge equivalent to $(A \bullet (\zeta + 1))^{\complement}$,
- II is in charge of B, and
- III is in charge of $A \bullet (\zeta + 2)$.

II applies a w.s. that reduces I and III applies a w.s. that reduces II. This means that if I plays x_1, II plays x_2 and III plays x_3 then II reduces I if $x_1 \in (A \bullet \zeta) + A^{\complement} \iff x_2 \in B$; and III reduces II if $x_3 \in A \bullet (\zeta + 2) \iff x_2 \in B$.

Assume now that player I remaining in the right tail $A^{\complement}$ (i.e. without going into $(A \bullet \zeta)$ or $(A \bullet \zeta)^{\complement}$) applies a winning strategy in the Wadge game $W(A^{\complement}, A)$ against Player III as long as III stays in the tail part A of $A \bullet (\zeta + 1) + A$. Necessarily after a finite number of moves player III exits the right most A and chooses $(A \bullet (\zeta + 1))^{\complement}$ – for the other choice $A \bullet (\zeta + 1)$ would be a losing one. We let v be the position of player II at that point, so that we obtain:

$$(A \bullet (\zeta + 1))^{\complement} \leq_w v^{-1}B \leq_w (A \bullet (\zeta + 1))^{\complement},$$

hence

$$v^{-1}B \equiv_w (A \bullet (\zeta + 1))^{\complement}.$$

If $\theta + 1 = \zeta + 1$ we are done. Otherwise we have

$$(A \bullet (\theta + 1))^{\complement} \leq_w v^{-1}B^{\complement} \leq_w A \bullet (\zeta + 1)$$

By induction hypothesis there exists u extending v such that

$$u^{-1}B^{\complement} \equiv_w A \bullet (\theta + 1) \text{ or } u^{-1}B^{\complement} \equiv_w (A \bullet (\theta + 1))^{\complement}$$

which gives

$$u^{-1}B \equiv_w A \bullet (\theta + 1) \text{ or } u^{-1}B \equiv_w (A \bullet (\theta + 1))^{\complement}.$$

(ii) **Assume $B \leq_w A \bullet (\zeta + 1)$.**
Since $\zeta + 1 < \delta$ holds, the result relies on the induction hypothesis for

$$A \bullet (\theta + 1) \leq_w B \leq_w A \bullet (\zeta + 1).$$

(b) **Assume $\delta = \zeta + 1$, ζ limit**:
If $B \leq_w A \bullet \zeta$ holds the result follows from the induction hypothesis. Thus we assume that the following holds:

$$(A \bullet \zeta)^{\complement} \leq_w B \leq_w (A \bullet \zeta) + A$$

Now consider the following combination of Wadge games with 3 players where:

- I is in charge of $(A \bullet \theta) + A + A + A^{\complement}$ (this is Wadge equivalent to $(A \bullet (\theta + 3))^{\complement}$),
- II is in charge of B, and

- *III* is in charge of $A \bullet \zeta + A$.

II applies a w.s. that reduces *I* and *III* applies a w.s. that reduces *II*. Player *I* applies a winning strategy in the Wadge game $W(A^{\complement}, A)$ against Player *III* as long as Player *III* remains in the tail part A of $(A \bullet \zeta) + A$. Necessarily after a finite number of moves player *III* exits the first A and chooses $(A \bullet \zeta)^{\complement}$ or $(A \bullet \zeta)$.
Now notice that since A is non-self dual, the set $\{w \in X^* \mid w^{-1}A \equiv_w A\}$ is a tree – it is closed under prefixes – that contains an infinite branch. We let x be such an infinite branch.

- If $x \in A$, then player *I* chooses to go into $(A \bullet \theta + A + A)^{\complement}$ if *III* chooses $(A \bullet \zeta)^{\complement}$ and into $(A \bullet \theta + A + A)$ if *III* chooses $(A \bullet \zeta)$.
- If $x \notin A$, then player *I* chooses to go into $(A \bullet \theta + A + A)$ if *III* chooses $(A \bullet \zeta)^{\complement}$ and into $(A \bullet \theta + A + A)^{\complement}$ if *III* chooses $(A \bullet \zeta)$.

Then *I* plays along x, so that *III* is forced to choose $A \bullet \gamma$ for some $\gamma < \zeta$ (by definition $A \bullet \zeta = \sup_{\gamma<\zeta} A \bullet \gamma$). After *III* makes that choice, *II* is in a position v that satisfies

$$A \bullet (\theta + 2)^{\complement} \leq_w v^{-1}B \leq_w A \bullet \gamma \quad or \quad A \bullet (\theta + 2) \leq_w v^{-1}B \leq_w A \bullet \gamma.$$

Therefore we obtain

$$A \bullet (\theta + 1) \leq_w v^{-1}B \leq_w A \bullet \gamma$$

which gives the result by induction hypothesis since $\gamma < \delta$ holds.

Lemma 4. *We let $B \subseteq Y^\omega$, $A \subseteq X^\omega$ be any initializable set, and δ, θ be any countable ordinals. We consider any set of the form*

$$C = A \bullet \omega_1^n \bullet \nu_n + \ldots + A \bullet \omega_1^{n-1} \bullet \nu_{n-1} + \ldots + A \bullet \omega_1 \bullet \nu_1$$

for any non-zero integer n, and countable coefficients $\nu_n, \nu_{n-1}, \ldots, \nu_1$ with at least one of them being non-null.

$$C+A\bullet(\theta+1) \leq_w B \leq_w C+A\bullet\delta \implies \exists u \in Y^* \begin{cases} u^{-1}B \equiv_w C + A \bullet (\theta + 1) \\ or \\ u^{-1}B \equiv_w (C + A \bullet (\theta + 1))^{\complement}. \end{cases}$$

Proof. The proof is very similar to the one of Lemma 3, so we leave it to the reader.

We recall that for any set of ordinals $\mathcal{O}$, its order type – denoted $ot(\mathcal{O})$ – is the unique ordinal that is isomorphic to $\mathcal{O}$ ordered by membership (which is nothing but the usual ordering on ordinals).

Definition 13. *The function $\mathcal{H} : \omega^\omega \times \omega^\omega \longrightarrow On$ is defined by*

$$\mathcal{H}(\alpha, \beta) = \omega^k \cdot (l_k + m_k) + \omega^{k-1} \cdot (l_{k-1} + m_{k-1}) + \ldots + \omega^0 \cdot (l_0 + m_0).$$

Where (a variation of the) the Cantor normal form of base ω of α (resp. β) is

$$\alpha = \omega^k \cdot l_k + \omega^{k-1} \cdot l_{k-1} + \ldots + \omega^0 \cdot l_0$$

$$\beta = \omega^k \cdot m_k + \omega^{k-1} \cdot m_{k-1} + \ldots + \omega^0 \cdot m_0$$

with $l_k, m_k, l_{k-1}, m_{k-1}, \ldots, l_0, m_0 \in \mathbb{N}$. (Some of these integers may be null [13].)

Lemma 5. *Let $\mathcal{H} : \omega^\omega \times \omega^\omega \longrightarrow On$, $0 < \alpha', \alpha, \beta' \beta < \omega^\omega$ with $\alpha' \leq \alpha$, $\beta' \leq \beta$ but either $\alpha' < \alpha$ or $\beta' < \beta$, then*

$$\mathcal{H}(\alpha', \beta') < \mathcal{H}(\alpha, \beta).$$

Proof. This is tedious but straightforward, and left to the reader.

We make use of the mapping $\mathcal{H}$ to prove the following combinatorial result.

Lemma 6. *Let α, β, γ be non-null ordinals with $\alpha, \beta < \omega^\omega$, and $f : \gamma \longrightarrow \{0, 1\}$.*

$$\text{If both } \alpha = ot(f^{-1}[0]) \text{ and } \beta = ot(f^{-1}[1]) \text{ hold, then } \gamma \leq \mathcal{H}(\alpha, \beta).$$

Proof. The proof goes by induction on $(\max\{\alpha, \beta\}, \min\{\alpha, \beta\})$ ordered by lexicographic ordering.

(a) If $\alpha = \beta = 1$, then the result is immediate.

(b) We assume $\alpha \geq \beta$, and we let (α, β) be the $<_{lex}$-least pair such that there exists some ordinal γ together with $f : \gamma \longrightarrow \{0, 1\}$ that verify
- $\alpha = ot(f^{-1}[0])$,
- $\beta = ot(f^{-1}[1])$, and
- $\gamma > \mathcal{H}(\alpha, \beta)$.

We consider the order types of the following two sets of ordinals:
(i) $\alpha' = ot(\{\theta < \mathcal{H}(\alpha, \beta) \mid f(\theta) = 0\})$, and
(ii) $\beta' = ot(\{\theta < \mathcal{H}(\alpha, \beta) \mid f(\theta) = 1\})$,

together with f' the restriction of f to $\mathcal{H}(\alpha, \beta)$. Necessarily either $\alpha' < \alpha$ or $\beta' < \beta$ holds. Therefore we have

$$(\max\{\alpha', \beta'\}, \min\{\alpha', \beta'\}) <_{lex} (\max\{\alpha, \beta\}, \min\{\alpha, \beta\}).$$

Hence we get the ordinal $\gamma' = \mathcal{H}(\alpha, \beta)$, together with the mapping $f' : \mathcal{H}(\alpha, \beta) \longrightarrow \{0, 1\}$ such that $\alpha' = ot(\{\theta < \gamma' \mid f'(\theta) = 0\})$ and $\beta' = ot(\{\theta < \gamma' \mid f'(\theta) = 1\})$. But by Lemma 5, we obtain $\gamma' = \mathcal{H}(\alpha, \beta) > \mathcal{H}(\alpha', \beta')$ which contradicts the induction hypothesis.

Corollary 1. *Let k, n be non-null integers, γ be any ordinal, $0 \leq \alpha_0, \ldots, \alpha_k < \omega^n$, and $f : \gamma \longrightarrow \{0, \ldots, k\}$.*

$$\text{If } \forall i \leq k \;\; \alpha_i = ot(f^{-1}[i]) \text{ holds, then } \gamma < \omega^{\mathrm{n}}.$$

[13] In particular, $l_k, l_{k-1}, \ldots$ $m_k, m_{k-1}, \ldots$ might be null, but since $\alpha, \beta > 0$ holds, at least one of the l_i's, and one of the m_i's are different from zero.

Proof. This is immediate from Lemma 6.

Lemma 7. *Let k be some non-null integer, $(\mathbb{N}^k, \lesssim)$ be a well-ordering such that*

$$(a_0, \dots, a_{k-1}) \lesssim (b_0, \dots, b_{k-1}) \Longrightarrow \begin{cases} \forall i < k \quad a_i \leq b_i \\ \text{or} \\ \exists i, j < k \quad a_i < b_i \text{ and } a_j > b_j \end{cases}$$

holds for every k-tuples $(a_0, \dots, a_{k-1}), (b_0, \dots, b_{k-1}) \in \mathbb{N}^k$. Then, the order type of $(\mathbb{N}^k, \lesssim)$ is at most ω^k.

Proof. The proof goes by induction on $k \geq 1$.

(a) The initial case $k = 1$ is immediate since $(\mathbb{N}^1, \lesssim)$ is nothing but the usual ordering on integers.

(b) We assume the result holds for $k \geq 1$, and we show that it holds for $k + 1$.

Claim. For any integer n, the order type of the following set (ordered by $\lesssim$)

$$A_n = \{(a_0, a_1, \dots, a_k) \in \mathbb{N}^{k+1} \mid (a_0, a_1, \dots, a_k) < (n, n, \dots, n)\}$$

is strictly below $\omega^{(k+1)}$.

Proof. Notice that if $(a_0, a_1, \dots, a_k) < (n, n, \dots, n)$ holds then $a_i < n$ must hold for some $i \leq k + 1$.
For each $i \leq k + 1$ and each $j < n$ we consider

$$A_{(i,j)} = \{(a_0, a_1, \dots, a_k) \in \mathbb{N}^{k+1} \mid a_i = j\},$$

and $\alpha_{(i,j)} = ot(A_{(i,j)})$ its order type (ordered by $\lesssim$). Notice that for

$$(a_0, \dots, a_{i-1}, j, a_{i+1}, \dots, a_k) \lesssim (b_0, \dots, b_{i-1}, j, b_{i+1}, \dots, b_k)$$
$$\Longrightarrow$$
$$\begin{cases} \forall l \in \{0, \dots, i-1, i+1, \dots, k\} \quad a_l \leq b_l \\ \text{or} \\ \exists l, m \in \{0, \dots, i-1, i+1, \dots, k\} \quad a_l < b_l \text{ and } a_m > b_m \end{cases}$$

Therefore by induction hypothesis, $\alpha_{(i,j)} < \omega^{k+1}$ holds for all $i \leq k + 1$ and $j < n$. It follows from Corollary 1 that $ot(A_n) < \omega^{k+1}$ holds.

On the other hand for every integer $n \geq 0$ it holds that $(n, n, \dots, n) < (n + 1, n + 1, \dots, n + 1)$. Moreover if $n = max\{a_0, a_1, \dots, a_k\} + 1$ then $(a_0, a_1, \dots, a_k) < (n, n, \dots, n)$. Therefore the sequence of ordinals $(ot(A_n))_{n \geq 1}$ is cofinal in $ot(\mathbb{N}^{k+1})$ and thus the order type of $(\mathbb{N}^{k+1}, \lesssim)$ is at most ω^{k+1}.

Lemma 8. *We let k be any non-null integer, $B \in \mathbf{BC}(\mathbf{k})$, $A \subseteq X^\omega$ be any initializable set, and δ any countable ordinal.*

$$B \leq_w A \bullet \delta \quad \Longrightarrow \quad B \leq_w A \bullet \alpha \text{ dor some } \alpha < \omega^{k+1}.$$

Notice that an immediate consequence is that $B \equiv_w A \bullet \delta$ holds only for ordinals $\delta < \omega^{k+1}$.

Proof. First notice that for every $B \subseteq X^\omega$, and every $u \in X^*$, if $B \in \mathbf{BC}(\mathbf{k})$ holds, then $u^{-1}B \in \mathbf{BC}(\mathbf{k})$ holds too.

Towards a contradiction, we assume that $A \bullet \alpha <_w B \leq_w A \bullet \delta$ holds for all $\alpha < \omega^{k+1}$. We let $\mathcal{B}$ be a k-blind counter automaton that recognizes B. By Lemma 3, for each successor ordinal $\alpha < \omega^{k+1}$ there exists some $u_\alpha \in X^*$ such that $u_\alpha^{-1}B \equiv_w A \bullet \alpha$ or $u_\alpha^{-1}B \equiv_w (A \bullet \alpha)^{\complement}$. For each such u_α, we form $(q_\alpha, c_{\alpha,0}, c_{\alpha,1}, \ldots, c_{\alpha,k-1})$ where q_α denotes the control state that $\mathcal{B}$ is in after having read u_α, and $c_{\alpha,i}$ the height of its counter number i (any $i < k$).

Now there exists necessarily some control state q such that

$$\text{the order type of the set } S = \{\alpha < \omega^{k+1} \mid \alpha \text{ successor and } q_\alpha = q\} \text{ is } \omega^{k+1}.$$

Now, by Lemma 7 there exist $\alpha, \alpha' \in S$ such that $\alpha' < \alpha$ holds together with $c_{\alpha,i} \leq c'_{\alpha,i}$ holds for all $i < k$. (Without loss of generality, we may even assume that $\omega \leq \alpha' < \alpha$ holds.) Let us denote $\mathcal{B}_\alpha$ the k-blind counter automaton $\mathcal{B}$ that starts in state $(q_{\alpha'}, c_{\alpha',0}, c_{\alpha',1}, \ldots, c_{\alpha',k-1})$, and $\mathcal{B}_\alpha$ the one that starts in state $(q_\alpha, c_{\alpha,0}, c_{\alpha,1}, \ldots, c_{\alpha,k-1})$. Notice that since $c_{\alpha,i} \leq c'_{\alpha,i}$ holds for all $i < k$, $\mathcal{B}_{\alpha'}$ performs exactly the same as $\mathcal{B}_\alpha$ except when the latter crashes for it tries to decrease a counter that is already empty. But it is then not difficult to see that given the above assumption – that $\omega \leq \alpha' < \alpha$ holds – $u_\alpha^{-1}B \leq_w u_{\alpha'}^{-1}B$ holds which leads to either $A \bullet \alpha \leq_w A \bullet \alpha'$ or $(A \bullet \alpha)^{\complement} \leq_w A \bullet \alpha'$. In both cases, it contradicts $\alpha' < \alpha$.

Notice that the set $\emptyset \bullet \omega_1^n$ is initializable, so we have in particular the following result.

Lemma 9. *For k, n any integers, A any non-self dual ω-language in $\mathbf{BC}(\mathbf{k})$, and any non-zero countable ordinal α,*

$$A \quad or \quad A^{\complement} \equiv_w (\emptyset \bullet \omega_1^n) \bullet \alpha \Longrightarrow \alpha < \omega^{k+1}.$$

In a similar way, we can now state the following lemma.

Lemma 10. *We let k be any non-null integer, $B \in \mathbf{BC}(\mathbf{k})$, $A \subseteq X^\omega$ be any initializable set, δ be any countable ordinal, and C be any set of the form*

$$C = A \bullet \omega_1^n \bullet \nu_n + \ldots + A \bullet \omega_1^{n-1} \bullet \nu_{n-1} + \ldots + A \bullet \omega_1 \bullet \nu_1$$

for any non-zero integer n, and countable multiplicative coefficients $\nu_n, \nu_{n-1}, \ldots, \nu_1$ with at least one of them being non-null.

$$B \leq_w C + A \bullet \delta \quad \Longrightarrow \quad B \leq_w C + A \bullet \alpha \text{ dor some } \alpha < \omega^{k+1}.$$

Proof. The proof is very similar to the one of Lemma 8, so we leave it to the reader.

Theorem 6. *Let k be any non-null integer, $B \subseteq X^\omega$ be non-self dual. If $B \in \mathbf{BC}(\mathbf{k})$, then either B or $B^{\complement}$ is Wadge equivalent to some*

$$\Omega(\alpha) = (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \ldots + (\emptyset \bullet \omega_1^{n_0}) \bullet \delta_0.$$

where $j \in \mathbb{N}$, $n_j > n_{j-1} > \ldots > n_0$ and $\omega^{k+1} > \delta_j, \delta_{j-1}, \ldots, \delta_0 > 0$.

Proof. This is an almost immediate consequence of Lemmas 8 and 10.

This settles the case of the non-self dual ω-languages in $\mathbf{BC}(\mathbf{k})$. For the self-dual ones, it is enough to notice the easy following:

(a) Given any $A \subseteq X^\omega$, if $A \in \mathbf{BC}(\mathbf{k})$ is self dual, then there exists two non-self dual sets $B, C \subseteq X^\omega$ such that both B and C belong to $\mathbf{BC}(\mathbf{k})$, $B \equiv_w C^{\complement}$, and $A \equiv_w X_0 B \cup X_1 C$, where $\{X_0, X_1\}$ is any partition of X in two non-empty sets.

(b) If $A \subseteq X^\omega$ and $B \subseteq X^\omega$ are non-self dual, verify $A \equiv_w B^{\complement}$, and both belong to $\mathbf{BC}(\mathbf{k})$, then, given any partition of X in two non-empty sets $\{X_0, X_1\}$, $X_0 A \cup X_1 B$ is self-dual, and also belongs to $\mathbf{BC}(\mathbf{k})$.

As a consequence, we obtain the following general result if we come back to the original definition of the Wadge degree of a set (denoted d°) – from which we slightly departed from to define d_W – namely:

Definition 14. *For $A \subseteq X^\omega$, we set*

$$d^\circ(A) = sup\{d^\circ(B) + 1 \mid B <_W A\}.$$

(Notice that this definition implies $d^\circ(\emptyset) = d^\circ(\emptyset^{\complement}) = 0$.)

Theorem 7. *For any $A \subseteq X^\omega$, there exists an ω-language $B \subseteq X^\omega$ recognized by some deterministic Petri net, such that $A \equiv_w B$ if and only if $d^\circ A$ is of the form*

$$\alpha = \omega_1^n \cdot \delta_n + \ldots + \omega_1^0 \cdot \delta_0.$$

for some $n \in \mathbb{N}$, and $\omega^\omega > \delta_n, \ldots, \delta_0 \geq 0$.

From where we immediately obtain the following:

Corollary 2. *The height of the Wadge hierarchy of ω-language recognized deterministic Petri net is $(\omega^\omega)^\omega = \omega^{\omega^2}$.*

7 Conclusions

We provided a description of the extension of the Wagner hierarchy from automata to deterministic Petri Nets with Muller acceptance conditions. Of course the results would be rigorously the same if we replace Muller acceptance conditions with parity acceptance conditions. But with Büchi acceptance conditions instead, it becomes even simpler since the ω-languages are no more boolean combinations of $\mathbf{\Sigma}_2^0$-sets, but $\mathbf{\Pi}_2^0$-sets. So, the whole hierarchy comes down to the following:

Corollary 3. *For any $A \subseteq X^\omega$, there exists an ω-language $B \subseteq X^\omega$ recognized by some deterministic Petri net with* Büchi acceptance conditions*, such that $A \equiv_w B$ if and only if either*

- $d^\circ A = \omega_1$*, and A is $\mathbf{\Pi}^0_2$-complete, or*
- $d^\circ A < \omega^\omega$*.*

Deciding the degree of a given ω-language in $\mathbf{BC}(\mathbf{k})$, for $k \geq 2$, recognized by some deterministic Petri net – either with *Büchi* or *Muller* acceptance conditions, remains an open question. Notice that for $k = 1$ this decision problem has been shown to be decidable in [11].

Another rather interesting open direction of research is to go from deterministic to non-deterministic Petri nets. It is clear that this step forward brings new Wadge classes – for instance there exist ω-languages recognized by non-deterministic Petri nets with *Büchi acceptance conditions* that are $\mathbf{\Sigma}^0_3$-complete, hence not $\mathbf{\Delta}^0_3$, [19] – but the description of this whole hierarchy still requires more investigations.

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