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## APPROXIMATING CURVES ON REAL RATIONAL SURFACES

JÁNOS KOLLÁR AND FRÉDÉRIC MANGOLTE

ABSTRACT. We give necessary and sufficient topological conditions for a simple closed curve on a rational surface to be approximable by embedded smooth rational curves. We also study approximation by embedded smooth rational curves with given complex self-intersection number.

### 1. INTRODUCTION

As a generalization of the Weierstrass approximation theorem, every  $C^\infty$  map to a rational variety  $\mathbb{S}^1 \rightarrow X$  can be approximated, in the  $C^\infty$ -topology, by a real algebraic map  $\mathbb{R}\mathbb{P}^1 \rightarrow X$ ; see [BK99] and Definition 4. In particular, any simple closed curve on a rational surface  $S$  can be approximated by a rational curve on  $S$ . Note that the usual result is about *maps* of rational curves, so the image may have some extra isolated points. For example, consider the parametrization of the nodal plane cubic curve  $((x^2+y^2)z = x^3)$  given by  $f: (u, v) \rightarrow (v(u^2+v^2), u(u^2+v^2), v^3)$ . Clearly  $f(\mathbb{R}\mathbb{P}^1)$  is a simple closed curve in  $\mathbb{R}\mathbb{P}^2$  but its Zariski closure has an extra isolated real point at  $(0, 0, 1)$ . One can remove this point either by perturbing the equation to  $(z(x^2+y^2+\epsilon^2 z^2) = x^3)$  or by blowing up the point  $(0, 0, 1)$ . In the first case the curve becomes elliptic, in the second case the topology of the real surface changes.

In this note we get rid of these extra real singular points, using the methods introduced in [KM09]. The main result is the following.

**Theorem 1.** *Let  $S(\mathbb{R})$  be the underlying topological surface of the real points of a smooth rational surface and  $L \subset S(\mathbb{R})$  a simple, connected, closed curve. The following are equivalent.*

- (1)  *$L$  can be approximated by smooth rational curves in the  $C^\infty$ -topology.*
- (2) *There is a smooth rational surface  $S'$  and a smooth rational curve  $C' \subset S'$  such that  $(S, L)$  is diffeomorphic to  $(S'(\mathbb{R}), C'(\mathbb{R}))$ .*
- (3)  *$(S(\mathbb{R}), L)$  is **not** diffeomorphic to the pair (torus, null homotopic curve).*

Note that complex algebraic surfaces contain very few smooth rational curves; for instance only lines and conics in  $\mathbb{P}^2$ . Thus our approximating curves almost always have many singular points, but they come in complex conjugate pairs.

It is clear that (1.1)  $\Rightarrow$  (1.2) and (1.2)  $\Rightarrow$  (1.1) can be derived from the main result of [KM09]. The implication (1.2)  $\Rightarrow$  (1.3) turns out to be a straightforward genus computation in Proposition 19. The main result is (1.3)  $\Rightarrow$  (1.2), which is proved by enumerating all possible topological pairs  $(S(\mathbb{R}), L)$  and then exhibiting each for a suitable rational surface, with one exception as in (1.3).

In order to state a more precise version, we fix our topological notation.

**Notation 2.** Let  $\mathbb{S}^1$  denote the circle,  $\mathbb{S}^2$  the 2-sphere,  $\mathbb{R}\mathbb{P}^2$  the real projective plane,  $\mathbb{T}^2 \sim \mathbb{S}^1 \times \mathbb{S}^1$  the 2-torus and  $\mathbb{K}^2$  the Klein bottle.

We also use some standard curves on these surfaces.  $\mathbb{L} \subset \mathbb{S}^2$  denotes a circle and  $\mathbb{L} \subset \mathbb{RP}^2$  a line. We think of both  $\mathbb{T}^2$  and  $\mathbb{K}^2$  as an  $\mathbb{S}^1$ -bundle over  $\mathbb{S}^1$ . Then  $\mathbb{L}$  denotes a section and  $\mathbb{F}$  a fiber. Note that  $(\mathbb{T}^2, \mathbb{L})$  is diffeomorphic to  $(\mathbb{T}^2, \mathbb{F})$  but  $(\mathbb{K}^2, \mathbb{L})$  is not diffeomorphic to  $(\mathbb{K}^2, \mathbb{F})$ .

Diffeomorphism of two surfaces  $S_1, S_2$  is denoted by  $S_1 \sim S_2$ . Connected sum with  $r$  copies of  $\mathbb{RP}^2$  (resp.  $\mathbb{T}^2$ ) is denoted by  $\#r\mathbb{RP}^2$  (resp.  $\#\mathbb{T}^2$ ).

Let  $(S_1, L_1)$  be a surface and a curve on it. Its connected sum with a surface  $S_2$  is denoted by  $(S_1, L_1)\#S_2$ , whose underlying surface is  $S_1\#S_2$ . We assume that the connected sum operation is disjoint from  $L_1$ ; then we get  $L_1 \subset S_1\#S_2$ . This operation is well defined if  $S_1 \setminus L_1$  is connected. If  $S_1 \setminus L_1$  is disconnected, then it matters to which side we attach  $S_2$ . In the latter case we distinguish these by putting  $\#S_2$  on the left or right of  $(S_1, L_1)$ . Thus

$$r_1\mathbb{RP}^2\#(\mathbb{S}^2, L)\#r_2\mathbb{RP}^2$$

indicates that we attach  $r_1$  copies of  $\mathbb{RP}^2$  to one side of  $\mathbb{S}^2 \setminus L$  and  $r_2$  copies of  $\mathbb{RP}^2$  to the other side.

We also need to take connected sums of the form  $(S_1, L_1)\#(S_2, L_2)$ . From both surfaces we remove a disc that intersects the curves  $L_i$  in an interval. Then we glue so as to get a simple closed curve on  $S_1\#S_2$ . In general there are several ways of doing this. However, when one of the pairs is  $(\mathbb{RP}^2, \mathbb{L})$ , then the end result is unique. This is the only case that we use.

The following result lists the topological pairs  $(S(\mathbb{R}), C(\mathbb{R}))$ , depending on the complex self-intersection of the rational curve. In the table below we ignore the trivial cases when  $C(\mathbb{R}) = \emptyset$ . Furthermore, since every topological type that occurs for  $(C^2) = e + 2$  also occurs for  $(C^2) = e$ , line  $e$  lists only those types that do not appear for  $e + 2, \dots, e + 8$ .

**Theorem 3.** *Let  $S$  be a smooth, projective surface defined and rational over  $\mathbb{R}$ . Let  $C \subset S$  be a rational curve that is smooth (even over  $\mathbb{C}$ ). For  $(C^2) \geq -2$  the following table lists the possible topological types of the pair  $(S(\mathbb{R}), C(\mathbb{R}))$ .*

$(C^2)$	topology
even $\geq 6$	$(\mathbb{T}^2, \mathbb{L})\#r\mathbb{RP}^2 : r = 0, 1, \dots$
odd $\geq 5$	$(\mathbb{K}^2, \mathbb{L})\#r\mathbb{RP}^2 : r = 0, 1, \dots$
4	$r_1\mathbb{RP}^2\#(\mathbb{S}^2, \mathbb{L})\#r_2\mathbb{RP}^2 : r_1 + r_2 \geq 1$
3	nothing new
2	$(\mathbb{S}^2, \mathbb{L})$
1	$(\mathbb{RP}^2, \mathbb{L})$
0	$(\mathbb{K}^2, \mathbb{F})$
-1	$(\mathbb{RP}^2, \mathbb{L})\#\mathbb{T}^2$
-2	$(\mathbb{K}^2, \mathbb{F})\#\mathbb{T}^2$ .

As we see in Section 2, the entries for  $e \geq -1$  follow from an application of the minimal model program to the pair  $(S, C)$ . Nothing unexpected happens for  $e = -2$  but this depends on some rather delicate properties of singular Del Pezzo surfaces of degrees 1, 2 and 4; see Lemma 9.

By contrast, we know very little about the cases  $e \leq -3$ . These lead to the study of rational surfaces with quotient singularities and ample canonical class. There are many such cases – see [Kol08, Sec.5] or Example 10 – but very few definitive results [Keu11, HK12, HK11a, HK11b].

The pairs  $(S, L)$  listed in Theorem 3 and the pairs easily derivable from them give almost all examples needed to prove (1.3)  $\Rightarrow$  (1.2). The only exceptions are pairs  $(S, L)$  where  $S \setminus L$  is the disjoint union of a Möbius strip and of an orientable surface of genus  $\geq 2$ . These are constructed by hand in Example 18.

**Definition 4.** Given a compact real algebraic variety  $X$ , we consider the set of embeddings of nonsingular rational curves to be a subset of the space  $C^\infty(\mathbb{S}^1, X)$  of all  $C^\infty$  maps of  $\mathbb{S}^1$  into  $X$ , endowed with the  $C^\infty$ -topology.

Let  $L \hookrightarrow X$  be an embedded circle on a real algebraic variety  $X$ , we say that  $L$  can be *approximated* by curves of a certain kind if every neighborhood of  $L$  in  $C^\infty(\mathbb{S}^1, X)$  contains a curve of that kind.

**5 (Real algebraic surfaces).** For us a *real algebraic variety* is an algebraic variety defined over  $\mathbb{R}$ . If  $X$  is a real algebraic variety then  $X(\mathbb{C})$  denotes the set of complex points and  $X(\mathbb{R})$  the set of real points. (Note that frequently – for instance in the book [BCR98] –  $X(\mathbb{R})$  itself is called a real algebraic variety.) Thus for us  $\mathbb{P}^n$  is a real algebraic variety whose real points  $\mathbb{P}^n(\mathbb{R})$  can be identified with  $\mathbb{R}\mathbb{P}^n$  and whose complex points  $\mathbb{P}^n(\mathbb{C})$  can be identified with  $\mathbb{C}\mathbb{P}^n$ .

For many purposes, the behavior of a real variety at its complex points is not relevant, but in this paper it is crucial to consider complex points as well. When we talk about a smooth, projective, real algebraic variety, it is important that smoothness hold at all complex points and  $X(\mathbb{C})$  be compact.

We say that a real algebraic variety  $X$  of dimension  $n$  is *rational* if it is birational to  $\mathbb{P}^n$ ; that is, the birational map is also defined over  $\mathbb{R}$ . If such a birational map exists with complex coefficients, we say that  $X$  is *geometrically rational*.

The intersection number of two real curves  $(C_1 \cdot C_2)$  on a surface is the intersection number of the underlying complex curves. The intersection number of the real parts  $(C_1(\mathbb{R}) \cdot C_2(\mathbb{R}))$  is only defined modulo 2 and

$$(C_1(\mathbb{R}) \cdot C_2(\mathbb{R})) \equiv (C_1 \cdot C_2) \pmod{2}.$$

In particular, if  $C \subset S$  is a rational curve such that  $C$  is smooth at its real points, then  $S(\mathbb{R})$  is orientable along  $C(\mathbb{R}) \Leftrightarrow (C(\mathbb{R}) \cdot C(\mathbb{R})) \equiv 0 \pmod{2} \Leftrightarrow (C \cdot C)$  is even.

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## 2. MINIMAL MODELS FOR PAIRS

Let  $\mathcal{S}$  be a class of smooth, projective surfaces defined over  $\mathbb{R}$  that is closed under birational equivalence. We would like to understand the possible topological types  $(S(\mathbb{R}), C(\mathbb{R}))$  where  $S \in \mathcal{S}$  and  $C \subset S$  is a smooth, rational curve.

We are mostly interested in the cases when  $\mathcal{S}$  consists of rational or geometrically rational surfaces. It is not a priori obvious, but the answer turns out to have an interesting dependence on the self-intersection number  $e := (C^2)$ .

Our approach is to run the  $(S, K_S + C)$ -minimal model program (abbreviated as MMP). (For a general introduction over any field, see [KM98, Sec.1.4]. The real case is discussed for smooth surfaces in [Kol01, Sec.2] and for surfaces with Du Val singularities in [Kol00, Sec.2].) Then we need to understand how the topology of

$(S(\mathbb{R}), C(\mathbb{R}))$  changes with the steps of the program and describe the possible last steps. At the end we try to work backwards to get our final answer.

Since  $(C \cdot (K_S + C)) = -2$ , the divisor  $K_S + C$  is not nef, so the minimal model program always produces a nontrivial contraction  $\pi : S \rightarrow S_1$ . If  $\pi$  is birational and  $C$  is not  $\pi$ -exceptional, set  $C_1 := \pi(C) \subset S_1$ .

**6** (List of the possible steps of the MMP).

Note that if  $E \subset S$  is an irreducible curve such that  $(K_S + C) \cdot E < 0$  then  $(K_S \cdot E) < 0$ , except possibly when  $E = C$  and  $(C^2) < 0$ . Thus – aside from the latter case which we discuss in (6.5) – all steps of the  $(K_S + C)$ -MMP are also steps on the traditional MMP.

In what follows, we ignore the few cases where  $S(\mathbb{R}) = \emptyset$  since these are not relevant for us.

*Elementary birational contractions.*

(6.1)  $\pi$  contracts a conjugate pair of  $(-1)$ -curves that are disjoint from  $C$ . Then

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (S_1(\mathbb{R}), C_1(\mathbb{R})) \quad \text{and} \quad (C^2) = (C_1^2).$$

(6.2)  $\pi$  contracts a conjugate pair of  $(-1)$ -curves that intersect  $C$  with multiplicity 1 each. Then

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (S_1(\mathbb{R}), C_1(\mathbb{R})) \quad \text{and} \quad (C^2) = (C_1^2) - 2.$$

The inverse shows that every topological type that occurs for  $(C^2) = e$  also occurs for  $(C^2) = e - 2$ .

(6.3)  $\pi$  contracts a real  $(-1)$ -curve that is disjoint from  $C$ . Then

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (S_1(\mathbb{R}), C_1(\mathbb{R})) \# \mathbb{R}\mathbb{P}^2 \quad \text{and} \quad (C_1^2) = (C^2).$$

The inverse shows that for every topological type that occurs, its connected sum with  $\mathbb{R}\mathbb{P}^2$  also occurs.

(6.4)  $\pi$  contracts a real  $(-1)$ -curve that intersects  $C$  with multiplicity 1. Then

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (S_1(\mathbb{R}), C_1(\mathbb{R})) \# (\mathbb{R}\mathbb{P}^2, \mathbb{L}) \quad \text{and} \quad (C^2) = (C_1^2) - 1.$$

We describe these completely in (7.1–2).

With these birational contractions,  $(S_1, C_1)$  is again a pair in our class  $\mathcal{S}$  and we can continue running the minimal model program to get

$$(S, C) = (S_0, C_0) \xrightarrow{\pi_0} (S_1, C_1) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{m-1}} (S_m, C_m)$$

until no such contractions are possible. We call such pairs  $(S_m, C_m)$  *classically minimal*. Note also that in any sequence of these steps, the value of  $(C_i^2)$  is non-decreasing.

*Singular birational contraction.*

(6.5)  $\pi$  contracts  $C$  to a point. This can happen only if  $(C^2) < 0$ . If  $(C^2) \leq -2$ , the resulting  $S_1$  is singular. For  $(C^2) \leq -3$  these are very difficult cases and we try to avoid them if possible.

*Non-birational contractions.*

(6.6)  $\pi : S \rightarrow \mathbb{P}^1$  is a  $\mathbb{P}^1$ -bundle and  $C$  is a section. In this case

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (\mathbb{T}^2, \mathbb{L}) \quad \text{or} \quad (\mathbb{K}^2, \mathbb{L}) \quad \text{with} \quad (C^2) \text{ arbitrary.}$$

(More precisely,  $(C^2)$  is even for  $\mathbb{T}^2$  and odd for  $\mathbb{K}^2$ .)

(6.7)  $S = \mathbb{P}^2$  and  $C$  is a conic. Thus

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (\mathbb{S}^2, \mathbb{L}) \# \mathbb{R}\mathbb{P}^2 \quad \text{and} \quad (C^2) = 4.$$

(6.8)  $S = (x^2 + y^2 + z^2 = t^2) \subset \mathbb{P}^3$  and  $C = (x = 0)$  is a plane section. Thus

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (\mathbb{S}^2, \mathbb{L}) \quad \text{and} \quad (C^2) = 2.$$

(6.9)  $S = \mathbb{P}^2$  and  $C$  is a line. Thus

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L}) \quad \text{and} \quad (C^2) = 1.$$

(6.9)  $\pi : S \rightarrow \mathbb{P}^1$  is a conic bundle and  $C$  is a smooth fiber. If  $S$  is rational then we have three possibilities

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (\mathbb{T}^2, \mathbb{F}), (\mathbb{K}^2, \mathbb{F}) \quad \text{or} \quad (\mathbb{S}^2, \mathbb{L}) \quad \text{and} \quad (C^2) = 0.$$

If  $S$  is geometrically rational then we can also have the disjoint union of  $(\mathbb{S}^2, \mathbb{L})$  with copies of  $\mathbb{S}^2$ .

Putting these together, we get the following.

**Corollary 7.** *Let  $S$  be a smooth, projective surface defined over  $\mathbb{R}$  and  $C \subset S$  a smooth, rational curve. Run the  $(K_S + C)$ -MMP to get*

$$(S, C) = (S_0, C_0) \xrightarrow{\pi_0} (S_1, C_1) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{m-1}} (S_m, C_m) \xrightarrow{\tau} T.$$

Assume that the  $\pi_i$  are elementary contractions as in (6.1–4) and  $\tau$  is a non-birational contraction as in (6.6–9).

Then  $S_m, C_m$  are smooth and  $(S(\mathbb{R}), C(\mathbb{R}))$  can be described as follows.

(1) In all cases we can have

$$(S(\mathbb{R}), C(\mathbb{R})) \sim (S_m(\mathbb{R}), C_m(\mathbb{R})) \# r_1(\mathbb{R}\mathbb{P}^2, \mathbb{L}) \# r_2 \mathbb{R}\mathbb{P}^2$$

for some  $r_1, r_2 \geq 0$ . Here  $(C^2) \leq (C_m^2) - r_1$ .

(2) Other possibilities occur only when  $(S_m(\mathbb{R}), C_m(\mathbb{R}))$  is  $(\mathbb{S}^2, \mathbb{L})$  or  $(\mathbb{S}^2, \mathbb{L}) \# \mathbb{R}\mathbb{P}^2$  and all exceptional curves of  $S \rightarrow S_m$  are disjoint from  $C(\mathbb{R})$ . In this case  $C(\mathbb{R})$  separates  $S(\mathbb{R})$  and in taking connected sums we need to keep track on which side we blow up. Thus

$$(S(\mathbb{R}), C(\mathbb{R})) \sim r_1 \mathbb{R}\mathbb{P}^2 \# (S_m(\mathbb{R}), C_m(\mathbb{R})) \# r_2 \mathbb{R}\mathbb{P}^2$$

for some  $r_1, r_2 \geq 0$ . Here  $(C^2) \leq (C_m^2)$ . □

It remains to understand what happens if the MMP ends with a singular birational contraction. We start with the simplest,  $(C_m^2) = -1$  case; here the adjective “singular” is not warranted.

**8** (Case  $(C_m^2) = -1$ ). Let  $C_m \subset S_m$  be a  $(-1)$ -curve and  $\tau : S_m \rightarrow S_{m+1}$  its contraction. Then  $S_{m+1}$  is again a surface in  $\mathcal{S}$ . Thus

$$(S_m(\mathbb{R}), C_m(\mathbb{R})) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L}) \# S_{m+1}(\mathbb{R}).$$

If  $S$  is a rational surface then so is  $S_{m+1}$  thus, by Comessatti’s theorem [Com14], we have only the cases

$$(S_m(\mathbb{R}), C_m(\mathbb{R})) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L}) \# r \mathbb{R}\mathbb{P}^2 \quad \text{or} \quad (\mathbb{R}\mathbb{P}^2, \mathbb{L}) \# \mathbb{T}^2. \quad (8.1)$$

We can then obtain  $(S(\mathbb{R}), C(\mathbb{R}))$  from  $(S_m(\mathbb{R}), C_m(\mathbb{R}))$  as in (7.1).



Thus the composite  $\tau \circ \pi: S \rightarrow T_1$  consist of two smooth blow ups. This again shows that  $(S, C)$  is not classically minimal.

Case 2:  $\tau: T \rightarrow T_1$  is a conic bundle. Then  $\tau \circ \pi: S \rightarrow T_1$  is a non-minimal conic bundle, hence there is a  $(-1)$ -curve  $E$  contained in a fiber.  $C$  is also contained in a fiber thus  $(E \cdot C) \leq 1$  since any 2 irreducible curves in a fiber of a conic bundle intersect in at most 1 point. Thus again  $(S, C)$  is not classically minimal.

Case 3:  $T$  is a Del Pezzo surface of Picard number 1.

In this case  $S$  itself is a weak Del Pezzo surface (that is  $-K_S$  is nef) of Picard number 2. Thus  $S$  has another extremal ray giving a contraction  $\rho: S \rightarrow S_1$ . Next we study the possible types of  $\rho$ .

Case 3.1:  $\rho$  is a  $\mathbb{P}^1$ -bundle. Then  $C$  has to be a negative section, giving the first possibility.

Case 3.2:  $\rho$  is birational so  $S_1$  is a Del Pezzo surface of Picard number 1. Since  $(S, C)$  is classically minimal, the exceptional curve of  $\rho$  has intersection number  $\geq 2$  with  $C$ . In particular  $C_1 := \rho(C)$  is singular.

Since  $|-K_S|$  has dimension  $\geq 1$  and  $(C \cdot K_S) = 0$ , there is a divisor  $D \in |-K_S|$  whose support contains  $C$ .

Thus  $C_1 := \rho(C)$  is singular and is contained in a member of  $|-K_{S_1}|$ . A general member of  $|-K_{S_1}|$  is smooth, elliptic; special members are either irreducible, rational with a single node or cusp or reducible with smooth rational components. Thus  $C_1$  is a member of  $|-K_{S_1}|$  and has a node or cusp at a point  $p_1$ .

From  $-2 = (C^2) = (C_1^2) - 4$  we see that  $S_1$  is a smooth Del Pezzo surface of degree 2. We obtain  $S$  by blowing up the singular point of  $C_1$  and so  $(K_S^2) = (K_{S_1}^2) = (K_{S_1}^2) - 1 = 1$ .

Case 3.3.  $\rho$  is a conic bundle.

Let  $F \subset S$  be a general fiber of  $\rho$ . Then  $(\pi(F) \cdot K_T) = (F \cdot K_S) = -2$ .

There are several types of Del Pezzo surfaces of Picard number 1 with an ordinary node that contain a pencil of curves with anticanonical degree 2.

Case 3.3.1.  $T$  is a quadric cone; this was discussed in Case 3.1.

In other cases  $-K_T$  generates the Picard group. Thus  $2\pi(F) \sim -aK_T$  for some  $a \geq 1$ . This gives  $4 = a(K_T^2)$  hence  $(K_T^2) \in \{1, 2, 4\}$ . We consider these next.

Case 3.3.2.  $T$  has degree 2 and  $|\pi(F)|$  is a pencil in  $|-K_T|$ . Thus  $(K_S^2) = 2$ , hence  $\rho$  is a conic bundle and  $C$  is a double section.

Case 3.3.3.  $T$  has degree 1 and  $|\pi(F)|$  is a pencil in  $|-2K_T|$ . Note that  $|-2K_T|$  maps  $T$  onto a quadric cone  $Q \subset \mathbb{P}^3$  and so every pencil in  $|-2K_T|$  is obtained as the pull-back of a pencil of planes in  $\mathbb{P}^3$ . The fixed line intersects  $Q$  in 2 points, thus every such pencil on  $T$  has at least 2 base points. (The exception is when the line passes through the vertex, in this case every curve in the pencil is reducible.) Hence this case can not happen.

Case 3.3.4.  $T$  has degree 4 and  $|\pi(F)|$  is a pencil in  $|\frac{1}{2}K_T|$ . Thus a general member of  $|\pi(F)|$  is a smooth rational curve and so  $C$  is a section of  $\rho$ . This happens only for the quadric cone.

(It is interesting to note that this case can happen if  $T$  has 2 ordinary nodes. In  $\mathbb{P}^2$  pick points  $p_1, \dots, p_4$  such that  $p_2, p_3, p_4$  are on a line  $L$  but  $p_1$  is not. Then  $2p_1, p_2, p_3, p_4$  determine a degree 4 Del Pezzo surface. The line  $L$  and the exceptional curve of the first blow-up of  $p_1$  are contracted to nodes. The lines through  $p_1$  form a half-canonical pencil.)

Finally note that a Del Pezzo surface of degree 2 and of Picard number 1 or a conic bundle of degree 2 and of Picard number 2 is never rational over the ground field  $k$  by the Segre–Manin theorem; see [Seg51, Man66] or [KSC04, Chap.2] for an introduction to these results.  $\square$

The main difficulty with the  $(C^2) \leq -3$  cases is that contracting such a curve can yield a rational surface with trivial or ample canonical class. Here are some simple examples of this. For  $d = 6$  the example below has  $(C^2) = -4$ ; we do not know such pairs with  $(C^2) = -3$ .

**Example 10.** Let  $\bar{C}_d \subset \mathbb{P}^2$  be a rational curve of degree  $d$  whose singularities are nodes. Thus we have  $\binom{d-1}{2}$  nodes forming a set  $N_d$ . Let  $p_d: S_d := B_d\mathbb{P}^2 \rightarrow \mathbb{P}^2$  denote the blow-up of all the nodes with exceptional curves  $E_d$  and  $C_d \subset S_d$  the birational transform of  $\bar{C}_d$ . We compute that  $(C_d^2) = d^2 - 4\binom{d-1}{2}$  and

$$K_{S_d} + \frac{3}{d}C_d - \left(1 - \frac{6}{d}\right)E_d \sim_{\mathbb{Q}} p_d^*(K_{\mathbb{P}^2} + \frac{3}{d}\bar{C}_d) \sim_{\mathbb{Q}} 0.$$

If  $d \geq 6$  then  $(C_d^2) < 0$ ; let  $\pi: S_d \rightarrow T_d$  be its contraction. Then

$$K_{T_d} \sim_{\mathbb{Q}} \left(1 - \frac{6}{d}\right)\pi_*E_d$$

is trivial for  $d = 6$  and ample for  $d \geq 7$ .

### 3. TOPOLOGY OF PAIRS $(S, L)$

In this section let  $S$  denote the real part of a smooth, projective, real algebraic surface that is rational over  $\mathbb{R}$ . By Comessatti's theorem [Com14],  $S$  is either  $\mathbb{S}^2$ ,  $\mathbb{T}^2$  or  $\#r\mathbb{R}\mathbb{P}^2$  for some  $r \geq 1$ . Let  $L \subset S$  be a connected, simple, closed curve. We aim to classify the pairs  $(S, L)$  up to diffeomorphism. We distinguish 4 main cases.

**11** ( $S$  is orientable). Thus  $S \sim \mathbb{S}^2$  or  $S \sim \mathbb{T}^2$ . There are three possibilities

- (1)  $(S, L) \sim (\mathbb{S}^2, \mathbb{L})$ ,
- (2)  $(S, L) \sim (\mathbb{T}^2, \mathbb{L})$  and
- (3)  $(S, L) \sim (\mathbb{T}^2, \text{null-homotopic curve})$ .

**12** ( $S$  is not orientable along  $L$ ). Contracting  $L$  we get another topological surface  $S'$  thus  $(S, L) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L})\#S'$ . This gives two possibilities

- (1)  $(S, L) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L})\#r\mathbb{R}\mathbb{P}^2$  for some  $r \geq 0$  or
- (2)  $(S, L) \sim (\mathbb{R}\mathbb{P}^2, \mathbb{L})\#g\mathbb{T}^2$  for some  $g > 0$ .

In the remaining 2 cases  $S$  is non-orientable but orientable along  $L$ .

**13** ( $L$  is non-separating). Then we have another simple closed curve  $L' \subset S$  such that  $S$  is non-orientable along  $L'$  and  $L$  meets  $L'$  at a single point transversally. Then a neighborhood of  $L \cup L'$  is a punctured Klein bottle and  $(S, L)$  is the connected sum of  $(\mathbb{K}^2, \mathbb{F})$  with another surface. This gives two possibilities

- (1)  $(S, L) \sim (\mathbb{K}^2, \mathbb{F})\#r\mathbb{R}\mathbb{P}^2$  for some  $r \geq 0$  or
- (2)  $(S, L) \sim (\mathbb{K}^2, \mathbb{F})\#g\mathbb{T}^2$  for some  $g > 0$ .

**14** ( $L$  is separating). Then  $S \setminus L$  has 2 connected components and at least one of them is non-orientable. This gives two possibilities

- (1)  $(S, L) \sim r_1\mathbb{R}\mathbb{P}^2\#\mathbb{S}^2\#\mathbb{L}\#r_2\mathbb{R}\mathbb{P}^2$  for some  $r_1 + r_2 \geq 1$  or
- (2)  $(S, L) \sim r_1\mathbb{R}\mathbb{P}^2\#\mathbb{S}^2\#\mathbb{L}\#g\mathbb{T}^2$  for some  $r_1, g > 0$ .

Since we can always create a connected sum with  $\mathbb{R}\mathbb{P}^2$  by blowing up a point, for construction purposes the only new case that matters is

(3)  $(S, L) \sim \mathbb{RP}^2 \# (\mathbb{S}^2, \mathbb{L}) \# g\mathbb{T}^2$  for some  $g > 0$ .

By the formula (7.1), we need to understand connected sum with  $(\mathbb{RP}^2, \mathbb{L})$ . This is again easy, but usually not treated in topology textbooks, so we state the formulas for ease of reference.

**15** (Some diffeomorphisms). We start with the list of elementary steps.

$$\begin{aligned}
(\mathbb{RP}^2, \mathbb{L}) \# \mathbb{RP}^2 &\sim (\mathbb{K}^2, \mathbb{L}) \\
(\mathbb{T}^2, \mathbb{L}) \# \mathbb{RP}^2 &\sim (\mathbb{K}^2, \mathbb{L}) \# \mathbb{RP}^2 \\
(\mathbb{T}^2, \mathbb{L}) \# (\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{L}) \# \mathbb{RP}^2 \\
(\mathbb{K}^2, \mathbb{L}) \# (\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{T}^2, \mathbb{L}) \# \mathbb{RP}^2 \\
(\mathbb{S}^2, \mathbb{L}) \# (\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{RP}^2, \mathbb{L}) \\
(\mathbb{RP}^2, \mathbb{L}) \# (\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{F}) \\
(\mathbb{K}^2, \mathbb{F}) \# (\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{RP}^2, \mathbb{L}) \# \mathbb{T}^2
\end{aligned} \tag{15.1}$$

There are – probably many – elementary topological ways to see these. An approach using algebraic geometry is the following.

For the first, blow up a point in  $\mathbb{P}^2$  not on the line  $\mathbb{L}$ . We get a minimal ruled surface over  $\mathbb{P}^1$  and the line becomes a section.

For the next three, take a minimal ruled surface  $S$  over  $\mathbb{P}^1$  with negative section  $E$ . If  $(E^2)$  is even then  $(S(\mathbb{R}), E(\mathbb{R})) \sim (\mathbb{T}^2, \mathbb{L})$  and if  $(E^2)$  is odd then  $(S(\mathbb{R}), E(\mathbb{R})) \sim (\mathbb{K}^2, \mathbb{L})$ . Blowing up a point on  $E$  changes the parity of  $(E^2)$ . Also, the fiber through that point becomes a  $(-1)$ -curve  $F'$  disjoint from the birational transform  $E'$  of  $E$ . We can contract  $F'$  to get a minimal ruled surface  $S'$  over  $\mathbb{P}^1$ .

Blowing up a point  $p \in \mathbb{L} \subset \mathbb{S}^2$  we get  $(\mathbb{S}^2, \mathbb{L}) \# (\mathbb{RP}^2, \mathbb{L})$ . The conjugate lines through  $p$  become conjugate  $(-1)$ -curves and contracting them gives  $(\mathbb{RP}^2, \mathbb{L})$ .

Blowing up a point  $p \in \mathbb{L} \subset \mathbb{RP}^2$  we get a minimal ruled surface  $S$  over  $\mathbb{P}^1$ . The exceptional curve  $E$  is the negative section  $E$  and the birational transform  $\mathbb{L}'$  of  $\mathbb{L}$  is a fiber; this is  $(\mathbb{K}^2, \mathbb{F})$ .

Blowing up a point  $p \in \mathbb{F} \subset \mathbb{K}^2$ , the birational transform  $\mathbb{F}'$  of  $\mathbb{F}$  is a  $(-1)$ -curve. As discussed at the beginning, contracting it we get  $\mathbb{T}^2$ , giving the last diffeomorphism.

Iterating these, we get the following

$$\begin{aligned}
(\mathbb{T}^2, \mathbb{L}) \# 2r(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{T}^2, \mathbb{L}) \# 2r\mathbb{RP}^2 \\
(\mathbb{T}^2, \mathbb{L}) \# (2r+1)(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{L}) \# (2r+1)\mathbb{RP}^2 \\
(\mathbb{K}^2, \mathbb{L}) \# 2r(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{L}) \# 2r\mathbb{RP}^2 \\
(\mathbb{K}^2, \mathbb{L}) \# (2r+1)(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{T}^2, \mathbb{L}) \# (2r+1)\mathbb{RP}^2 \\
(\mathbb{S}^2, \mathbb{L}) \# 2r(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{F}) \# (r-1)\mathbb{T}^2 \quad (r \geq 1) \\
(\mathbb{S}^2, \mathbb{L}) \# (2r+1)(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{RP}^2, \mathbb{L}) \# r\mathbb{T}^2 \\
(\mathbb{RP}^2, \mathbb{L}) \# 2r(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{RP}^2, \mathbb{L}) \# r\mathbb{T}^2 \\
(\mathbb{RP}^2, \mathbb{L}) \# (2r+1)(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{F}) \# r\mathbb{T}^2 \\
(\mathbb{K}^2, \mathbb{F}) \# 2r(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{K}^2, \mathbb{F}) \# r\mathbb{T}^2 \\
(\mathbb{K}^2, \mathbb{F}) \# (2r+1)(\mathbb{RP}^2, \mathbb{L}) &\sim (\mathbb{RP}^2, \mathbb{L}) \# (r+1)\mathbb{T}^2
\end{aligned} \tag{15.2}$$

#### 4. PROOFS OF THE THEOREMS

**16** (Proof of Theorem 3). Let  $S$  be a smooth, projective real algebraic surface that is rational over  $\mathbb{R}$  and  $C \subset S$  a smooth rational curve such that  $(C^2) \geq -2$ . We

run the  $(K_S + C)$ -MMP while we can perform elementary contractions to get

$$(S, C) = (S_0, C_0) \xrightarrow{\pi_0} (S_1, C_1) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{m-1}} (S_m, C_m).$$

We saw that  $(C_m^2) \geq (C^2) \geq -2$ . Let

$$\tau : (S_m, C_m) \rightarrow T$$

be the next step of the MMP. If  $(C_m^2) \geq 0$  then  $\tau$  is a non-birational contraction as in (6.6–9). The topology of  $(S_m(\mathbb{R}), C_m(\mathbb{R}))$  is fully understood and (7.1) shows how to get  $(S(\mathbb{R}), C(\mathbb{R}))$  from  $(S_m(\mathbb{R}), C_m(\mathbb{R}))$ . The first seven lines of the table in Theorem 3 follow from these using the formulas (15.2).

If  $(C_m^2) = -1$  then we use (8.1) and (7.1).

Finally, if  $(C_m^2) = -2$  then  $(S_m(\mathbb{R}), C_m(\mathbb{R})) \sim (\mathbb{T}^2, \mathbb{L})$  by Lemma 9, but this is already listed. The only new example comes from

$$(\mathbb{S}^2, \mathbb{L}) \# 4(\mathbb{R}\mathbb{P}^2, \mathbb{L}) \sim (\mathbb{K}^2, \mathbb{F}) \# \mathbb{T}^2. \quad \square$$

**17** (Proof of Theorem 1). We already noted that (1.1)  $\Rightarrow$  (1.2) is clear.

The converse, (1.2)  $\Rightarrow$  (1.1) involves two steps. First, if  $S_1, S_2$  are smooth, projective real algebraic surfaces that are rational over  $\mathbb{R}$  and  $S_1(\mathbb{R}) \sim S_2(\mathbb{R})$  then there is a birational map  $\phi : S_1 \dashrightarrow S_2$  that is an isomorphism between suitable Zariski open neighborhoods of  $S_1(\mathbb{R})$  and  $S_2(\mathbb{R})$ . This is [BH07, Thm.1.2]; see also [HM09] for a more direct proof.

Thus we have  $L \subset S(\mathbb{R})$  and a rational curve  $C \subset S$  that is smooth at its real points and a diffeomorphism

$$\phi : (S(\mathbb{R}), L) \sim (S(\mathbb{R}), C(\mathbb{R})).$$

By [KM09], the diffeomorphism  $\phi^{-1}$  can be approximated in the  $C^\infty$ -topology by birational maps  $\psi_n : S \dashrightarrow S$  that are isomorphisms between suitable Zariski open neighborhoods of  $S(\mathbb{R})$ . Thus

$$C_n := \psi_n(C) \subset S$$

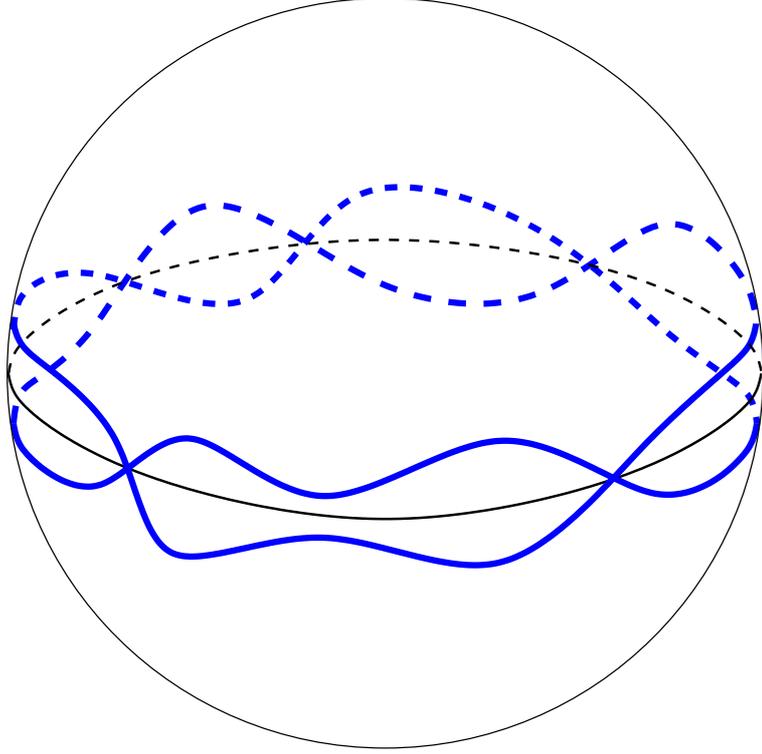
is a sequence of rational curves that are smooth at their real points and  $C_n(\mathbb{R}) \rightarrow L$  in the  $C^\infty$ -topology. One can resolve the complex singular points of  $C_n$  to get approximation of  $L$  by smooth rational curves ( $C'_n \subset S_n$ ). Here the surfaces  $S_n$  are isomorphic near their real points but not everywhere.

Again using [BH07, Thm.1.2], in order to show (1.2)  $\Rightarrow$  (1.3), it is enough to prove that on  $\mathbb{P}^1 \times \mathbb{P}^1$  there is a no rational curve  $C$  defined over  $\mathbb{R}$  such that  $C$  is smooth at its real points and  $C(\mathbb{R})$  is null-homotopic. This follows from a genus computation done in Proposition 19.

It remains to show that (1.3)  $\Rightarrow$  (1.2). All possible topological pairs  $(S(\mathbb{R}), L)$  were enumerated in (12–14). With the exception of cases (11.3) and (14.1–2), the examples listed in Theorem 3, and their descendants using the formulas (15.2), cover everything. We already proved that (11.3) never occurs. This leaves us with the task of exhibiting examples for (14.1–2). As noted there, we only need to find examples for (14.3); these are constructed next.  $\square$

**Example 18.** Let  $L_1, \dots, L_{g+1}$  be distinct lines through the origin in  $\mathbb{R}^2$  and  $H(x, y)$  the equation of their union. For some  $0 < \epsilon \ll 1$  let  $\bar{C}^\pm \subset \mathbb{P}^2$  be the Zariski closure of the image of the unit circle ( $x^2 + y^2 = 1$ ) under the map

$$(x, y) \mapsto (1 \pm \epsilon H(x, y))(x, y).$$

FIGURE 1. The  $g = 2$  case

The curves  $\bar{C}^\pm$  are rational and intersect each other at the  $2g + 2$  points where the unit circle intersects one of the lines  $L_i$  and also at the conjugate point pair  $(1 : \pm i : 0)$ . Note further that  $(1 : \pm i : 0)$  are the only points of  $\bar{C}^\pm$  at infinity.

It is better to use the inverse of the stereographic projection from the south pole to compactify  $\mathbb{R}_{xy}^2$  as the quadric  $Q^2 := (z_1^2 + z_2^2 + z_3^2 = z_0^2) \subset \mathbb{P}^3$ . From  $\mathbb{P}^2$  this is obtained by blowing up the conjugate point pair  $(1 : \pm i : 0)$  and contracting the birational transform of the line at infinity. We think of the image of the unit circle as the equator. Thus we get rational curves  $C^\pm \subset Q^2$ . Since  $(1 : \pm i : 0)$  are the only points of  $\bar{C}^\pm$  at infinity, the south pole is not on the curves  $C^\pm$  and so the real points of the curves  $C^\pm$  are all smooth and they intersect each other at  $2g + 2$  points on the equator.

Pick one of these points  $p$  and view  $C_0 := C^+ \cup C^-$  as the image of a map  $\phi_0$  from the reducible curve  $B_0 := (uv = 0) \subset \mathbb{P}_{uvw}^2$  to  $Q^2$  that sends the point  $(0 : 0 : 1)$  to  $p$ . As in [AK03, Sec.4],  $\phi_0$  can be deformed to morphisms

$$\phi_t : B_t := (uv = tw^2) \rightarrow Q^2.$$

Let  $C_t \subset Q^2$  denote the image of  $B_t$ . For  $t$  near the origin and with suitable sign,  $C_t(\mathbb{R}) \subset S^2 = Q(\mathbb{R})$  goes around the equator twice and has  $2g + 1$  self intersections; see Figure 1.

Finally we blow up the  $2g + 1$  real singular points of  $C_t$  to get a rational surface  $S_g$ . The birational transform of  $C_t$  gives a rational curve  $C_g \subset S_g$  which is smooth at its real points.

The  $2g + 1$  regions of  $\mathbb{S}^2 \setminus C_t(\mathbb{R})$  near the equator become a Möbius band on  $S_g(\mathbb{R}) \setminus C_g(\mathbb{R})$  and the northern and southern hemispheres become  $\#g\mathbb{T}^2$  (with one puncture). Thus

$$(S_g(\mathbb{R}), C_g(\mathbb{R})) \sim \mathbb{R}\mathbb{P}^2 \# (\mathbb{S}^2, \mathbb{L}) \# g\mathbb{T}^2.$$

**Proposition 19.** *Let  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a rational curve defined over  $\mathbb{R}$  that is smooth at all points of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \sim \mathbb{T}^2$ . Then  $[C(\mathbb{R})] \in H_1(\mathbb{T}^2, \mathbb{Z}/2)$  is nonzero.*

Proof. Let  $E_1, E_2$  denote a horizontal (resp. vertical) complex line on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Every complex algebraic curve  $C$  has homology class  $a_1E_1 + a_2E_2$  for some  $a_1, a_2 \geq 0$ . Furthermore, if  $C$  is defined over  $\mathbb{R}$  then

$$a_i = (C \cdot E_{3-i}) \equiv (C(\mathbb{R}) \cdot E_{3-i}(\mathbb{R})) \pmod{2}.$$

Thus if  $[C(\mathbb{R})] \in H_1(\mathbb{T}^2, \mathbb{Z}/2)$  is zero then  $a_1, a_2$  are even. By the adjunction formula

$$2p_a(C) - 2 = (a_1E_1 + a_2E_2) \cdot ((a_1 - 2)E_1 + (a_2 - 2)E_2) = a_1(a_2 - 2) + a_2(a_1 - 2),$$

hence  $p_a(C) = (a_1 - 1)(a_2 - 1)$ . Thus, if the  $a_i$  are even then  $p_a(C)$  is odd. Therefore, if  $C$  is rational then it has an odd number of singular points and at least one of them has to be real.  $\square$

## 5. RELATED APPROXIMATION PROBLEMS

**20** (Approximation of curves on other surfaces). When  $S$  is a non-rational surface, we can ask for several possible analogs of Theorem 1.

On many surfaces there are no rational curves at all, thus the best one can hope for is approximation by higher genus curves. Even for this, there are several well known obstructions.

First of all, given a real algebraic variety  $X$ , a necessary condition for a smooth curve  $C$  to admit an approximation by an algebraic curve is that its fundamental class  $[C]$  belong to the group of algebraic cycles  $H_{\text{alg}}^1(X, \mathbb{Z}/2)$ . The latter group is generally a proper subgroup of the cohomology group  $H^1(X, \mathbb{Z}/2)$ . See [BH61] and [BCR98, Sec.12.4] for details.

The structure of these groups for various real algebraic surfaces of special type is computed in [Man94, Man97, MvH98, Man00, Man03]. These papers contain the classification of totally algebraic surfaces, that is surfaces such that  $H_{\text{alg}}^1(S, \mathbb{Z}/2) = H^1(S, \mathbb{Z}/2)$ , among K3, Enriques, bi-elliptic, and properly elliptic surfaces. In particular, if  $S$  is a non-orientable surface underlying an Enriques surface or a bi-elliptic surface, then there are simple, connected, closed curves on  $S$  with no approximation by any algebraic curve, see [MvH98, Thm.1.1] and [Man03, Thm.0.1].

If  $S$  is orientable, there can be further obstructions involving  $H^1(S, \mathbb{Z})$ . For instance, let  $S \subset \mathbb{R}\mathbb{P}^3$  be a very general K3 surface. By the Noether–Lefschetz theorem, the Picard group of  $S(\mathbb{C})$  is generated by the hyperplane class. If  $S$  is contained in  $\mathbb{R}^3$  then the restriction of  $\mathcal{O}_{\mathbb{P}^3}(1)$  to  $S$  is trivial, thus only null-homotopic curves can be approximated by algebraic curves.

Note also that if  $S$  is a real K3 surface, then by [Man97], there is a totally algebraic real K3 surface real deformation equivalent to  $S$  (at least if  $S$  is a non-maximal

surface) thus in general there is no purely topological obstruction to approximability for real K3 surfaces.

Another interesting class to study is surfaces that are geometrically rational but not rational. These contain many rational curves, so approximation by smooth rational curves could be possible. Any geometrically rational surface is totally algebraic but there are not enough automorphisms to approximate all diffeomorphisms, at least if the number of connected components is greater than 2; see [BM11].

Another obstruction arises from the genus formula. For example, let  $S$  be a degree 2 Del Pezzo surface with Picard number  $\rho(S) = 1$  and  $C \subset S$  a curve on it. Then  $C \sim -aK_S$  for some positive integer  $a$  and so  $C(C + K_S) = 2a(a - 1)$  is divisible by 4. Thus the arithmetic genus  $p_a(C)$  is odd hence every real rational curve on  $S$  has an odd number of singular points on  $S(\mathbb{C})$ . These can not all be complex conjugate, thus every rational curve on  $S$  has a real singular point.

It seems, however, that this type of parity obstruction for approximation does not occur on any other geometrically rational surface. We hazard the hope that if  $S$  is a geometrically rational surface then every simple, connected, closed curve can be approximated by smooth rational curves, save when either  $S \sim \mathbb{T}^2$  or  $S$  is isomorphic to a degree 2 Del Pezzo surface with Picard number 1.

As another generalization, one can study such problems for singular rational surfaces as in [HM10]. See also the series [CM08], [CM09] for the classification of geometrically rational surfaces with Du Val singularities.

**21** (Approximation of curves on higher dimensional varieties). As for surfaces, we can hope to approximate every simple, connected, closed curve on a real variety  $X$  by a nonsingular rational curve over  $\mathbb{R}$  only if there are many rational curves on the corresponding complex variety  $X(\mathbb{C})$ . First one should consider rational varieties.

We claim that if  $X$  is a smooth, real variety of dimension  $\geq 3$  that is rational, then every simple, connected, closed curve on  $X$  can be approximated by smooth rational curves. Indeed, the proof of [BK99] essentially automatically produces approximations by rational curves  $g : \mathbb{P}^1 \rightarrow X$  such that  $g^*T_X$  is ample. Then an easy lemma (cf. [Kol96, II.3.14]) shows that a general small perturbation of  $g$  is an embedding.

The next class to consider is geometrically rational varieties, or, more generally, rationally connected varieties [Kol96, Chap.IV].

Let  $X$  be a smooth, real variety such that  $X(\mathbb{C})$  is rationally connected. By a combination of [Kol99, Cor.1.7] and [Kol04, Thm.23], if  $p_1, \dots, p_s \in X(\mathbb{R})$  are in the same connected component then there is a rational curve  $g : \mathbb{P}^1 \rightarrow X$  passing through all of them. By the previous argument, we can even choose  $g$  to be an embedding if  $\dim X \geq 3$ . Thus  $X$  contains plenty of smooth rational curves.

Nonetheless, we believe that usually not every homotopy class of  $X(\mathbb{R})$  can be represented by rational curves. The following example illustrates some of the possible obstructions.

*Example 21.1.* Let  $q_1, q_2, q_3$  be quadrics such that  $C := (q_1 = q_2 = q_3 = 0) \subset \mathbb{P}^4$  is a smooth curve with  $C(\mathbb{R}) \neq \emptyset$ . Consider the family of 3-folds

$$X_t := (q_1^2 + q_2^2 + q_3^2 - t(x_0^4 + \dots + x_4^4) = 0) \subset \mathbb{P}^4$$

For  $0 < t \ll 1$ , the real points  $X_t(\mathbb{R})$  form an  $\mathbb{S}^2$ -bundle over  $C(\mathbb{R})$ . We conjecture that if  $0 < t \ll 1$ , then every rational curve  $g : \mathbb{P}^1 \rightarrow X_t$  gives a null-homotopic map  $g : \mathbb{R}\mathbb{P}^1 \rightarrow X_t(\mathbb{R})$ .

We do not know how to prove this, but the following argument shows that if  $g_t : \mathbb{P}^1 \rightarrow X_t$  is a continuous family of rational curves defined for every  $0 < t \ll 1$ , then  $g_t : \mathbb{R}\mathbb{P}^1 \rightarrow X_t(\mathbb{R})$  is null-homotopic. More precisely, the images  $g_t(\mathbb{R}\mathbb{P}^1) \subset X_t(\mathbb{R})$  shrink to a point as  $t \rightarrow 0$ .

Indeed, otherwise by taking the limit as  $t \rightarrow 0$ , we get a non-constant map  $\tilde{g}_0 : \mathbb{P}^1 \rightarrow C$ . However, the genus of  $C$  is 5, hence every map  $\mathbb{P}^1 \rightarrow C$  is constant. (A priori, the limit, taken in the moduli space of stable maps as in [FP97], is a morphism  $g_0 : B \rightarrow X_0$  where  $B$  is a (usually reducible) real curve with only nodes as singularities such that  $h^1(B, \mathcal{O}_B) = 0$ . For such curves, the set of real points  $B(\mathbb{R})$  is a connected set. Thus the image of  $B(\mathbb{R})$  is a connected subset of  $X_0(\mathbb{R})$  that contains at least 2 distinct points. Since  $X_0(\mathbb{R}) = C(\mathbb{R})$ , one of the irreducible components of  $B$  gives a non-constant map  $\tilde{g}_0 : \mathbb{P}^1 \rightarrow C$ .)

Unfortunately, this only implies that if we have a sequence  $t_i \rightarrow 0$  and a sequence of homotopically nontrivial rational curves  $g_{t_i} : \mathbb{P}^1 \rightarrow X$  then their degree must go to infinity. We did not exclude the possibility that, as  $t_i \rightarrow 0$ , we have higher and higher degree maps approximating non null-homotopic loops.

We do not have a conjecture about which homotopy classes give obstructions. On the other hand, while we do not have much evidence, the following could be true.

*Conjecture 21.2.* Let  $X$  be a smooth, rationally connected variety defined over  $\mathbb{R}$ . Then a  $C^\infty$  map  $\mathbb{S}^1 \rightarrow X(\mathbb{R})$  can be approximated by rational curves iff it is homotopic to a rational curve  $\mathbb{R}\mathbb{P}^1 \rightarrow X(\mathbb{R})$ .

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