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APPROXIMATING CURVES ON REAL RATIONAL SURFACES

JÁNOS KOLLÁR AND FRÉDÉRIC MANGOLTE

1. Introduction

As a generalization of the Weierstrass approximation theorem, every C^∞ map to a rational variety $\mathbb{S}^1 \to X$ can be approximated, in the C^∞-topology, by an algebraic map $\mathbb{RP}^1 \to X$, see [Bochnak-Kucharz99] and Definition 3. In particular, any simple closed curve on a rational surface S can be approximated by a rational curve on S. Note that the usual result is about maps of rational curves, so the image may have some extra isolated points. For example, consider the classical plane cubic curve $f\colon (u,v) \longrightarrow \bigl(v(u^2+v^2), u(u^2+v^2), v^3\bigr)$. Clearly $f(\mathbb{RP}^1)$ is a simple closed curve in $\mathbb{RP}^2$ but its Zariski closure, given by $zy^2 - x^2(x-z) = 0$, has an extra isolated real point at $(0,0,1)$. If we drop this point by smoothing, the curve becomes smooth elliptic. If we blow-up the point, the topology of the real surface changes.

In this note we get rid of these extra points, using the methods introduced in [Kollár-Mangolte09].

Theorem 1. *Let S be the underlying topological surface of the real points of a nonsingular rational surface and let $L \subset S$ be a simple, connected, closed curve.*

Then L can be approximated by a nonsingular rational curve in the C^∞-topology.

Recall that a real rational surface is a real algebraic surface real birational to $\mathbb{S}^2$ and that by Comessatti's theorem, such a surface is diffeomorphic to $\mathbb{S}^2, \mathbb{S}^1 \times \mathbb{S}^1$ or to a nonorientable surface. Thus we obtain the following.

Corollary 2. *Let S be a closed topological surface and $L \subset S$ a simple, connected, closed curve. Assume that S is either nonorientable or of genus ≤ 1.*

Then $L \subset S$ is diffeomorphic to the embedding of a nonsingular rational curve into a nonsingular rational surface.

Our proof actually goes the other way round. We prove Corollary 2 by direct constructions and then use the main result of [Kollár-Mangolte09] to show that this implies Theorem 1.

We also consider what kind of nonsingular rational curves can be used in these approximations. In Theorems 10 and 11 we give necessary and sufficient topological conditions for a simple, closed curve on a rational surface S to be approximable by a smooth, rational curve whose (complex) self-intersection is -1 or 0. We call these curves (-1)-*curves* resp. 0-*curves*.

Both of these have a geometric characterization. A real algebraic curve $L \subset S$ is a (-1)-curve iff there exists a birational morphism $\pi\colon S \to T$ such that $\pi(L)$ is a smooth point on T and π induces an isomorphism $S \setminus L \to T \setminus \pi(L)$. A real algebraic curve $L \subset S$ is a 0-curve iff there exists a morphism $\rho\colon S \to B$ to a curve B such that $L = \rho^{-1}(x)$ for some real point $x \in B$. Thus, in the latter case, L is a fiber of a *conic bundle*.

Definition 3. Given a compact real algebraic variety X, we consider the set of embeddings of nonsingular rational curves to be a subset of the space $C^\infty(\mathbb{RP}^1, X)$ of all C^∞ maps of $\mathbb{RP}^1$ into X, endowed with the C^∞-topology.

Let $L \hookrightarrow X$ be an embedded circle on a real algebraic variety X, we say that L can be *approximated by* a curve of a certain kind if every neighborhood of L in $C^\infty(\mathbb{RP}^1, X)$ contains a curve of that kind.

4 (Self-intersection number over $\mathbb{C}$ and over $\mathbb{R}$)**.** For many purposes, the behavior of a real variety at its imaginary points is not relevant, but in this paper it is crucial to consider that a smooth real curve is smooth at imaginary points as well.

Likewise, the intersection number of two real curves $\big(C_1 \cdot C_2\big)$ is the intersection number of the underlying complex curves. The intersection number of the real parts $\big(C_1(\mathbb{R}) \cdot C_2(\mathbb{R})\big)$ is only defined modulo 2 and

$$\big(C_1(\mathbb{R}) \cdot C_2(\mathbb{R})\big) \equiv \big(C_1 \cdot C_2\big) \mod 2.$$

Note that the self-intersection number of a real algebraic curve is not invariant by real algebraic isomorphism (here we understand real algebraic isomorphisms as birational maps which are biregular at real points, see e.g. [Blanc-Mangolte11, Introduction]). Indeed, let $C \subset S$ be a real algebraic curve. By blowing-up a pair of conjugated imaginary points, we get a real algebraic curve $C' \subset S'$ real isomorphic to $C \subset S$ such that $(C' \cdot C') = (C \cdot C) - 2$.

In some cases we can also increase the complex self-intersection number. For instance, let S be a Hirzebruch surface and $C \subset S$ a section. Let $S \dashrightarrow S'$ be a real elementary transformation obtained by blowing up a pair of conjugated non real points away from C. We get $C' \subset S'$ and $(C' \cdot C') = (C \cdot C) + 2$. We can iterate this to get higher and higher self-intersection numbers. This is, however, rare and all cases where $(C \cdot C)$ can be made ≥ 0 are enumerated in (16).

Thus in general the interesting question is to approximate by smooth rational curves whose complex self-intersection number (C^2) is negative but as large as possible.

Theorem 5. *Let S be a real algebraic surface birational to $\mathbb{S}^2$ and $L \subset S$ a simple, connected, closed curve.*

1. *If the topological genus of S is $g(S) > 2$ then L can be approximated by smooth rational curves whose self intersection satisfies $0 \leq -(C^2) \leq g - 2$.*
2. *If $g(S) \leq 2$ then L can be approximated by smooth rational curves C such that*
   - (a) *$(C^2) = 0$ if S is orientable along L and*
   - (b) *$(C^2) = -1$ if S is nonorientable along L.*

(As there exists at least two irreconcilable definitions for the genus of a nonorientable surface in the literature, it is worthwhile to state our convention. The genus of a compact connected topological surface of Euler number $e := e(S)$ is given by $g = 2 - e$ if S is nonorientable and by $g = \frac{2-e}{2}$ if S is orientable.)

Recall that a simple closed curve L on a nonorientable closed surface S is an *essential circle* if it is two-sided, nonseparating and $S \setminus L$ is nonorientable, see e.g. [Korkmaz02]. If S is orientable, L is essential if and only if it is not is contractible. The motivation for this name is that Dehn twists along essential circles are primitive elements in the modular group $\mathrm{Mod}(S)$. We get the following corollary to Theorem 11.

Corollary 6. *Any essential circle on a rational surface S can be approximated by a fiber of a conic bundle structure.*

7 (Approximation of curves on other surfaces)**.** When S is a non-rational surface, we can ask for several possible analogs of Theorem 1.

On many surfaces there are no rational curves at all, thus the best one can hope for is approximation by higher genus curves. Even for this, there are several well known obstructions.

First of all, given a real algebraic variety X, a necessary condition for a smooth curve C to admit an approximation by an algebraic curve is that its fundamental class $[C]$ belong to the group of algebraic cycles $\mathrm{H}^1_{\mathrm{alg}}(X, \mathbb{Z}/2)$. Now this group is generally a proper subgroup of the cohomology group $\mathrm{H}^1(X, \mathbb{Z}/2)$. See [Borel-Haefliger61] and [Bochnak-Coste-Roy98, § 12.4] for details.

The structure of these groups for various real algebraic surfaces of special type is computed in [Mangolte94], [Mangolte97], [Mangolte-van Hamel98], [Mangolte00], [Mangolte03]. These papers contain in particular the classification of totally algebraic surfaces, that is surfaces such that $\mathrm{H}^1_{\mathrm{alg}}(S, \mathbb{Z}/2) = \mathrm{H}^1(S, \mathbb{Z}/2)$, among K3, Enriques, bi-elliptic, and properly elliptic surfaces. In particular, if S is a nonorientable surface underlying an Enriques surface or a bi-elliptic surface, then there are simple, connected, closed curves on S with no approximation by any algebraic curve [Mangolte-van Hamel98, Theorem 1.1] and [Mangolte03, Theorem 0.1].

If S is orientable, there can be further obstructions involving $\mathrm{H}^1(S, \mathbb{Z})$.

For instance, let $S \subset \mathbb{RP}^3$ be a very general K3 surface. By the Noether–Lefschetz theorem, the Picard group of $S(\mathbb{C})$ is generated by the hyperplane class. If S is contained in $\mathbb{R}^3$ then the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ to S is trivial, thus only contractible curves can be approximated by algebraic curves.

Note also that if S is a real K3 surface, then by [Mangolte97], there is a totally algebraic real K3 surface real deformation equivalent to S (at least if S is a non-maximal surface) thus in general there is no purely topological obstruction to approximability for real K3 surfaces.

Another interesting class to study is surfaces that are geometrically rational but not rational. These contain many rational curves, so approximation by smooth rational curves could be possible. Any geometrically rational surface is totally algebraic but Lemma 13 fails for these surfaces, at least if the number of connected components is greater than 2; see [Blanc-Mangolte11].

Another obstruction arises from the genus formula. For example, let S be a degree 2 Del Pezzo surface with Picard number $\rho(S) = 1$ and $C \subset S$ a curve on it. Then $C \sim -aK_S$ for some positive integer a and so $C(C + K_S) = 2a(a - 1)$ is divisible by 4. Thus the arithmetic genus $p_a(C)$ is odd hence every real rational curve on S has an odd number of singular points on $S(\mathbb{C})$. These can not all be complex conjugate, thus there are no smooth rational curves on S at all.

It seems, however, that this type of parity obstruction for the genus does not occur on any other geometrically rational surface. We hazard the conjecture that if S is a geometrically rational surface that is not isomorphic to a degree 2 Del Pezzo surface with Picard number 1, then every simple, connected, closed curve can be approximated by smooth rational curves.

As another generalization, one can study such problems for singular rational surfaces as in [Huisman-Mangolte10]. See also the series [Catanese-Mangolte08],

[Catanese-Mangolte09] for the classification of geometrically rational surfaces with Du Val singularities.

2. Embedded circles in surfaces

Let S be a smooth compact surface without boundary and $L \subset S$ an embedded circle, that is a simple connected closed curve. Let $e(S)$ denote the Euler number of S.

If S is nonorientable then $g = 2 - e(S) \geq 1$ and we can choose g embedded circles $L_1, \ldots, L_g$ whose normal bundles give disjoint embedded Möbius bands and such that L intersects each L_i at most once and transversally. Then S is orientable along L if and only if L intersects an even number of the L_i.

We want to classify topologically the pair (S, L) when S is either nonorientable or of genus ≤ 1. Let $F := S \setminus L$ be the complement of L in S, F is a surface with boundary such that $e(F) = e(S)$.

8 ($N_{L|S}$ non trivial: is L topologically a (-1)-curve on a rational surface?)**.** Assume that S is not orientable along L. In particular, S is a nonorientable surface. In this case, F is connected and its boundary ∂F is connected. Cutting a Möbius band along a non homotopically trivial simple closed curve give rise to a cylinder.

We can contract L to a point p to get a compact surface S' of Euler characteristic $e(S') = e(S) + 1$. Thus S is the connected sum $S' \# \mathbb{RP}^2$.

If S' is a rational real algebraic surface and if S' is orientable, then $e(S') \geq 0$ by Comessatti's theorem [Comessatti14]. Hence $F \sim S' \setminus \{p\}$ is a sphere with one puncture or a torus with one puncture.

Conversely, if F is orientable and $e(S') < 0$, which implies in particular that $g = 2 - e(S) > 3$ is odd, then L intersects each L_i, $1 \leq i \leq g$. Indeed, if F contains one of the L_i, then F contains a Möbius band, hence it is nonorientable.

In conclusion, if $L \subset S$ is not topologically a (-1)-curve on a rational surface, then L intersects each L_i, $1 \leq i \leq g$ and g is odd, and $g \geq 5$.

9 ($N_{L|S}$ trivial: is L topologically a fiber of a conic bundle?)**.** Assume that S is orientable along L, then ∂F has two connected components.

If $F = S \setminus L$ is connected (then $e(S) \leq 0$), assume that F is not a cylinder. If F is disconnected, assume that L is not contractible. With these assumptions, if L is isotopic to a fiber of a conic bundle, then the connected components of F are nonorientable. Indeed, every real conic bundle is obtained from $\mathbb{S}^2$ or from an $\mathbb{S}^1$-bundle over $\mathbb{S}^1$ by blowing up points. Thus, if the complement of a fiber is orientable, then there are no real singular fibers.

Conversely, assume that L is not topologically a fiber of a conic bundle. There are two possibilities:

(1) F is a connected orientable surface. Then as above, L intersects each L_i, $1 \leq i \leq g$ and g is even, and $g \geq 4$.
(2) $F = F_1 \sqcup F_2$, where F_1 is connected orientable of nonzero genus and F_2 is connected nonorientable. We get that L is a non homotopically trivial simple closed curve which is homologically trivial. Furthermore, $g \geq 3$.

3. Real rational models of pairs

In this section, we prove theorems 1 and 5. Furthermore, we prove that the necessary topological conditions for an embedded curve to be approximated by a (-1)-curve (resp. by a fiber of a conic bundle structure) are in fact sufficient.

Theorem 10. *Let S be a real algebraic surface birational to $\mathbb{S}^2$ and $L \subset S$ a smooth connected curve. Then the following are equivalent*

1. *L is homotopic to a (-1)-curve.*
2. *L can be approximated by a (-1)-curve.*
3. *S is not orientable along L and one of the following 3 possibilities holds*
   - (a) *$S \setminus L$ is a sphere with 1 puncture*
   - (b) *$S \setminus L$ is a torus with 1 puncture, or*
   - (c) *$S \setminus L$ is nonorientable.*

Theorem 11. *Let S be a real algebraic surface birational to $\mathbb{S}^2$ and $L \subset S$ a smooth connected curve. Then the following are equivalent*

1. *L is homotopic to a fiber of a conic bundle structure.*
2. *L can be approximated by a fiber of a conic bundle structure.*
3. *S is orientable along L and one of the following 3 possibilities holds*
   - (a) *L is contractible (it bounds a disc),*
   - (b) *$S \setminus L$ is a cylinder, or*
   - (c) *every connected component of $S \setminus L$ is nonorientable.*

Corollary 12. *Let S be a real algebraic surface birational to $\mathbb{S}^2$, and $L \subset S$ be a smooth connected curve.*

1. *If $e(S) \equiv 0 \mod 2$, then L can be approximated by a (-1)-curve if and only if S is not orientable along L.*
2. *If $e(S) \equiv 1 \mod 2$, then L can be approximated by a fiber of a conic bundle structure if and only if S is orientable along L and no connected component of $S \setminus L$ is orientable of positive genus.*
3. *If S is diffeomorphic to one the following surfaces: $\mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{RP}^2$, $\mathbb{RP}^2 \# \mathbb{RP}^2$, then L can be approximated by a fiber of a conic bundle or a (-1)-curve according to the orientability of S along L.*

Thanks to the following lemma, the proofs of theorems 1, 10, 11 and 5 are reduced to the construction of an algebraic model for each possible topological pair whose list is given in Section 2.

Lemma 13. *Let $L \subset S$, $L' \subset S'$ be embedded algebraic curves in real algebraic surfaces and $\Phi\colon (L' \subset S') \to (L \subset S)$ be a diffeomorphism.*

If S and S' are rational, and if L' is the real part of a rational curve whose self-intersection is m, then L can be approximated in the C^∞-topology by rational curves whose self-intersection is m.

Proof. The main result of [Biswas-Huisman07, Theorem 1.2] asserts that diffeomorphic rational surfaces are real isomorphic (see also [Huisman-Mangolte09] for a more direct proof). Let $\Psi\colon S' \to S$ be such a real algebraic isomorphism. By [Kollár-Mangolte09], the self-diffeomorphism $\Phi \circ \Psi^{-1} \in \mathrm{Diff}(S)$ can be approximated in the C^∞-topology by an automorphism $\eta \in \mathrm{Aut}(S)$.

Then $\Phi' = \eta \circ \Psi$ is an algebraic approximation of Φ which is a C^∞-approximation along L. Hence $\Phi'(L')$ is the required C^∞-approximation of L.

□

14 (Proof of theorems 10 and 11)**.** In both theorems the implication (2) ⇒ (1) is clear and (1) ⇒ (3) is proved in Section 2.

In order to prove (3) ⇒ (2) we construct some algebraic models.

Up to diffeomorphism, all possible pairs $L \subset S$ allowed by (10.3) are the following:

(1) $S = B_{p_1,\dots,p_g}\mathbb{S}^2$ and $L = E_1$, the exceptional curve over p_1.
(2) $S = B_p(\mathbb{P}^1 \times \mathbb{P}^1)$ and $L = E$, the exceptional curve over p.

Up to diffeomorphism, all possible pairs
allowed by (11.3) are the following:

(3) ($S \setminus L$ is nonconnected) For g even and $0 \le b \le g/2$, blow up b points on the sphere $(x^2 + y^2 + z^2 = 1)$ with negative z-coordinate and $g - b$ points with positive z-coordinate to get S. Let $\pi\colon S \to \mathbb{P}^1$ be projection to the z-axis and $L = S \cap (z = 0)$.
(4) ($S \setminus L$ is connected) For $g \ge 2$, blow up $g - 2$ points on the sphere $(x^2 + y^2 + z^2 = 1)$ which do not lie on $(z = 0)$ and also blow up twice $(-1, 0, 0)$ to get S. Let $\pi\colon S \to \mathbb{P}^1$ be projection from the line $(x + 1 = z = 0)$. Let L be the birational transform of $\mathbb{S}^2 \cap (z = 0)$.
(5) $S = \mathbb{P}^1 \times \mathbb{P}^1$, π the first projection and L any fiber.

Take now $(L \subset S)$ as in (11.3) or (10.3) and choose a model $(L' \subset S')$ as in (14.1–5) and a diffeomorphism $\Phi\colon (L' \subset S') \to (L \subset S)$. The conclusion follows from Lemma 13.

□

15 (Proof of theorems 1 and 5)**.** Let R be a nonorientable topological surface of genus g. Up to diffeomorphism, the remaining topological pairs to prove the theorems are the following:

(6) Let again $\mathbb{S}^2 := (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$ be the usual sphere and $S = B_{p_1,\dots,p_g}\mathbb{S}^2$ be the real algebraic model of R obtained from $\mathbb{S}^2$ by blowing-up g points. Let $L = S \cap (z = 0)$ be the birational transform of a great circle. Let g' be any natural integer such that $0 \le g' \le g$, choose g' points on the great circle $C := \mathbb{S}^2 \cap (z = 0)$ and $g - g'$ points in $\mathbb{S}^2 \setminus C$. By blowing-up the g points we get a nonsingular rational curve such that $L \cdot L = 2 - g'$.

If $0 < g'$, then the surface $S \setminus L$ is connected. If $g' < g$ or if $g \le 2$, we use the models (1)–(5) to get that L can be approximated by a (-1)-curve or the fiber of a conic bundle depending on the nontriviality of its normal bundle. If $g' = g > 2$, then L has self-intersections $2 - g$. By Lemma 13, we get the conclusion of theorems 1 and 5 for all topological pairs up to one kind.

(7) Let m be a positive integer and let L_1 be a high odd degree plane rational curve with no real singular point. Then wiggle L_1 to get L_2 intersecting it $2m + 1$ times transversally in $\mathbb{P}^2(\mathbb{R})$. We have L_1, L_2 braid around each other. By Brusotti's theorem, we can smooth one of the real nodes away such that we have just one irreducible curve that goes around twice and self-intersects $2m$ times. This resulting nodal rational curve has $2m$ real nodes $p_1, \dots, p_{2m}$. After resolution of these real nodes, the birational transform L of the nodal curve cut the surface $B_{p_1,\dots,p_{2m}}\mathbb{RP}^2$ in two connected components: a Möbius band and an orientable surface of genus m.

Blow-up $g-2m-1$ points on the Möbius band, we get the real algebraic model $S = B_{p_1,\dots,p_{g-1}}\mathbb{RP}^2$ of R and $S \setminus L = F_1 \sqcup F_2$, where F_1 is connected orientable of genus m and F_2 is connected nonorientable. Thanks to Lemma 13, this ends the proof of Theorem 1.

To see that Theorem 5 is true also for this kind of topological pairs, blow-up pairs of conjugated imaginary points to provide that the birational transform L of C is a 0-curve on S.

□

4. Minimal pairs

The previous results suggest that it is of interest to understand pairs (S, C) where S is a nonsingular projective surface and C a nonsingular curve. Here we are mainly interested in the real case, but the problem makes sense over any field. Given a pair (S, C) and a point $p \in S$ by blowing up we get a new pair (B_pS, C') where $C' \subset B_pS$ is the birational transform of C.

As in the theory of smooth surfaces, a pair (S, C) is called *classically minimal* if it can not be obtained from another pair by blowing-up. In the real case this means that

(1) there is no real (-1)-curve E whose intersection number with C is ≤ 1 and
(2) there is no conjugate pair of disjoint (-1)-curves E, E' such that $(E \cdot C) = (E' \cdot C) \leq 1$.

(The terminology "classically minimal" was chosen to distinguish it from being "minimal" in the sense of Mori's program; see [Kollár-Mori98] for an introduction. The two are closely related; the differences are discussed below.)

If the Kodaira dimension of S is ≥ 0 then the study of pairs (S, C) is essentially equivalent to the study of the minimal model $S^{\min}$ of S and of the singularities of the image of C on $S^{\min}$.

By contrast, when S is a geometrically rational surface and C is a smooth rational curve, the question of understanding pairs (S, C) is especially interesting and quite difficult. We have been unable to make much progress on it, thus this section is devoted to some preliminary remarks and examples.

16 (Basic numerical restrictions)**.** Let S be a smooth surface over a field k and $C \subset S$ a smooth, rational curve.

Then $(C \cdot (K_S + C)) = -2$, thus $K_S + C$ is not nef. Hence there is a $(K_S + C)$-extremal contraction $\pi\colon S \to T$. There are 3 possibilities.

(Del Pezzo) If $\dim T = 0$ then S has Picard number 1, thus $-K_S$ is ample, hence S is a Del Pezzo surface of Picard number 1. It contains a smooth, rational curve, thus S is either $\mathbb{P}^2$ or a quadric. (This holds over any field.)

(Conic bundle) If $\dim T = 1$ then S has Picard number 2 and the fibers of π are conics. C can be either a fiber or a section. In the latter case π is a $\mathbb{P}^1$-bundle.

(Birational contraction) If $\dim T = 2$ then π is birational and it contracts a curve $E \subset S$ to a point. If $E \neq C$ then E is a conjugate set of disjoint (-1)-curves $E = \cup_i E_i$. Since $((K_S + C) \cdot E_i) < 0$, the E_i are disjoint from C and so (S, C) is not classically minimal.

The main difference between the two notions of minimality arises when $E = C$. This is allowed by Mori's program but not by our notion of classically minimal.

Note that $(C^2) < 0$ since C is contracted. One can try to classify these cases by considering the different values of (C^2) separately.

From now on we concentrate on the case when S is geometrically rational.

17 (Case $(C^2) = -1$)**.** Since S is a geometrically rational surface, it can be obtained as a sequence of blow-ups

$$S = S_m \to S_{m-1} \to \cdots \to S_0$$

where S_0 is either a Del Pezzo surface of Picard number 1 or a minimal conic bundle. Since (S, C) is classically minimal, S does not contain disjoint (-1)-curves, hence each $S_{i+1} \to S_i$ is obtained by blowing up a point on the exceptional curve of $S_i \to S_{i-1}$.

The latter condition is not sufficient to ensure that S be classically minimal, but it is easy to write down series of examples.

For example, start with $S_0 = \mathbb{P}^2$. a line $L \subset S_0$ and a point $p \in L$. Blow up p repeatedly to obtain S_m with $C_m \subset S_m$ the last exceptional curve. We claim that C_m is the only (-1)-curve on S_m for $m \geq 3$, thus (S_m, C_m) is classically minimal.

We can fix coordinates on S_0 such that $L = (y = 0)$ and $p = (0 : 0 : 1)$. Then the $(\mathbb{C}^*)^2$-action $(x : y : z) \mapsto (\lambda x : \mu y : z)$ lifts to S_m, hence the only possible curves with negative self-intersection on S_m are the preimages of the coordinate axes and the exceptional curves of $S_m \to S_0$. These are easy to compute explicitly. Their dual graph is a cycle of rational curves

$$\begin{tikzcd}
(-2) \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & (-2) \arrow[d, dash] \\
(-1) \arrow[r, dash] & (1-m) \arrow[r, dash] & (1) \arrow[r, dash] & (0)
\end{tikzcd}$$

where (r) denotes a curve of self-intersection r, each curve intersects only the two neighbors connected to it by a solid line and there are $m - 1$ curves with self-intersection -2 in the top row. Thus C_m is the only (-1)-curve for $m \geq 3$.

There are probably many more series of such surfaces.

18 (Case $(C^2) = -2$)**.** To our surprise this case behaves quite differently and a complete enumeration is possible.

Lemma 19. *Let S be a smooth, geometrically rational surface and $C \subset S$ a smooth, rational curve. Assume that the pair (S, C) is classically minimal and $(C^2) = -2$.*

Then there is a degree 2 Del Pezzo surface S_1 with Picard number 1 and a rational curve $C_1 \in |-K_{S_1}|$ with a unique singular point $p_1 \in C_1$ such that $S = B_{p_1} S_1$ and C is the birational transform of C_1. In particular, S is not rational.

Proof. Let $\pi \colon S \to T$ be the contraction of C. Then T has an ordinary node $q \in T$. The special feature of the $(C^2) = -2$ case is that $K_S \sim \pi^* K_T$, thus K_T is not nef. So there is an extremal contraction $\tau \colon T \to T_1$.

There are 3 possibilities for τ.

Case 1: τ is birational with exceptional curve $E \subset T$.

If q does not lie on E then E gives a (-1)-curve on S which is disjoint from C, a contradiction. If q lies on E then E is irreducible and T_1 is smooth. Indeed, on a surface with DuVal singularities, every extremal contraction results in a smooth point, see [Morrison85].

Thus the composite $\tau \circ \pi \colon S \to T_1$ consist of two smooth blow ups. This again shows that (S, C) is not classically minimal.

Case 2: $\tau \colon T \to T_1$ is a conic bundle. Then $\tau \circ \pi \colon S \to T_1$ is a nonminimal conic bundle hence it contains a (-1)-curve E contained in a fiber. C is also contained in a fiber thus $(E \cdot C) \leq 1$ since any 2 irreducible curves in a fiber of a conic bundle intersect in at most 1 point.

Case 3: T is a Del Pezzo surface of Picard number 1.

In this case S itself is a weak Del Pezzo surface (that is $-K_S$ is nef) of Picard number 2. Thus S has another extremal ray giving a contraction $\rho \colon S \to S_1$ and S_1 is a Del Pezzo surface of Picard number 1.

Moreover, since $|-K_S|$ has dimension ≥ 1 and $(C \cdot K_S) = 0$, we see that there is a divisor $D \in |-K_S|$ whose support contains C.

Thus $C_1 := \rho(C)$ is singular and is contained in a member of $|-K_{S_1}|$. A general member of $|-K_{S_1}|$ is smooth, elliptic; special members are either irreducible, rational with a single node or cusp or reducible with smooth rational components. Thus C_1 is a member of $|-K_{S_1}|$ and has a node or cusp at a point p_1.

From $-2 = (C^2) = (C_1^2)$, we obtain that S_1 is a smooth Del Pezzo surface of degree 2. We obtain S by blowing up the singular point of C_1.

Finally note that a Del Pezzo surface of degree 2 and of Picard number 1 is never rational over the ground field k by the Segre–Manin theorem; see [Segre51, Manin66] or [Kollár-Smith-Corti04, Chap.2] for an introduction to these results. □

20 (Case $(C^2) = -3$)**.** We know very little about these surfaces.

Starting with a degree 1 Del Pezzo surface S_1 in Lemma 19 gives classically minimal examples with $(C^2) = -3$. We do not have other examples.

21 (Case $(C^2) \leq -4$)**.** As we see below, there are 3 types of such pairs. Two of them fit into the earlier patterns. The most interesting case is (22.1) but we know very little about it.

Lemma 22. *Let S be a smooth, projective surface and $C \subset S$ a smooth rational curve such that $(C^2) \leq -4$ and (S, C) is classically minimal. Let $\pi \colon S \to T$ be the contraction of C. Then one of the following holds.*

1. *K_T is nef.*
2. *S is a minimal ruled surface and $C \subset S$ is the (unique) negative section.*
3. *S is a minimal conic bundle and $C \subset S$ is the (unique) negative double section.*

Proof. Set $m := -(C^2)$. By direct computation

$$\pi^* K_T \sim_{\mathbb{Q}} K_S + \bigl(1 - \tfrac{2}{m}\bigr) C.$$

Thus if K_T is not nef then $K_S + \bigl(1 - \frac{2}{m}\bigr)C$ is also not nef and there is a $K_S + \bigl(1 - \frac{2}{m}\bigr)C$-extremal contraction $\tau \colon S \to S_1$.

Since C is not an extremal ray for $K_S + \bigl(1 - \frac{2}{m}\bigr)C$, τ is also a K_S-extremal contraction.

S can not be a Del Pezzo surface with Picard number 1 since every curve on S would have ≥ 0 self-intersection.

If $\tau \colon S \to S_1$ is a conic bundle then C is either a section, so $\tau \colon S \to S_1$ is a $\mathbb{P}^1$-bundle and $C \subset S$ is the unique negative section, or C is a double section.

Since the Picard number of S is 2, its cone of curves has 2 extremal rays. The fibers of the conic bundle generate one of the extremal rays and any curve with

negative self-intersection has to generate the other. In particular, such a curve is unique.

We are left with the case when $\tau\colon S \to S_1$ is birational and contracts a conjugate set of (-1)-curves E_i. By assumption

$$(K_S \cdot E_i) = 1 \quad \text{and} \quad \Big(\big(K_S + \big(1 - \tfrac{2}{m}\big)C\big) \cdot E_i\Big) < 0.$$

By assumption $m \geq 4$, thus $1 - \frac{2}{m} \geq \frac{1}{2}$ and these imply that $(C \cdot E_i) \leq 1$. This is impossible if (S, C) is classically minimal. □

In case (22.1) the surface T is rational with a unique singular point but K_T is nef. Uniqueness of the minimal models (in the sense of Mori's program) of a pair (S, C) implies that in case (22.1), the classically minimal pair (S, C) is unique up to isomorphism. We can not say much more about these cases, but here is a series of examples where S is rational.

Example 23. Let $\bar{C}_d \subset \mathbb{P}^2$ be a rational curve of degree d whose singularities are nodes. Thus we have $\binom{d-1}{2}$ nodes forming a set N_d. Let $p_d\colon S_d := B_d\mathbb{P}^2 \to \mathbb{P}^2$ denote the blow-up of all the nodes with exceptional curves E_d and $C_d \subset S_d$ the birational transform of $\bar{C}_d$. We compute that $\big(C_d^2\big) = d^2 - 4\binom{d-1}{2}$ and

$$K_{S_d} + \tfrac{3}{d}C_d - \big(1 - \tfrac{6}{d}\big)E_d \sim_{\mathbb{Q}} p_d^*\big(K_{\mathbb{P}^2} + \tfrac{3}{d}\bar{C}_d\big) \sim_{\mathbb{Q}} 0.$$

If $d \geq 6$ then $\big(C_d^2\big) < 0$; let $\pi\colon S_d \to T_d$ be its contraction. Then

$$K_{T_d} \sim_{\mathbb{Q}} \big(1 - \tfrac{6}{d}\big)\pi_* E_d$$

is trivial for $d = 6$ and ample for $d \geq 7$.

References

[Biswas-Huisman07] I. Biswas, J. Huisman, Rational real algebraic models of topological surfaces, Doc. Math. **12** (2007), 549–567.

[Blanc-Mangolte11] J. Blanc, F. Mangolte, Geometrically rational real conic bundles and very transitive actions, Compositio Mathematica **147** (2011), 161–187.

[Bochnak-Coste-Roy98] J. Bochnak, M. Coste, M.-F. Roy, Real algebraic geometry, Ergeb. Math. Grenzgeb. (3), vol. 36, Springer Verlag, 1998.

[Bochnak-Kucharz99] J. Bochnak, W. Kucharz, The Weierstrass approximation theorem for maps between real algebraic varieties, Math. Ann. **314** (1999), 601–612.

[Borel-Haefliger61] A. Borel, A. Haefliger, La classe d'homologie fondamentale d'un espace analytique, Bull. Soc. Math. France **89** (1961), 461–513.

[Catanese-Mangolte08] F. Catanese, F. Mangolte, Real singular Del Pezzo surfaces and 3-folds fibred by rational curves, I, Michigan Math. J. **56** (2008), 357–373.

[Catanese-Mangolte09] F. Catanese, F. Mangolte, Real singular Del Pezzo surfaces and 3-folds fibred by rational curves, II, Ann. Sci. E. N. S. **42** (2009), 531–557.

[Comessatti14] A. Comessatti, Sulla connessione delle superficie razionali reali, Annali di Math. **23** (1914), 215–283.

[Huisman-Mangolte09] J. Huisman, F. Mangolte, The group of automorphisms of a real rational surface is n-transitive, Bull. London Math. Soc. **41** (2009), 563–568.

[Huisman-Mangolte10] J. Huisman, F. Mangolte, *Automorphisms of real rational surfaces and weighted blow-up singularities*, manuscripta math. **132** (2010), 1–17.

[Kollár99] J. Kollár, Real algebraic threefolds. III. Conic bundles, J. Math. Sci. (New York) **94** (1999) 996–1020.

[Kollár-Mangolte09] J. Kollár, F. Mangolte, Cremona transformations and diffeomorphisms of surfaces, Adv. in Math. **222** (2009), 44–61.

[Kollár-Mori98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kollár-Smith-Corti04] Kollár, János, Smith, Karen E., and Corti, Alessio. 2004. *Rational and nearly rational varieties*. Cambridge Studies in Advanced Mathematics, vol. 92. Cambridge University Press.

[Korkmaz02] M. Korkmaz, Mapping Class Groups of Nonorientable Surfaces, Geometriae Dedicata **89** (2002), 109–133.

[Mangolte94] F. Mangolte, Une surface réelle de degré 5 dont l'homologie est entièrement engendrée par des cycles algébriques, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), 343–346.

[Mangolte97] F. Mangolte, Cycles algébriques sur les surfaces $K3$ réelles, Math. Z. **225** (1997) 559–576.

[Mangolte00] F. Mangolte, Surfaces elliptiques réelles et inégalité de Ragsdale-Viro, Math. Z. **235** (2000), 213–226.

[Mangolte03] F. Mangolte, Cycles algébriques et topologie des surfaces bielliptiques réelles, Comment. Math. Helv. **78** (2003), 385–393.

[Mangolte-van Hamel98] F. Mangolte and J. van Hamel, Algebraic cycles and topology of real Enriques surfaces, Compositio Math. **110** (1998), 215–237.

[Manin66] Yu. I. Manin, Rational surfaces over perfect fields, Inst. Hautes Études Sci. Publ. Math. **30** (1966), 55–113.

[Morrison85] , David R. Morrison, The birational geometry of surfaces with rational double points, Math. Ann. **271** (1985), no. 3, 415–438.

[Segre51] Beniamino Segre, On the rational solutions of homogeneous cubic equations in four variables, Math. Notae **11** (1951), 1–68.

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