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Sequential δ -optimal consumption and investment for stochastic volatility markets with unknown parameters

Belkacem Berdjane*and Serguei Pergamenshchikov†

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Abstract

We consider an optimal investment and consumption problem for a Black-Scholes financial market with stochastic volatility and unknown stock price appreciation rate. The volatility parameter is driven by an external economic factor modeled as a diffusion process of Ornstein-Uhlenbeck type with unknown drift. We use the dynamical programming approach and find an optimal financial strategy which depends on the drift parameter. To estimate the drift coefficient we observe the economic factor Y in an interval $[0, T_0]$ for fixed $T_0 > 0$, and use sequential estimation. We show that the consumption and investment strategy calculated through this sequential procedure is δ -optimal.

Key words: Sequential analysis, Truncate sequential estimate, Black-Scholes market model, Stochastic volatility, Optimal Consumption and Investment, Hamilton-Jacobi-Bellman equation.

MS Classification (2010) 62L12, 62L20, 91B28, 91G80, 93E20.

JEL Classification G11, C13.

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1 Introduction

We deal with the finite-time optimal consumption and investment problem in a Black-Scholes financial market with stochastic volatility (see, e.g., [7]). We consider the same power utility function for both consumption and terminal wealth. The volatility parameter in our situation depends on some economic factor, modeled as a diffusion process of Ornstein-Uhlenbeck type. The classical approach to this problem goes back to Merton [23].

By applying results from the stochastic control, explicit solutions have been obtained for financial markets with nonrandom coefficients (see, e.g. [13], [16] and references therein). Since then, the consumption and investment problems has been extended in many directions [27]. One of the important generalizations considers financial models with stochastic volatility, since empirical studies of stock-price returns show that the estimated volatility exhibits random characteristics (see e.g., [28] and [10]).

The pure investment problem for such models is considered in [29] and [26]. In these papers, authors use the dynamic programming approach and show that the nonlinear HJB (Hamilton-Jacobi-Bellman) equation can be transformed into a quasilinear PDE. The similar approach has been used in [17] for optimal consumption-investment problems with the default risk for financial markets with non random coefficients. Furthermore, in [5], by making use of the Girsanov measure transformation the authors study a pure optimal consumption problem for stochastic volatility markets. In [2] and [9] the authors use dual methods.

Usually, the classical existence and uniqueness theorem for the HJB equation is shown by the linear PDE methods (see, for example, chapter VI.6 and appendix E in [6]). In this paper we use the approach proposed in [4] and used in [1]. The difference between our work and these two papers is that, in [4], authors consider a pure jump process as the driven economic factor. The HJB equation in this case is an integro-differential equation of the first order. In our case it is a highly non linear PDE of the second order. In [1] the same problem is considered where the market coefficients are known, and depend on a diffusion process with bounded parameters. The result therein does not allow the Gaussian Ornstein-Uhlenbeck process. Similarly to [4] and [1] we study the HJB equation through the Feynman - Kac representation. We introduce a special metric space in which the

Feynman - Kac mapping is a contraction. Taking this into account we show the fixed-point theorem for this mapping and we show that the fixed-point solution is the classical unique solution for the HJB equation in our case.

In the second part of our paper, we consider both the stock price appreciation rate and the drift of the economic factor to be unknown. To estimate the drift of a process of Ornstein-Uhlenbeck type we require sequential analysis methods (see [24] and [20], Sections 17.5-6). The drift parameter will be estimated from the observations of the process Y , in some interval $[0, T_0]$. It should be noted that in this case the usual likelihood estimator for the drift parameter is a nonlinear function of observations and it is not possible to calculate directly a non-asymptotic upper bound for its accuracy. To overcome this difficulty we use the truncated sequential estimate from [15] which enables us a non-asymptotic upper bound for mean accuracy estimation. After that we deal with the optimal strategy in the interval $[T_0, T]$, under the estimated parameter. We show that the expected absolute deviation of the objective function for the given strategy is less than some known fixed level δ i.e. the strategy calculated through the sequential procedure is δ -optimal. Moreover, in this paper we find the explicit form for this level. This allows to keep small the deviation of the objective function from the optimal one by controlling the initial endowment.

The paper is organized as follows: In Sections 2-3 we introduce the market model, state the optimization problem and give the related HJB equation. Section 4 is set for definitions. The solution of the optimal consumption and investment problem is given in Sections 5-7. In Section 8 we consider unknown the drift parameter α for the economic factor Y and use a truncated sequential method to construct its estimate $\hat{\alpha}$. We obtain an explicit upper for the deviation $\mathbf{E} |\hat{\alpha} - \alpha|$ for any fixed $T_0 > 0$. Moreover considering the optimal consumption investment problem in the finite interval $[T_0, T]$, we show that the strategy calculated through this truncation procedure is δ -optimal. Similar results are given in Section 8.3 when, in addition of using $\hat{\alpha}$, we consider an estimate $\hat{\mu}$ of the unknown stock price appreciation rate. A numerical example is given in Section 9 and auxiliary results are reported into the appendix.

2 Market model

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ be a standard and filtered probability space with two standard independent $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted Wiener processes $(W_t)_{0 \leq t \leq T}$ and $(\mathbf{U}_t)_{0 \leq t \leq T}$ taking their values in \mathbb{R} . Our financial market consists of one *riskless money market account* $(S_0(t))_{0 \leq t \leq T}$ and one *risky stock* $(S(t))_{0 \leq t \leq T}$ governed by the following equations:

$$\begin{cases} dS_0(t) = r S_0(t) dt, \\ dS(t) = S(t)\mu dt + S(t)\sigma(Y_t) dW_t, \end{cases} \quad (2.1)$$

with $S_0(0) = 1$ and $S(0) = s > 0$. In this model $r \in \mathbb{R}_+$ is the *riskless bond interest rate*, μ is the *stock price appreciation rate* and $\sigma(y)$ is *stock-volatility*. For all $y \in \mathbb{R}$ the coefficient $\sigma(y) \in \mathbb{R}_+$ is a nonrandom continuous bounded function and satisfies

$$\inf_{y \in \mathbb{R}} \sigma(y) = \sigma_1 > 0.$$

We assume also that $\sigma(y)$ is differentiable and has bounded derivative. Moreover we assume that the stochastic factor Y valued in \mathbb{R} is of Ornstein-Uhlenbeck type. It has a dynamics governed by the following stochastic differential equation:

$$dY_t = \alpha Y_t dt + \beta d\mathbf{U}_t, \quad (2.2)$$

where the initial value Y_0 is a non random constant, $\alpha < 0$ and $\beta > 0$ are fixed parameters. We denote by $(Y_s^{t,y})_{s \geq t}$ the process Y starts at $Y_t = y$, i.e.

$$Y_s^{t,y} = ye^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} d\mathbf{U}_v.$$

We note, that for the model (2.1) the risk premium is the $\mathbb{R} \rightarrow \mathbb{R}$ function defined as

$$\theta(y) = \frac{\mu - r}{\sigma(y)}. \quad (2.3)$$

Similarly to [14] we consider the fractional portfolio process $\varphi(t)$, i.e. $\varphi(t)$, is the fraction of the wealth process X_t invested in the stock at the time t . The fractions for the consumption is denoted by $c = (c_t)_{0 \leq t \leq T}$. In this case the wealth process satisfies the following stochastic equation

$$dX_t = X_t(r + \pi_t \theta(Y_t) - c_t) dt + X_t \pi_t dW_t, \quad (2.4)$$

where $\pi_t = \sigma(Y_t) \varphi_t$ and the initial endowment $X_0 = x$.

Now we describe the set of all admissible strategies. A portfolio control (financial strategy) $\vartheta = (\vartheta_t)_{t \geq 0} = ((\pi_t, c_t))_{t \geq 0}$ is said to be *admissible* if it is $(\mathcal{F}_t)_{0 \leq t \leq T}$ - progressively measurable with values in $\mathbb{R} \times [0, \infty)$, such that

$$\|\pi\|_T := \int_0^T |\pi_t|^2 dt < \infty \quad \text{and} \quad \int_0^T c_t dt < \infty \quad \text{a.s.} \quad (2.5)$$

and equation (2.4) has a unique strong a.s. positive continuous solution $(X_t^\vartheta)_{0 \leq t \leq T}$ on $[0, T]$. We denote the set of *admissible portfolios controls* by \mathcal{V} .

In this paper we consider an agent using the power utility function x^γ for $0 < \gamma < 1$. The goal is to maximize the expected utilities from the consumption on the time interval $[T_0, T]$, for fixed T_0 , and from the terminal wealth at maturity T . Then for any $x, y \in \mathbb{R}$, and $\vartheta \in \mathcal{V}$ the value function is defined by

$$J(T_0, x, y, \vartheta) := \mathbf{E}_{T_0, x, y} \left(\int_{T_0}^T c_t^\gamma (X_t^\vartheta)^\gamma dt + (X_T^\vartheta)^\gamma \right),$$

where $\mathbf{E}_{T_0, x, y}$ is the conditional expectation $\mathbf{E}(\cdot | X_{T_0} = x, Y_{T_0} = y)$. Our goal is to maximize this function, i.e. to calculate

$$J(T_0, x, y, \vartheta^*) = \sup_{\vartheta \in \mathcal{V}} J(T_0, x, y, \vartheta). \quad (2.6)$$

For the sequel we will use the notations $J^*(T_0, x, y)$ or simply $J_{T_0}^*$ instead of $J(T_0, x, y, \vartheta^*)$, moreover we set $\tilde{T} = [T - T_0]$.

Remark 2.1. *Note that the same problem as (2.6) is solved in [1], but the economic factor Y considered there is a general diffusion process with bounded coefficients. In the present paper Y is an Ornstein-Uhlenbeck process, so the drift is not bounded, but we take advantage of the fact that Y is Gaussian and not correlated to the market, which is not the case in [1].*

3 Hamilton-Jacobi-Bellman equation

Now we introduce the HJB equation for the problem (2.6). To this end, for any differentiable function f we denote by $\mathbf{D}_y f(t, y)$ and

$\mathbf{D}_{x,y}f(t, x, y)$ its partial derivatives i.e.

$$\mathbf{D}_y f(t, y) = \frac{\partial}{\partial y} f(t, y) \quad \text{and} \quad \mathbf{D}_{x,y} f(t, x, y) = \frac{\partial^2 f(t, x, y)}{\partial x \partial y}. \quad (3.1)$$

Moreover we denote by $\mathbf{D}_{x,y}^2 f(t, x, y)$ the Hessian of f , that is the square matrix of second order partial derivatives with respect to x and y .

Let now $(\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{R}^2$ be fixed parameters and

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix}, \quad \mathbf{M}_{ij} \in \mathbb{R}.$$

For these parameters with $\mathbf{q}_1 > 0$ we define the Hamilton function as

$$\begin{aligned} H(t, x, y, \mathbf{q}_1, \mathbf{q}_2, \mathbf{M}) &= x r \mathbf{q}_1 + \alpha \mathbf{q}_2 + \frac{1}{q_*} \left(\frac{\gamma}{\mathbf{q}_1} \right)^{q_* - 1} \\ &\quad + \frac{|\theta(y) \mathbf{q}_1|^2}{2|\mathbf{M}_{11}|} + \frac{\beta^2}{2} \mathbf{M}_{22}, \end{aligned} \quad (3.2)$$

where $q_* = (1 - \gamma)^{-1}$. The HJB equation is given by

$$\begin{cases} z_t(t, x, y) + H(t, x, y, \mathbf{D}_x z(t, x, y), \mathbf{D}_y z(t, x, y), \mathbf{D}_{x,y}^2 z(t, x, y)) = 0 \\ z(T, x, y) = x^\gamma. \end{cases} \quad (3.3)$$

To study this equation we represent $z(t, x, y)$ as

$$z(t, x, y) = x^\gamma h(t, y). \quad (3.4)$$

It is easy to deduce that the function h satisfies the following quasi-linear PDE:

$$\begin{cases} h_t(t, y) + Q(y) h(t, y) + \alpha y \mathbf{D}_y h(t, y) + \frac{\beta^2}{2} \mathbf{D}_{y,y} h(t, y) \\ \quad + \frac{1}{q_*} \left(\frac{1}{h(t, y)} \right)^{q_* - 1} = 0; \\ h(T, y) = 1. \end{cases} \quad (3.5)$$

We recall that $q_* = 1/(1 - \gamma)$ and we define

$$Q(y) = \gamma \left(r + \frac{|\theta(y)|^2}{2(1 - \gamma)} \right). \quad (3.6)$$

Note that, using the conditions on $\sigma(y)$; the function $Q(y)$ is bounded differentiable and has bounded derivative. Therefore, we can set

$$Q_* = \sup_{y \in \mathbb{R}} Q(y) \quad \text{and} \quad Q_1^* = \sup_{y \in \mathbb{R}} |\mathbf{D}_y Q(y)|. \quad (3.7)$$

Our goal is to study equation (3.5). By making use of the probabilistic representation for the linear PDE (the Feynman-Kac formula) we show in Proposition 5.4, that the solution of this equation is the fixed-point solution for a special mapping of the integral type which will be introduced in the next section.

4 Useful definitions

First, to study equation (3.5) we introduce a special functional space. Let \mathcal{X} be the set of continuous functions defined on $\mathcal{K} := [T_0, T] \times \mathbb{R}$ with values in $[1, \infty)$ such that

$$\|f\|_\infty = \sup_{(t,y) \in \mathcal{K}} |f(t,y)| \leq \mathbf{r}^*, \quad (4.1)$$

where

$$\mathbf{r}^* = (\tilde{T} + 1) e^{Q_* \tilde{T}}. \quad (4.2)$$

Now, we define a metric $\varrho_*(\cdot, \cdot)$ in \mathcal{X} as follows: for any f, g in \mathcal{X}

$$\varrho_*(f, g) = \|f - g\|_*, \quad \|f\|_* = \sup_{(t,y) \in \mathcal{K}} e^{-\varkappa(T-t)} |f(t,y)|, \quad (4.3)$$

where

$$\varkappa = Q_* + \zeta + 1. \quad (4.4)$$

Here ζ is any positive parameter which will be specified later.

We define now the process η by its dynamics

$$d\eta_s = \alpha \eta_s ds + \beta d\tilde{\mathbf{U}}_s \quad \text{with} \quad \eta_0 = Y_0 \quad (4.5)$$

so that η_t has the same distribution as Y_t . Here $(\tilde{\mathbf{U}}_t)$ is a standard Brownian motion independent of (\mathbf{U}_t) . Let's now define the $\mathcal{X} \rightarrow \mathcal{X}$ Feynman-Kac mapping \mathcal{L} :

$$\mathcal{L}_f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathcal{H}_f(t, s, y) ds, \quad (4.6)$$

where $\mathcal{G}(t, s, y) = \exp\left(\int_t^s Q(\eta_u^{t,y}) du\right)$ and

$$\mathcal{H}_f(t, s, y) = \mathbf{E}\left(f(s, \eta_s^{t,y})\right)^{1-q_*} \mathcal{G}(t, s, y). \quad (4.7)$$

and $(\eta_s^{t,y})_{t \leq s \leq T}$ is the process η starting at $\eta_t = y$. To solve the HJB equation we need to find the fixed-point solution for the mapping \mathcal{L} in \mathcal{X} , i.e.

$$\mathcal{L}_h = h. \quad (4.8)$$

To this end we construct the following iterated scheme. We set $h_0 \equiv 1$

$$h_n(t, y) = \mathcal{L}_{h_{n-1}}(t, y) \quad \text{for } n \geq 1. \quad (4.9)$$

and study the convergence of this sequence in \mathcal{K} . Actually, we will use the existence argument of a fixed point, for a contracting operator in a complete metrical space.

5 Solution of the HJB equation

We give in this section the existence and uniqueness result, of a solution for the HJB equation (3.5). For this, we show some properties of the Feynman-Kac operator \mathcal{L} .

Proposition 5.1. *The operator \mathcal{L} is "stable" in \mathcal{X} that is*

$$\mathcal{L}_f \in \mathcal{X}, \quad \forall f \in \mathcal{X}.$$

Moreover, $\mathcal{L}_f \in \mathbf{C}^{1,2}(\mathcal{K})$ for any $f \in \mathcal{X}$.

Proof. Obviously, that for any $f \in \mathcal{X}$ the mapping \mathcal{L}_f is continuous and $\mathcal{L}_f \geq 1$. Moreover, setting

$$\tilde{f}_s = f(s, \eta_s^{t,y}), \quad (5.1)$$

we represent $\mathcal{L}_f(t, y)$ as

$$\mathcal{L}_f(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathbf{E} \left(\tilde{f}_s\right)^{1-q_*} \mathcal{G}(t, s, y) ds. \quad (5.2)$$

Therefore, taking into account that $\tilde{f}_s \geq 1$ and $q_* \geq 1$ we get

$$\mathcal{L}_f(t, y) \leq e^{Q_*(T-t)} + \int_t^T \frac{1}{q_*} e^{Q_*(s-t)} ds \leq \mathbf{r}^*, \quad (5.3)$$

where the upper bound \mathbf{r}^* is defined in (4.2). Now we have to show that $\mathcal{L}_f \in \mathbf{C}^{1,2}(\mathcal{K})$, for any $f \in \mathcal{X}$. Indeed, to this end we consider for any f from \mathcal{X} the equation (3.5), i.e.

$$\begin{cases} u_t(t, y) + Q(y)u(t, y) + \alpha \mathbf{D}_y u(t, y) \\ + \frac{\beta^2}{2} \mathbf{D}_{y,y} u(t, y) + \frac{1}{q_*} \left(\frac{1}{f(t, y)} \right)^{q_*-1} = 0; \\ u(T, y) = 1. \end{cases} \quad (5.4)$$

Setting here $\tilde{u}(t, y) = u(T_0 + T - t, y)$ we obtain a uniformly parabolic equation for \tilde{u} with initial condition $\tilde{u}(T_0, y) = 1$. Moreover, we know that Q has bounded derivative. Therefore, for any f from \mathcal{X} Theorem 5.1 from [18] (p. 320) with $0 < l < 1$ provides the existence of the unique solution of (5.4) belonging to $\mathbf{C}^{1,2}(\mathcal{K})$. Applying the Itô formula to the process

$$\left(u(s, \eta_s^{t,y}) e^{\int_t^s Q(\eta_v^{t,y}) dv} \right)_{t \leq s \leq T}$$

and taking into account equation (5.4) we get

$$u(t, y) = \mathcal{L}_f(t, y). \quad (5.5)$$

Therefore, the function $\mathcal{L}_f(t, y) \in \mathbf{C}^{1,2}(\mathcal{K})$, i.e. $\mathcal{L}_f \in \mathcal{X}$ for any $f \in \mathcal{X}$. Hence Proposition 5.1. \square

Proposition 5.2. *The mapping \mathcal{L} is a contraction in the metric space (\mathcal{X}, ϱ_*) , i.e. for any f, g from \mathcal{X}*

$$\varrho_*(\mathcal{L}_f, \mathcal{L}_g) \leq \lambda \varrho_*(f, g), \quad (5.6)$$

where the parameter $0 < \lambda < 1$ is given by

$$\lambda = \frac{1}{\zeta + 1}, \quad \zeta > 0. \quad (5.7)$$

Actually, as shown in Corollary 6.2, an appropriate choice of ζ gives a super-geometric convergence rate for the sequence $(h_n)_{n \geq 1}$ defined in (4.9), to the limit function $h(t, y)$, which is the fixed point of the operator \mathcal{L} .

Proof. First note that, for any f and g from \mathcal{X} and for any $y \in \mathbb{R}$

$$\begin{aligned} |\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| &\leq \frac{1}{q_*} \mathbf{E} \int_t^T \mathcal{G}(t, s, y) \left| (\tilde{f}_s)^{1-q_*} - (\tilde{g}_s)^{1-q_*} \right| ds \\ &\leq \gamma \mathbf{E} \int_t^T \mathcal{G}(t, s, y) \left| \tilde{f}_s - \tilde{g}_s \right| ds. \end{aligned}$$

We recall that $\tilde{f}_s = f(s, \eta_s^{t,y})$ and $\tilde{g}_s = g(s, \eta_s^{t,y})$. Taking into account here that $\mathcal{G}(t, s, y) \leq e^{Q_*(s-t)}$ we obtain

$$|\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)| \leq \int_t^T e^{Q_*(s-t)} \mathbf{E} |\tilde{f}_s - \tilde{g}_s| ds.$$

Taking into account in the last inequality, that

$$|\tilde{f}_s - \tilde{g}_s| \leq e^{\varkappa(T-s)} \varrho_*(f, g) \quad \text{a.s.}, \quad (5.8)$$

we get for all (t, y) in \mathcal{K}

$$\left| e^{-\varkappa(T-t)} (\mathcal{L}_f(t, y) - \mathcal{L}_g(t, y)) \right| \leq \frac{1}{\varkappa - Q_*} \varrho_*(f, g). \quad (5.9)$$

Taking into account the definition of \varkappa in (4.4), we obtain inequality (5.6). Hence Proposition 5.2. \square

Proposition 5.3. *The fixed point equation $\mathcal{L}_h = h$ has a unique solution in \mathcal{X} .*

Proof. Indeed, using the contraction of the operator \mathcal{L} in \mathcal{X} and the definition of the sequence $(h_n)_{n \geq 1}$ in (4.9) we get, that for any $n \geq 1$

$$\varrho_*(h_n, h_{n-1}) \leq \lambda^{n-1} \varrho_*(h_1, h_0), \quad (5.10)$$

i.e. the sequence $(h_n)_{n \geq 1}$ is fundamental in (\mathcal{X}, ϱ_*) . The metric space (\mathcal{X}, ϱ_*) is complete since it is included in the Banach space $C^{0,0}(\mathcal{K})$, and $\|\cdot\|_\infty$ is equivalent to $\|\cdot\|_*$ defined in (4.3). Therefore, this sequence has a limit in \mathcal{X} , i.e. there exists a function h from \mathcal{X} for which

$$\lim_{n \rightarrow \infty} \varrho_*(h, h_n) = 0.$$

Moreover, taking into account that $h_n = \mathcal{L}_{h_{n-1}}$ we obtain, that for any $n \geq 1$

$$\varrho_*(h, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \varrho_*(\mathcal{L}_{h_{n-1}}, \mathcal{L}_h) \leq \varrho_*(h, h_n) + \lambda \varrho_*(h, h_{n-1}).$$

The last expression tends to zero as $n \rightarrow \infty$. Therefore $\varrho_*(h, \mathcal{L}_h) = 0$, i.e. $h = \mathcal{L}_h$. Proposition 5.2 implies immediately that this solution is unique. \square

We are ready to state the result about the solution of the HJB equation:

Proposition 5.4. *The HJB equation (3.5) has a unique solution which is the solution h of the fixed-point problem $\mathcal{L}_h = h$.*

Proof. Choosing in (5.4) the function $f = u$ and taking into account the representation (5.5) and the fixed point equation $\mathcal{L}_h = h$ we obtain, that the solution of equation (5.4)

$$u = \mathcal{L}_h = h.$$

Therefore, the function h satisfies equation (3.5). Moreover, this solution is unique since h is the unique solution of the fixed point problem. \square

Remark 5.1. 1. *We can find in [22] an existence and uniqueness proof for a more general quasilinear equation but therein, authors did not give a way to calculate this solution, whereas in our case, the solution is the fixed point function for the Feynman-Kac operator. Moreover our method allows to obtain the super geometric convergence rate for the sequence approximating the solution, which is a very important property in practice. In [3] author shows an existence and uniqueness result where the global result is deduced from a local existence and uniqueness theorem.*

2. *The application of contraction mapping or fixed-point theorem to solve nonlinear PDE is not new see, e.g. [8] and [25] where the term "generalised solution" is used for quasilinear/semilinear PDE, and the fixed point of the Feynman-Kac representation is discussed.*

6 Super-geometrical convergence rate

For the sequence $(h_n)_{n \geq 1}$ defined in (4.9), and h the fixed point solution for $h = \mathcal{L}_h$, we study the behavior of the deviation

$$\Delta_n(t, y) = h(t, y) - h_n(t, y).$$

In the following theorem we make an appropriate choice of ζ for the contraction parameter λ to get the super-geometric convergence rate for the sequence $(h_n)_{n \geq 1}$.

Theorem 6.1. *The fixed point problem $\mathcal{L}_h = h$ admits a unique solution h in \mathcal{X} such that for any $n \geq 1$ and $\zeta > 0$*

$$\sup_{(t,y) \in \mathcal{K}} |\Delta_n(t,y)| \leq \mathbf{B}^* \lambda^n, \quad (6.1)$$

where $\mathbf{B}^* = e^{\varkappa \tilde{T}} (1 + \mathbf{r}^*) / (1 - \lambda)$ and \varkappa is given in (4.4).

Proof. Proposition 5.3 implies the first part of this theorem. Moreover, from (5.10) it is easy to see, that for each $n \geq 1$

$$\varrho_*(h, h_n) \leq \frac{\lambda^n}{1 - \lambda} \varrho_*(h_1, h_0).$$

Thanks to Proposition 5.1 all the functions h_n belong to \mathcal{X} , i.e. by the definition of the space \mathcal{X}

$$\varrho_*(h_1, h_0) \leq \sup_{(t,y) \in \mathcal{K}} |h_1(t,y) - 1| \leq 1 + \mathbf{r}^*.$$

Taking into account that

$$\sup_{(t,y) \in \mathcal{K}} |\Delta_n(t,y)| \leq e^{\varkappa \tilde{T}} \varrho_*(h, h_n),$$

we obtain the inequality (6.1). Hence Theorem 6.1. \square

Now we can minimize the upper bound (6.1) over $\zeta > 0$. Indeed,

$$\mathbf{B}^* \lambda^n = \mathbf{C}^* \exp\{\mathbf{g}_n(\zeta)\},$$

where $\mathbf{C}^* = (1 + \mathbf{r}^*) e^{(Q_* + 1)\tilde{T}}$ and

$$\mathbf{g}_n(x) = x \tilde{T} - \ln x - (n - 1) \ln(1 + x).$$

Now we minimize this function over $x > 0$, i.e.

$$\min_{x > 0} \mathbf{g}_n(x) = x_n^* \tilde{T} - \ln x_n^* - (n - 1) \ln(1 + x_n^*),$$

where

$$x_n^* = \frac{\sqrt{(\tilde{T} - n)^2 + 4\tilde{T}} + n - \tilde{T}}{2\tilde{T}}.$$

Therefore, for

$$\zeta = \zeta_n^* = x_n^*$$

we obtain the optimal upper bound (6.1).

Corollary 6.2. *The fixed point problem has a unique solution h in \mathcal{X} such that for any $n \geq 1$*

$$\sup_{(t,y) \in \mathcal{K}} |\Delta_n(t,y)| \leq \mathbf{U}_n^*, \quad (6.2)$$

where $\mathbf{U}_n^* = \mathbf{C}^* \exp\{\mathbf{g}_n^*\}$. Moreover one can check directly that for any $0 < \delta < 1$

$$\mathbf{U}_n^* = O(n^{-\delta n}) \quad \text{as } n \rightarrow \infty.$$

This means that the convergence rate is more rapid than any geometric one, i.e. it is super-geometric.

7 Known parameters

We consider our optimal consumption and investment problem in the case of markets with known parameters. The next theorem is the analogous of theorem 3.4 in [1]. The main difference between the two results is that the drift coefficient of the process Y in [1] must be bounded and so does not allow the Ornstein-Uhlenbeck process. Moreover the economic factor Y is correlated to the market by the Brownian motion \mathbf{U} , which is not the case in the present paper, since we consider the process \mathbf{U} independent of W .

Theorem 7.1. *The optimal value of $J(T_0, x, y, \vartheta)$ for the optimization problem (2.6) is given by*

$$J_{T_0}^* = J(T_0, x, y, \vartheta^*) = \sup_{\vartheta \in \mathcal{V}} J(T_0, x, y, \vartheta) = x^\gamma h(T_0, y)$$

where $h(t, y)$ is the unique solution of equation (3.5). Moreover, for all $T_0 \leq t \leq T$ an optimal financial strategy $\vartheta^* = (\pi^*, c^*)$ is of the form

$$\begin{cases} \pi_t^* = \pi^*(Y_t) = \frac{\theta(Y_t)}{1 - \gamma}; \\ c_t^* = c^*(t, Y_t) = (h(t, Y_t))^{-q^*}. \end{cases} \quad (7.1)$$

The optimal wealth process $(X_t^*)_{T_0 \leq t \leq T}$ satisfies the following stochastic equation

$$dX_t^* = \mathbf{a}^*(t, Y_t) X_t^* dt + X_t^* \mathbf{b}^*(Y_t) dW_t, \quad X_{T_0}^* = x, \quad (7.2)$$

where

$$\begin{cases} \mathbf{a}^*(t, y) &= \frac{|\theta(y)|^2}{1-\gamma} + r - (h(t, y))^{-q_*} ; \\ \mathbf{b}^*(y) &= \frac{\theta(y)}{1-\gamma} . \end{cases} \quad (7.3)$$

The solution X_t^* can be written as

$$X_s^* = X_t^* e^{\int_t^s \mathbf{a}^*(v, Y_v) dv} \mathcal{E}_{t,s}, \quad (7.4)$$

where $\mathcal{E}_{t,s} = \exp \left\{ \int_t^s \mathbf{b}^*(Y_v) dW_v - \frac{1}{2} \int_t^s |\mathbf{b}^*(Y_v)|^2 dv \right\}$.

The proof of the theorem follows the same arguments, as Theorem 3.4 in [1], so it is omitted.

8 Unknown parameters

In this section we consider the Black-Scholes market with unknown stock price appreciation rate μ . Moreover, we consider unknown the drift parameter α of the economic factor Y . We observe the process Y in the interval $[0, T_0]$, and use sequential methods to estimate the drift. After that, we will deal with the consumption-investment optimization problem on the finite interval $[T_0, T]$ and look for the behavior of the optimal value function $J^*(T_0, x, y)$ under the estimated parameters.

8.1 Sequential procedure

We assume the unknown parameter α taking values in some bounded interval $[\alpha_2, \alpha_1]$, with $\alpha_2 < \alpha < \alpha_1 < 0$. We define the function $\epsilon(\cdot)$, which will serve later for the δ -optimality:

$$\epsilon(T_0) = \sqrt{\frac{\beta^2}{H} + \frac{\alpha_2^2}{\beta^{12}} \left(\frac{\mathbf{k}(3)}{T_0^2} \right)}. \quad (8.1)$$

Here $H = \beta_2 (T_0 - T_0^\varepsilon)$, $\beta_2 = \beta^2/2|\alpha_2|$, $\varepsilon = 5/6$ and

$$\mathbf{k}(m) = 3^{2m-1} (Y_0^{2m} + (1 + (m(2m-1))^m (2\beta)^{2m}) \mathbf{k}_1(m)),$$

with $\mathbf{k}_1(m) = 2^{2m-1} (Y_0^{2m} + (2m-1)!! \beta_1^m)$ and $\beta_1 = \beta^2/2|\alpha_1|$. The proposition below gives $\hat{\alpha}$ the truncated sequential estimate of α and gives a bound for the expected deviation $\mathbf{E}|\hat{\alpha} - \alpha|$. We set for the sequel $\bar{\alpha} = \hat{\alpha} - \alpha$.

Proposition 8.1. *We can find $\hat{\alpha}$ an estimate for α , such that*

$$\mathbf{E}|\hat{\alpha} - \alpha| \leq \epsilon(T_0).$$

More precisely we define $\hat{\alpha}$ as the projection onto the interval $[\alpha_2, \alpha_1]$ of the sequential estimate α^ .*

$$\hat{\alpha} = Proj_{[\alpha_2, \alpha_1]} \alpha^*, \quad \alpha^* = \left(\frac{\int_0^{\tau_H} Y_t dY_t}{H} \right) 1_{\{\tau_H \leq T_0\}} \quad (8.2)$$

where $\tau_H = \inf \left\{ t \geq 0, \int_0^t Y_s^2 ds \geq H \right\}$.

Proof. Note first that $\mathbf{E}|\hat{\alpha} - \alpha| \leq \mathbf{E}|\alpha^* - \alpha|$, so it is enough to show that $\mathbf{E}|\alpha^* - \alpha| \leq \epsilon(T_0)$. Moreover, we know from [20] chapter 17, that the maximum likelihood estimate of α is given by

$$\frac{\int_0^{T_0} Y_t dY_t}{\int_0^{T_0} Y_t^2 dt} \quad \text{with} \quad \int_0^\infty Y_t^2 dt = +\infty \text{ a.s.}$$

We define by $\tilde{\alpha}$ the α -sequential that is

$$\tilde{\alpha} = \frac{\int_0^{\tau_H} Y_t dY_t}{\int_0^{\tau_H} Y_t^2 dt} = \alpha + \beta \frac{\int_0^{\tau_H} Y_t d\mathbf{U}_t}{H},$$

so that $\tilde{\alpha} \rightsquigarrow \mathcal{N}(\alpha, \beta^2/H)$ and hence $\mathbf{E}|\tilde{\alpha} - \alpha|^2 = \beta^2/H$.

The problem with the previous estimate is that τ_H may be greater than T_0 . To overcome this difficulty we define the truncated sequential estimate α^* as in the theorem ie: $\alpha^* = \tilde{\alpha} 1_{\{\tau_H \leq T_0\}}$. We observe that

$$\begin{aligned} \alpha^* - \alpha &= (\alpha^* - \alpha) 1_{\{\tau_H \leq T_0\}} + (\alpha^* - \alpha) 1_{\{\tau_H > T_0\}} \\ &= \beta \frac{\int_0^{\tau_H} Y_t d\mathbf{U}_t}{H} 1_{\{\tau_H \leq T_0\}} - \alpha 1_{\{\tau_H > T_0\}}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{E}(\alpha^* - \alpha)^2 &= \frac{\beta^2}{H^2} \mathbf{E} \left(\int_0^{\tau_H} Y_t d\mathbf{U}_t 1_{\{\tau_H \leq T_0\}} \right)^2 + \alpha^2 \mathbf{P}(\tau_H > T_0) \\ &\leq \frac{\beta^2}{H^2} \mathbf{E} \left(\int_0^{\tau_H} Y_t d\mathbf{U}_t \right)^2 + \alpha^2 \mathbf{P}(\tau_H > T_0) \\ &\leq \frac{\beta^2}{H} + \alpha^2 \mathbf{P} \left(\int_0^{T_0} Y_t^2 dt < H \right). \end{aligned} \quad (8.3)$$

Moreover, by the Itô formula

$$dY_t^2 = 2Y_t dY_t + \beta^2 dt = (2\alpha Y_t^2 + \beta^2) dt + 2\beta Y_t d\mathbf{U}_t.$$

From there we deduce that

$$\int_0^{T_0} (2\alpha Y_t^2 + \beta^2) dt = Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t.$$

Taking into account that $\alpha_2 \leq \alpha \leq \alpha_1 < 0$ and using the Markov's inequality, we get for any integer $m > 0$

$$\begin{aligned} \mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) &= \mathbf{P}\left(\int_0^{T_0} (2\alpha Y_t^2 + \beta^2) dt > 2\alpha H + \beta^2 T_0\right) \\ &= \mathbf{P}\left(Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t > 2\alpha H + \beta^2 T_0\right) \\ &\leq \frac{\mathbf{E}\left(Y_{T_0}^2 - Y_0^2 - 2\beta \int_0^{T_0} Y_t d\mathbf{U}_t\right)^{2m}}{(2\alpha_2 H + \beta^2 T_0)^{2m}} \end{aligned}$$

Here $2\alpha H + \beta^2 T_0 > 0$, ie: $0 < H < \beta_2 T_0$. With the centered Gaussian variable $\xi_t = \int_0^t \beta e^{\alpha(t-v)} d\mathbf{U}_v$ we get for any $m \in \mathbb{N}_*$

$$\mathbf{E}(\xi_t^{2m}) = (2m - 1)!! [\mathbf{E}(\xi_t^2)]^m \leq (2m - 1)!! \beta_1^m.$$

Furthermore, in view of $Y_{T_0} = Y_0 e^{\alpha T_0} + \xi_{T_0}$ we obtain

$$\begin{aligned} \mathbf{E}Y_{T_0}^{2m} &\leq 2^{2m-1} \left(\mathbf{E}(Y_0 e^{\alpha T_0})^{2m} + \mathbf{E}(\xi_{T_0}^{2m}) \right) \\ &\leq \mathbf{k}_1(m). \end{aligned}$$

Moreover, we have (see e.g. [19] Lemma 4.12):

$$\begin{aligned} \mathbf{E}\left(\int_0^{T_0} Y_t d\mathbf{U}_t\right)^{2m} &\leq (m(2m - 1))^m T_0^{m-1} \int_0^{T_0} \mathbf{E}Y_s^{2m} ds \\ &\leq \mathbf{k}_2(m) T_0^m. \end{aligned}$$

where $\mathbf{k}_2(m) = (m(2m - 1))^m \mathbf{k}_1(m)$. We conclude that

$$\mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) \leq \frac{3^{2m-1} (Y_0^{2m} + \mathbf{k}_1(m) + (2\beta)^{2m} \mathbf{k}_2(m) T_0^m)}{(2\alpha_2 H + \beta^2 T_0)^{2m}}.$$

We set $H = \beta_2 (T_0 - T_0^\varepsilon)$ for some ε , we obtain

$$\mathbf{P}\left(\int_0^{T_0} Y_t^2 dt < H\right) \leq \frac{1}{(\beta^2)^{2m}} \left(\frac{\mathbf{k}(m)}{T_0^{m(2\varepsilon-1)}} \right)$$

Replacement in (8.3) gives

$$\mathbf{E}(\alpha^* - \alpha)^2 \leq \frac{\beta^2}{\beta_2(T_0 - T_0^\varepsilon)} + \frac{\alpha^2}{\beta^4 m} \left(\frac{\mathbf{k}(m)}{T_0^{m(2\varepsilon-1)}} \right).$$

We fixe $\varepsilon = 5/6$ and $m = 3$ so that $m(2\varepsilon - 1) = 2$, which gives $\varepsilon^2(T_0)$ and then the desired result. \square

8.2 Known stock price appreciation rate μ

We consider in this section the consumption-investment problem for markets with known μ and unknown α . We define the value function $\widehat{J}_{T_0}^*$ the estimate of $J_{T_0}^*$

$$\widehat{J}_{T_0}^* := \mathbf{E}_{T_0} \left(\int_{T_0}^T (\widehat{c}_t^*)^\gamma (\widehat{X}_t^*)^\gamma dt + (\widehat{X}_T^*)^\gamma \right). \quad (8.4)$$

\mathbf{E}_{T_0} is the conditional expectation $\mathbf{E}(\cdot | \mathcal{F}_{T_0})$. \widehat{X}_t^* is a simplified notation for $X_t^{\widehat{\vartheta}^*}$ and from 7.4 we write

$$\widehat{X}_s^* = \widehat{X}_t^* e^{\int_t^s \widehat{\mathbf{a}}^*(v, Y_v) dv} \widehat{\mathcal{E}}_{t,s}, \quad (8.5)$$

where $\widehat{\mathcal{E}}_{t,s} = \exp \left\{ \int_t^s \widehat{\mathbf{b}}^*(Y_v) dW_v - \frac{1}{2} \int_t^s |\widehat{\mathbf{b}}^*(Y_v)|^2 dv \right\}$. Here

$$\begin{cases} \widehat{\mathbf{a}}^*(t, y) = \frac{|\widehat{\theta}(y)|^2}{1-\gamma} + r - \left(\widehat{h}(t, y) \right)^{-q_*}; \\ \widehat{\mathbf{b}}^*(y) = \frac{\widehat{\theta}(y)}{1-\gamma}. \end{cases} \quad (8.6)$$

The estimated consumption process is $\widehat{c}_t^* = \widehat{c}^*(t, Y_t) = \left(\widehat{h}(t, Y_t) \right)^{-q_*}$ and $\widehat{h}(t, y)$ is the unique solution for $h = \widehat{\mathcal{L}}h$. The operator $\widehat{\mathcal{L}}$ is defined by:

$$\widehat{\mathcal{L}}_f(t, y) = \mathbf{E} \widehat{\mathcal{G}}(t, T, y) + \frac{1}{q_*} \int_t^T \mathbf{E} \left((f(s, \widehat{\eta}_s^{t,y}))^{1-q_*} \widehat{\mathcal{G}}(t, s, y) \right) ds, \quad (8.7)$$

where $\widehat{\mathcal{G}}(t, s, y) = \exp \left(\int_t^s \widehat{Q}(\widehat{\eta}_u^{t,y}) du \right)$. The process $(\widehat{\eta}_s^{t,y})_{t \leq s \leq T}$ has the following dynamics:

$$d\widehat{\eta}_s^{t,y} = \widehat{\alpha} \widehat{\eta}_s^{t,y} ds + \beta d\widetilde{\mathbf{U}}_s, \quad \widehat{\eta}_t^{t,y} = y. \quad (8.8)$$

To state the approximation result we set

$$\begin{cases} \mathbf{h}_1 &= \frac{1 + 2\gamma + \zeta_0}{1 + \zeta_0} \frac{\tilde{T}}{|\alpha_1|} \left(2Q_1^* \tilde{T} + \gamma h_1^* \right), \\ \Gamma &= \left(q_* \tilde{T} (\tilde{d})^\gamma + (\tilde{T} + 1) \left(\sqrt{\tilde{c} q_*} \right)^\gamma \right) \frac{1}{x^\gamma} e^{\gamma \varkappa \tilde{T}}. \end{cases} \quad (8.9)$$

Here $\zeta_0 > 0$, $\tilde{c} = 4\tilde{T} e^{c_0 \tilde{T}} \tilde{d}^2$, $c_0 = 2 \sup_{(s,y) \in \mathcal{K}} (|\mathbf{a}^*(s,y)|^2 + |\mathbf{b}^*(s,y)|^2)$. Moreover, \tilde{d} is the upper bound (8.13) and h_1^* is the bound for $|\partial h(t,y)/\partial y|$ which is given in Lemma A.2.

We notice that in the estimation interval $[0, T_0]$, we don't invest in the risky stock. We chose the strategy $(c_t, \pi_t) = (r, 0)$ for $0 \leq t \leq T_0$, so that $\forall 0 \leq t \leq T_0$, $X_t = X_0 = x$, *a.s.*

Theorem 8.2. *For any deterministic time $0 < T_0 < T$ and any $m \geq 1$ we have the following estimate*

$$\mathbf{E} |\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \delta, \quad (8.10)$$

where

$$\delta = \delta(x, T_0) = \Gamma \mathbf{h}_1^\gamma x^\gamma \left((2\iota_0)^\gamma + \tilde{c}_m \right) \epsilon(T_0)^\gamma.$$

Here $\tilde{c}_m = ((2m-1)!! \beta^{2m} / (2|\alpha_1|)^m)^{\gamma/2m}$. Recall that $\iota_0 = \beta / \sqrt{2|\alpha_1|}$ and $\epsilon(T_0)$ is defined in (8.1).

Proof. We observe that for a deterministic time $T_0 < T$

$$\begin{aligned} |\widehat{J}_{T_0}^* - J_{T_0}^*| &\leq \mathbf{E}_{T_0} \left(\int_{T_0}^T |(\widehat{c}_t^*)^\gamma (\widehat{X}_t^*)^\gamma - (c_t^*)^\gamma (X_t^*)^\gamma| dt \right) \\ &\quad + \mathbf{E}_{T_0} |(\widehat{X}_T^*)^\gamma - (X_T^*)^\gamma| \\ &\leq \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) \\ &\quad + \mathbf{E}_{T_0} |\widehat{X}_T^* - X_T^*|^\gamma \end{aligned} \quad (8.11)$$

where we used in the last inequality the fact that

$$|a^\gamma - b^\gamma| \leq |a - b|^\gamma \quad \text{when } a \geq b \geq 0 \quad \text{and } \gamma < 1$$

and then we use Lemma 8.3 below to get

$$|\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \Gamma \mathbf{h}_1^\gamma x^\gamma (2\iota_0 + |Y_{T_0}|)^\gamma |\widehat{\alpha} - \alpha|^\gamma$$

The expectation yields to,

$$\begin{aligned} \mathbf{E} |\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| &\leq \Gamma x^\gamma \mathbf{h}_1^\gamma (2\iota_0)^\gamma \mathbf{E} |\widehat{\alpha} - \alpha|^\gamma \\ &\quad + \Gamma x^\gamma \mathbf{h}_1^\gamma \mathbf{E} (|Y_{T_0}|^\gamma |\widehat{\alpha} - \alpha|^\gamma). \end{aligned}$$

By Holder's and Jensen's inequalities for $m' = m(2-\gamma)/\gamma > 1$ with $m \geq 1$

$$\begin{aligned} \mathbf{E} (|Y_{T_0}|^\gamma |\widehat{\alpha} - \alpha|^\gamma) &\leq \left(\mathbf{E} |Y_{T_0}|^{\frac{2\gamma}{2-\gamma}} \right)^{(2-\gamma)/2} (\mathbf{E} |\widehat{\alpha} - \alpha|^2)^{\gamma/2} \\ &\leq \left(\mathbf{E} |Y_{T_0}|^{\frac{2\gamma m'}{2-\gamma}} \right)^{(2-\gamma)/2 m'} \epsilon(T_0)^\gamma \\ &\leq \left(\mathbf{E} |Y_{T_0}|^{2m} \right)^{\gamma/2 m} \epsilon(T_0)^\gamma. \end{aligned}$$

From [11], Lemma 1.1.1 we get $\mathbf{E} |Y_{T_0}|^{2m} \leq c_m(T_0) \leq c_m(0)$ where

$$c_m(T_0) = (2m-1)!! \beta^{2m} \left(\frac{1 - e^{2\alpha T_0}}{2|\alpha|} \right)^m.$$

We conclude that for any $m \geq 1$

$$\mathbf{E} (|Y_{T_0}|^\gamma |\widehat{\alpha} - \alpha|^\gamma) \leq \tilde{c}_m \epsilon^\gamma(T_0), \quad (8.12)$$

which gives the desired result. \square

Remark 8.1. We observe in Theorem 8.2, that the expected deviation $\mathbf{E} |\widehat{J}_{T_0}^* - J^*(T_0, x, y)|$ can be arbitrary small, if either we observe the process Y in a wide interval $[0, T_0]$ so that $\mathbf{E} |\widehat{\alpha} - \alpha|$ be small enough, or we invest a small capital x at the initial time. That means, when the estimation interval is not wide enough, which is the case in practice, we can always find a consumption-investment strategy that belongs closer to the optimal one. For this aim, we need to be cautious in choosing the initial endowment.

Lemma 8.3. For any deterministic $T_0 \leq T$:

$$\mathbf{E}_{T_0} \left(\sup_{T_0 \leq s \leq T} (\widehat{X}_s^*)^2 \right) < x^2 \tilde{d}^2, \quad \text{where } \tilde{d}^2 = 4e^{2\tilde{T}(A^* + (B^*)^2)}. \quad (8.13)$$

Here $A^* = \sup_{(s,y) \in \mathcal{K}} \widehat{\mathbf{a}}^*(s, y)$, $B^* = \sup_{(s,y) \in \mathcal{K}} \widehat{\mathbf{b}}^*(s, y)$.

Moreover we have

$$\sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma \leq k_1 x^\gamma (\mathbf{h}_1 (2\iota_0 + |Y_{T_0}|))^\gamma |\widehat{\alpha} - \alpha|^\gamma, \quad (8.14)$$

where $k_1 = (\sqrt{\widetilde{c}q_*})^\gamma e^{\gamma \varkappa \widetilde{T}} / \varkappa^\gamma$. We have also

$$\begin{aligned} \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) \\ \leq k_2 x^\gamma (\mathbf{h}_1 (2\iota_0 + |Y_{T_0}|))^\gamma |\widehat{\alpha} - \alpha|^\gamma, \end{aligned} \quad (8.15)$$

where $k_2 = \left(\widetilde{T}(\sqrt{\widetilde{c}q_*})^\gamma + \widetilde{d}^\gamma q_* \widetilde{T} \right) e^{\gamma \varkappa \widetilde{T}} / \varkappa^\gamma$.

Proof. It is clear from (8.5), that for the bounded function $\widehat{\mathbf{b}}^*(y)$ the process $(\widehat{\mathcal{E}}_{t,s})_{t \leq s \leq T}$ is a quadratic integrable martingale and by the Doob inequality

$$\begin{aligned} \mathbf{E}_{T_0} \left(\sup_{T_0 \leq s \leq T} (\widehat{X}_s^*)^2 \right) &\leq x^2 e^{2\widetilde{T}A^*} \mathbf{E} \sup_{t \leq s \leq T} \widehat{\mathcal{E}}_{t,s}^2 \leq x^2 4 e^{2\widetilde{T}A^*} \mathbf{E} \widehat{\mathcal{E}}_{t,T}^2 \\ &\leq 4x^2 e^{2\widetilde{T}A^*} e^{\widetilde{T}(B^*)^2}. \end{aligned}$$

this gives (8.13).

We set $\Delta_t = \widehat{X}_t^* - X_t^*$, $A_s = \mathbf{a}^*(s, Y_s)$ and $B_s = \mathbf{b}^*(Y_s)$. Moreover we define $\varphi_1(s) = \widehat{A}_s \widehat{X}_s^* - A_s X_s^*$ and $\varphi_2(s) = \widehat{B}_s \widehat{X}_s^* - B_s X_s^*$. So, from (7.2) we get

$$\begin{aligned} \Delta_t^2 &= \left(\int_{T_0}^t \varphi_1(s) ds + \int_{T_0}^t \varphi_2(s) dW_s \right)^2 \\ &\leq 2(t - T_0) \int_{T_0}^t \varphi_1^2(s) ds + 2 \left(\int_{T_0}^t \varphi_2(s) dW_s \right)^2. \end{aligned}$$

We observe that

$$\begin{aligned} \varphi_1(s)^2 &\leq \left(|\widehat{A}_s - A_s| |\widehat{X}_s^*| + |A_s| |\Delta_s| \right)^2 \\ &\leq 2|\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 + 2|A_s|^2 |\Delta_s|^2, \end{aligned}$$

and since $\widehat{B}_s - B_s = 0$ we have

$$\varphi_2(s)^2 \leq \left(|\widehat{B}_s - B_s| |\widehat{X}_s^*| + |B_s| |\Delta_s| \right)^2 \leq |B_s|^2 |\Delta_s|^2.$$

We define $g(t) = \mathbf{E}_{T_0}(\Delta_t^2)$ so

$$g(t) \leq c_0 \int_{T_0}^t g(s) ds + \psi(t),$$

where

$$\psi(t) = 4\tilde{T} \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 ds.$$

From the Gronwall-Bellman inequality

$$\begin{aligned} g(t) &\leq \psi(t)e^{c_0 t} \\ &\leq x^2 4\tilde{T} e^{c_0 T} \int_{T_0}^t \mathbf{E}_{T_0} \left(|\widehat{A}_s - A_s|^2 |\widehat{X}_s^*|^2 \right) ds \\ &\leq \tilde{c} x^2 \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{A}_s - A_s|^2 ds \\ &\leq \tilde{c} x^2 \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s)^{-q_*} - h(s, Y_s)^{-q_*}|^2 ds \\ &\leq \tilde{c} x^2 q_* \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^2 ds. \end{aligned}$$

Here $\tilde{c} = 4\tilde{T} e^{c_0 \tilde{T}} \tilde{d}^2$. Using (A.12) and Lemma A.5 we obtain, that for any $T_0 \leq s \leq T$

$$\begin{aligned} \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)| &\leq \mathbf{h}_1 \mathbf{E}_{T_0} \left(e^{\varkappa(T-s)} (\iota_0 + |Y_s|) \right) |\widehat{\alpha} - \alpha| \\ &\leq \mathbf{h}_1 (\iota_0 + \mathbf{E}_{T_0} |Y_s|) e^{\varkappa(T-s)} |\widehat{\alpha} - \alpha| \\ &\leq \mathbf{h}_1 (2\iota_0 + |Y_{T_0}|) e^{\varkappa(T-s)} |\widehat{\alpha} - \alpha|. \end{aligned} \tag{8.16}$$

Therefore,

$$g(t) \leq x^2 \tilde{c} q_* (\mathbf{h}_1 (2\iota_0 + |Y_{T_0}|))^2 \frac{e^{2\varkappa \tilde{T}}}{\varkappa^2} |\widehat{\alpha} - \alpha|^2.$$

Hence, (8.14) holds.

We show now inequality (8.15). We have

$$\begin{aligned}
& \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) \\
& \leq \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma |\widehat{X}_t^*|^\gamma dt \right) + \mathbf{E}_{T_0} \left(\int_{T_0}^T (c_t^*)^\gamma |\widehat{X}_t^* - X_t^*|^\gamma dt \right) \\
& \leq \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma |\widehat{X}_t^*|^\gamma dt \right) + \int_{T_0}^T \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma dt \\
& \leq x^\gamma \widetilde{d}^\gamma \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma dt \right) + \widetilde{T} \sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\widehat{X}_t^* - X_t^*|^\gamma.
\end{aligned}$$

The definition of the optimal consumption c_t^* given in (7.1), the fact that $q_* > 1$, $h(t, y) \geq 1$ for any $(t, y) \in \mathcal{K}$ and (8.16) give:

$$\begin{aligned}
\mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* - c_t^*|^\gamma dt \right) & \leq q_* \int_{T_0}^T \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^\gamma dt \\
& \leq q_* \widetilde{T} (\mathbf{h}_1 (2\iota_0 + |Y_{T_0}|))^\gamma \frac{e^{\gamma \times \widetilde{T}}}{\varkappa^\gamma} \mathbf{E}_{T_0} |\widehat{\alpha} - \alpha|^\gamma.
\end{aligned}$$

Then we conclude

$$\begin{aligned}
& \mathbf{E}_{T_0} \left(\int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) \\
& \leq k_2 x^\gamma (\mathbf{h}_1 (2\iota_0 + |Y_{T_0}|))^\gamma \mathbf{E}_{T_0} |\widehat{\alpha} - \alpha|^\gamma,
\end{aligned}$$

which gives (8.15) and then Lemma 8.3. \square

8.3 Unknown stock price appreciation rate μ

In practice, it is not realistic to consider known the stock price appreciation rate μ . In this section, in addition to the unknown drift parameter α of the economic factor process, we consider an unknown stock price appreciation rate μ such that $0 < \mu_1 < \mu < \mu_2$. We recall that the dynamics of the risky stock is given in (2.1). Let $\widehat{\mu}$ its estimate defined by

$$\widehat{\mu} = \frac{Z_{T_0}}{T_0} \quad \text{with} \quad Z_t = \int_0^t \frac{1}{S_t} dS_t. \quad (8.17)$$

Lemma 8.4. *With the previous definition of $\widehat{\mu}$ we have*

$$\mathbf{E}|\widehat{\mu} - \mu| \leq \epsilon_1(T_0), \quad (8.18)$$

where $\epsilon_1(T_0) = \sigma^* / \sqrt{T_0}$ and $\sigma^* = \sup_{y \in \mathbb{R}} \sigma(y)$.

Proof. From the definition of the process Z we get

$$Z_{T_0} = \mu T_0 + \int_0^{T_0} \sigma(Y_t) dW_t,$$

end then

$$\widehat{\mu} - \mu = \frac{1}{T_0} \int_0^{T_0} \sigma(Y_t) dW_t.$$

The calculus of $\mathbf{E}(\widehat{\mu} - \mu)^2$ gives the desired result. \square

Let the optimal value functions $J^*(T_0, x, y)$ and $\widehat{J}_{T_0}^*$ its estimate given in (8.4), and let define the constants

$$k'_1 = 2 \sqrt{\widetilde{c}T} \left(\frac{2\mu_2 + r + \sigma_1}{\sigma_1^2(1-\gamma)} \right) \quad \text{and} \quad k'_2 = \frac{e^{\varkappa\widetilde{T}}}{\varkappa}.$$

Moreover, we define

$$\Gamma_1 = k_3 + k_5 \quad \text{and} \quad \Gamma_2 = k_4 + k_6,$$

where $k_3 = (k'_1)^\gamma + \left(\sqrt{2\widetilde{c}q_*} k'_2 \mathbf{h}_2 \right)^\gamma$, $k_4 = \left(\sqrt{2\widetilde{c}q_*} k'_2 \mathbf{h}_1 \right)^\gamma$,

$k_5 = \widetilde{T} (k'_1)^\gamma + k_7 (k'_2 \mathbf{h}_2)^\gamma$, $k_6 = k_7 (k'_2 \mathbf{h}_1)^\gamma$, $k_7 = \left(\sqrt{2\widetilde{c}q_*} + q_* \widetilde{d}^\gamma \right)$.

recall that $\widetilde{c} = 4e^{c_0 t} \widetilde{d}^2$ and \widetilde{d} is given in (8.13). The constants \mathbf{h}_1 and \mathbf{h}_2 are given in (8.9) and (A.16) respectively. We are dealing with the following result

Theorem 8.5. *We have*

$$\begin{aligned} |\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| &\leq x^\gamma \Gamma_1 (2\iota_0 + |Y_{T_0}|)^\gamma |\widehat{\mu} - \mu|^\gamma \\ &+ x^\gamma \Gamma_2 (2\iota_0 + |Y_{T_0}|)^\gamma |\widehat{\alpha} - \alpha|^\gamma. \end{aligned} \quad (8.19)$$

Moreover we have for any $m \geq 1$

$$\mathbf{E}|\widehat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| \leq \delta_2, \quad (8.20)$$

with $\delta_2 = \delta_2(x, T_0) = x^\gamma \left(\tilde{\Gamma}_1 \epsilon_1(T_0)^\gamma + \tilde{\Gamma}_2 \epsilon(T_0)^\gamma \right)$,

$$\tilde{\Gamma}_1 = \Gamma_1(3\iota_0^\gamma + |Y_0|^\gamma) \quad \text{and} \quad \tilde{\Gamma}_2 = \Gamma_2((2\iota_0)^\gamma + \tilde{c}_m).$$

Here $\tilde{c}_m = ((2m-1)!! \beta^{2m} / (2|\alpha_1|)^m)^{\gamma/2m}$. Recall that $\iota_0 = \beta / \sqrt{2|\alpha_1|}$, $\epsilon_1(T_0)$ is the bound (8.18) and $\epsilon(T_0)$ is defined in (8.1).

Proof. We follow the same arguments as in the proof of Theorem 8.2, and use Lemma 8.6 below to conclude for (8.19).

Now, to show 8.20, we observe from (8.19) that

$$\begin{aligned} \mathbf{E} |\hat{J}_{T_0}^* - J^*(T_0, x, Y_{T_0})| &\leq x^\gamma \Gamma_1 ((2\iota_0)^\gamma + (\mathbf{E}|Y_{T_0}|)^\gamma) \epsilon_1(T_0)^\gamma \\ &\quad + x^\gamma \Gamma_2 (2\iota_0)^\gamma \epsilon(T_0)^\gamma + \mathbf{E}(|Y_{T_0}|^\gamma |\hat{\alpha} - \alpha|^\gamma). \end{aligned}$$

Then we use (A.7) and 8.12 to conclude. \square

Lemma 8.6. *We have*

$$\begin{aligned} \sup_{T_0 \leq t \leq T} \mathbf{E}_{T_0} |\hat{X}_t - X_t|^\gamma &\leq x^\gamma k_3 (2\iota_0 + |Y_{T_0}|)^\gamma |\hat{\mu} - \mu|^\gamma \\ &\quad + x^\gamma k_4 (2\iota_0 + |Y_{T_0}|)^\gamma |\hat{\alpha} - \alpha|^\gamma. \end{aligned} \quad (8.21)$$

Moreover,

$$\begin{aligned} \mathbf{E}_{T_0} \left(\int_{T_0}^T |\hat{c}_t^* \hat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) &\leq x^\gamma k_5 (2\iota_0 + |Y_{T_0}|)^\gamma |\hat{\mu} - \mu|^\gamma \\ &\quad + x^\gamma k_6 (2\iota_0 + |Y_{T_0}|)^\gamma |\hat{\alpha} - \alpha|^\gamma. \end{aligned} \quad (8.22)$$

Proof. We follow the arguments in Lemma 8.3 we set $\Delta_t = \hat{X}_t^* - X_t^*$, $g(t) = \mathbf{E}_{T_0}(\Delta_t^2)$ we get

$$g(t) \leq c_0 \int_{T_0}^t g(s) ds + \psi(t),$$

where

$$\psi(t) = 4 \mathbf{E}_{T_0} \int_{T_0}^t \left(|\hat{A}_s - A_s|^2 + |\hat{B}_s - B_s|^2 \right) |\hat{X}_s^*|^2 ds.$$

From the Gronwall-Bellman inequality

$$\begin{aligned}
g(t) &\leq \psi(t)e^{c_0 t} \\
&\leq x^2 \tilde{c} \int_{T_0}^t \mathbf{E}_{T_0} \left(|\widehat{A}_s - A_s|^2 + |\widehat{B}_s - B_s|^2 \right) ds \\
&\leq x^2 \tilde{c} \int_{T_0}^t \frac{2(2\mu_2 + r)^2 + \sigma_1^2}{\sigma_1^4(1 - \gamma)^2} (\widehat{\mu} - \mu)^2 \\
&\quad + 2x^2 \tilde{c} \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s)^{-q_*} - h(s, Y_s)^{-q_*}|^2 ds \\
&\leq 2x^2 \tilde{c} \widetilde{T} \left(\frac{2\mu_2 + r + \sigma_1}{\sigma_1^2(1 - \gamma)} \right)^2 (\widehat{\mu} - \mu)^2 \\
&\quad + 2x^2 \tilde{c} q_* \int_{T_0}^t \mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)|^2 ds.
\end{aligned}$$

We use then proposition A.8 to get the analogous of (8.16):

$$\mathbf{E}_{T_0} |\widehat{h}(s, Y_s) - h(s, Y_s)| \leq e^{z(T-s)} (2\iota_0 + |Y_{T_0}|) \Theta_{\widehat{\alpha}, \widehat{\mu}}, \quad (8.23)$$

where

$$\Theta_{\widehat{\alpha}, \widehat{\mu}} = \mathbf{h}_2 |\widehat{\mu} - \mu| + \mathbf{h}_1 |\widehat{\alpha} - \alpha|.$$

Then

$$\begin{aligned}
g(t) &\leq 2x^2 \tilde{c} \widetilde{T} \left(\frac{2\mu_2 + r + \sigma_1}{\sigma_1^2(1 - \gamma)} \right)^2 (\widehat{\mu} - \mu)^2 \\
&\quad + 2x^2 \tilde{c} q_* \frac{e^{2z\widetilde{T}}}{z^2} (2\iota_0 + |Y_{T_0}|)^2 (\Theta_{\widehat{\alpha}, \widehat{\mu}})^2 \\
&\leq x^2 (k'_1 |\widehat{\mu} - \mu| + k'_2 (2\iota_0 + |Y_{T_0}|) \Theta_{\widehat{\alpha}, \widehat{\mu}})^2.
\end{aligned}$$

The concavity of z^γ , for $0 < \gamma < 1$ and the Chebyshev's inequality let have the result.

Now, we show 8.22. We follow the same arguments used in Lemma 8.3 to arrive at

$$\begin{aligned}
\mathbf{E} \left(\int_{T_0}^T |\widehat{c}_t^* \widehat{X}_t^* - c_t^* X_t^*|^\gamma dt \right) &\leq x^\gamma q_* \widetilde{d}^\gamma \int_{T_0}^T \mathbf{E} |\widehat{h}(s, Y_s) - h(s, Y_s)|^\gamma ds \\
&\quad + \widetilde{T} \sup_{T_0 \leq t \leq T} \mathbf{E} |\widehat{X}_t^* - X_t^*|^\gamma.
\end{aligned}$$

Then, we use (8.21) and (8.23) to conclude. \square

9 Simulation

In this section we use Scilab for simulations. In Fig 1. we simulate the truncated sequential estimate $\hat{\alpha}$ for different values of T_0 , through 30 paths of the driving process Y . The sequential estimates are represented by \times for $T_0 = 5$ and $*$ for $T_0 = 10$. The true drift value of the process Y is $\alpha = -5$. We take the bounds $\alpha \in [-0.15, -10]$ and set $\beta = 1$.

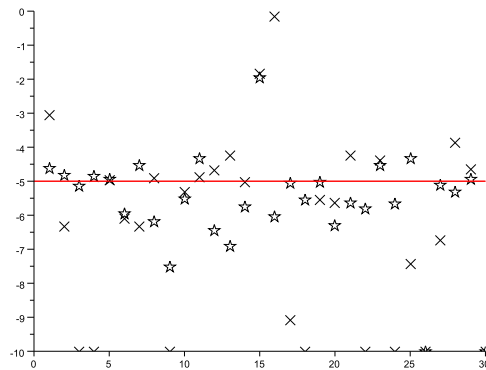


Fig 1: The truncated sequential estimate for $T_0 = 5$, $T_0 = 10$

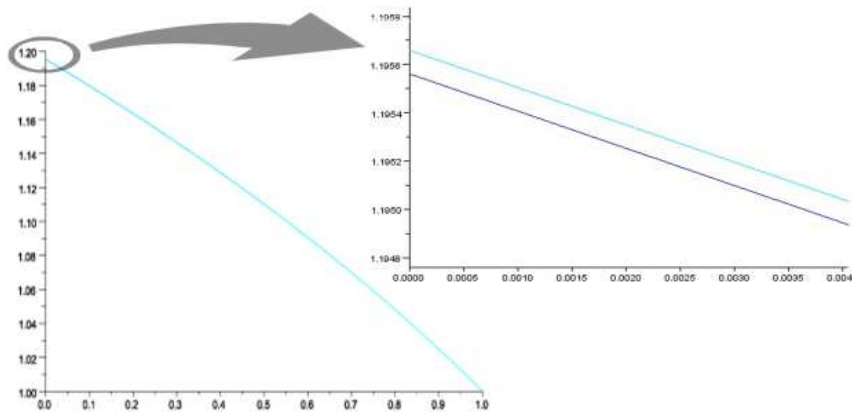


Fig 2: The limit functions $h(t, 0)$ and $\hat{h}(t, 0)$

In Fig 2. we simulate the limit functions $h(t, y)$ and $\hat{h}(t, y)$, under the following market settings: we set $T_0 = 5$ and $\tilde{T} = T - T_0 = 1$, $r = 0.01$, $\mu = 0.02$. The volatility is defined by $\sigma(y) = 0.5 + \sin^2(y)$.

The utility parameter is $\gamma = 0.75$. To simulate $\widehat{h}(t, y)$, we use a very pessimistic realization of the truncated estimate ie; $\widehat{\alpha} = -0.5$. The true value is $\alpha = -5$. We see that, even in this extreme situation, the estimated function $\widehat{h}(t, y)$ does not deviate significantly from the real value $h(t, y)$.

10 Appendix

A.1 Bounds for f and \mathcal{H}_f

Let f the fixed point solution for $f = \mathcal{L}_f$ and recall the definition

$$\mathcal{H}_f(t, s, y) = \mathbf{E} \left(f(s, \eta_s^{t,y}) \right)^{1-q_*} \mathcal{G}(t, s, y).$$

where $\mathcal{G}(t, s, y) = \exp \left(\int_t^s Q(\eta_u^{t,y}) du \right)$

Lemma A.1. For any (t, s) such that $T_0 < t \leq s \leq T$

$$\sup_{y \in \mathbb{R}} \sup_{f \in \mathcal{X}} \left| \frac{\partial}{\partial y} \mathcal{H}_f(t, s, y) \right| \leq Q_1^* \tilde{T} e^{Q_* \tilde{T}} + \frac{e^{Q_* \tilde{T}}}{\nu_s} \quad (\text{A.1})$$

where $\nu_s^2 = \beta^2(1 - e^{2\alpha(s-t)})/2|\alpha|$.

Proof. To calculate this conditional expectation note, first that

$$\begin{aligned} \eta_s &= ye^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} d\tilde{\mathbf{U}}_v \\ &= ye^{\alpha(s-t)} + \xi_s. \end{aligned}$$

Since η it is a *gaussian process*, for any $t < v_1 < \dots < v_k < s$ and for any bounded $\mathbb{R}^k \rightarrow \mathbb{R}$ function G

$$\mathbf{E} \left(G(\eta_{v_1}, \dots, \eta_{v_k}) | \eta_s = z \right) = \mathbf{E} G(\mathbf{B}_{v_1}, \dots, \mathbf{B}_{v_k}), \quad (\text{A.2})$$

where \mathbf{B}_v is the Gaussian process defined by $\mathbf{B}_v = \eta_v - k(v) \eta_s + k(v) z$ and $k(v)$ is chosen so that

$$\mathbf{E} (\xi_v - k(v) \xi_s) \xi_s = 0$$

ie:

$$k(v) = \frac{\mathbf{E} \xi_v \xi_s}{\mathbf{E} \xi_s^2} = e^{\alpha(s-v)} \frac{1 - e^{2\alpha(v-t)}}{1 - e^{2\alpha(s-t)}} \leq 1$$

The conditional expectation with respect to η_s lets represent \mathcal{H}_f as

$$\mathcal{H}_f(t, s, y) = \int_{\mathbb{R}} \widehat{\mathcal{H}}_f(s, y, z) \mathbf{p}(z, y) dz, \quad (\text{A.3})$$

where

$$\mathbf{p}(z, y) = \frac{1}{\nu_s \sqrt{2\pi}} \exp\left(-\frac{(z - \mu(y))^2}{2\nu_s^2}\right).$$

Here $\mu(y) = \mathbf{E} \eta_s = y e^{\alpha(s-t)}$, $\nu_s^2 = \text{Var} \eta_s$. So since $\mathbf{B}_s = z$

$$\begin{aligned} \widehat{\mathcal{H}}_f(s, y, z) &= \mathbf{E} \left((f(s, \eta_s^{t,y}))^{1-q_*} \exp\left(\int_t^s Q(\eta_u^{t,y}) du\right) \middle| \eta_s = z \right) \\ &= \mathbf{E} \left((f(s, z))^{1-q_*} \exp\left(\int_t^s Q(\mathbf{B}_u) du\right) \right) \\ &\leq e^{Q_*(s-t)}. \end{aligned} \quad (\text{A.4})$$

From there we deduce

$$\begin{aligned} \left| \frac{\partial}{\partial y} \widehat{\mathcal{H}}_f(s, y, z) \right| &\leq \left| \int_t^s \frac{\partial Q(\mathbf{B}_u)}{\partial y} du \right| \widehat{\mathcal{H}}_f(s, y, z) \\ &\leq Q_1^*(s-t) e^{Q_*(s-t)} \leq Q_1^* \tilde{T} e^{Q_* \tilde{T}}. \end{aligned} \quad (\text{A.5})$$

Now from (A.3) we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_f(t, s, y)}{\partial y} &= \int_{\mathbb{R}} \frac{\partial \widehat{\mathcal{H}}_f(s, y, z)}{\partial y} \mathbf{p}(z, y) dz \\ &\quad + \int_{\mathbb{R}} \widehat{\mathcal{H}}_f(s, y, z) \frac{(z - \mu(y)) \mu'(y)}{\nu_s^2} \mathbf{p}(z, y) dz. \end{aligned}$$

then

$$\begin{aligned} \left| \frac{\partial \mathcal{H}_f(t, s, y)}{\partial y} \right| &\leq Q_1^*(s-t) e^{Q_*(s-t)} + e^{Q_*(s-t)} \frac{\mu'(y)}{\nu_s^2} \int_{\mathbb{R}} |z - \mu(y)| \mathbf{p}(z, y) dz \\ &\leq Q_1^*(s-t) e^{Q_*(s-t)} + \frac{e^{(Q_* + \alpha)(s-t)}}{\nu_s^2} \frac{2\nu_s}{\sqrt{2\pi}} \\ &\leq Q_1^* \tilde{T} e^{Q_* \tilde{T}} + \frac{e^{Q_* \tilde{T}}}{\nu_s}. \end{aligned}$$

□

Lemma A.2. For any $y \in \mathbb{R}$, the unique solution of the fixed point equation $f = \mathcal{L}_f$ is differentiable with respect to y , and its partial derivative is bounded:

$$\sup_{T_0 \leq t \leq T, y \in \mathbb{R}} \left| \frac{\partial}{\partial y} f(t, y) \right| \leq h_1^*,$$

where

$$h_1^* = \left(\tilde{T} Q_1^* + \frac{Q_1^* \tilde{T}^2}{q_*} \right) e^{Q_* \tilde{T}} + \frac{3}{q_*} \sqrt{\frac{2|\alpha_2|}{\beta^2(1 - e^{2\alpha_2})}} e^{Q_* \tilde{T}} \tilde{T}.$$

Proof. It is obviously sufficient to show that $\mathcal{L}_f(t, y)$ is differentiable with respect to y , and its partial derivative is bounded:

$$\sup_{T_0 \leq t \leq T, y \in \mathbb{R}} \left| \frac{\partial}{\partial y} \mathcal{L}_f(t, y) \right| \leq h_1^*.$$

From the definition of \mathcal{L}_f in (4.6), for all $f \in \mathcal{X}$ and for all $t \in [T_0, T]$ and $y \in \mathbb{R}$ we get

$$\frac{\partial}{\partial y} \mathcal{L}_f(t, y) = \mathbf{E} \frac{\partial}{\partial y} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \frac{\partial}{\partial y} \mathcal{H}_f(t, s, y) ds.$$

So that, using lemmas A.1 and A.4 we get

$$\begin{aligned} \sup_{T_0 \leq t \leq T, y \in \mathbb{R}} \left| \frac{\partial}{\partial y} \mathcal{L}_f(t, y) \right| &\leq \tilde{T} Q_1^* e^{Q_* \tilde{T}} + \frac{1}{q_*} \int_t^T Q_1^* \tilde{T} e^{Q_* \tilde{T}} ds \\ &+ \frac{1}{q_*} \int_t^T \frac{e^{Q_* \tilde{T}}}{\nu_s} ds \\ &\leq \tilde{T} Q_1^* e^{Q_* \tilde{T}} + \frac{Q_1^* \tilde{T}^2}{q_*} e^{Q_* \tilde{T}} \\ &+ \frac{e^{Q_* \tilde{T}}}{q_*} \int_t^T \frac{1}{\nu_s} ds. \end{aligned}$$

To estimate $\int_t^T (1/\nu_s) ds$ we observe that $2|\alpha|(s-t) \leq 2|\alpha|\tilde{T}$ so

$$\nu_s^2 = \beta^2 \frac{(1 - e^{2\alpha(s-t)})}{2|\alpha|(s-t)} (s-t) \geq \beta^2 \frac{(1 - e^{2\alpha})}{2|\alpha|} (s-t) \quad \text{if } (s-t) \leq 1$$

and

$$\nu_s^2 = \beta^2 \frac{(1 - e^{2\alpha(s-t)})}{2|\alpha|} \geq \beta^2 \frac{(1 - e^{2\alpha})}{2|\alpha|} \quad \text{if } (s-t) \geq 1$$

and then

$$\begin{aligned} \int_t^T \frac{1}{\nu_s} ds &\leq \sqrt{\frac{2|\alpha|}{\beta^2(1 - e^{2\alpha})}} \int_t^{t+1} \frac{1}{\sqrt{s-t}} ds + \sqrt{\frac{2|\alpha|}{\beta^2(1 - e^{2\alpha})}} \int_{t+1}^T ds \\ &\leq 2\sqrt{\frac{2|\alpha|}{\beta^2(1 - e^{2\alpha})}} + \sqrt{\frac{2|\alpha|}{\beta^2(1 - e^{2\alpha})}} \tilde{T} \\ &\leq 3\sqrt{\frac{2|\alpha|}{\beta^2(1 - e^{2\alpha})}} \tilde{T}. \end{aligned}$$

We recall that $\alpha_2 \leq \alpha \leq \alpha_1 < 0$ which gives the desired result. \square

A.2 Properties of the function \mathcal{G}

Now we study the partial derivatives of the function $\mathcal{G}(t, s, y)$ defined in (4.6). To this end we need the following general result.

Lemma A.3. *Let $F = F(y, \omega)$ be a $\mathbb{R} \times \Omega \rightarrow \mathbb{R}$ random bounded function such that for some nonrandom constant c^**

$$\left| \frac{d}{dy} F(y, \omega) \right| \leq c^* \quad \text{a.s. .}$$

Then

$$\frac{d}{dy} \mathbf{E} F(y, \omega) = \mathbf{E} \frac{d}{dy} F(y, \omega).$$

This Lemma follows immediately from the Lebesgue dominated convergence theorem.

Lemma A.4. *The partial derivatives $(\partial \mathcal{G}(t, s, y) / \partial y)$ exists and*

$$\sup_{y \in \mathbb{R}} \left| \frac{\partial \mathcal{G}(t, s, y)}{\partial y} \right| \leq (s-t) Q_1^* e^{Q_*(s-t)}, \quad (\text{A.6})$$

where

$$\frac{\partial}{\partial y} \mathbf{E} \mathcal{G}(t, s, y) = \mathbf{E} \frac{\partial}{\partial y} \mathcal{G}(t, s, y).$$

Proof. We have immediately

$$\frac{\partial \mathcal{G}(t, s, y)}{\partial y} = \mathcal{G}(t, s, y) \mathbf{G}(t, s, y),$$

where $\mathbf{G}(t, s, y) = \int_t^s Q_0(\eta_u^{t,y}) (\partial \eta_u^{t,y} / \partial y) du$ and $Q_0(z) = \mathbf{D}_z Q(z)$. Now Lemma A.3 imply directly this lemma. \square

A.3 Properties of the process η

We recall that to the process $(\eta_s)_{0 \leq s \leq T}$ is defined in (4.5) and $(\hat{\eta}_s)_{0 \leq s \leq T}$ defined in (8.8), and let $\bar{\eta}_t = \hat{\eta}_t - \eta_t$.

Lemma A.5. *For any $T_0 \leq t \leq s \leq T$, we have the following estimate*

$$\mathbf{E}_{T_0} |\hat{\eta}_s^{t,y}| \leq \tilde{m}(y) \quad \text{where} \quad \tilde{m}(y) = \iota_0 + |y| = \frac{\beta}{\sqrt{2|\alpha_1|}} + |y|, \quad (\text{A.7})$$

and

$$\mathbf{E}_{T_0} \left| \int_t^T \bar{\eta}_s^{t,0} ds \right| \leq \mathbf{E}_{T_0} \int_t^T |\bar{\eta}_t^{t,0}| dt \leq \frac{\tilde{T} \tilde{m}(y)}{|\alpha_1|} |\hat{\alpha} - \alpha|. \quad (\text{A.8})$$

We have also for known μ and unknown α

$$\mathbf{E}_{T_0} |\hat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \leq \tilde{T} Q_1^* e^{Q_*(T-t)} \frac{\tilde{m}(y)}{|\alpha_1|} |\hat{\alpha} - \alpha|. \quad (\text{A.9})$$

We recall that Q^* and Q_1^* are defined in (3.7), and $\hat{\mathcal{G}}(t, s, y)$ is given in (8.7).

Proof. Since $\eta_s = \eta_t e^{\alpha(s-t)} + \int_t^s \beta e^{\alpha(s-v)} d\tilde{\mathbf{U}}_v$ we have for any fixed α such that $\alpha_2 \leq \alpha \leq \alpha_1 < 0$

$$\begin{aligned} \mathbf{E}((\eta_s^{t,y})^2) &= y^2 e^{2\alpha(s-t)} + \beta^2 \int_t^s e^{2\alpha(t-v)} dv \leq y^2 + \frac{\beta^2}{2|\alpha_1|} \\ &\leq \left(|y| + \frac{\beta}{\sqrt{2|\alpha_1|}} \right)^2, \end{aligned}$$

which gives (A.7). Moreover we have

$$\begin{aligned} d(\hat{\eta}_s^{t,y} - \eta_s^{t,y}) &= (\hat{\alpha} \hat{\eta}_s^{t,y} - \alpha \eta_s^{t,y}) ds + 0 \\ &= \alpha (\hat{\eta}_s^{t,y} - \eta_s^{t,y}) ds + (\hat{\alpha} - \alpha) \hat{\eta}_s^{t,y} ds. \end{aligned}$$

The explicit solution $\bar{\eta}_s^{t,0}$ is given by $\bar{\eta}_s^{t,0} = \int_t^s \bar{\alpha} e^{\alpha(s-u)} \hat{\eta}_u^{t,y} du$, so

$$|\bar{\eta}_s^{t,0}| \leq |\bar{\alpha}| \int_t^s |\hat{\eta}_u^{t,y}| e^{\alpha(s-u)} du.$$

Since $\hat{\alpha}$ is independent of the Brownian motion $(\tilde{\mathbf{U}}_t)$, we get

$$\begin{aligned} \mathbf{E}_{T_0} |\hat{\eta}_s^{t,y} - \eta_s^{t,y}| &\leq |\bar{\alpha}| \mathbf{E}_{T_0} \int_t^s |\hat{\eta}_u^{t,y}| e^{\alpha(s-u)} du \\ &\leq |\bar{\alpha}| \int_t^s e^{\alpha(s-u)} \mathbf{E}_{T_0} |\hat{\eta}_u^{t,y}| du \\ &\leq \frac{\tilde{m}(y)}{|\alpha_1|} |\bar{\alpha}|. \end{aligned} \quad (\text{A.10})$$

Moreover for all $T_0 \leq t \leq T$

$$\begin{aligned} \mathbf{E}_{T_0} \int_t^T |\bar{\eta}_s^{t,0}| ds &\leq \mathbf{E}_{T_0} \left(\int_t^T |\bar{\alpha}| \int_t^s e^{\alpha(s-u)} |\hat{\eta}_u^{t,y}| du ds \right) \\ &\leq \tilde{T} |\bar{\alpha}| \int_t^T e^{\alpha(s-u)} \mathbf{E}_{T_0} |\hat{\eta}_u^{t,y}| du \\ &\leq \frac{\tilde{m}(y) \tilde{T}}{|\alpha_1|} |\bar{\alpha}|, \end{aligned}$$

which gives (A.8). To get inequality (A.9) we see that

$$\begin{aligned} |\hat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| &= \left| \exp \left(\int_t^s Q(\hat{\eta}_u^{t,y}) du \right) - \exp \left(\int_t^s Q(\eta_u^{t,y}) du \right) \right| \\ &\leq \sup_{0 \leq z \leq Q_*(T-t)} e^z \left| \int_t^s Q(\hat{\eta}_u^{t,y}) du - \int_t^s Q(\eta_u^{t,y}) du \right| \\ &\leq e^{Q_*(T-t)} \int_t^s \sup_{y \in \mathbb{R}} \left| \frac{\partial Q(y)}{\partial y} \right| |\hat{\eta}_u^{t,y} - \eta_u^{t,y}| du \\ &\leq Q_1^* e^{Q_*(T-t)} \int_t^T |\hat{\eta}_u^{t,y} - \eta_u^{t,y}| du. \end{aligned}$$

Then

$$\mathbf{E}_{T_0} |\hat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| \leq Q_1^* e^{Q_*(T-t)} \int_t^T \mathbf{E}_{T_0} |\hat{\eta}_u^{t,y} - \eta_u^{t,y}| du.$$

Inequality (A.10) lets conclude. \square

We study in the next proposition the behavior of $h(t, y)$, the solution of the fixed point problem $h = \mathcal{L}_h$, when using the estimate $\widehat{\alpha}$ of the parameter α . We look for a bound for the deviation $|\widehat{h}(t, y) - h(t, y)|$ where $\widehat{h} = \widehat{\mathcal{L}}_{\widehat{h}}$. The operator $\widehat{\mathcal{L}}$ is defined in (8.7). Similarly to (4.3) we define on \mathcal{X} the metric $\widetilde{\varrho}_*$ as follows:

$$\widetilde{\varrho}_*(f, g) = \sup_{(t, y) \in \mathcal{K}} e^{-\varkappa(T-t)} \frac{|f(t, y) - g(t, y)|}{\iota_0 + |y|}, \quad (\text{A.11})$$

where we set $\iota_0 = \beta/\sqrt{2\alpha_1}$ and $\varkappa = Q_* + \zeta + 1$ and set $\zeta = \zeta_0 + 2\gamma$ for some $\zeta_0 > 0$.

Proposition A.6. *For known μ and unknown α , and for any deterministic time $T_0 \in (0, T)$, we have*

$$\widetilde{\varrho}_*(\widehat{h}, h) \leq \mathbf{h}_1 |\widehat{\alpha} - \alpha|. \quad (\text{A.12})$$

Here $\varkappa = Q_* + 1 + 2\gamma + \zeta_0$, $\zeta_0 > 0$ and

$$\mathbf{h}_1 = \frac{1 + 2\gamma + \zeta_0}{1 + \zeta_0} \left(2Q_1^* \widetilde{T} + \gamma h_1^* \right) \frac{\widetilde{T}}{|\alpha_1|}. \quad (\text{A.13})$$

h_1^* is the bound of the derivative $\partial h(t, y)/\partial y$ given in Lemma A.2.

Proof. We use the definition of the operator \mathcal{L} in (4.6):

$$h(t, y) = \mathcal{L}_h(t, y) = \mathbf{E} \mathcal{G}(t, T, y) + \frac{1}{q_*} \int_t^T \mathcal{H}_h(t, s, y) ds,$$

and set $\widehat{h}(t, y) = \widehat{\mathcal{L}}_{\widehat{h}}(t, y)$. We can write

$$|\overline{h}(t, y)| := |\widehat{h}(t, y) - h(t, y)| \leq \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)| + I(\widehat{\alpha}),$$

where (from the definition of $\mathcal{H}_f(t, s, y)$ in (4.7)):

$$\begin{aligned} I(\widehat{\alpha}) &:= \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left| \left(\widehat{h}(s, \widehat{\eta}_s^{t, y}) \right)^{1-q_*} \widehat{\mathcal{G}}(t, s, y) - \left(h(s, \eta_s^{t, y}) \right)^{1-q_*} \mathcal{G}(t, s, y) \right| ds \\ &\leq \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left(h(s, \eta_s^{t, y}) \right)^{1-q_*} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| ds \\ &\quad + \frac{1}{q_*} \int_t^T \mathbf{E}_{T_0} \left| \left(\widehat{h}(s, \widehat{\eta}_s^{t, y}) \right)^{1-q_*} - \left(h(s, \eta_s^{t, y}) \right)^{1-q_*} \right| e^{Q_*(s-t)} ds \\ &\leq \int_t^T \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| ds \\ &\quad + \frac{|1 - q_*|}{q_*} \int_t^T \mathbf{E}_{T_0} |\widehat{h}(s, \widehat{\eta}_s^{t, y}) - h(s, \eta_s^{t, y})| e^{Q_*(s-t)} ds. \end{aligned}$$

We use the fact that $q_* = 1/(1 - \gamma) > 1$ and the bounds (A.9) and (A.10) to deduce

$$\begin{aligned} |\bar{h}(t, y)| &\leq (1 + \tilde{T}) \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)| \\ &\quad + \gamma \int_t^T \mathbf{E}_{T_0} |h(s, \widehat{\eta}_s^{t,y}) - h(s, \eta_s^{t,y})| e^{Q_*(s-t)} ds \\ &\quad + \gamma \int_t^T \mathbf{E}_{T_0} |\widehat{h}(s, \widehat{\eta}_s^{t,y}) - h(s, \widehat{\eta}_s^{t,y})| e^{Q_*(s-t)} ds. \end{aligned}$$

We use the bound h_1^* of the partial derivative of $h(t, y)$ to get $|h(s, \widehat{\eta}_s^{t,y}) - h(s, \eta_s^{t,y})| \leq h_1^* |\widehat{\eta}_s^{t,y} - \eta_s^{t,y}|$ the definition of the metric $\tilde{\varrho}_*$ in (A.11) and the fact that $\varkappa > Q_*$ where \varkappa is given in (4.4) we get

$$\begin{aligned} \tilde{\varrho}_*(\widehat{h}, h) &\leq \frac{(1 + \tilde{T}) \tilde{T} Q_1^*}{|\alpha_1|} \sup_{(t,y) \in \mathcal{K}} \left(\frac{\tilde{m}(y)}{\iota_0 + |y|} e^{(Q_* - \varkappa)(T-t)} \right) |\widehat{\alpha} - \alpha| \\ &\quad + \gamma \sup_{(t,y) \in \mathcal{K}} \int_t^T h_1^* \mathbf{E}_{T_0} |\widehat{\eta}_s^{t,y} - \eta_s^{t,y}| e^{(Q_* - \varkappa)(T-t)} ds \\ &\quad + \gamma \sup_{(t,y) \in \mathcal{K}} \int_t^T \mathbf{E}_{T_0} \frac{|\widehat{h}(s, \widehat{\eta}_s^{t,y}) - h(s, \widehat{\eta}_s^{t,y})| e^{-\varkappa(T-s)} \iota_0 + |\widehat{\eta}_s^{t,y}|}{\iota_0 + |\widehat{\eta}_s^{t,y}|} e^{(Q_* - \varkappa)(s-t)} ds. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\varrho}_*(\widehat{h}, h) &\leq C_{\tilde{T}} |\widehat{\alpha} - \alpha| + \gamma \tilde{\varrho}_*(\widehat{h}, h) \sup_{(t,y) \in \mathcal{K}} \int_t^T \frac{\iota_0 + \mathbf{E}_{T_0} |\widehat{\eta}_s^{t,y}|}{\iota_0 + |y|} e^{(Q_* - \varkappa)(s-t)} ds \\ &\leq C_{\tilde{T}} |\widehat{\alpha} - \alpha| + \gamma \tilde{\varrho}_*(\widehat{h}, h) \sup_{(t,y) \in \mathcal{K}} \left(\frac{\iota_0 + \tilde{m}(y)}{\iota_0 + |y|} \int_t^T e^{(Q_* - \varkappa)(s-t)} ds \right) \\ &\leq C_{\tilde{T}} |\widehat{\alpha} - \alpha| + \frac{2\gamma}{\varkappa - Q_*} \tilde{\varrho}_*(\widehat{h}, h). \end{aligned}$$

Here $C_{\tilde{T}} = (2Q_1^* \tilde{T} + \gamma h_1^*) \tilde{T} / |\alpha_1|$. Hence we get

$$\tilde{\varrho}_*(\widehat{h}, h) \leq \frac{\varkappa - Q_*}{\varkappa - Q_* - 2\gamma} C_{\tilde{T}} |\widehat{\alpha} - \alpha|$$

Recall the definition of $\varkappa = Q_* + \zeta_0 + 2\gamma + 1$ we obtain (A.12) hence Proposition A.6. \square

We consider unknown both the stock price appreciation rate $\mu \in [\mu_1, \mu_2]$, and the drift α of the economic factor Y . The next lemma gives the analogous of equation (A.9).

Lemma A.7. For μ and α unknown, and for any deterministic time $T_0 \in (0, T)$, we have the following estimate

$$\begin{aligned} \mathbf{E}_{T_0} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| &\leq \gamma \frac{(\mu_2 + r)}{(1 - \gamma)\sigma_1^2} \widetilde{T} e^{Q_*(T-t)} |\widehat{\mu} - \mu| \\ &+ \widetilde{T} Q_1^* e^{Q_*(T-t)} \frac{\widetilde{m}(y)}{|\alpha_1|} |\widehat{\alpha} - \alpha|. \end{aligned} \quad (\text{A.14})$$

Proof. We observe first that for the function Q defined in (3.6)

$$\begin{aligned} \widehat{Q}(z) - Q(z) &= \gamma \left(\frac{\widehat{\theta}(z)^2 - \theta(z)^2}{2(1 - \gamma)} \right) = \frac{\gamma}{2(1 - \gamma)\sigma(z)} \left(\widehat{\theta}(z)^2 - \theta(z)^2 \right) \\ &\leq \frac{\gamma(\mu_2 + r)}{(1 - \gamma)\sigma_1^2} |\widehat{\mu} - \mu|. \end{aligned}$$

We deduce then

$$\begin{aligned} \left| e^{\int_t^s \widehat{Q}(z_u) du} - e^{\int_t^s Q(z_u) du} \right| &\leq \left| \int_t^s (\widehat{Q}(z_u) - Q(z_u)) du \right| \sup_{0 \leq z \leq Q_*(T-t)} e^z \\ &\leq \widetilde{T} e^{Q_*(T-t)} \frac{\gamma(\mu_2 + r)}{(1 - \gamma)\sigma_1^2} |\widehat{\mu} - \mu|. \end{aligned}$$

Hence, for any $T_0 \leq t \leq s \leq T$

$$\begin{aligned} |\widehat{\mathcal{G}}(t, s, y) - \mathcal{G}(t, s, y)| &= \left| \exp \left(\int_t^s \widehat{Q}(\widehat{\eta}_u^{t,y}) du \right) - \exp \left(\int_t^s Q(\eta_u^{t,y}) du \right) \right| \\ &\leq \left| \exp \left(\int_t^s \widehat{Q}(\widehat{\eta}_u^{t,y}) du \right) - \exp \left(\int_t^s Q(\widehat{\eta}_u^{t,y}) du \right) \right| \\ &+ \sup_{0 \leq z \leq Q_*(T-t)} e^z \left| \int_t^s Q(\widehat{\eta}_u^{t,y}) du - \int_t^s Q(\eta_u^{t,y}) du \right| \\ &\leq \widetilde{T} e^{Q_*(T-t)} \frac{\gamma(\mu_2 + r)}{(1 - \gamma)\sigma_1^2} |\widehat{\mu} - \mu| \\ &+ Q_1^* e^{Q_*(T-t)} \int_t^T |\widehat{\eta}_u^{t,y} - \eta_u^{t,y}| du \end{aligned}$$

Lemma A.5 lets conclude. \square

The next proposition is the analogous of Proposition A.6. The difference is that, in the proposition bellow, both μ and α are unknown.

Proposition A.8. For μ and α unknown, and for any deterministic time $T_0 \in (0, T)$, we have

$$\tilde{\varrho}_*(\hat{h}, h) \leq \mathbf{h}_2 |\hat{\mu} - \mu| + \mathbf{h}_1 |\hat{\alpha} - \alpha|, \quad (\text{A.15})$$

with

$$\mathbf{h}_2 = \frac{\gamma(\mu_2 + r) 2\tilde{T}^2}{(1 - \gamma)\sigma_1^2 \iota_0}. \quad (\text{A.16})$$

\mathbf{h}_1 is defined in 8.9, and the metric $\tilde{\varrho}_*$ is given in (A.11).

Proof. We follow the same arguments as in the proof of Proposition A.6 and use Lemma A.7 for the bound of $\mathbf{E}_{T_0} |\hat{\mathcal{G}}(t, T, y) - \mathcal{G}(t, T, y)|$.
□

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