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Realizability algebras III : some examples

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Introduction

The notion of *realizability algebra*, which was introduced in [17, 18], is a tool to study the proof-program correspondence and to build models of set theory.

It is a variant of the well known notion of *combinatory algebra*, with a new instruction `cc`, and a new type for the *environments*.

The *sets of forcing conditions*, in common use in set theory, are (very) particular cases of realizability algebras.

We show here how to extend an arbitrary realizability algebra, by means of a certain set of conditions, so that the axiom DC of dependent choice is realized.

In order to avoid introducing new instructions, we use an idea of A. Miquel [19].

This technique has applications of two kinds :

1. Construction of models of ZF + DC.

When the initial realizability algebra is not trivial (that is if the associated Boolean algebra \mathfrak{B} is $\neq \{0, 1\}$, in other words, if we are not in the case of forcing), then we obtain always a model of ZF + DC *in which \mathbb{R} is not well orderable*.

We show in this way, for instance, the relative consistency over ZF, of the following two theories :

i) ZF + DC + there exists an increasing function $i \mapsto X_i$, from the countable atomless Boolean algebra \mathcal{B} into $\mathcal{P}(\mathbb{R})$ such that :

$X_0 = \{0\}$; $i \neq 0 \Rightarrow X_i$ is not countable ;

$X_{i \wedge j} = X_i \cap X_j$;

$i \wedge j = 0 \Rightarrow X_{i \vee j}$ is equipotent with $X_i \times X_j$;

$X_i \times X_i$ is equipotent with X_i ;

there exists a surjection from X_1 onto \mathbb{R} ;

if there exists a surjection from X_j onto X_i , then $i \leq j$;

more generally, if $A \subset \mathcal{B}$ and if there exists a surjection from $\bigcup_{j \in A} X_j$ onto X_i , then $i \leq j$ for some $j \in A$.

ii) ZF + DC + there exists $X \subset \mathbb{R}$ such that :

X is not countable and there is no surjection from X onto \aleph_1

(and therefore, every well orderable subset of X is countable) ;

$X \times X$ is equipotent with X ;
 there exists a total order on X , every proper initial segment of which is countable ;
 there exists a surjection from $X \times \aleph_1$ onto \mathbb{R} ;
 there exists an injection from \aleph_1 (thus also from $X \times \aleph_1$) into \mathbb{R} .

2. Curry-Howard correspondence.

With this technique of extension of realizability algebras, we can obtain a program from a proof, in $ZF + DC$, of an arithmetical formula F , which is a λ_c -term, that is, a λ -term containing cc , *but no other new instruction*.

This is a notable difference with the method given in [14, 15], where we use the instruction `quote` and which is, on the other hand, simpler and not limited to arithmetical formulas.

It is important to observe that the program we get in this way does not really depend on the given proof of $DC \rightarrow F$ in ZF , but only on the *program P extracted from this proof*, which is a closed λ_c -term. We get it by means of an operation of *compilation* applied to P (look at the remark at the end of the introduction of [17]).

Finally, apart from applications 1 and 2, we may notice theorem 19, which gives an interesting property of *every* realizability model : as soon as the Boolean algebra \mathfrak{B} is not trivial (i.e. if the model is not a forcing model), there exists a non well orderable individual.

1 Generalities

Realizability algebras

It is a first order structure, which is defined in [17]. We recall here briefly the definition and some essential properties :

A *realizability algebra* is made up of three sets : Λ (the set of *terms*), Π (the set of *stacks*), $\Lambda \star \Pi$ (the set of *processes*) with the following operations :

$(\xi, \eta) \mapsto (\xi)\eta$ from Λ^2 into Λ (*application*) ;
 $(\xi, \pi) \mapsto \xi \bullet \pi$ from $\Lambda \times \Pi$ into Π (*push*) ;
 $(\xi, \pi) \mapsto \xi \star \pi$ from $\Lambda \times \Pi$ into $\Lambda \star \Pi$ (*process*) ;
 $\pi \mapsto k_\pi$ from Π into Λ (*continuation*).

There is, in Λ , distinguished elements B, C, E, I, K, W, cc , called *elementary combinators* or *instructions*.

Notation.

The term $(\dots(((\xi)\eta_1)\eta_2)\dots)\eta_n$ will be also written $(\xi)\eta_1\eta_2\dots\eta_n$ or $\xi\eta_1\eta_2\dots\eta_n$.

For instance : $\xi\eta\zeta = (\xi)\eta\zeta = (\xi\eta)\zeta = ((\xi)\eta)\zeta$.

We define a preorder on $\Lambda \star \Pi$, denoted by \succ , which is called *execution* ;

$\xi \star \pi \succ \xi' \star \pi'$ is read as : *the process $\xi \star \pi$ reduces to $\xi' \star \pi'$* .

It is the smallest reflexive and transitive binary relation, such that, for any $\xi, \eta, \zeta \in \Lambda$ and $\pi, \varpi \in \Pi$, we have :

$(\xi)\eta \star \pi \succ \xi \star \eta \bullet \pi$.

$I \star \xi \bullet \pi \succ \xi \star \pi$.

$\mathbf{K} \star \xi \cdot \eta \cdot \pi \succ \xi \star \pi.$
 $\mathbf{E} \star \xi \cdot \eta \cdot \pi \succ (\xi)\eta \star \pi.$
 $\mathbf{W} \star \xi \cdot \eta \cdot \pi \succ \xi \star \eta \cdot \eta \cdot \pi.$
 $\mathbf{C} \star \xi \cdot \eta \cdot \zeta \cdot \pi \succ \xi \star \zeta \cdot \eta \cdot \pi.$
 $\mathbf{B} \star \xi \cdot \eta \cdot \zeta \cdot \pi \succ (\xi)(\eta)\zeta \star \pi.$
 $\mathbf{cc} \star \xi \cdot \pi \succ \xi \star \mathbf{k}_\pi \cdot \pi.$
 $\mathbf{k}_\pi \star \xi \cdot \varpi \succ \xi \star \pi.$

We are also given a subset \perp of $\Lambda \star \Pi$ such that :

$$\xi \star \pi \succ \xi' \star \pi', \xi' \star \pi' \in \perp \Rightarrow \xi \star \pi \in \perp.$$

Given two processes $\xi \star \pi, \xi' \star \pi'$, the notation $\xi \star \pi \succ \xi' \star \pi'$ means :

$$\xi \star \pi \notin \perp \Rightarrow \xi' \star \pi' \notin \perp.$$

Given two terms $\xi, \xi' \in \Lambda$, the notation $\xi \succ \xi'$ means :

$$(\forall \pi \in \Pi)(\xi \star \pi \notin \perp \Rightarrow \xi' \star \pi \notin \perp).$$

Therefore, obviously, $\xi \star \pi \succ \xi' \star \pi' \Rightarrow \xi \star \pi \succ \xi' \star \pi'$.

Finally, we choose a set of terms $\mathbf{QP}_{\mathcal{A}} \subset \Lambda$, containing the elementary combinators :

$\mathbf{B}, \mathbf{C}, \mathbf{E}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{cc}$ and closed by application. They are called the *proof-like terms of the algebra \mathcal{A}* . We write also \mathbf{QP} instead of $\mathbf{QP}_{\mathcal{A}}$ if there is no ambiguity about \mathcal{A} .

The algebra \mathcal{A} is called *coherent* if, for every proof-like term $\theta \in \mathbf{QP}_{\mathcal{A}}$, there exists a stack π such that $\theta \star \pi \notin \perp$.

c-terms and λ -terms

The terms of the language of combinatory algebra, built with variables, elementary combinators and the application (binary operation) will be called *combinatory terms* or *c-terms*, in order to distinguish them from the terms of the algebra \mathcal{A} , which are elements of Λ . Each closed c-term (i.e. without variable) takes a value in the algebra \mathcal{A} , which is a proof-like term of \mathcal{A} .

Given a c-term t and a variable x , we define inductively on t , a new c-term denoted by $\lambda x t$, which does not contain x . To this aim, we apply the first possible case in the following list :

1. $\lambda x t = (\mathbf{K})t$ if t does not contain x .
2. $\lambda x x = \mathbf{I}$.
3. $\lambda x tu = (\mathbf{C}\lambda x(\mathbf{E})t)u$ if u does not contain x .
4. $\lambda x tx = (\mathbf{E})t$ if t does not contain x .
5. $\lambda x tx = (\mathbf{W})\lambda x(\mathbf{E})t$ (if t contains x).
6. $\lambda x(t)(u)v = \lambda x(\mathbf{B})tuv$ (if uv contains x).

In [17], it is shown that this definition is correct. This allows us to translate every λ -term into a c-term. In the following, almost every c-term will be written as a λ -term.

The fundamental property of this translation is given by theorem 1, which is proved in [17] :

Theorem 1. *Let t be a c-term with the only variables x_1, \dots, x_n ; let $\xi_1, \dots, \xi_n \in \Lambda$ and $\pi \in \Pi$. Then $\lambda x_1 \dots \lambda x_n t \star \xi_1 \cdot \dots \cdot \xi_n \cdot \pi \succ t[\xi_1/x_1, \dots, \xi_n/x_n] \star \pi$.*

Realizability models

The language we use in order to formalise set theory is made up of the three binary relation symbols $\notin, \subseteq, \subset$ and of some function symbols. The only logical symbols are $\perp, \rightarrow, \forall$.

Notations. Let A_1, \dots, A_n, A, B be some formulas. Then :

$A \rightarrow \perp$ is written $\neg A$;

$A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow B) \dots)$ is written $A_1, A_2, \dots, A_n \rightarrow B$;

$\neg A_1, \dots, \neg A_n \rightarrow \perp$ is written $A_1 \vee \dots \vee A_n$;

$(A_1, \dots, A_n \rightarrow \perp) \rightarrow \perp$ is written $A_1 \wedge \dots \wedge A_n$;

$\neg \forall x (A_1, \dots, A_n \rightarrow \perp)$ is written $\exists x \{A_1, \dots, A_n\}$;

$a \varepsilon b$ is the formula $a \notin b \rightarrow \perp$; $a \in b$ is $a \notin b \rightarrow \perp$.

In this language, we write the axioms of a theory named ZF_ε , which are given in [18].

The usual set theory ZF is supposed written with the only symbols \notin, \subset .

Then, ZF_ε is a *conservative extension* of ZF.

Let us consider a *coherent* realizability algebra \mathcal{A} , defined in a model \mathcal{M} of ZFL, which is called the *ground model*. The elements of \mathcal{M} will be called *individuals* (in order to avoid the word *set*, as far as possible).

We defined, in [18], a *realizability model*, denoted by $\mathcal{N}_\mathcal{A}$ (or even \mathcal{N} , if there is no ambiguity about the algebra \mathcal{A}).

It has the same domain (the same individuals) as \mathcal{M} and the interpretation of the function symbols is the same as in \mathcal{M} .

Each closed formula F of ZF_ε with parameters in \mathcal{M} , has *two truth values* in \mathcal{N} , which are denoted by $\|F\| \subseteq \Pi$ and $|F| \subseteq \Lambda$. Let us give their definitions :

$|F|$ is defined immediately from $\|F\|$ as follows :

$$\xi \in |F| \Leftrightarrow (\forall \pi \in \|F\|) \xi \star \pi \in \perp.$$

We shall write $\xi \Vdash F$ (read “ ξ realizes F ”) for $\xi \in |F|$.

$\|F\|$ is now defined by recurrence on the length of F :

- F is atomic ;

then F has one of the forms $\top, \perp, a \notin b, a \subseteq b, a \not\subseteq b$ where a, b are parameters in \mathcal{M} .

We set :

$$\|\top\| = \emptyset ; \quad \|\perp\| = \Pi ; \quad \|a \notin b\| = \{\pi \in \Pi ; (a, \pi) \in b\}.$$

$\|a \subseteq b\|, \|a \not\subseteq b\|$ are defined simultaneously by induction on $(\text{rk}(a) \cup \text{rk}(b), \text{rk}(a) \cap \text{rk}(b))$ ($\text{rk}(a)$ being the rank of a in \mathcal{M}).

$$\|a \subseteq b\| = \bigcup_c \{\xi \cdot \pi ; \xi \in \Lambda, \pi \in \Pi, (c, \pi) \in a, \xi \Vdash c \notin b\} ;$$

$$\|a \not\subseteq b\| = \bigcup_c \{\xi \cdot \xi' \cdot \pi ; \xi, \xi' \in \Lambda, \pi \in \Pi, (c, \pi) \in b, \xi \Vdash a \subseteq c, \xi' \Vdash c \subseteq a\}.$$

- $F \equiv A \rightarrow B$; then $\|F\| = \{\xi \cdot \pi ; \xi \Vdash A, \pi \in \|B\|\}$.

- $F \equiv \forall x A$: then $\|F\| = \bigcup_a \|A[a/x]\|$.

Given a set of terms X and a formula F , we shall use the notation $X \rightarrow F$ as an *extended formula* ; its truth value is $\|X \rightarrow F\| = \{\xi \cdot \pi ; \xi \in X, \pi \in \|F\|\}$.

The following theorem, proved in [18], is an essential tool :

Theorem 2 (Adequacy lemma).

Let A_1, \dots, A_n, A be closed formulas of ZF_ε , and suppose that $x_1 : A_1, \dots, x_n : A_n \vdash t : A$.
 If $\xi_1 \Vdash A_1, \dots, \xi_n \Vdash A_n$ then $t[\xi_1/x_1, \dots, \xi_n/x_n] \Vdash A$.
 In particular, if $\vdash t : A$, then $t \Vdash A$.

Let F be a closed formula of ZF_ε , with parameters in \mathcal{M} . We say that $\mathcal{N}_\mathcal{A}$ realizes F or that F is realized in $\mathcal{N}_\mathcal{A}$ (which is written $\mathcal{N}_\mathcal{A} \Vdash F$ or even $\Vdash F$), if there exists a proof-like term θ such that $\theta \Vdash F$.

Two formulas $F[x_1, \dots, x_n]$ and $G[x_1, \dots, x_n]$ of ZF_ε will be called *interchangeable* if the formula $\forall x_1 \dots \forall x_n (F[x_1, \dots, x_n] \leftrightarrow G[x_1, \dots, x_n])$ is realized.

It is, for instance, the case if $\|F[a_1, \dots, a_n]\| = \|G[a_1, \dots, a_n]\|$
 or also if $\|F[a_1, \dots, a_n]\| = \|\neg\neg G[a_1, \dots, a_n]\|$

for every $a_1, \dots, a_n \in \mathcal{M}$.

It is shown in [18] that *all the axioms of ZF_ε are realized in $\mathcal{N}_\mathcal{A}$* , and thus also all the axioms of ZF.

Type-like sets and equality

We set $\mathfrak{J}X = X \times \Pi$ for every individual X of \mathcal{M} ; we define the quantifier $\forall x^{\mathfrak{J}X}$ as follows :

$$\|\forall x^{\mathfrak{J}X} F[x]\| = \bigcup_{a \in X} \|F[a]\|.$$

Of course, we set $\exists x^{\mathfrak{J}X} F[x] \equiv \neg \forall x^{\mathfrak{J}X} \neg F[x]$.

The quantifier $\forall x^{\mathfrak{J}X}$ has the intended meaning, which is that the formulas $\forall x^{\mathfrak{J}X} F[x]$ and $\forall x (x \in \mathfrak{J}X \rightarrow F[x])$ are interchangeable. This is shown by the :

Lemma 3.

$$\begin{aligned} \text{CI} \quad & \Vdash \forall x^{\mathfrak{J}X} F[x] \rightarrow \forall x^{\mathfrak{J}X} \neg\neg F[x] ; \\ \text{cc} \quad & \Vdash \forall x^{\mathfrak{J}X} \neg\neg F[x] \rightarrow \forall x^{\mathfrak{J}X} F[x] ; \\ & \|\forall x^{\mathfrak{J}X} \neg\neg F[x]\| = \|\forall x (\neg F[x] \rightarrow x \notin \mathfrak{J}X)\|. \end{aligned}$$

Immediate.

Q.E.D.

Each *functional* $f : \mathcal{M}^n \rightarrow \mathcal{M}$, defined in \mathcal{M} by a formula of ZF with parameters, gives a function symbol, that we denote also by f , and which has the same interpretation in the realizability model $\mathcal{N}_\mathcal{A}$.

If $f : X_1 \times \dots \times X_n \rightarrow Y$ is a function in \mathcal{M} , its interpretation in $\mathcal{N}_\mathcal{A}$ is also a function $f : \mathfrak{J}X_1 \times \dots \times \mathfrak{J}X_n \rightarrow \mathfrak{J}Y$.

The formula $x = y$ is, by definition, $\forall z (x \notin z \rightarrow y \notin z)$ (*Leibniz equality*).

Si t, u are terms of the language of ZF and F is a formula of ZF_ε , with parameters in \mathcal{M} , we define the formula $t = u \leftrightarrow F$. When it is closed, its truth value is :

$$\|t = u \leftrightarrow F\| = \|\top\| = \emptyset \text{ if } \mathcal{M} \models t \neq u ; \|t = u \leftrightarrow F\| = \|F\| \text{ if } \mathcal{M} \models t = u.$$

The formula $t = u \leftrightarrow \perp$ is written $t \neq u$.

The formula $t_1 = u_1 \leftrightarrow (t_2 = u_2 \leftrightarrow \dots \leftrightarrow (t_n = u_n \leftrightarrow F) \dots)$ is written :

$$t_1 = u_1, t_2 = u_2, \dots, t_n = u_n \leftrightarrow F.$$

The formulas $t = u \rightarrow F$ and $t = u \leftrightarrow F$ are interchangeable, as is shown in the :

Lemma 4.

$\lambda x x I \Vdash \forall x \forall y ((x = y \rightarrow F) \rightarrow (x = y \leftrightarrow F))$;
 $\lambda x \lambda y (\text{cc}) \lambda k (y)(k)x \Vdash \forall x \forall y ((x = y \leftrightarrow F) \rightarrow (x = y \rightarrow F))$.

Immediate.

Q.E.D.

Proposition 5.

Let $t, t_1, \dots, t_n, u, u_1, \dots, u_n$ be terms which are built with variables x_1, \dots, x_k and functional symbols of \mathcal{M} .

If $\mathcal{M} \models \forall x_1 \dots \forall x_k (t_1 = u_1, \dots, t_k = u_k \rightarrow t = u)$, then :

$I \Vdash \forall x_1 \dots \forall x_k (t_1 = u_1, \dots, t_k = u_k \leftrightarrow t = u)$.

If $\mathcal{M} \models (\forall x_1 \in X_1) \dots (\forall x_k \in X_k) (t_1 = u_1, \dots, t_k = u_k \rightarrow t = u)$, then :

$I \Vdash \forall x_1^{\mathfrak{J}X_1} \dots \forall x_k^{\mathfrak{J}X_k} (t_1 = u_1, \dots, t_k = u_k \leftrightarrow t = u)$.

Trivial.

Q.E.D.

The set $2 = \{0, 1\}$ is equipped with the trivial boolean functions, written \wedge, \vee, \neg .

The extension to $\mathcal{N}_{\mathcal{A}}$ of these operations gives a structure of Boolean algebra on $\mathfrak{J}2$.

It will be called the *characteristic Boolean algebra* of the model $\mathcal{N}_{\mathcal{A}}$.

Conservation of well-foundedness

Theorem 6 says that every well founded relation on a set X , in the ground model \mathcal{M} , gives a well founded relation on $\mathfrak{J}X$ in the realizability model \mathcal{N} .

Theorem 6. Let $f : X^2 \rightarrow 2$ be such that $f(x, y) = 1$ is a well founded relation on X , in the ground model \mathcal{M} . Then, for every formula $F[x]$ of ZF_ε with parameters in \mathcal{M} :

$\mathbf{Y} \Vdash \forall x^{\mathfrak{J}X} (\forall y^{\mathfrak{J}X} (f(y, x) = 1 \leftrightarrow F[y]) \rightarrow F[x]) \rightarrow \forall x^{\mathfrak{J}X} F[x]$

with $\mathbf{Y} = AA$ and $A = \lambda a \lambda f (f)(a)af$.

Let us fix $a \in X$ and let $\xi \Vdash \forall x^{\mathfrak{J}X} (\forall y^{\mathfrak{J}X} (f(y, x) = 1 \leftrightarrow F[y]) \rightarrow F[x])$. We show, by induction on a , following the well founded relation $f(x, y) = 1$, that $\mathbf{Y} \star \xi \cdot \pi \in \perp$ for every $\pi \in \Vdash F[a]$.

Thus, suppose that $\pi \in \Vdash F[a]$; since $\mathbf{Y} \star \xi \cdot \pi \succ \xi \star \mathbf{Y}\xi \cdot \pi$, we need to show that $\xi \star \mathbf{Y}\xi \cdot \pi \in \perp$. By hypothesis, we have $\xi \Vdash \forall y^{\mathfrak{J}X} (f(y, a) = 1 \leftrightarrow F[y]) \rightarrow F[a]$; thus, it suffices to show that :

$\mathbf{Y}\xi \Vdash f(b, a) = 1 \leftrightarrow F[b]$ for every $b \in X$. This is clear if $f(b, a) \neq 1$, by definition of \leftrightarrow .

If $f(b, a) = 1$, we must show $\mathbf{Y}\xi \Vdash F[b]$, i.e. $\mathbf{Y} \star \xi \cdot \rho \in \perp$ for every $\rho \in \Vdash F[b]$. But this follows from the induction hypothesis.

Q.E.D.

Integers

Let $\xi, \eta \in \Lambda$ and $n \in \mathbb{N}$; we define $(\xi)^n \eta \in \Lambda$ by setting $(\xi)^0 \eta = \eta$; $(\xi)^{n+1} \eta = (\xi)(\xi)^n \eta$.

For $n \in \mathbb{N}$, we define $\underline{n} = (\sigma)^n \underline{0}$ with $\underline{0} = \text{Kl}$ and $\sigma = (\text{BW})(\text{B})\text{B}$;

\underline{n} is “the integer n ” and σ the “successor” in combinatory logic.

The essential property of σ is : $\sigma \star \nu \cdot \xi \cdot \eta \cdot \pi \succ \nu \star \xi \cdot \xi \eta \cdot \pi$.

We set $\mathbb{N}_{\mathcal{A}} = \{(n, \underline{n} \cdot \pi) ; n \in \mathbb{N}, \pi \in \Pi\}$; it is shown below that $\mathbb{N}_{\mathcal{A}}$ is the set of integers of the realizability model $\mathcal{N}_{\mathcal{A}}$.

We define the quantifier $\forall x^{\text{int}}$ as follows :

$$\|\forall x^{\text{int}} F[x]\| = \{\underline{n} \cdot \pi ; n \in \mathbb{N}, \pi \in \|F[n]\|\}.$$

that is also :

$$\|\forall x^{\text{int}} F[x]\| = \|\forall n^{\mathbb{N}}(\{\underline{n}\} \rightarrow F[n])\|.$$

The formulas $\forall x^{\text{int}} F[x]$ and $\forall x(x \in \mathbb{N}_{\mathcal{A}} \rightarrow F[x])$ are interchangeable, as is shown in the :

Lemma 7.

$$\lambda x \lambda n \lambda y(y)(x)n \Vdash \forall x^{\text{int}} F[x] \rightarrow \forall x^{\text{int}} \neg \neg F[x] ;$$

$$\lambda x \lambda n(\text{cc})(x)n \Vdash \forall x^{\text{int}} \neg \neg F[x] \rightarrow \forall x^{\text{int}} F[x] ;$$

$$\|\forall x^{\text{int}} \neg \neg F[x]\| = \|\forall x(\neg F[x] \rightarrow x \notin \mathbb{N}_{\mathcal{A}})\|.$$

Immediate

Q.E.D.

Lemma 8.

$$i) \ K \Vdash \forall x(x \notin \mathbb{J}\mathbb{N} \rightarrow x \notin \mathbb{N}_{\mathcal{A}}).$$

$$ii) \ \lambda x(x)\underline{0} \Vdash 0 \notin \mathbb{N}_{\mathcal{A}} \rightarrow \perp ; \lambda f \lambda x(f)(\sigma)x \Vdash \forall y^{\mathbb{J}\mathbb{N}}((y+1) \notin \mathbb{N}_{\mathcal{A}} \rightarrow y \notin \mathbb{N}_{\mathcal{A}}).$$

$$iii) \ I \Vdash \forall x^{\text{int}} (\forall y^{\mathbb{J}\mathbb{N}}(F[y] \rightarrow F[y+1]), F[0] \rightarrow F[x]) \text{ for every formula } F[x] \text{ of } ZF_{\varepsilon}.$$

i) and ii) Immediate.

iii) We show that $I \Vdash \forall x^{\text{int}} (\forall y^{\mathbb{J}\mathbb{N}}(F[y] \rightarrow F[y+1]), F[i] \rightarrow F[x+i])$ for every $i \in \mathbb{N}$.

Let $n \in \mathbb{N}$, $\phi \Vdash \forall y^{\mathbb{J}\mathbb{N}}(F[y] \rightarrow F[y+1])$, $\alpha \Vdash F[i]$ et $\pi \in \|F[n+i]\|$. We must show :

$\underline{n} \star \phi \cdot \alpha \cdot \pi \in \perp$, which is done by recurrence on n .

If $n = 0$, we have $\underline{0} \star \phi \cdot \alpha \cdot \pi \succ \alpha \star \pi \in \perp$ since $\pi \in \|F[i]\|$.

Now, we have $\underline{n+1} \star \phi \cdot \alpha \cdot \pi \succ \sigma \star \underline{n} \cdot \phi \cdot \alpha \cdot \pi \succ \underline{n} \star \phi \cdot \phi \alpha \cdot \pi$. But we have $\alpha \Vdash F[i]$ and $\phi \Vdash F[i] \rightarrow F[i+1]$. Thus, we have $\phi \alpha \Vdash F[i+1]$; but we have $\pi \in \|F[n+i+1]\|$.

It follows that $\underline{n} \star \phi \cdot \phi \alpha \cdot \pi \in \perp$ by the recurrence hypothesis.

Q.E.D.

Lemma 8(i) shows that $\mathbb{N}_{\mathcal{A}}$ is a subset of $\mathbb{J}\mathbb{N}$.

But it is clear that $\mathbb{J}\mathbb{N}$ contains 0 and is closed by the function $n \mapsto n+1$.

Now, by lemma 8(ii) and (iii), $\mathbb{N}_{\mathcal{A}}$ is the smallest subset of $\mathbb{J}\mathbb{N}$ which contains 0 and is closed by the function $n \mapsto n+1$. Therefore :

$\mathbb{N}_{\mathcal{A}}$ is the set of integers of the model $\mathcal{N}_{\mathcal{A}}$.

The following lemma will be used in section 3.

Lemma 9 (Storage lemma).

Let $\varsigma, O, \Theta \in \Lambda$ be such that $\varsigma \succ \sigma, O \succ \underline{0}$ and $\Theta = \lambda f \lambda n(n) \lambda g \lambda z(g)(f)z$.

For every $n \in \mathbb{N}, \pi \in \Pi$ and $\xi, \phi, \alpha \in \Lambda$, we have :

$$\Theta \star \phi \cdot (\varsigma)^n O \cdot \xi \cdot \alpha \cdot \pi \succ \xi \star (\phi)^n \alpha \cdot \pi.$$

It suffices to show that $(\varsigma)^n O \star \lambda g \lambda z(g)(\phi)z \cdot \xi \cdot \alpha \cdot \pi \succ \xi \star (\phi)^n \alpha \cdot \pi$, which is done by recurrence on n . For $n = 0$, we have :

$O \star \lambda g \lambda z (g)(\phi)z \cdot \xi \cdot \alpha \cdot \pi \succ \underline{O} \star \lambda g \lambda z (g)(\phi)z \cdot \xi \cdot \alpha \cdot \pi \succ \xi \star \alpha \cdot \pi$.

Now, we have :

$(\varsigma)^{n+1} O \star \lambda g \lambda z (g)(\phi)z \cdot \xi \cdot \alpha \cdot \pi \succ \sigma \star (\varsigma)^n O \cdot \lambda g \lambda z (g)(\phi)z \cdot \xi \cdot \alpha \cdot \pi$
 $\succ (\varsigma)^n O \star \lambda g \lambda z (g)(\phi)z \cdot (\lambda g \lambda z (g)(\phi)z) \xi \cdot \alpha \cdot \pi$
 $\succ (\lambda g \lambda z (g)(\phi)z) \xi \star (\phi)^n \alpha \cdot \pi$ by the recurrence hypothesis
 $\succ \xi \star (\phi)(\phi)^n \alpha \cdot \pi$ which is $\xi \star (\phi)^{n+1} \alpha \cdot \pi$.

Q.E.D.

2 The characteristic Boolean algebra $\mathbb{J}2$

Function symbols

Let us now define the principal function symbols commonly used in the sequel :

- The projections $pr_0 : X \times Y \rightarrow X$ and $pr_1 : X \times Y \rightarrow Y$ defined by :

$$pr_0(x, y) = x, pr_1(x, y) = y$$

give, in $\mathcal{N}_{\mathcal{A}}$ a *bijection* from $\mathbb{J}(X \times Y)$ onto $\mathbb{J}X \times \mathbb{J}Y$.

- The function $app : Y^X \times X \rightarrow Y$ (read *application*) defined in \mathcal{M} by $app(f, x) = f(x)$ gives, in $\mathcal{N}_{\mathcal{A}}$, an *injection* from $\mathbb{J}(Y^X)$ into $(\mathbb{J}Y)^{\mathbb{J}X}$. Indeed, we have :

$$I \Vdash \forall f^{\mathbb{J}(Y^X)} \forall g^{\mathbb{J}(Y^X)} (\forall x^{\mathbb{J}X} (app(f, x) = app(g, x)) \rightarrow f = g).$$

We shall write $f(x)$ for $app(f, x)$.

- Let $sp : \mathcal{M} \rightarrow \{0, 1\}$ (read *support*) the unary function symbol defined by :
 $sp(\emptyset) = 0$; $sp(x) = 1$ if $x \neq \emptyset$.

In the realizability model $\mathcal{N}_{\mathcal{A}}$, we have $sp : \mathcal{N} \rightarrow \mathbb{J}2$.

- Let $P : \{0, 1\} \times \mathcal{M} \rightarrow \mathcal{M}$ (read *projection*) the binary function symbol defined by :
 $P(0, x) = \emptyset$; $P(1, x) = x$.

In the realizability model $\mathcal{N}_{\mathcal{A}}$, we have $P : \mathbb{J}2 \times \mathcal{N} \rightarrow \mathcal{N}$.

We shall write $P_i(x)$ instead of $P(i, x)$.

When t, u are terms with values in $\mathbb{J}2$, we write $t \leq u$ for $t \wedge u = t$.

Proposition 10.

- $I \Vdash \forall i^{\mathbb{J}2} \forall x (P_i(P_j(x)) = P_{i \wedge j}(x))$.
- $I \Vdash \forall i^{\mathbb{J}2} \forall x (P_i(x) = x \Leftrightarrow sp(x) \leq i)$.
- If $\emptyset \in E$, then $I \Vdash \forall x^{\mathbb{J}E} \forall i^{\mathbb{J}2} (P_i(x) \varepsilon \mathbb{J}E)$.
- If $f : \mathcal{M}^n \rightarrow \mathcal{M}$ is a function symbol such that $f(\emptyset, \dots, \emptyset) = \emptyset$, then :
 $I \Vdash \forall i^{\mathbb{J}2} \forall x_1 \dots \forall x_n (P_i(f(x_1, \dots, x_n)) = f(P_i(x_1), \dots, P_i(x_n)))$.

Trivial.

Q.E.D.

Because of property (iv), we shall define, as far as possible, each function symbol f in \mathcal{M} , so that to have $f(\emptyset, \dots, \emptyset) = \emptyset$.

- Thus the ordered pair (x, y) is changed by setting $(\emptyset, \emptyset) = \emptyset$. Then, we have :
 $I \Vdash \forall i^{\mathbb{J}2} \forall x \forall y (P_i((x, y)) = (P_i(x), P_i(y)))$.

- We define the binary function symbol $\sqcup : \mathcal{M}^2 \rightarrow \mathcal{M}$ by setting : $a \sqcup b = a \cup b$.

Remark. The extension to \mathcal{N} of this operation is *not* the union \cup .

The operation \mathfrak{J}_i

Let $E \in \mathcal{M}$ be such that $\emptyset \in E$. In \mathcal{M} , we define $\mathfrak{J}_i E$ for $i \in 2$ by setting :
 $\mathfrak{J}_0 E = \mathfrak{J}\{\emptyset\} = \{\emptyset\} \times \Pi$; $\mathfrak{J}_1 E = \mathfrak{J}E = E \times \Pi$.

In this way, we have now defined $\mathfrak{J}_i E$ in \mathcal{N} , for every $i \in \mathfrak{J}2$.

Proposition 11.

- i) $\vdash \forall i^{\mathfrak{J}2} \forall x \forall y (P_i(x \sqcup y) = P_i(x) \sqcup P_i(y))$.
- ii) $\vdash \forall i^{\mathfrak{J}2} \forall j^{\mathfrak{J}2} \forall x (P_{i \vee j}(x) = P_i(x) \sqcup P_j(x))$.
- iii) $\vdash \forall i^{\mathfrak{J}2} \forall j^{\mathfrak{J}2} \forall x \forall y \forall z (i \wedge j = 0, z = P_i(x) \sqcup P_j(y) \leftrightarrow P_i(z) = P_i(x))$.
 $\vdash \forall i^{\mathfrak{J}2} \forall j^{\mathfrak{J}2} \forall x \forall y \forall z (i \wedge j = 0, z = P_i(x) \sqcup P_j(y) \leftrightarrow P_j(z) = P_j(y))$.
- iv) $\vdash \forall i^{\mathfrak{J}2} \forall j^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} \forall y^{\mathfrak{J}E} \forall z (i \wedge j = 0, z = P_i(x) \sqcup P_j(y) \leftrightarrow z \in \mathfrak{J}_{i \vee j} E)$.

Trivial.

Q.E.D.

Proposition 12.

If $\emptyset \in E, E'$, the following formulas are realized :

- i) $\mathfrak{J}_i E$ increases with i . In particular, $\mathfrak{J}_i E \subseteq \mathfrak{J}E$.
- ii) The ε -elements of $\mathfrak{J}_i E$ are the $P_i(x)$ for $x \in \mathfrak{J}E$.
- iii) The ε -elements of $\mathfrak{J}_i E$ are those of $\mathfrak{J}E$ such that $\text{sp}(x) \leq i$.
- iv) The only ε -element common to $\mathfrak{J}_i E$ and $\mathfrak{J}_{1-i} E$ is \emptyset .
- v) If $i \wedge j = 0$, then the application $x \mapsto (P_i(x), P_j(x))$ is a bijection from $\mathfrak{J}_{i \vee j} E$ onto $\mathfrak{J}_i E \times \mathfrak{J}_j E$. The inverse function is $(x, y) \mapsto x \sqcup y$.
- vi) $\mathfrak{J}_i(E \times E') = \mathfrak{J}_i E \times \mathfrak{J}_i E'$.

We check immediately i), ii), iii), iv) below :

- i) $\vdash \forall i^{\mathfrak{J}2} \forall j^{\mathfrak{J}2} \forall x (i \wedge j = i \leftrightarrow (x \notin \mathfrak{J}_j E \rightarrow x \notin \mathfrak{J}_i E))$.
- ii) $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} (P_i(x) \notin \mathfrak{J}_i E \rightarrow \perp)$; $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} (P_i(x) \neq x \rightarrow x \notin \mathfrak{J}_i E)$.
- iii) $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} (x \notin \mathfrak{J}_i E \rightarrow \text{sp}(x) i \neq \text{sp}(x))$; $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} (\text{sp}(x) i \neq \text{sp}(x) \rightarrow x \notin \mathfrak{J}_i E)$;
- iv) $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} \forall y^{\mathfrak{J}E} (P_i(x) = P_{1-i}(y) \leftrightarrow P_i(x) = \emptyset)$.
- v) By proposition 10(iv), we have $\vdash \forall i^{\mathfrak{J}2} \forall x \forall y (P_i((x, y)) = (P_i(x), P_i(y)))$.
 By proposition 11(iii,iv), if $x, y \in \mathfrak{J}E$, there exists $z \in \mathfrak{J}_{i \vee j} E$ such that :
 $P_i(z) = P_i(x), P_j(z) = P_j(y)$, namely $z = P_i(x) \sqcup P_j(y)$.
- vi) By proposition 11(ii), we have $P_i(x) \sqcup P_j(x) = P_{i \vee j}(x) = x$ if $x \in \mathfrak{J}_{i \vee j} E$.

Q.E.D.

Proposition 13. Let $E, E' \in \mathcal{M}$ be such that $\emptyset \in E, E'$ and E is equipotent with E' .
 Then $\vdash \forall i^{\mathfrak{J}2} (\mathfrak{J}_i E \text{ is equipotent with } \mathfrak{J}_i E')$.

Let ϕ be, in \mathcal{M} , a bijection from E onto E' , such that $\phi(\emptyset) = \emptyset$. Then ϕ is, in \mathcal{N} , a bijection from $\mathfrak{J}E$ onto $\mathfrak{J}E'$. But we have immediately : $\vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}E} (\phi(P_i x) = P_i \phi(x))$.
 This shows that ϕ is a bijection from $\mathfrak{J}_i E$ onto $\mathfrak{J}_i E'$.

Q.E.D.

Some general theorems

Theorems 14 to 22, which are shown in this section, are valid *in every realizability model*. In the ground model \mathcal{M} , which satisfies ZFL, we denote by κ the cardinal of $\Lambda \cup \Pi \cup \mathbb{N}$ (which we shall also call the *cardinal of \mathcal{A}*) and by $\kappa_+ = \mathcal{P}(\kappa)$ the power set of κ .

Theorem 14.

Let $\forall \vec{x} \forall y F[\vec{x}, y]$ be a closed formula of ZF_ε with parameters in \mathcal{M} (where $\vec{x} = (x_1, \dots, x_n)$).

Then, there exists in \mathcal{M} , a functional $f_F : \kappa \times \mathcal{M}^n \rightarrow \mathcal{M}$ such that :

- i) If $\vec{a}, b \in \mathcal{M}$ and $\xi \Vdash F[\vec{a}, b]$, then there exists $\alpha \in \kappa$ such that $\xi \Vdash F[\vec{a}, f_F(\alpha, \vec{a})]$.
- ii) $\mathbf{CI} \Vdash \forall \vec{x} \forall y (F[\vec{x}, y] \rightarrow \exists \nu^{\mathbb{J}\kappa} F[\vec{x}, f_F(\nu, \vec{x})])$.

i) Let $\xi \mapsto \alpha_\xi$ be an injection from Λ into κ . Using the principle of choice in \mathcal{M} (which satisfies $V = L$), we can define a functional $f_F : \kappa \times \mathcal{M}^n \rightarrow \mathcal{M}$ such that, in \mathcal{M} , we have : $\forall \vec{x} \forall y (\forall \xi \in \Lambda) (\xi \Vdash F[\vec{x}, y] \Rightarrow \xi \Vdash F[\vec{x}, f_F(\alpha_\xi, \vec{x})])$.

ii) Let $\xi \Vdash F[\vec{a}, b]$, $\eta \Vdash \forall \nu^{\mathbb{J}\kappa} \neg F[\vec{a}, f_F(\nu, \vec{a})]$ and $\pi \in \Pi$.

Thus, we have $\eta \Vdash \neg F[\vec{a}, f_F(\alpha_\xi, \vec{a})]$; by definition of f_F , we have $\xi \Vdash F[\vec{a}, f_F(\alpha_\xi, \vec{a})]$. Therefore $\eta \star \xi \cdot \pi \in \perp$, and $\mathbf{CI} \star \xi \cdot \eta \cdot \pi \in \perp$.

Q.E.D.

Subsets of $\mathbb{J}\kappa_+$

Theorem 15. Let $\forall x \forall y \forall z F[x, y, z]$ be a closed formula of ZF_ε , with parameters in \mathcal{M} .

Then, there exists, in \mathcal{M} , a functional $\beta_F : \mathcal{M} \rightarrow \kappa_+$ such that :

$$\mathbf{W} \Vdash \forall z (\forall x \forall y \forall y' (F[x, y, z], F[x, y', z] \rightarrow y = y') \rightarrow \forall i^{\mathbb{J}2} \forall x (F[x, P_i(\beta_F(z)), z] \rightarrow \text{sp}(x) \geq i)).$$

In particular, for $i = 1$, we have :

$$\mathbf{W} \Vdash \forall z (\forall x \forall y \forall y' (F[x, y, z], F[x, y', z] \rightarrow y = y') \rightarrow \forall x (F[x, \beta_F(z), z] \rightarrow \text{sp}(x) = 1)).$$

By theorem 14(i), there exists, in \mathcal{M} , a functional $g : \kappa \times \mathcal{M}^2 \rightarrow \mathcal{M}$ such that :

(*) For $a, b, c \in \mathcal{M}$ and $\xi \Vdash F[a, b, c]$, there exists $\alpha \in \kappa$ such that $\xi \Vdash F[a, g(\alpha, a, c), c]$.

Using the principle of choice in \mathcal{M} , we define a functional $\beta_F : \mathcal{M} \rightarrow \kappa_+$ such that :

for every $\alpha \in \kappa$ and $c \in \mathcal{M}$, we have $\beta_F(c) \neq g(\alpha, \emptyset, c)$.

This is possible since κ_+ is of cardinal $> \kappa$.

Now let : $a, c \in \mathcal{M}$, $i \in \{0, 1\}$, $\phi \Vdash \forall x \forall y \forall y' (F[x, y, c], F[x, y', c], y \neq y' \rightarrow \perp)$,
 $\xi \Vdash F[a, P_i(\beta_F(c)), c]$, $\eta \Vdash \text{sp}(a)i \neq i$ and $\pi \in \Pi$.

We must show that $\mathbf{W} \star \phi \cdot \xi \cdot \eta \cdot \pi \in \perp$, that is $\phi \star \xi \cdot \xi \cdot \eta \cdot \pi \in \perp$.

We set $b = P_i(\beta_F(c))$ and therefore, we have $\xi \Vdash F[a, b, c]$.

Thus, by (*), we have $\xi \Vdash F[a, g(\alpha, a, c), c]$ for some $\alpha \in \kappa$.

Let us show that $\|b \neq g(\alpha, a, c)\| \subset \|\text{sp}(a)i \neq i\|$; there are three possible cases :

If $i = 0$, then $\|\text{sp}(a)i \neq i\| = \|0 \neq 0\| = \Pi$, hence the result.

If $i = 1$ and $a \neq \emptyset$, then $\|\text{sp}(a)i \neq i\| = \|1 \neq 1\| = \Pi$, hence the result.

If $i = 1$ and $a = \emptyset$, then :

$\|b \neq g(\alpha, a, c)\| = \|P_i(\beta_F(c)) \neq g(\alpha, a, c)\| = \|\beta_F(c) \neq g(\alpha, \emptyset, c)\| = \|\top\| = \emptyset$, by definition of β_F , hence the result.

It follows that $\eta \Vdash b \neq g(\alpha, a, c)$. Now, we have seen that :
 $\xi \Vdash F[a, b, c]$ and $\xi \Vdash F[a, g(\alpha, a, c), c]$.

Therefore, by hypothesis on ϕ , we have $\phi \star \xi \cdot \xi \cdot \eta \cdot \pi \in \perp$.

Q.E.D.

Corollary 16. *The following formulas are realized :*

$\forall E \forall i \exists j \exists j^2$ (if there exists a surjection from $\mathfrak{J}_j E$ onto $\mathfrak{J}_i \kappa_+$ then $j \geq i$).

More generally :

$\forall E \forall i \exists j^2$ (there exists no surjection from $\bigcup \{ \mathfrak{J}_j E ; j \in \mathfrak{J}2, j \not\geq i \}$ onto $\mathfrak{J}_i \kappa_+$).

In particular, with $i = 1$:

$\forall E$ (there exists no surjection from $\bigcup \{ \mathfrak{J}_j E ; j \in \mathfrak{J}2, j \neq 1 \}$ onto $\mathfrak{J}_1 \kappa_+$).

Remark. The notation $\bigcup \{ \mathfrak{J}_j E ; j \in \mathfrak{J}2, j \not\geq i \}$ denotes any individual X of \mathcal{N} such that :

$\forall x (x \in X \leftrightarrow \exists j \exists j^2 (j \not\geq i \wedge x \in \mathfrak{J}_j E))$.

We apply theorem 15, with the formula $F[x, y, z] \equiv (x, y) \varepsilon z$.

In the realizability model \mathcal{N} , we have $\beta_F : \mathcal{N} \rightarrow \mathfrak{J}_1 \kappa_+$.

Let z_0 be, in \mathcal{N} , a surjective function onto $\mathfrak{J}_1 \kappa_+$.

We have $\beta_F(z_0) \varepsilon \kappa_+$, and therefore $P_i(\beta_F(z_0)) \varepsilon \mathfrak{J}_i \kappa_+$.

If x_0 is such that $(x_0, P_i(\beta_F(z_0))) \varepsilon z_0$, then $\text{sp}(x_0) \geq i$ by theorem 15. Therefore, for any individual E , we have $x_0 \varepsilon \mathfrak{J}_j E \Rightarrow j \geq i$.

Q.E.D.

Theorem 17. *The formula : (there exists a surjection from $\mathfrak{J}_1 \kappa_+$ onto $2^{\mathfrak{J}\kappa}$) is realized.*

In the ground model \mathcal{M} , there exists a bijection from $\kappa_+ = 2^\kappa$ onto $2^{\kappa \times \Pi}$. Therefore, in \mathcal{N} , there exists a bijection from $\mathfrak{J}_1 \kappa_+$ onto $\mathfrak{J}2^{\kappa \times \Pi}$.

We now need a surjection from $\mathfrak{J}2^{\kappa \times \Pi}$ onto $2^{\mathfrak{J}\kappa}$.

Let $\phi : \mathcal{M} \rightarrow 2^{\kappa \times \Pi}$ be the unary function symbol defined by $\phi(x) = x \cap (\kappa \times \Pi)$.

In \mathcal{N} , we have $\phi : \mathcal{N} \rightarrow \mathfrak{J}2^{\kappa \times \Pi}$. Now, we check immediately that :

- i) $\Vdash \forall \nu \forall x \exists x^2 (\nu \notin \mathfrak{J}\kappa \rightarrow \nu \notin x)$ (because $\| \nu \notin a \| \subset \| \nu \notin \mathfrak{J}\kappa \|$ for all $a \in \mathcal{P}(\kappa \times \Pi)$).
- ii) $\Vdash \forall x \forall \nu \exists \nu^2 (\nu \notin x \Leftrightarrow \nu \notin \phi(x))$ (because $\| \nu \notin a \| = \| \nu \notin \phi(a) \|$ for all $\nu \in \kappa$).

From (i), it follows that $\mathfrak{J}2^{\kappa \times \Pi}$ is, in \mathcal{N} , a set of subsets of $\mathfrak{J}\kappa$;

from (ii), it follows that it contains at least one representative for each equivalence class of extensionality.

Thus the desired surjection simply associates, with each ε -element of $\mathfrak{J}2^{\kappa \times \Pi}$, its equivalence class of extensionality.

Q.E.D.

Theorem 18. *Let $E \in \mathcal{M}$ be infinite and such that $\emptyset \in E$. Then we have :*

$\Vdash \forall i \exists j^2 (i \neq 0 \rightarrow \text{there exists an injection from } \mathbb{N} \text{ into } \mathfrak{J}_i E)$.

In \mathcal{M} , let $\phi : \mathbb{N} \rightarrow (E \setminus \{\emptyset\})$ be injective. In \mathcal{N} , we have $\phi : \mathfrak{J}\mathbb{N} \rightarrow \mathfrak{J}E$. The desired function is $n \mapsto P(i, \phi(n))$. Indeed, we have :

$\Vdash \forall i \exists j^2 \forall m \exists m^2 \forall n \exists n^2 (i \neq 0 \rightarrow P_i(\phi(m+n+1)) \neq P_i(\phi(m)))$.

This shows that the restriction of this function to $\mathbb{N}_{\mathcal{A}}$ (the set of integers of $\mathcal{N}_{\mathcal{A}}$) is injective.

Q.E.D.

Theorem 19. $\Vdash \forall i^{\mathfrak{J}2} (i \neq 0, i \neq 1 \rightarrow (\mathfrak{J}\kappa_+ \text{ cannot be well ordered}))$.

Let $i \in \mathfrak{J}2, i \neq 0, 1$; then, $\mathfrak{J}_i\kappa_+$ and $\mathfrak{J}_{1-i}\kappa_+$ are infinite (theorem 18) and $\subseteq \mathfrak{J}\kappa_+$ by proposition 12(i). But there exists no surjection from $\mathfrak{J}_i\kappa_+$ onto $\mathfrak{J}_{1-i}\kappa_+$, neither from $\mathfrak{J}_{1-i}\kappa_+$ onto $\mathfrak{J}_i\kappa_+$, by corollary 16.

Q.E.D.

Remark. By theorem 19, if the Boolean algebra $\mathfrak{J}2$ is not trivial, then $\mathfrak{J}\kappa_+$ is not well orderable. On the other hand, it can be shown that, if this Boolean algebra is trivial, then the realizability model \mathcal{N} is an extension by forcing of the ground model \mathcal{M} . In this case, \mathcal{N} itself can be well ordered, since we suppose that the ground model \mathcal{M} satisfies ZFL.

A strict order on $\mathfrak{J}\kappa_+$

A binary relation $<$ on X is a *strict order* if it is transitive ($x < y, y < z \Rightarrow x < z$) and antireflexive ($x \not< x$). This strict order is called *total* if we have : $x < y$ or $y < x$ or $x = y$. If $(X_0, <_0), (X_1, <_1)$ are two strictly ordered sets, then the *strict order product* $<$ on $X_0 \times X_1$ is defined by : $(x_0, x_1) < (y_0, y_1) \Leftrightarrow x_0 <_0 y_0$ and $x_1 <_1 y_1$.

Lemma 20. *If the strict order product of $(X_0, <_0), (X_1, <_1)$ is well founded, then one of the strict orders $<_0, <_1$ is well founded.*

Proof by contradiction : if $E_0 \subseteq X_0, E_1 \subseteq X_1$ are non void and have no minimal ε -element, then $E_0 \times E_1 \subseteq X_0 \times X_1$ has the same properties.

Q.E.D.

We denote by \triangleleft a strict well ordering on κ_+ , in \mathcal{M} ; we suppose that its least element is \emptyset and that every proper initial segment is of cardinal κ .

This gives a binary function from κ_+^2 into $\{0, 1\}$, denoted by $(x \triangleleft y)$, which is defined as follows : $(x \triangleleft y) = 1 \Leftrightarrow x \triangleleft y$.

We can extend it to the realizability model $\mathcal{N}_{\mathcal{A}}$, which gives a function from $(\mathfrak{J}\kappa_+)^2$ into $\mathfrak{J}2$.

Lemma 21. *The following propositions are realized :*

If $i \in \mathfrak{J}2$, then $(x \triangleleft y) = i$ is a strict ordering of $\mathfrak{J}_i\kappa_+$, which we denote by \triangleleft_i .

If i is an atom of the Boolean algebra $\mathfrak{J}2$, then this ordering is total.

We have immediately :

i) $\Vdash \forall x^{\mathfrak{J}\kappa_+} \forall y^{\mathfrak{J}\kappa_+} \forall z^{\mathfrak{J}\kappa_+} ((x \triangleleft y)(y \triangleleft z) \leq (x \triangleleft z))$; $\Vdash \forall x^{\mathfrak{J}\kappa_+} ((x \triangleleft x) = 0)$.

ii) $\Vdash \forall i^{\mathfrak{J}2} \forall x^{\mathfrak{J}\kappa_+} \forall y^{\mathfrak{J}\kappa_+} ((P_i x \triangleleft P_i y) \leq i)$.

iii) $\Vdash \forall x^{\mathfrak{J}\kappa_+} \forall y^{\mathfrak{J}\kappa_+} ((x \triangleleft y) = 0, (y \triangleleft x) = 0 \leftrightarrow x = y)$.

It follows from (i) that, if $i \neq 0$, then $(x \triangleleft y) \geq i$ is a strict ordering relation on $\mathfrak{J}\kappa_+$.

It follows from (ii), that this relation, restricted to $\mathfrak{J}_i\kappa_+$, is equivalent to $(x \triangleleft y) = i$.

Finally, it follows from (iii), that the relation $(x \triangleleft y) = i$, restricted to $\mathfrak{J}_i\kappa_+$, is total when i is an atom of $\mathfrak{J}2$.

Q.E.D.

Lemma 22. *The following propositions are realized :*

i) $\forall i^{\mathfrak{J}2}$ (the application $x \mapsto (P_i x, P_{1-i} x)$ is an isomorphism of strictly ordered sets from $(\mathfrak{J}\kappa_+, \triangleleft)$ onto $(\mathfrak{J}_i\kappa_+, \triangleleft_i) \times (\mathfrak{J}_{1-i}\kappa_+, \triangleleft_{1-i})$).

ii) $\forall i^{\mathfrak{J}2}$ (either $\mathfrak{J}_i\kappa_+$ or $\mathfrak{J}_{1-i}\kappa_+$ is a well founded ordered set).

i) It follows from proposition 12(v), that the application $x \mapsto (P_i x, P_{1-i} x)$ is a bijection from $\mathfrak{J}\kappa_+$ onto $\mathfrak{J}_i \kappa_+ \times \mathfrak{J}_{1-i} \kappa_+$.

In fact, it is an *isomorphism* of ordered sets, since we have :

$$\begin{aligned} & \vdash \Vdash \forall i \mathfrak{J}^2 \forall x \mathfrak{J}^{\kappa_+} \forall y \mathfrak{J}^{\kappa_+} ((x \triangleleft y) = (P_i x \triangleleft P_i y) \vee (P_{1-i} x \triangleleft P_{1-i} y)) \text{ and therefore :} \\ & \vdash \Vdash \forall i \mathfrak{J}^2 \forall x \mathfrak{J}^{\kappa_+} \forall y \mathfrak{J}^{\kappa_+} ((x \triangleleft y) = 1 \leftrightarrow (P_i x \triangleleft P_i y) = i \wedge (P_{1-i} x \triangleleft P_{1-i} y) = 1 - i). \end{aligned}$$

ii) By theorem 6, the relation $(x \triangleleft y) = 1$ is well founded on $\mathfrak{J}\kappa_+$. Thus, the result follows immediately from (i) and lemma 20.

Q.E.D.

$\mathfrak{J}\kappa$ countable

In this section, we consider the consequences of the hypothesis : ($\mathfrak{J}\kappa$ is countable).

Non extensional and dependent choice

The formula $\forall z(z \notin y \rightarrow z \notin x)$ will be written $x \subseteq y$.

The formula $\forall x \forall y \forall y'((x, y) \varepsilon f, (x, y') \varepsilon f \rightarrow y = y')$ will be written $\text{Func}(f)$ (read : *f is a function*).

The formula $\forall z \exists f (f \subseteq z \wedge \text{Func}(f) \wedge \forall x \forall y \exists y'((x, y) \varepsilon z \rightarrow (x, y') \varepsilon f))$ is called the *non extensional axiom of choice* and denoted by NEAC.

It is easily shown [18] that $\text{ZF}_\varepsilon + \text{NEAC} \vdash \text{DC}$ (axiom of dependent choice). On the other hand, we have built, in [18], a model of $\text{ZF}_\varepsilon + \text{NEAC} + \neg \text{AC}$; and other such models will be given in the present paper. In all these models, \mathbb{R} is not well orderable.

Theorem 23.

There exists a closed c-term H such that $H \Vdash (\mathfrak{J}\kappa \text{ is countable}) \rightarrow \text{NEAC}$.

We apply theorem 14(ii) to the formula $(x, y) \varepsilon z$. We get a function symbol g such that $\text{C} \Vdash \forall x \forall y \forall z ((x, y) \varepsilon z \rightarrow \exists \nu \mathfrak{J}^\kappa (x, g(\nu, x, z)) \varepsilon z)$.

Therefore, it suffices to prove NEAC in ZF_ε , by means of this formula and the additional hypothesis : ($\mathfrak{J}\kappa$ is countable). Now, from this hypothesis, it follows that there exists a strict well ordering $<$ on $\mathfrak{J}\kappa$. Then, we can define the desired function f by means of the comprehension scheme :

$$(x, y) \varepsilon f \leftrightarrow (x, y) \varepsilon z \wedge \exists \nu \mathfrak{J}^\kappa (y = g(\nu, x, z) \wedge \forall \alpha \mathfrak{J}^\kappa (\alpha < \nu \rightarrow (x, g(\alpha, x, z)) \notin z)).$$

Intuitively, $f(x) = g(\nu, x, z)$ for the least $\nu \varepsilon \mathfrak{J}\kappa$ such that $(x, g(\nu, x, z)) \varepsilon z$.

Q.E.D.

Subsets of \mathbb{R}

Theorem 24. $\Vdash (\mathfrak{J}\kappa \text{ is countable}) \rightarrow$

every bounded above subset of the ordered set $(\mathfrak{J}\kappa_+, \triangleleft)$ is countable.

Every proper initial segment of the well ordering \triangleleft on κ_+ is of cardinal κ . Thus, there exists a function $\phi : \kappa \times \kappa_+ \rightarrow \kappa_+$ such that, for $x \neq \emptyset$, the function $\alpha \mapsto \phi(x, \alpha)$ is a surjection from κ onto $\{y \in \kappa_+ ; y \triangleleft x\}$. Then, we have immediately :

$$\vdash \Vdash \forall x \mathfrak{J}^{\kappa_+} \forall y \mathfrak{J}^{\kappa_+} ((y \triangleleft x) = 1 \leftrightarrow (\forall \alpha \mathfrak{J}^\kappa (y \neq \phi(x, \alpha)) \rightarrow \perp)).$$

This shows that, in \mathcal{N} , there exists a surjection from $\mathfrak{J}\kappa$, onto every subset of $\mathfrak{J}\kappa_+$ which is bounded from above for the strict ordering \triangleleft .

Thus, all these subsets of $\mathfrak{J}\kappa_+$ are countable, since $\mathfrak{J}\kappa$ is.

Q.E.D.

Theorem 25. $\Vdash (\mathfrak{J}\kappa \text{ is countable}) \rightarrow \text{there exists an injection from } \mathfrak{J}\kappa_+ \text{ into } \mathbb{R}.$

The natural injection from $\mathfrak{J}\kappa_+ = \mathfrak{J}(2^\kappa)$ into $(\mathfrak{J}2)^{\mathfrak{J}\kappa}$ gives an injection from $\mathfrak{J}\kappa_+$ into $\mathbb{N}^{\mathbb{N}} = \mathbb{R}$, since $\mathfrak{J}\kappa$, and thus also $\mathfrak{J}2$, is countable.

Q.E.D.

Theorem 26. *The following formula is realized : $(\mathfrak{J}\kappa \text{ is countable}) \rightarrow$ there exists an application $i \mapsto X_i$ from the countable Boolean algebra $\mathfrak{J}2$ into $\mathcal{P}(\mathbb{R})$ such that :*

- i) $X_0 = \{\emptyset\}$; $i \neq 0 \rightarrow X_i$ is uncountably infinite ;
- ii) $X_i \times X_i$ is equipotent with X_i ;
- iii) $X_i \cap X_j = X_{i \wedge j}$ and therefore $i \leq j \rightarrow X_i \subset X_j$;
- iv) $i \wedge j = 0 \rightarrow X_{i \vee j}$ is equipotent with $X_i \times X_j$;
- v) if there is a surjection from X_j onto X_i , then $i \leq j$;
more generally, if A is a subset of $\mathfrak{J}2$ and if there is a surjection from $\bigcup_{j \in A} X_j$ onto X_i , then $i \leq j$ for some $j \in A$.
- vi) there exists a surjection from X_1 onto \mathbb{R} .

For each $i \in \mathfrak{J}2$, let us denote by X_i the image of $\mathfrak{J}_i\kappa_+$ by the injection from $\mathfrak{J}\kappa_+$ into \mathbb{R} , given by theorem 25.

i) The fact that X_i is infinite for $i \neq 0$ is a consequence of theorem 18.

If $i = 1$, X_i is uncountable by (vi). If $i \neq 0, 1$ and X_i is countable, then X_{1-i} is infinite and thus, there exists a surjection from X_{1-i} onto X_i . This contradicts corollary 16.

ii) by proposition 12(vi), $\mathfrak{J}_i\kappa_+ \times \mathfrak{J}_i\kappa_+$ is equipotent with $\mathfrak{J}_i(\kappa_+^2)$, thus also with $\mathfrak{J}_i\kappa_+$ by proposition 13.

iii) If $a \in \mathfrak{J}_i\kappa_+$ and $a \in \mathfrak{J}_j\kappa_+$, then $P_i a = a$, and therefore $P_{i \wedge j} a = P_j a = a$.

iv) This is proposition 12(v).

v) Application of corollary 16.

vi) Application of theorem 17.

Q.E.D.

Theorem 26 is interesting only if the countable Boolean algebra $\mathfrak{J}2$ is not trivial. In this case, \mathbb{R} cannot be well ordered, by theorems 19 and 25.

In the following section, given an *arbitrary* realizability algebra \mathcal{A} , we build an algebra \mathcal{B} such that :

- $\mathcal{N}_{\mathcal{B}}$ realizes the formula : $(\mathfrak{J}\kappa \text{ is countable})$.
- The (countable) Boolean algebra $\mathfrak{J}2$ of the model $\mathcal{N}_{\mathcal{B}}$ is elementarily equivalent to the algebra $\mathfrak{J}2$ of $\mathcal{N}_{\mathcal{A}}$.

In the sequel, we shall consider two interesting cases :

$\mathfrak{J}2$ is atomless ; $\mathfrak{J}2$ has four ε -elements.

3 Collapsing \beth_κ

In the ground model \mathcal{M} , we consider a realizability algebra \mathcal{A} , the elementary combinators of which are denoted by $\mathbf{B}, \mathbf{C}, \mathbf{E}, \mathbf{I}, \mathbf{K}, \mathbf{W}, \mathbf{cc}$ and the continuations \mathbf{k}_π for $\pi \in \Pi$.

We define the combinators $\mathbf{B}^*, \mathbf{C}^*, \mathbf{E}^*, \mathbf{I}^*, \mathbf{K}^*, \mathbf{W}^*, \mathbf{cc}^*$, and the continuations \mathbf{k}_π^* as follows :

$$\mathbf{B}^* = \lambda n \lambda x \lambda y \lambda z (((\mathbf{C}x)(\mathbf{C}y)zn) ;$$

$$\mathbf{C}^* = \lambda n \lambda x \lambda y \lambda z (x) n z y ;$$

$$\mathbf{E}^* = \lambda n \lambda x \lambda y (\mathbf{C}) x y n ;$$

$$\mathbf{I}^* = \lambda n \lambda x (x) n ;$$

$$\mathbf{K}^* = \lambda n \lambda x \lambda y (x) n ;$$

$$\mathbf{W}^* = \lambda n \lambda x \lambda y (x) n y y ;$$

$$\mathbf{k}_\pi^* = \lambda n \lambda x (\mathbf{k}_\pi)(x) n ;$$

$$\mathbf{cc}^* = \lambda n \lambda x (\mathbf{cc}) \lambda k (x n) \lambda n \lambda x (k)(x) n.$$

Therefore, we have :

$$\mathbf{B}^* \star \nu . \xi . \eta . \zeta . \pi \succ (\mathbf{C}\xi)(\mathbf{C}\eta)\zeta \star \nu . \pi ;$$

$$\mathbf{C}^* \star \nu . \xi . \eta . \zeta . \pi \succ \xi \star \nu . \zeta . \eta . \pi ;$$

$$\mathbf{E}^* \star \nu . \xi . \eta . \pi \succ \mathbf{C}\xi\eta \star \nu . \pi ;$$

$$\mathbf{I}^* \star \nu . \xi . \pi \succ \xi \star \nu . \pi ;$$

$$\mathbf{K}^* \star \nu . \xi . \eta . \pi \succ \xi \star \nu . \pi ;$$

$$\mathbf{W}^* \star \nu . \xi . \eta . \pi \succ \xi \star \nu . \eta . \eta . \pi ;$$

$$\mathbf{k}_\pi^* \star \nu . \xi . \varpi \succ \xi \star \nu . \pi ;$$

$$\mathbf{cc}^* \star \nu . \xi . \pi \succ \xi \star \nu . \mathbf{k}_\pi^* . \pi.$$

(reminder : the notation $\xi \star \pi \succ \xi' \star \pi'$ means $\xi \star \pi \notin \perp \Rightarrow \xi' \star \pi' \notin \perp$).

Let κ be an infinite cardinal of \mathcal{M} , $\kappa \geq \text{card}(\Lambda \cup \Pi)$; we consider the tree (usually called $\kappa^{<\omega}$) of functions, the domain of which is an integer, with values in κ .

Let P be the ordered set obtained by adding a least element \mathbb{O} to this tree.

P is an inf-semi-lattice, the greatest element $\mathbf{1}$ of which is the function \emptyset .

The greatest lower bound of $p, q \in P$, denoted by pq , is p (resp. q) if $p, q \neq \mathbb{O}$ and $q \subset p$ (resp. $p \subset q$). It is \mathbb{O} in every other case.

Remark. $P \setminus \{\mathbb{O}\} = \kappa^{<\omega}$ is the ordered set used, in the method of forcing, to *collapse* (i.e. make countable) the cardinal κ .

We define a new realizability algebra \mathcal{B} by setting :

$$\mathbf{\Lambda} = \Lambda \times P ; \mathbf{\Pi} = \Pi \times P ; \mathbf{\Lambda} \star \mathbf{\Pi} = (\Lambda \star \Pi) \times P ;$$

$$(\xi, p) \cdot (\pi, q) = (\xi \cdot \pi, pq) ; (\xi, p) \star (\pi, q) = (\xi \star \pi, pq) ; (\xi, p)(\eta, q) = (\mathbf{C}\xi\eta, pq).$$

$$\mathbf{B} = (\mathbf{B}^*, \mathbf{1}) ; \mathbf{C} = (\mathbf{C}^*, \mathbf{1}) ; \mathbf{E} = (\mathbf{E}^*, \mathbf{1}) ; \mathbf{I} = (\mathbf{I}^*, \mathbf{1}) ; \mathbf{K} = (\mathbf{K}^*, \mathbf{1}) ; \mathbf{W} = (\mathbf{W}^*, \mathbf{1}) ;$$

$$\mathbf{cc} = (\mathbf{cc}^*, \mathbf{1}) ; \mathbf{k}_{(\pi, p)} = (\mathbf{k}_\pi^*, p).$$

We define, in \mathcal{M} , a function symbol from $P \times \mathbb{N}$ into $\{0, 1\}$, denoted by $(p \ll n)$, by setting : $(p \ll n) = 1 \Leftrightarrow p \neq \mathbb{O}$ and the domain of p is an integer $\leq n$.

We define $\perp_{\mathcal{B}}$, that we shall denote also by \perp , as follows :

$$(\xi \star \pi, p) \in \perp \Leftrightarrow (\forall n \in \mathbb{N})((p \ll n) = 1 \Rightarrow \xi \star \underline{n} \cdot \pi \in \perp) \text{ for } p \in P, \xi \in \Lambda \text{ and } \pi \in \Pi.$$

In particular, we have $(\xi \star \pi, \mathbb{O}) \in \perp$ for any $\xi \in \Lambda, \pi \in \Pi$.

We check now that \mathcal{B} is a realizability algebra.

- $(\xi, p)(\eta, q) \star (\pi, r) \notin \mathbb{L} \Rightarrow (\xi, p) \star (\eta, q) \cdot (\pi, r) \notin \mathbb{L}$:

Suppose that $(\xi \star \eta \cdot \pi, pqr) \in \mathbb{L}$; we must show $(C\xi\eta \star \pi, pqr) \in \mathbb{L}$ i.e. $C\xi\eta \star \underline{n} \cdot \pi \in \mathbb{L}$ for $(pqr \ll n) = 1$. Now, we have $C\xi\eta \star \underline{n} \cdot \pi \succ \xi \star \underline{n} \cdot \eta \cdot \pi$ which is in \mathbb{L} by hypothesis.

- $(B^*, \mathbf{1}) \star (\xi, p) \cdot (\eta, q) \cdot (\zeta, r) \cdot (\pi, s) \notin \mathbb{L} \Rightarrow ((\xi, p))((\eta, q))(\zeta, r) \star (\pi, s) \notin \mathbb{L}$:

Suppose that $((C\xi)(C\eta)\zeta \star \pi, pqrst) \in \mathbb{L}$; we must show :

$(B^* \star \xi \cdot \eta \cdot \zeta \cdot \pi, pqrst) \in \mathbb{L}$ i.e. $B^* \star \underline{n} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \in \mathbb{L}$ for $(pqrst \ll n) = 1$.

Now, we have $B^* \star \underline{n} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \succ (C\xi)(C\eta)\zeta \star \underline{n} \cdot \pi$ which is in \mathbb{L} by hypothesis.

- $(C^*, \mathbf{1}) \star (\xi, p) \cdot (\eta, q) \cdot (\zeta, r) \cdot (\pi, s) \notin \mathbb{L} \Rightarrow (\xi, p) \star (\zeta, r) \cdot (\eta, q) \cdot (\pi, s) \notin \mathbb{L}$:

Suppose that $(\xi \star \zeta \cdot \eta \cdot \pi, pqrst) \in \mathbb{L}$; we must show :

$(C^* \star \xi \cdot \eta \cdot \zeta \cdot \pi, pqrst) \in \mathbb{L}$ i.e. $C^* \star \underline{n} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \in \mathbb{L}$ for $(pqrst \ll n) = 1$.

Now, we have $C^* \star \underline{n} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \succ \xi \star \underline{n} \cdot \zeta \cdot \eta \cdot \pi$ which is in \mathbb{L} by hypothesis.

- $(E^*, \mathbf{1}) \star (\xi, p) \cdot (\eta, q) \cdot (\pi, r) \notin \mathbb{L} \Rightarrow (\xi, p)(\eta, q) \star (\pi, r) \notin \mathbb{L}$:

Suppose that $(C\xi\eta \star \pi, pqrst) \in \mathbb{L}$; we must show :

$(E^* \star \xi \cdot \eta \cdot \pi, pqr) \in \mathbb{L}$ i.e. $E^* \star \underline{n} \cdot \xi \cdot \eta \cdot \pi \in \mathbb{L}$ for $(pqr \ll n) = 1$.

Now, we have $E^* \star \underline{n} \cdot \xi \cdot \eta \cdot \pi \succ C\xi\eta \star \underline{n} \cdot \pi$ which is in \mathbb{L} by hypothesis.

- $(cc^*, \mathbf{1}) \star (\xi, p) \cdot (\pi, q) \notin \mathbb{L} \Rightarrow (\xi, p) \star (k_\pi^*, q) \cdot (\pi, q) \notin \mathbb{L}$:

Suppose that $(\xi \star k_\pi^* \cdot \pi, pq) \in \mathbb{L}$; we must show :

$(cc^* \star \xi \cdot \pi, pq) \in \mathbb{L}$ i.e. $cc^* \star \underline{n} \cdot \xi \cdot \pi \in \mathbb{L}$ for $(pq \ll n) = 1$.

Now, we have $cc^* \star \underline{n} \cdot \xi \cdot \pi \succ \xi \star \underline{n} \cdot k_\pi^* \cdot \pi$ which is in \mathbb{L} by hypothesis.

- $(k_\pi^*, p) \star (\xi, q) \cdot (\varpi, r) \notin \mathbb{L} \Rightarrow (\xi, q) \star (\pi, p) \notin \mathbb{L}$:

Suppose that $(\xi \star \pi, pq) \in \mathbb{L}$; we must show :

$(k_\pi^* \star \xi \cdot \varpi, pqr) \in \mathbb{L}$ i.e. $k_\pi^* \star \underline{n} \cdot \xi \cdot \varpi \in \mathbb{L}$ for $(pqr \ll n) = 1$.

Now, we have $k_\pi^* \star \underline{n} \cdot \xi \cdot \varpi \succ \xi \star \underline{n} \cdot \pi$ which is in \mathbb{L} by hypothesis.

For each closed c-term τ (built with the elementary combinators and the application), we define τ^* by recurrence :

this is already done if τ is an elementary combinator ; we set $(tu)^* = Ct^*u^*$.

In the algebra \mathcal{B} , the value of the combinator τ is $\tau_{\mathcal{B}} = (\tau_{\mathcal{A}}^*, \mathbf{1})$.

In particular, the integer n of the algebra \mathcal{B} is $\underline{n}_{\mathcal{B}} = (\underline{n}^*, \mathbf{1})$.

We have $\underline{0}_{\mathcal{B}} = (\underline{0}^*, \mathbf{1}) = (\mathbf{K}^*, \mathbf{1})(\mathbf{I}^*, \mathbf{1})$; therefore :

$$\underline{0}^* = \mathbf{CK}^*\mathbf{I}^*.$$

We have $(\underline{n+1})_{\mathcal{B}} = ((\underline{n+1})^*, \mathbf{1}) = (\sigma^*, \mathbf{1})(\underline{n}^*, \mathbf{1})$; therefore :

$$(\underline{n+1})^* = \mathbf{C}\sigma^*\underline{n}^*.$$

Thus, we have $\underline{n}^* = (\mathbf{C}\sigma^*)^n \underline{0}^*$ for every $n \in \mathbb{N}$.

We define the proof-like terms of the algebra \mathcal{B} as terms of the form $(\theta, \mathbf{1})$ where θ is a proof-like term of the algebra \mathcal{A} . The condition of coherence for \mathcal{B} is therefore :

If θ is a proof-like term of \mathcal{A} , there exists $n \in \mathbb{N}$ and $\pi \in \Pi$ such that $\theta \star \underline{n} \cdot \pi \notin \mathbb{L}$.

If \mathcal{A} is coherent, so is \mathcal{B} : indeed, if θ is a proof-like term of \mathcal{A} , then so is $\theta \underline{0}$; this gives a stack π such that $\theta \underline{0} \star \pi \notin \mathbb{L}$.

The truth value of a formula F in the algebra \mathcal{B} will be denoted by $\|F\|_{\mathcal{B}}$ or also $\|F\|$.

The realizability models associated with the algebras \mathcal{A} and \mathcal{B} are denoted respectively by $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$.

We now define $\mathcal{X} \in \mathcal{M}$ by setting :

$$\mathcal{X} = \{((m, a), (\pi, p)) \ ; \ \pi \in \Pi \ ; \ p \in P \setminus \{\mathbf{0}\}, p(m) \text{ is defined and } p(m) = a\}.$$

Theorem 27.

The formula $(\mathcal{X}$ is a surjection from \mathbb{N} onto $\mathbb{J}\kappa$) is realized in the model $\mathcal{N}_{\mathcal{B}}$.

More precisely, we have :

- i) $(\theta_0, \mathbf{1}) \Vdash \forall x \forall y \forall y' ((x, y) \in \mathcal{X}, y \neq y' \rightarrow (x, y') \notin \mathcal{X})$ with $\theta_0 = \lambda n \lambda k \lambda x(x)n$;
- ii) $(\theta_1, \mathbf{1}) \Vdash \forall y^{\mathfrak{J}\kappa} [\forall x^{\text{int}}((x, y) \notin \mathcal{X}) \rightarrow \perp]$ with $\theta_1 = \lambda n \lambda x(\Theta)(C\sigma^*)n(Cx)\underline{0}^*(\sigma)n$,
 $\Theta = \lambda f \lambda n(n) \lambda g \lambda z(g)(f)z$ and $\sigma = (BW)(B)B$ (successor).

i) Let $m \in \mathbb{N}, \alpha, \alpha' \in \kappa, (\pi, p) \in \Vdash(m, \alpha) \notin \mathcal{X}\Vdash, (\pi', p') \in \Vdash(m, \alpha') \notin \mathcal{X}\Vdash$
and $(\xi, q) \Vdash \alpha \neq \alpha'$.

Thus, we have $m \in \text{dom}(p), m \in \text{dom}(p'), p(m) = \alpha$ and $p'(m) = \alpha'$.

We show that : $(\theta_0, \mathbf{1}) \star \mathfrak{k}_{(\pi, p)} \cdot (\xi, q) \cdot (\pi', p') \in \mathbb{L}$.

We have to show that $(\theta_0 \star \mathfrak{k}_\pi^* \cdot \xi \cdot \pi', pp'q) \in \mathbb{L}$; this is obvious if $pp'q = \mathbb{O}$.

Otherwise, p and p' are compatible, thus $\alpha = \alpha'$.

Let n be such that $(pp'q \ll n) = 1$; we must show that $\theta_0 \star \underline{n} \cdot \mathfrak{k}_\pi^* \cdot \xi \cdot \pi' \in \mathbb{L}$ i.e.
 $\xi \star \underline{n} \cdot \pi' \in \mathbb{L}$.

Now, we have $(\xi, q) \Vdash \perp$ by hypothesis on (ξ, q) , thus $(\xi, q) \star (\pi', \mathbf{1}) \in \mathbb{L}$.

Since $(q \ll n) = 1$, it follows that $\xi \star \underline{n} \cdot \pi' \in \mathbb{L}$.

ii) Let us first show that $\theta_1 \star \underline{n} \cdot \eta \cdot \varpi \succ \eta \star \underline{n+1} \cdot \underline{n}^* \cdot \varpi$

for each $n \in \mathbb{N}, \eta \in \Lambda$ and $\varpi \in \Pi$.

We have $\theta_1 \star \underline{n} \cdot \eta \cdot \varpi \succ \Theta \star C\sigma^* \cdot \underline{n} \cdot C\eta \cdot \underline{0}^* \cdot \underline{n+1} \cdot \varpi$.

By lemma 9, in which we set $\xi = C\eta, \phi = C\sigma^*, \alpha = \underline{0}^*, \varsigma = \sigma, O = \underline{0}$ and $\pi = \underline{n+1} \cdot \varpi$,
we obtain :

$$\begin{aligned} \theta_1 \star \underline{n} \cdot \eta \cdot \varpi &\succ C\eta \star \underline{n}^* \cdot \underline{n+1} \cdot \varpi \quad (\text{since } \underline{n}^* = (C\sigma^*)^n \underline{0}^*) \\ &\succ \eta \star \underline{n+1} \cdot \underline{n}^* \cdot \varpi. \end{aligned}$$

We prove now that $(\theta_1, \mathbf{1}) \Vdash \forall y^{\mathfrak{J}\kappa} [\forall x^{\text{int}}((x, y) \notin \mathcal{X}) \rightarrow \perp]$.

Let $\alpha \in \kappa, (\eta, p_0) \Vdash \forall x^{\text{int}}((x, \alpha) \notin \mathcal{X})$ and $(\varpi, q_0) \in \Pi \times P$;

we show that $(\theta_1, \mathbf{1}) \star (\eta, p_0) \cdot (\varpi, q_0) \in \mathbb{L}$.

This is trivial if $p_0 q_0 = \mathbb{O}$; otherwise, let $n \in \mathbb{N}$ be such that $(p_0 q_0 \ll n) = 1$.

We must show that $\theta_1 \star \underline{n} \cdot \eta \cdot \varpi \in \mathbb{L}$, that is $\eta \star \underline{n+1} \cdot \underline{n}^* \cdot \varpi \in \mathbb{L}$.

But we have $(\eta, p_0) \Vdash \{(\underline{n}^*, \mathbf{1})\} \rightarrow (n, \alpha) \notin \mathcal{X}$ by hypothesis on η .

Since $(p_0 q_0 \ll n) = 1$, we can define $q \in P$ with domain $n+1$ such that $q \supset p_0 q_0$ and
 $q(n) = \alpha$. Then, we have $(\varpi, q) \in \Vdash(n, \alpha) \notin \mathcal{X}\Vdash$ by definition of \mathcal{X} .

We have thus $(\eta, p_0) \star (\underline{n}^*, \mathbf{1}) \cdot (\varpi, q) \in \mathbb{L}$ that is $(\eta \star \underline{n}^* \cdot \varpi, p_0 q) \in \mathbb{L}$.

But we have $p_0 q = q$, and therefore $(\eta \star \underline{n}^* \cdot \varpi, q) \in \mathbb{L}$.

Since $(q \ll n+1) = 1$, it follows that $\eta \star \underline{n+1} \cdot \underline{n}^* \cdot \varpi \in \mathbb{L}$.

Q.E.D.

Corollary 28. $\mathcal{N}_{\mathcal{B}}$ realizes the non extensional axiom of choice and thus also DC.

Indeed, by theorem 27, the model $\mathcal{N}_{\mathcal{B}}$ realizes the formula : $(\mathfrak{J}\kappa$ is countable).

But we have $\kappa = \text{card}(\Lambda \cup \Pi \cup \mathbb{N})$, since $\kappa \geq \text{card}(\Lambda \cup \Pi \cup \mathbb{N})$ and $\kappa = \text{card}(P)$.

Therefore $\mathcal{N}_{\mathcal{B}}$ realizes NEAC, by theorem 23.

Q.E.D.

Remark. Intuitively, the model $\mathcal{N}_{\mathcal{B}}$ is an extension of the model $\mathcal{N}_{\mathcal{A}}$ obtained by forcing, by collapsing $\mathfrak{J}\kappa$. We cannot apply directly the usual theory of forcing, because $\mathfrak{J}\kappa$ is not defined in ZF.

Elementary formulas

Elementary formulas are defined as follows, where t, u are terms of the language of ZF_ε (built with variables, individuals, and function symbols defined in \mathcal{M}) :

- \top, \perp are elementary formulas ;
- if U is an elementary formula, then $t = u \leftrightarrow U$ and $\forall x U$ are too ;
- if U, V are elementary formulas, then $U \rightarrow V$ too ;
- if U is an elementary formula, then $\forall n^{\text{int}} U$ too.

Remark. $t \neq u$ is an elementary formula, and also $t \neq \exists u$ (which can be written $f(t, u) \neq 1$ where f is the function symbol defined in \mathcal{M} by : $f(a, b) = 1$ iff $a \in b$).

If U is an elementary formula, then $\forall x^{\exists t} U$ is too : indeed, it is written $\forall x(f(x, t) = 1 \leftrightarrow U)$.

For each elementary formula U , we define, by recurrence, two formulas U_p and U^p , with one additional free variable p :

1. $U^p \equiv \forall q^{\exists P} \forall n^{\text{int}} ((pq \ll n) = 1 \leftrightarrow U_q)$;
2. $\perp_p \equiv \perp$ and $\top_p \equiv \top$;
3. $(t = u \leftrightarrow U)_p \equiv (t = u \leftrightarrow U_p)$; $(\forall x U[x])_p \equiv \forall x U_p[x]$;
4. $(U \rightarrow V)_p \equiv \forall q^{\exists P} \forall r^{\exists P} (p = qr \leftrightarrow (U^q \rightarrow V_r))$;
5. $(\forall n^{\text{int}} U[n])_p \equiv \forall n^{\exists \mathbb{N}} (\{\underline{n}^*\} \rightarrow U_p[n])$, in other words :
 $\|(\forall n^{\text{int}} U[n])_p\| = \{\underline{n}^* \cdot \pi ; n \in \mathbb{N}, \pi \in \|U_p[n]\|\}$.

Lemma 29. For each closed elementary formula U , we have :

$$(\pi, p) \in \|U\| \Leftrightarrow \pi \in \|U_p\| ; (\xi, p) \Vdash U \Leftrightarrow \xi \Vdash U^p.$$

Proof by recurrence on the length of the formula U .

1. We have $(\xi, p) \Vdash U \Leftrightarrow (\xi, p) \star (\pi, q) \in \perp$ for $(\pi, q) \in \|U\|$, that is :
 $(\xi \star \pi, pq) \in \perp$ for every $\pi \in \|U_q\|$, by the recurrence hypothesis, or also :
 $(\forall q \in P)(\forall \pi \in \|U_q\|)(\forall n \in \mathbb{N})(pq \ll n) = 1 \Rightarrow \xi \star \underline{n} \cdot \pi \in \perp$ which is equivalent to :
 $\xi \Vdash \forall q^{\exists P} \forall n^{\text{int}} ((pq \ll n) = 1 \leftrightarrow U_q)$ that is $\xi \Vdash U^p$.

2 and 3. Obvious.

4. Any element of $\|U \rightarrow V\|$ has the form $(\xi, q) \cdot (\pi, r)$, i.e. $(\xi \cdot \pi, p)$, with $p = qr$,
 $(\xi, q) \Vdash U$ and $(\pi, r) \in \|V\|$;

by the recurrence hypothesis, this is equivalent to $\xi \cdot \pi \in \|U^q \rightarrow V_r\|$.

5. We have $\|\forall n^{\text{int}} U[n]\| = \|\forall n^{\exists \mathbb{N}} (\{\underline{n}^*, \mathbf{1}\} \rightarrow U[n])\|$
 $= \{(\underline{n}^*, \mathbf{1}) \cdot (\pi, p) ; n \in \mathbb{N}, (\pi, p) \in \|U[n]\|\} = \{(\underline{n}^* \cdot \pi, p) ; n \in \mathbb{N}, (\pi, p) \in \|U[n]\|\}$.

Thus, by the recurrence hypothesis, it is $\{(\underline{n}^* \cdot \pi, p) ; n \in \mathbb{N}, \pi \in \|U_p[n]\|\}$.

Q.E.D.

Lemma 30. For each elementary formula U , there exist proof-like terms θ_U^0, θ_U^1 , such that :

- i) $\theta_U^0 \Vdash \forall p^{\exists P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (U \rightarrow U_p))$;
 - ii) $\theta_U^1 \Vdash \forall p^{\exists P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (U_p \rightarrow U))$;
 - iii) $\tau_U^0 \Vdash \forall p^{\exists P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (U \rightarrow U^p))$;
 - iv) $\tau_U^1 \Vdash \forall p^{\exists P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (U^p \rightarrow U))$;
- with $\tau_U^0 = \lambda n \lambda x \lambda m (\theta_U^0) m x$ and $\tau_U^1 = \lambda n \lambda x (\theta_U^1 n)(x) n$.

We first show (iii) and (iv) from (i) and (ii).

(i) \Rightarrow (iii)

Let $p \in P$ and $n \in \mathbb{N}$ be such that $(p \ll n) = 1$; let $\xi \Vdash U$ and $\pi \in \ll U^p \ll$.

We have to show : $\lambda n \lambda x \lambda m (\theta_U^0) m x \star \underline{n} \cdot \xi \cdot \pi \in \perp$.

Now, by the definition (1) of U^p , there exist $q \in P$, $m \in \mathbb{N}$ and $\varpi \in \ll U_q \ll$ such that $(pq \ll m) = 1$ and $\pi = \underline{m} \cdot \varpi$. Therefore, we have $(q \ll m) = 1$ and, by (i) :

$\theta_U^0 \star \underline{m} \cdot \xi \cdot \varpi \in \perp$, hence $\lambda n \lambda x \lambda m (\theta_U^0) m x \star \underline{n} \cdot \xi \cdot \underline{m} \cdot \varpi \in \perp$.

(ii) \Rightarrow (iv)

Let $p \in P$, $n \in \mathbb{N}$, $\xi \in \Lambda$ and $\pi \in \ll U \ll$ such that $(p \ll n) = 1$ and $\xi \Vdash U^p$.

We have to show : $\lambda n \lambda x (\theta_U^1 n)(x) n \star \underline{n} \cdot \xi \cdot \pi \in \perp$ i.e. $\theta_U^1 \star \underline{n} \cdot \xi \cdot \pi \in \perp$.

But, by the definition (1) of U^p , in which we set $q = p$, we have $\xi \underline{n} \Vdash U_p$; therefore, the desired result follows from (ii).

We now show (i) and (ii) by recurrence on the length of U .

- If U is \perp or \top , we take $\theta_U^0 = \theta_U^1 = \lambda n \lambda x x$.
- If $U \equiv (t = u \rightarrow V)$ or $U \equiv \forall x^{\text{Int}} V$, then $\theta_U^0 = \theta_V^0$ and $\theta_U^1 = \theta_V^1$ by (3).
- If $U \equiv V \rightarrow W$, let $q, r \in \mathbb{N}$ and $p = qr$; let $n \in \mathbb{N}$ such that $(p \ll n) = 1$. We have : $\tau_V^0 \underline{n} \Vdash V \rightarrow V^q$; $\tau_V^1 \underline{n} \Vdash V^q \rightarrow V$; $\theta_W^0 \underline{n} \Vdash W \rightarrow W_r$; $\theta_W^1 \underline{n} \Vdash W_r \rightarrow W$.

Let $\xi \Vdash V \rightarrow W$; then, by the recurrence hypothesis, we have :

$(\theta_W^0 \underline{n}) \circ \xi \Vdash V \rightarrow W_r$ and $(\theta_W^0 \underline{n}) \circ \xi \circ (\tau_V^1 \underline{n}) \Vdash V^q \rightarrow W_r$.

Thus, by (4), we obtain $\theta_U^0 = \lambda n \lambda x \lambda y (\theta_W^0 n)(x) (\tau_V^1 n) y$.

Now, let $\xi \Vdash V^q \rightarrow W_r$; then, by the recurrence hypothesis, we have :

$(\theta_W^1 \underline{n}) \circ \xi \Vdash V^q \rightarrow W$ and $(\theta_W^1 \underline{n}) \circ \xi \circ (\tau_V^0 \underline{n}) \Vdash V \rightarrow W$.

Thus, by (4), we obtain $\theta_U^1 = \lambda n \lambda x \lambda y (\theta_W^1 n)(x) (\tau_V^0 n) y$.

- If $U \equiv \forall n^{\text{Int}} V[n]$, we first prove :

Lemma 31.

There exist two proof-like terms T_0, T_1 such that, for every closed formula F of ZF_ε :

i) $T_0 \Vdash \forall n^{\text{Int}} ((\{\underline{n}^*\} \rightarrow F) \rightarrow (\{\underline{n}\} \rightarrow F))$.

ii) $T_1 \Vdash \forall n^{\text{Int}} ((\{\underline{n}\} \rightarrow F) \rightarrow (\{\underline{n}^*\} \rightarrow F))$.

iii) For every elementary formula $V[n]$, we have :

$T_0 \Vdash (\forall n^{\text{Int}} V[n])_p \rightarrow \forall n^{\text{Int}} V_p[n]$ and $T_1 \Vdash \forall n^{\text{Int}} V_p[n] \rightarrow (\forall n^{\text{Int}} V[n])_p$.

i) We apply lemma 9 to the realizability algebra \mathcal{A} , with :

$\varsigma = \sigma$, $O = \underline{0}$, $\phi = C\sigma^*$ and $\alpha = \underline{0}^*$; we have $\Theta = \lambda f \lambda n (n) \lambda g \lambda z (g)(f) z$.

For every $n \in \mathbb{N}$, $\xi \in \Lambda$ and $\pi \in \Pi$, we obtain :

$\Theta \star C\sigma^* \cdot \underline{n} \cdot \xi \cdot \underline{0}^* \cdot \pi \succ \xi \star \underline{n}^* \cdot \pi$ for $n \in \mathbb{N}$, $\xi \in \Lambda$, $\pi \in \Pi$, since $\underline{n}^* = (C\sigma^*)^n \underline{0}^*$.

Therefore, if we set $T_0 = \lambda f \lambda n ((\Theta)(C)\sigma^*) n f \underline{0}^*$, we have $T_0 \star \xi \cdot \underline{n} \cdot \pi \succ \xi \star \underline{n}^* \cdot \pi$.

Thus, we have $T_0 \Vdash \forall n^{\text{Int}} ((\{\underline{n}^*\} \rightarrow F) \rightarrow (\{\underline{n}\} \rightarrow F))$.

ii) We apply now lemma 9, to the realizability algebra \mathcal{B} , with :

$\varsigma = \sigma_{\mathcal{B}}$, $O = \underline{0}_{\mathcal{B}}$, $\phi = (C\sigma, \mathbf{1})$ and $\alpha = (\underline{0}, \mathbf{1})$.

Since $\Theta_{\mathcal{B}} = (\Theta^*, \mathbf{1})$, $\underline{n}_{\mathcal{B}} = (\sigma_{\mathcal{B}})^n \underline{0}_{\mathcal{B}} = (\underline{n}^*, \mathbf{1})$, we get, by setting $\sigma_2 = (C)^2 \sigma$:

$(\Theta^*, \mathbf{1}) \star (C\sigma, \mathbf{1}) \cdot (\underline{n}^*, \mathbf{1}) \cdot (\xi, \mathbf{1}) \cdot (\underline{0}, \mathbf{1}) \cdot (\pi, \mathbf{1}) \succ (\xi, \mathbf{1}) \star ((\sigma_2)^n \underline{0}, \mathbf{1}) \cdot (\pi, \mathbf{1})$

since $((C\sigma, \mathbf{1}))^n (\underline{0}, \mathbf{1}) = (((C)^2 \sigma)^n \underline{0}, \mathbf{1})$. We write this as :

$(\Theta^* \star C\sigma \cdot \underline{n}^* \cdot \xi \cdot \underline{0} \cdot \pi, \mathbf{1}) \succ (\xi \star (\sigma_2)^n \underline{0} \cdot \pi, \mathbf{1})$.

It follows that $\Theta^* \star \underline{0} \cdot C\sigma \cdot \underline{n}^* \cdot \xi \cdot \underline{0} \cdot \pi \succ \xi \star \underline{m} \cdot (\sigma_2)^n \underline{0} \cdot \pi$ for some $m \in \mathbb{N}$.

Now, we set $\xi = (K)(T)\eta$, with $T = \lambda f \lambda n (\Theta) \sigma n f \underline{0}$. Thus, we have :

$\Theta^* \star \underline{0} \cdot C\sigma \cdot \underline{n}^* \cdot (K)(T)\eta \cdot \underline{0} \cdot \pi \succ T \star \eta \cdot (\sigma_2)^n \underline{0} \cdot \pi \succ \Theta \star \sigma \cdot (\sigma_2)^n \underline{0} \cdot \eta \cdot \underline{0} \cdot \pi$.

We apply now lemma 9, to the realizability algebra \mathcal{A} , with :

$\varsigma = \sigma_2, O = \underline{0}, \phi = \sigma$ and $\alpha = \underline{0}$. Thus, we obtain :

$\Theta \star \sigma \cdot (\sigma_2)^n \underline{0} \cdot \eta \cdot \underline{0} \cdot \pi \succ \eta \star \underline{n} \cdot \pi$ for every $n \in \mathbb{N}, \eta \in \Lambda$ and $\pi \in \Pi$.

Finally, if we set $T_1 = \lambda f \lambda n (((\Theta^* \underline{0})(C)\sigma)n)(K)(T)f \underline{0}$, we have :

$T_1 \star \eta \cdot \underline{n}^* \cdot \pi \succ \eta \star \underline{n} \cdot \pi$ and therefore $T_1 \Vdash \forall n^{\mathbb{N}}((\{\underline{n}\} \rightarrow F) \rightarrow (\{\underline{n}^*\} \rightarrow F))$.

iii) This follows immediately from (i) and (ii), by definition of $(\forall n^{\text{int}} V[n])_p$.

Q.E.D.

We can now finish the proof of lemma 30, considering the last case which is :

- $U \equiv \forall m^{\text{int}} V[m]$.

We show that $\theta_V^0 = \lambda n \lambda x (T_1) \lambda m (\theta_V^0 n)(x)m$.

By the recurrence hypothesis, we have $\theta_V^0 \Vdash \forall p^{\mathbb{1}P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (V[m] \rightarrow V_p[m]))$.

Let $p \in P, n \in \mathbb{N}, \xi \in \Lambda$ be such that $(p \ll n) = 1$ and $\xi \Vdash \forall m^{\text{int}} V[m]$.

Then, for every $m \in \mathbb{N}$, we have $\xi \underline{m} \Vdash V[m]$; thus $(\theta_V^0 \underline{n})(\xi)m \Vdash V_p[m]$ and therefore $\lambda m (\theta_V^0 \underline{n})(\xi)m \Vdash \forall m^{\text{int}} V_p[m]$. By lemma 31(iii), we get $(T_1) \lambda m (\theta_V^0 \underline{n})(\xi)m \Vdash (\forall m^{\text{int}} V[m])_p$ and therefore : $\lambda x (T_1) \lambda m (\theta_V^0 \underline{n})(x)m \Vdash \forall m^{\text{int}} V[m] \rightarrow (\forall m^{\text{int}} V[m])_p$. Finally :

$\lambda n \lambda x (T_1) \lambda m (\theta_V^0 n)(x)m \Vdash \forall p^{\mathbb{1}P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (\forall m^{\text{int}} V[m] \rightarrow (\forall m^{\text{int}} V[m])_p))$.

We show now that $\theta_V^1 = \lambda n \lambda x \lambda m (\theta_V^1 n)(T_0)xm$.

By the recurrence hypothesis, we have $\theta_V^1 \Vdash \forall p^{\mathbb{1}P} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (V_p[m] \rightarrow V[m]))$;

Let $p \in P, n \in \mathbb{N}, \xi \in \Lambda$ be such that $(p \ll n) = 1$ and $\xi \Vdash (\forall m^{\text{int}} V[m])_p$.

By lemma 31(iii), we have $T_0 \xi \Vdash \forall m^{\text{int}} V_p[m]$, thus $T_0 \xi \underline{m} \Vdash V_p[m]$.

Therefore $(\theta_V^1 \underline{n})(T_0) \xi \underline{m} \Vdash V[m]$, and $\lambda m (\theta_V^1 \underline{n})(T_0) \xi m \Vdash \forall m^{\text{int}} V[m]$, hence the result.

Q.E.D.

Theorem 32.

The same closed elementary formulas, with parameters in \mathcal{M} , are realized in the models $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$.

Let U be a closed elementary formula, which is realized in $\mathcal{N}_{\mathcal{A}}$ and let θ be a proof-like term such that $\theta \Vdash U$. Then, we have $(\tau_U^0) \underline{n} \theta \Vdash U^p$ for $(p \ll n) = 1$, by lemma 30(iii) ; therefore, setting $p = \emptyset = \mathbf{1}$, we have $((\tau_U^0) \underline{0} \theta, \mathbf{1}) \Vdash U$ by lemma 29.

Therefore, the formula U is also realized in the model $\mathcal{N}_{\mathcal{B}}$.

Conversely, if $(\theta, \mathbf{1}) \Vdash U$ with $\theta \in \text{QP}$, we have $\theta \Vdash U^1$, by lemma 29. Thus $\tau_U^1 \underline{0} \theta \Vdash U$ by lemma 30(iv).

Q.E.D.

Remark. For instance :

- If the Boolean algebra $\mathbb{12}$ has four ε -elements or if it is atomless, in the model $\mathcal{N}_{\mathcal{A}}$, it is the same in the model $\mathcal{N}_{\mathcal{B}}$.
- Arithmetical formulas are elementary. Therefore, by theorem 32, the models $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$ realize the same arithmetical formulas. In fact, this was already known, because they are the same as the arithmetical formulas which are true in \mathcal{M} [15, 16].

Arithmetical formulas and dependent choice

In this section, we obtain, by means of the previous results, a technique to transform into a program, a given proof, in $ZF + DC$, of an arithmetical formula F .

We notice that this program is a closed c -term, written with the elementary combinators B, C, E, I, K, W, cc *without any other instruction*.

Thus, let us consider a proof of $ZF_e \vdash NEAC \rightarrow F$. It gives us a closed c -term Φ_0 such that $\Phi_0 \Vdash NEAC \rightarrow F$, in every realizability algebra.

We now describe a rewriting on closed c -terms, which will transform Φ_0 into a closed c -term Φ such that $\Phi \Vdash F$ in every realizability algebra \mathcal{A} .

By theorem 23, we have $\Phi_1 \Vdash (\aleph_\kappa \text{ is countable}) \rightarrow F$ with $\Phi_1 = \lambda x(\Phi_0)(H)x$.

We apply this result in the algebra \mathcal{B} , which gives :

$(\Phi_1^*, \mathbf{1}) \Vdash (\aleph_\kappa \text{ is countable}) \rightarrow F$.

Now, theorem 27 gives a closed c -term Δ such that $(\Delta, \mathbf{1}) \Vdash (\aleph_\kappa \text{ is countable})$.

It follows that $(\Phi_1^*, \mathbf{1})(\Delta, \mathbf{1}) \Vdash F$, i.e. $(\Psi, \mathbf{1}) \Vdash F$, with $\Psi = C\Phi_1^*\Delta$.

Since F is an arithmetical formula, it is an elementary formula.

Therefore, by lemma 29, we have $\Psi \Vdash F^1$. Now, by lemma 30(iv), we have :

$\tau_F^1 \Vdash \forall p^{\aleph^p} \forall n^{\text{int}} ((p \ll n) = 1 \leftrightarrow (F^p \rightarrow F))$.

We set $p = \mathbf{1}$ and $n = 0$, and we obtain $\tau_F^1 \underline{0} \Vdash F^1 \rightarrow F$.

Finally, by setting $\Phi = (\tau_F^1 \underline{0})\Psi$, we have $\Phi \Vdash F$.

A relative consistency result

In [18], we have defined a countable realizability algebra \mathcal{A} such that the characteristic Boolean algebra \aleph_2 of the model $\mathcal{N}_{\mathcal{A}}$ is atomless (in this example, we have $\kappa = \mathbb{N}$).

If we apply the technique of section 3, in order to collapse \aleph_κ , we obtain a realizability algebra \mathcal{B} and a model $\mathcal{N}_{\mathcal{B}}$, the characteristic Boolean algebra of which is also atomless.

Indeed, the property : $(\aleph_2 \text{ is atomless})$ is expressed by an elementary formula.

But now \aleph_2 is *the countable atomless Boolean algebra* (they are all isomorphic). Therefore, by applying theorems 23 and 26, we obtain the relative consistency result (i) announced in the introduction.

Remark. We note that this method applies to every realizability algebra such that we have :
 $\Vdash (\aleph_2 \text{ is an atomless Boolean algebra})$.

4 A two threads model (\aleph_2 with four elements)

In this section, we suppose that \mathcal{A} is a *standard realisability algebra* [18].

This means, by definition, that the terms and the stacks are finite sequences, built with :

the alphabet $B, C, E, I, K, W, cc, k, \cdot, (,), [,]$

a countable set of *term constants* (also called *instructions*),

a countable set of *stack constants*

and that they are defined by the following rules :

B, C, E, I, K, W, cc and all the term constants are terms ;

if t, u are terms, the sequence $(t)u$ is a term ;

if π is a stack, the sequence $k[\pi]$ is a term (denoted by k_π) ;
each stack constant is a stack ;
if t is a term and π is a stack, then $t \cdot \pi$ is a stack.

If t is a term and π is a stack, then the ordered pair (t, π) is a *process*, denoted by $t \star \pi$.

A proof-like term of \mathcal{A} is a term which does not contain the symbol k .

We now build a realizability model in which $\mathbb{N}2$ has exactly 4 elements.

We suppose that there are exactly two stack constants π^0, π^1 and one term constant d .

For $i \in \{0, 1\}$, let Λ^i (resp. Π^i) be the set of terms (resp. stacks)

which contain the only stack constant π^i .

For $i, j \in \{0, 1\}$, define \perp_j^i as the least set $P \subset \Lambda^i \star \Pi^i$ of processes such that :

1. $d \star \underline{j} \cdot \pi \in P$ for every $\pi \in \Pi^i$.
2. $\xi \star \pi \in \Lambda^i \star \Pi^i, \xi' \star \pi' \in P, \xi \star \pi \succ \xi' \star \pi' \Rightarrow \xi \star \pi \in P$
3. If at least two out of three processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in P , then $d \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \in P$.

Remarks.

The preorder \succ on $\Lambda \star \Pi$ was defined at the beginning of section 1.

We express condition 2 by saying that P is *saturated* in $\Lambda^i \star \Pi^i$.

Following this definition of \succ , the constant d is a *halting instruction*. Indeed, we have :

$$d \star \pi \succ \xi \star \varpi \Leftrightarrow \xi \star \varpi = d \star \pi.$$

We define \perp by : $\Lambda \star \Pi \setminus \perp = (\Lambda^0 \star \Pi^0 \setminus \perp_0^0) \cup (\Lambda^1 \star \Pi^1 \setminus \perp_1^1)$

In other words, a process is in \perp if and only if

either it is in $\perp_0^0 \cup \perp_1^1$ or it contains both stack constants π^0, π^1 .

Lemma 33. *If $\xi \star \pi \in \perp_j^i$ and $\xi \star \pi \succ \xi' \star \pi'$ then $\xi' \star \pi' \in \perp_j^i$ (closure by reduction).*

Suppose that $\xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$, $\xi_0 \star \pi_0 \in \perp_j^i$ and $\xi'_0 \star \pi'_0 \notin \perp_j^i$. We may suppose that :

(*) $\xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$ in exactly one step of reduction.

Let us show that $\perp_j^i \setminus \{\xi_0 \star \pi_0\}$ has properties 1,2,3 defining \perp_j^i , which will contradict the definition of \perp_j^i :

1. If $\xi_0 \star \pi_0 = d \star \underline{j} \cdot \pi$, with $\pi \in \Pi^i$, then $d \star \underline{j} \cdot \pi \succ \xi'_0 \star \pi'_0$, thus $\xi'_0 \star \pi'_0 = d \star \underline{j} \cdot \pi$.
Therefore $\xi'_0 \star \pi'_0 \in \perp_j^i$, which is false.

2. Suppose $\xi \star \pi \in \Lambda^i \star \Pi^i, \xi \star \pi \succ \xi' \star \pi' \in \perp_j^i, \xi' \star \pi' \neq \xi_0 \star \pi_0$. Then $\xi \star \pi \in \perp_j^i$, by (2).
If $\xi \star \pi = \xi_0 \star \pi_0$, then $\xi_0 \star \pi_0 \succ \xi' \star \pi'$; since $\xi' \star \pi' \neq \xi_0 \star \pi_0$, it follows from (*) that $\xi'_0 \star \pi'_0 \succ \xi' \star \pi'$ and therefore $\xi'_0 \star \pi'_0 \in \perp_j^i$, which is false.

3. Suppose that two out of the processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in $\perp_j^i \setminus \{\xi_0 \star \pi_0\}$, but $d \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi$ is not. From (3), it follows that $d \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi = \xi_0 \star \pi_0$.

Thus, $d \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \succ \xi'_0 \star \pi'_0$, and therefore $\xi'_0 \star \pi'_0 = d \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi$.

Therefore $\xi'_0 \star \pi'_0 \in \perp_j^i$, which is false.

Q.E.D.

Lemma 34. $\perp_0^0 \cap \perp_1^1 = \emptyset$.

We prove that $(\Lambda^i \star \Pi^i \setminus \perp_1^i) \supset \perp_0^i$ by showing that $\Lambda^i \star \Pi^i \setminus \perp_1^i$ has properties 1, 2, 3 which define \perp_0^i .

1. $\mathbf{d} \star \underline{0} \cdot \pi \notin \perp_1^i$ because $\perp_1^i \setminus \{\mathbf{d} \star \underline{0} \cdot \pi\}$ has properties 1, 2, 3 defining \perp_1^i .
 2. Follows from lemma 33.
 3. Suppose $\xi_0 \star \pi_0, \eta_0 \star \pi_0 \notin \perp_1^i$; we show that $\mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0 \notin \perp_1^i$ by showing that $\perp_1^i \setminus \{\mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0\}$ has properties 1, 2, 3 defining \perp_1^i .
 1. Clearly, $\mathbf{d} \star \underline{1} \cdot \pi' \in (\perp_1^i \setminus \{\mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0\})$ for every $\pi' \in \Pi^i$.
 2. Suppose that $\xi \star \pi \in \Lambda^i \star \Pi^i, \xi \star \pi \succ \xi' \star \pi' \in \perp_1^i, \xi' \star \pi' \neq \mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0$ and that $\xi \star \pi \notin (\perp_1^i \setminus \{\mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0\})$.
From (2), it follows that $\xi \star \pi = \mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0$ which contradicts $\xi \star \pi \succ \xi' \star \pi'$.
 3. Suppose that two out of the processes $\xi \star \pi, \eta \star \pi, \zeta \star \pi$ are in $\perp_1^i \setminus \{\mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0\}$ but that $\mathbf{d} \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi$ is not.
It follows from (3) that $\mathbf{d} \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi = \mathbf{d} \star \underline{2} \cdot \xi_0 \cdot \eta_0 \cdot \zeta_0 \cdot \pi_0$, i.e. $\xi = \xi_0, \eta = \eta_0, \zeta = \zeta_0$ and $\pi = \pi_0$. But this contradicts the hypothesis :
 $\xi_0 \star \pi_0, \eta_0 \star \pi_0 \notin \perp_1^i$.
- Q.E.D.

Theorem 35. *This realizability algebra is coherent.*

Let $\theta \in \text{QP}$ be such that $\theta \star \pi^0 \in \perp_0^0$ and $\theta \star \pi^1 \in \perp_1^1$. Then $\theta \star \pi^0 \in \perp_0^0 \cap \perp_1^0$ which contradicts lemma 34.

Q.E.D.

Lemma 36. $\mathbf{d} \underline{2} \Vdash$ (the boolean algebra $\mathfrak{B}2$ has at most four ε -elements).

We show that $\mathbf{d} \underline{2} \Vdash \forall x^{\mathfrak{B}2} \forall y^{\mathfrak{B}2} (x \neq 0, y \neq 1, x \neq y \rightarrow x \wedge y \neq x)$.

Let $i, j \in \{0, 1\}, \xi \Vdash i \neq 0, \eta \Vdash j \neq 1, \zeta \Vdash i \neq j$ and $\pi \in \|\!| i \wedge j \neq i \|\!$.

Since $\|\!| i \wedge j \neq i \|\!| \neq \emptyset$, we have $i \leq j$. Thus, there are three possibilities for (i, j) :

$i = j = 0$; $i = j = 1$; $i = 0, j = 1$.

In each case, two out of the terms ξ, η, ζ realize \perp . Thus, we have $\mathbf{d} \star \underline{2} \cdot \xi \cdot \eta \cdot \zeta \cdot \pi \in \perp$.

Q.E.D.

Remark. If $\pi \in \Pi \setminus (\Pi_0 \cup \Pi_1)$, then $\xi \star \pi \in \perp$ for every term ξ . Thus, we can remove these stacks and consider only $\Pi^0 \cup \Pi^1$.

We define two individuals in this realizability model :

$\gamma_0 = (\{0\} \times \Pi^0) \cup (\{1\} \times \Pi^1)$; $\gamma_1 = (\{1\} \times \Pi^0) \cup (\{0\} \times \Pi^1)$.

Obviously, $\gamma_0, \gamma_1 \subseteq \mathfrak{B}2 = \{0, 1\} \times \Pi$. Now we have :

$\|\!| \forall x^{\mathfrak{B}2} (x \notin \gamma_0) \|\!| = \Pi^0 \cup \Pi^1 = \|\!| \perp \|\!|$ and therefore $1 \Vdash \neg \forall x^{\mathfrak{B}2} (x \notin \gamma_0)$.

$\mathbf{d} \underline{0} \Vdash 0 \notin \gamma_0$ and $\mathbf{d} \underline{1} \Vdash 1 \notin \gamma_0$.

It follows that γ_0, γ_1 are not ε -empty and that every ε -element of γ_0, γ_1 is $\neq 0, 1$.

Therefore :

The Boolean algebra $\mathfrak{B}2$ has exactly four ε -elements.

We have $\xi \Vdash \forall x^{\mathfrak{B}2} (x \varepsilon \gamma_0, x \varepsilon \gamma_1 \rightarrow \perp)$ for every term ξ :

Indeed, $|i \varepsilon \gamma_0| = \{\mathbf{k}_\pi ; \pi \in \Pi^i\}$ for $i = 0, 1$ and $\xi \star \mathbf{k}_{\rho_0} \cdot \mathbf{k}_{\rho_1} \cdot \pi \in \perp$ if $\rho_i \in \Pi^i$.

In the same way, we get :

$\lambda x \lambda y \lambda z z \Vdash \forall x \forall y (x \varepsilon \gamma_i, y \varepsilon \gamma_i, x \neq y \rightarrow \perp)$.

It follows that γ_0, γ_1 are singletons and that their ε -elements are the two atoms of $\mathfrak{B}2$.

$\mathbb{J}2$ has four ε -elements and $\mathbb{J}\kappa$ is countable

We now apply to the algebra \mathcal{A} the technique expounded in section 3, in order to make $\mathbb{J}\kappa$ countable ; this gives a realizability algebra \mathcal{B} .

In this case, we have $\kappa = \mathbb{N}$, and therefore $\kappa_+ = \mathcal{P}(\kappa) = \mathbb{R}$.

Now, there is an elementary formula which express that the Boolean algebra $\mathbb{J}2$ has four ε -elements, for instance : $\exists x^{\mathbb{J}2}\{x \neq 0, x \neq 1\} \wedge \forall x^{\mathbb{J}2}\forall y^{\mathbb{J}2}(x \neq 1, y \neq 1, x \neq y \rightarrow xy = 0)$.

Therefore, the realizability model $\mathcal{N}_{\mathcal{B}}$ realizes the following two formulas :

($\mathbb{J}2$ has four ε -elements) ; ($\mathbb{J}\kappa$ is countable).

Let us denote by i_0, i_1 the two atoms of $\mathbb{J}2$; thus, we have $i_1 = 1 - i_0$.

We suppose that $\mathcal{M} \models V = L$; thus, there exists on $\kappa_+ = \mathcal{P}(\mathbb{N}) = \mathbb{R}$ a strict well ordering \triangleleft of type \aleph_1 . This gives a function from \mathbb{R}^2 into $\{0, 1\}$, denoted by $(x \triangleleft y)$, which is defined as follows : $(x \triangleleft y) = 1 \Leftrightarrow x \triangleleft y$.

We can extend it to $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$, which gives a function from $(\mathbb{J}\mathbb{R})^2$ into $\mathbb{J}2$.

From lemmas 21 and 22, we get :

For $i = i_0$ or i_1 , the relation $(x \triangleleft y) = i$ is a strict total ordering on $\mathbb{J}_i\mathbb{R}$ and one of these two relations is a well ordering ;

in order to fix the ideas, we shall suppose that it is for $i = i_0$.

The relation $(x \triangleleft y) = 1$ is a strict order relation on $\mathbb{J}\mathbb{R}$, which is well founded.

The application $x \mapsto (P_{i_0}x, P_{i_1}x)$ from $\mathbb{J}\mathbb{R}$ onto $\mathbb{J}_{i_0}\mathbb{R} \times \mathbb{J}_{i_1}\mathbb{R}$ is an isomorphism of strictly ordered sets.

It follows from theorem 18, that each of the sets $\mathbb{J}_{i_0}\mathbb{R}$, $\mathbb{J}_{i_1}\mathbb{R}$ contain a countable subset.

By corollary 16, there is no surjection from each one of the sets $\mathbb{J}_{i_0}\mathbb{R}$, $\mathbb{J}_{i_1}\mathbb{R}$ onto the other.

Thus, there is no surjection from \mathbb{N} onto $\mathbb{J}_{i_0}\mathbb{R}$ or onto $\mathbb{J}_{i_1}\mathbb{R}$.

Therefore, the well ordering on $\mathbb{J}_{i_0}\mathbb{R}$ has, at least, the order type \aleph_1 in $\mathcal{N}_{\mathcal{B}}$.

Now, by theorem 24, every subset of $\mathbb{J}\mathbb{R}$, which is bounded from above for the ordering \triangleleft , is countable ; thus, it is the same for the proper initial segments of $\mathbb{J}_{i_0}\mathbb{R}$ and $\mathbb{J}_{i_1}\mathbb{R}$, since these sets are totally ordered and $\mathbb{J}\mathbb{R}$ is isomorphic to $\mathbb{J}_{i_0}\mathbb{R} \times \mathbb{J}_{i_1}\mathbb{R}$.

It follows that the well ordering on $\mathbb{J}_{i_0}\mathbb{R}$ is at most \aleph_1 , and therefore exactly \aleph_1 .

Moreover, *there exists, on $\mathbb{J}_{i_1}\mathbb{R}$, a total ordering, every proper initial segment of which is countable.*

Then, we can apply theorem 26, to the sets X_{i_0}, X_{i_1} which are the images of $\mathbb{J}_{i_0}\mathbb{R}, \mathbb{J}_{i_1}\mathbb{R}$ by the injection from $\mathbb{J}\kappa_+$ into \mathbb{R} , which is given by theorem 25. By setting $X = X_{i_1}$, we obtain exactly the result (ii) of relative consistency announced in the introduction.

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