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SEMI-LAGRANGIAN DISCONTINUOUS GALERKIN SCHEMES FOR SOME FIRST AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

OLIVIER BOKANOWSKI AND GIOREVINUS SIMARMATA

ABSTRACT. Explicit, unconditionally stable, high order schemes for the approximation of some first and second order linear, time-dependent partial differential equations (PDEs) are proposed. The schemes are based on a weak formulation of a semi-Lagrangian scheme using discontinuous Galerkin (DG) elements. It follows the ideas of the recent works of Crouseilles, Mehrenberger and Vecil (2010), Rossmannith and Seal (2011), for first order equations, based on exact integration, quadrature rules, and splitting techniques for the treatment of two-dimensional PDEs. For second order PDEs the idea of the scheme is a blending between weak Taylor approximations and projection on a DG basis. New and sharp error estimates are obtained for the fully discrete schemes and for variable coefficients. In particular we obtain high order schemes, unconditionally stable and convergent, in the case of linear first order PDEs, or linear second order PDEs with constant coefficients. In the case of non-constant coefficients, we construct, in some particular cases, "almost" unconditionally stable second order schemes and give precise convergence results. The schemes are tested on several academic examples.

1. INTRODUCTION

In this paper we consider equations of the form

$$u_t - \frac{1}{2} \text{Tr}(\sigma \sigma^T D^2 u) + b \cdot \nabla u + ru = 0, \quad x \in \Omega, \quad t \in (0, T), \quad (1)$$

where $\Omega \subset \mathbb{R}^d$ is a box (with some boundary conditions on $\partial\Omega$), σ (matrix), b (vector) and r (scalar) may be x -dependent, at least Lipschitz continuous, together with an initial condition

$$u(0, x) = u_0(x), \quad x \in \Omega, \quad (2)$$

with $u_0 \in L^2(\Omega)$. The matrix σ may be zero or degenerated. Unless otherwise precised, we will in general assume periodic boundary conditions for (1) in order to avoid difficulties on the boundary. We will assume sufficient regularity on the data in order to have existence and uniqueness of weak solutions of (1)-(2), and so that $t \rightarrow u(t, \cdot)$ be in $C^0([0, T], L^2(\Omega))$.

We study and propose new Semi-Lagrangian Discontinuous Galerkin schemes, also abbreviated "SLDG" in this work, in order to approximate the solutions of (1)-(2).

The Semi-Lagrangian (SL) approach [11] is based on the approximation of the "method of characteristics". By considering a weak formulation of this principle,

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an explicit SLDG scheme is obtained. In the case of first order PDEs, in the case of constant coefficient, our approach is based on a similar method than in the recent works of Crouseilles, Mehrenberger and Vecil [7] (for the Vlasov equation in plasma physics), Rossmannith and Seal [29]. However our approach seems not to have been considered for variable coefficients. It is slightly different from the work of Qiu and Shu [26] (see also Restelli et al [27]), where first a weak formulation of the PDE is considered, and then quadrature formula are used (see also [28] for the original approach). Here we will furthermore introduce new SLDG schemes for second order PDEs on which we prove stability and convergence results, and obtain higher-order accuracy when possible.

First, in Section 2, we revisit the one-dimensional first order advection equation with non-constant advection term $b(x)$ (case $\sigma = 0$ in (1)). We give a new unconditional stability result, and convergence proof, extending similar results of [7], [29] (or [26]) that was obtained for the case of a constant advection term. The unconditional stability property can be interesting when compared to a standard DG approach where a restrictive CFL condition (and a small enough time step) must in general be considered [5].

Based on the operator construction for first order advection, we then introduce, in Section 3, new schemes for linear second order PDEs of type (1), in the form of explicit high order SLDG schemes. These schemes are based, for the temporal discretization, on the use of "weak Taylor approximations", see in particular the review book by Kloeden and Platen [15] (see also Kushner [16] and the review book by Kushner and Dupuis [17], Platen [25], Milstein [20], Talay [32], Pardoux and Talay [23], Menaldi [19], Camilli and Falcone [3], Milstein and Tretyakov [21], [8]). Such approximations were used by R. Ferretti in [12] as well as in Debrabant and Jakobsen [9] in the context of semi-Lagrangian schemes, using interpolation methods for the space variable. The problem of coupling such approximation with a spatial grid approximation, in particular using an interpolation method, is the stability of the method. Indeed, only the P_1 interpolation is known to be L^∞ stable. However, it is only second order accurate in space (for regular data).

The schemes of the present paper can be seen as projections of these approximation on a discontinuous Galerkin basis. We will in particular propose a second order approximation (in time) corresponding to a Platen's scheme [15, Chapter 14], but higher order approximations (in time) could be obtained in the same way. The scheme will be proved to be also high-order in space, stable and convergent under a weak CFL condition (of the form $\Delta x^4 \leq \lambda \Delta t$ for some constant λ , where Δt and Δx denote the time and mesh steps).

For the more simple case of second order PDEs with constant coefficients, we also propose explicit and unconditionally stable schemes, high order in space and up to third order in time (higher order can be obtained [1]).

In section 4 we consider extensions to some linear two-dimensional PDEs. For first order PDEs, we show how to combine the scheme with higher-order splitting techniques, like Strang's splitting, but also Ruth's third order splitting [30], Forest's fourth-order splitting [13] and Yoshida's sixth order splitting [35] (see also [14] and [36]). A splitting strategy to treat general second order PDEs with constant coefficients is explained. The case of variable diffusion coefficients is discussed but only treated in some specific cases. (The general case will be treated in a forthcoming work).

Finally in Section 5 we show the relevance of our approach on several academic numerical examples in one and two dimensions (using Cartesian meshes), including also a Black and Scholes PDE in mathematical Finance.

The advantage of the proposed schemes is that they combine the DG framework which allows high-order spatial accuracy and the potential of degree adaptivity, together with unconditional stability properties in the L^2 norm from the weak formulation of the semi-Lagrangian scheme.

Note that our general strategy is to use a cartesian grid, a particular one-dimensional advection scheme, and splitting techniques (for more standard Discontinuous Galerkin approaches, see for instance [6] or [24]).

Ongoing works using the current approach concern the obtention of higher order schemes for general second order PDEs [1], extensions to nonlinear PDEs arising from deterministic control [2] or from stochastic control.

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2. ADVECTION EQUATION

We first consider the Semi-Lagrangian Discontinuous Galerkin scheme (SLDG for short) for the following one-dimensional first-order PDEs, as in [7]

$$\begin{cases} v_t + b(x)v_x = 0, & (t, x) \in (0, T) \times \Omega \\ v(0, x) = v_0(x), & x \in \Omega \end{cases} \quad (3)$$

where $\Omega = (x_{min}, x_{max})$, together with periodic boundary conditions on Ω .

In order to simplify the presentation and the proofs, we will assume that $\Omega = (0, 1)$ and that b is a 1-periodic function.

Let $y = y_x$ denote the solution of the differential equation

$$\begin{cases} \dot{y}(t) = b(y(t)), & t \in \mathbb{R} \\ y(0) = x \end{cases} \quad (4)$$

We will also assume that $b(\cdot)$ is Lipschitz continuous.

Let $N \in \mathbb{N}$, $N \geq 1$, $\Delta t = \frac{T}{N}$ a time step and $t_n = n\Delta t$ a time discretisation. Let

$$v^n(x) := v(t_n, x).$$

By the method of characteristics, the solution of (3) satisfies

$$v^{n+1}(x) = v^n(y_x(-\Delta t)). \quad (5)$$

Then we aim to obtain a fully discrete scheme.

Let us consider a space discretization that is considered uniform for simplification of presentation. Let $\Delta x = \frac{x_{max} - x_{min}}{M}$ for some integer $M \geq 1$, $x_{i-\frac{1}{2}} := x_{min} + i\Delta x$, $\forall i = 0, \dots, M$, and $I_i := (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. Let $k \in \mathbb{N}$. We define V_k as the space of discontinuous-Galerkin elements on Ω with polynomials of degree k , that is:

$$V_k = \{v \in L^2(\Omega, \mathbb{R}) : v|_{I_i} \in P_k, \forall i = 0, \dots, M-1\} \quad (6)$$

whereas P_k denotes the set of polynomials of degree at most k .

Remark 2.1. *In the classical Semi-Lagrangian approach, looking for $u^n(x)$, an approximation of $v(t_n, x)$, a first "direct" iterative scheme for (5) would be*

$$u^{n+1}(x_i) = [u^n](y_{x_i}(-\Delta t)) \quad (7)$$

where $[u^n](x)$ denotes some interpolation of the function u^n at point x . We could take for instance a set of $k+1$ values $(x_\alpha^i)_{\alpha=0, \dots, k}$ in each interval I_i , and define the new polynomial u^{n+1} such that $u^{n+1}(x_\alpha^i) := [u^n](x_\alpha^i - b\Delta t)$ for all $\alpha = 0, \dots, k$. However, given the discontinuities between the intervals I_i , this may lead to instabilities in the scheme ([26]). For instance, taking x_α^i to be the Gauss quadrature points on each interval I_i is in general unstable (see Appendix A, see also [22]).

Here we consider a Lagrange-Galerkin approach by taking the weak form of (5): for $n = 0, \dots, N-1$, find $u^{n+1} \in V_k$ such that

$$\int_{\Omega} u^{n+1}(x)\varphi(x)dx = \int_{\Omega} u^n(y_x(-\Delta t))\varphi(x)dx, \quad \forall \varphi \in V_k, \quad (8)$$

and for $n = 0$, find $u^0 \in V_k$ such that:

$$\int_{\Omega} u^0(x)\varphi(x)dx = \int_{\Omega} v_0(x)\varphi(x)dx, \quad \forall \varphi \in V_k. \quad (9)$$

From now on, we rewrite (8) in the following abstract form :

$$u^{n+1} = \mathcal{T}_{b\Delta t}(u^n).$$

In the case of a constant coefficient b , $y_x(-\Delta t) = x - b\Delta t$, and $u^n(x - b\Delta t)$ is a piecewise constant polynomial. The integral $\int_{I_i} u^n(x - b\Delta t)\varphi(x)dx$ will have in general two regular parts. Each part involves a polynomial of degree at most $2k$ and the Gaussian quadrature rule with $k+1$ points is applied and is exact. At this stage the method is the same as in [7], or [29]. Hence the new function u^{n+1} can be computed by solving exactly (8).

However, if $b(x)$ is not a constant, $x \rightarrow u^n(y_x(-\Delta t))$ is no more a piecewise polynomial. Therefore the computing procedure for the right-hand-side (R.H.S.) of (8) can no more be exact.

In order to obtain an implementable scheme, a precise EDO integration for the characteristics and a quadrature rule can be used. We follow an approach very similar to [26] for variable coefficients. It consists in using gaussian quadrature formula to approximate (8) in regions where the involved functions are smooth.

Remark 2.2. *Indeed, in [26], an other SLDG scheme is presented, but our form is equivalent to one form of SLDG as explained in [26, Proposition 4.5]. This may lead to different programming algorithms however.*

2.1. Preliminaries. Let $\{x_\alpha\}_{\alpha=0, \dots, k}$ be the set of Gauss points in the interval $(-1, 1)$, with its corresponding weights $\{w_\alpha\}_{\alpha=0, \dots, k}$ ($w_\alpha > 0$), such that:

$$\forall p \in P_{2k+1}, \quad \int_{-1}^1 p(x)dx = \sum_{\alpha=0}^k w_\alpha p(x_\alpha). \quad (10)$$

In particular, we get on the interval I_i ,

$$\forall p \in P_{2k+1}, \quad \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} p(x) dx = \sum_{\alpha=0}^k w_{\alpha}^i p(x_{\alpha}^i), \quad (11)$$

where $x_{\alpha}^i := x_i + x_{\alpha} \Delta x \equiv x_{i-\frac{1}{2}} + \frac{1}{2}(1 + x_{\alpha}) \Delta x$ and $w_{\alpha}^i := \frac{\Delta x}{2} w_{\alpha}$.

From each set of Gauss points $\{x_{\alpha}^i\}_{\alpha=0,\dots,k}$ in I_i , we can associate the corresponding Lagrange polynomials (dual basis) $\{\varphi_{\alpha}^i\}_{\alpha=0,\dots,k}$ defined by

$$\varphi_{\alpha}^i(x) := 1_{I_i}(x) \prod_{\substack{0 \leq \beta \leq k \\ \beta \neq \alpha}} \frac{x - x_{\beta}}{x_{\alpha} - x_{\beta}}. \quad (12)$$

For any $u^n \in V_k$, there exists coefficients $(u_{\alpha,i}^n)_{\alpha=0,\dots,k}^{i=0,\dots,M-1} \in \mathbb{R}$ such that:

$$u^n(x) = \sum_{i=0}^{M-1} \sum_{\alpha=0}^k u_{\alpha,i}^n \varphi_{\alpha}^i(x). \quad (13)$$

In particular, the left-hand side of (8) for $\varphi = \varphi_{\alpha}^i$ becomes

$$\int_{\Omega} u^{n+1}(x) \varphi_{\alpha}^i(x) dx = \int_{I_i} u^{n+1}(x) \varphi_{\alpha}^i(x) dx = u_{\alpha,i}^{n+1} w_{\alpha}^i.$$

2.2. Scheme definition in the general case. Due to the discontinuities of u^n , we separate the right-hand side of (8) into several integral parts involving only regular functions: the R.H.S. of (8) is approximated by the Gaussian quadrature rule on each sub-interval where $u^n(y_x(-\Delta t))$ is a regular function.

For a given mesh cell I_i , we first consider the points $(x_{i,q})_{1 \leq q \leq p_i}$ (in finite number) of the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, such that for $1 \leq q \leq p_i$, $y_{x_{i,q}}(-\Delta t) = x_{\ell_{i,q} - \frac{1}{2}}$ for some $\ell_{i,q} \in \mathbb{Z}$, and $x_{i,0} := x_{i-\frac{1}{2}}$, $x_{i,p_i+1} := x_{i+\frac{1}{2}}$ (see Figure 1). Then we apply the Gaussian quadrature rule on each interval $J_{i,q} = (x_{i,q}, x_{i,q+1})$ and obtain the following quadrature rule, for any polynomial $\varphi \in V_k$:

$$\int_{I_i} u^n(y_x(-\Delta t)) \varphi(x) dx = \sum_{q=0}^{p_i} \int_{x_{i,q}}^{x_{i,q+1}} u^n(y_x(-\Delta t)) \varphi(x) dx \quad (14)$$

$$\simeq \sum_{q=0}^{p_i} \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i u^n(y_{\tilde{x}_{q,\alpha}^i}(-\Delta t)) \varphi(\tilde{x}_{q,\alpha}^i), \quad (15)$$

with $\tilde{w}_{q,\alpha}^i := \frac{w_{\alpha}}{2}(x_{i,q+1} - x_{i,q})$ and $\tilde{x}_{q,\alpha}^i := x_{i,q} + \frac{1}{2}(1 + x_{\alpha})(x_{i,q+1} - x_{i,q}) \equiv \frac{x_{i,q} + x_{i,q+1}}{2} + x_{\alpha} \left(\frac{x_{i,q+1} - x_{i,q}}{2} \right)$.

Scheme definition (operator $\tilde{T}_{b,\Delta t}$): u^{n+1} is the unique element of V_k satisfying for all $\varphi \in V_k$,

$$\int_{\Omega} u^{n+1}(x) \varphi(x) dx = \sum_{i=0}^{M-1} \sum_{q=0}^{p_i} \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i u^n(y_{\tilde{x}_{q,\alpha}^i}(-\Delta t)) \varphi(\tilde{x}_{q,\alpha}^i). \quad (16)$$

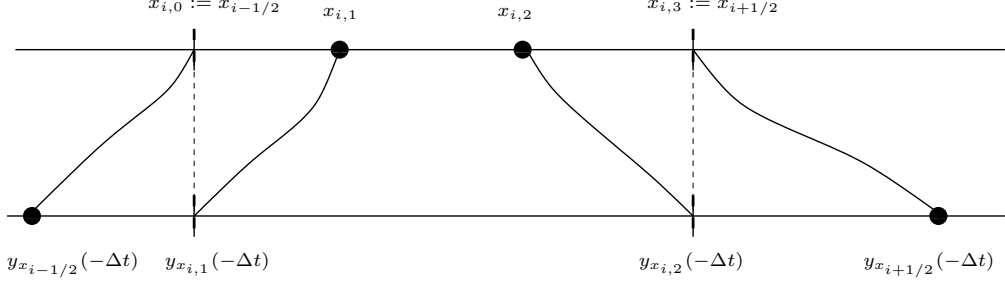


FIGURE 1. Determination of the point of discontinuity of the data.

The scheme is made explicit by using formula (16) on each $\varphi = \varphi_\beta^j$. The scheme equivalently defines an operator $\tilde{\mathcal{T}}_{b,\Delta t}$ such that

$$u^{n+1} = \tilde{\mathcal{T}}_{b,\Delta t} u^n.$$

In particular, if b is constant, then $\tilde{\mathcal{T}}_{b,\Delta t} = \mathcal{T}_{b,\Delta t}$, and this is no more true if b is non-constant.

Definition 1. For further analysis, let us introduce the following scalar product on V_k (where the index "G" stands for the use of the Gaussian quadrature rule):

$$(a, b)_G := \sum_{i=0}^{M-1} \sum_{q=0}^{p_i} \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i a(\tilde{x}_{q,\alpha}^i) b(\tilde{x}_{q,\alpha}^i). \quad (17)$$

Then the scheme (16) is equivalently defined by

$$(u^{n+1}, \varphi) = (u^n(y(-\Delta t)), \varphi)_G, \quad \forall \varphi \in V_k.$$

2.3. Stability and error estimate for constant drift coefficient. The weak form (8) gives the stability of the scheme in the L^2 norm, at least in the case when $b = \text{const}$. Indeed, taking $\varphi = u^{n+1}$ in (8) we get

$$\|u^{n+1}\|_{L^2}^2 = (u^n(\cdot - b\Delta t), u^{n+1}) \leq \|u^n(\cdot - b\Delta t)\|_{L^2} \|u^{n+1}\|_{L^2},$$

where $\|\cdot\|_{L^2}$ denotes the L^2 norm on Ω and (\cdot, \cdot) is the associated scalar product. Then, by the periodic boundary condition, $\|u^n(\cdot - b\Delta t)\|_{L^2} = \|u^n\|_{L^2}$ and therefore

$$\|u^{n+1}\|_{L^2} \leq \|u^n\|_{L^2}. \quad (18)$$

This proof works only for b constant, however.

For any $w \in L^2$, we denote its projection on V_k by Πw , corresponding to the unique element of V_k such that

$$\|w - \Pi w\|_{L^2} = \inf_{f \in V_k} \|w - f\|_{L^2}. \quad (19)$$

Remark 2.3. The function u^{n+1} defined by (8) corresponds to the projection of the function $x \rightarrow u^n(y_x(-\Delta t))$ on the space V_k :

$$u^{n+1} = \Pi(u^n(y(-\Delta t))),$$

and, in the same way, we have $u^0 = \Pi v_0$.

We now recall a simple estimate for the L^2 projection on V_k .

Lemma 2.1 (Projection error). *Let $k \geq 0$ and $\ell \leq k$. If $w \in \mathcal{C}^{\ell+1}$, then*

$$\|w - \Pi w\|_{L^2} \leq |\Omega|^{1/2} C_\ell(w) \Delta x^{\ell+1}$$

where $C_\ell(w) := \frac{1}{2^{\ell+1}(\ell+1)!} \|w^{(\ell+1)}\|_{L^\infty}$.

Proof. Let us write $w = P + R$ where P is the element of V_k corresponding, on each interval I_i , to the Taylor expansion of w centered at x_i and of degree ℓ . We have $\|w - \Pi w\|_{L^2} \leq \|w - P\|_{L^2} = \|R\|_{L^2} \leq |\Omega|^{1/2} \|R\|_{L^\infty}$. By the definition of R and usual Taylor estimates, we have $\|R\|_{L^\infty} \leq C_\ell \Delta x^{\ell+1}$. \square

Let $v^n(x) := v(t_n, x)$ where v denotes the exact solution of (3). Using the L^2 -stability of the projection, it is straightforward to show that $\|u^{n+1} - \Pi v^{n+1}\|_{L^2} = \|\Pi(u^n(\cdot - b\Delta t) - v^n(\cdot - b\Delta t))\|_{L^2} \leq \|u^n - v^n\|_{L^2}$, therefore we have

$$\|u^{n+1} - v^{n+1}\|_{L^2} \leq \|u^n - v^n\|_{L^2} + \|v^{n+1} - \Pi v^{n+1}\|_{L^2}.$$

By using Lemma 2.1, this leads to the following known convergence result [26].

Theorem 2.1. *Let $k \geq 0$ and b be a constant. Assume the initial condition v_0 is 1-periodic and in \mathcal{C}^{k+1} . The following estimate holds:*

$$\|u^n - v^n\|_{L^2} \leq \|u^0 - v^0\|_{L^2} + CT \frac{\Delta x^{k+1}}{\Delta t}, \quad \forall n \leq N, \quad (20)$$

where the constant C depends only of $|\Omega|$ and k .

Remark 2.4. *By taking $\Delta t = \Delta x$ this leads to an error estimate in $O(\Delta x^k)$. However the examples (such as in Example 1) will show a numerical behavior in $O(\Delta x^{k+1})$ (as already remarked also in [26]). We refer to the recent work in [31] for more insight about this gap.*

2.4. Non-constant b : preliminary results. For $u \in V_k$, the following approximation result is central. It controls the error between the desired formula (8) and the implementable scheme (16).

Proposition 2.1 (Gauss quadrature errors). *Let $k \geq 0$ and let b be of class \mathcal{C}^{2k+2} and 1-periodic.*

(i) *For all $u \in V_k$,*

$$\left| (u(y.(-\Delta t)), \varphi)_G - (u(y.(-\Delta t)), \varphi) \right| \leq C \Delta t \Delta x^2 \|u\|_{L^2} \|\varphi\|_{L^2}, \quad \forall \varphi \in V_k.$$

where $C \geq 0$ is a constant. In particular, we have, in the L^2 -norm:

$$\tilde{\mathcal{T}}_{b, \Delta t} u^n \equiv u^{n+1} = \mathcal{T}_{b, \Delta t} u^n + O(\Delta t \Delta x^2 \|u^n\|_{L^2}), \quad \forall n \geq 0. \quad (21)$$

(ii) *For all $u \in V_k$, for any ψ in \mathcal{C}^{k+1} , 1-periodic,*

$$\begin{aligned} & \left| (u(y.(-\Delta t)) - \psi(y.(-\Delta t)), \varphi)_G - (u(y.(-\Delta t)) - \psi(y.(-\Delta t)), \varphi) \right| \\ & \leq C \Delta t \Delta x^2 \|u - \psi\|_{L^2} \|\varphi\|_{L^2} + CM_{k+1}(\psi) \Delta x^{k+1} \|\varphi\|_{L^2}, \quad \forall \varphi \in V_k, \end{aligned} \quad (22)$$

where $C \geq 0$ is a constant which depends only of k , and

$$M_p(\psi) := \max_{0 \leq r \leq p} \|\psi^{(r)}\|_{L^\infty}. \quad (23)$$

(iii) *For any regular $\psi \in \mathcal{C}^{k+1}$, for any $\varphi \in V_k$,*

$$(\psi, \varphi)_G = (\psi, \varphi) + O(M_{k+1}(\psi) \Delta x^{k+1} \|\varphi\|_{L^2}). \quad (24)$$

(iv) Furthermore, $\exists C \geq 0$, for any $\psi \in \mathcal{C}^{k+1}$, 1-periodic,

$$\|\tilde{\mathcal{T}}_{b,\Delta t}\psi - \mathcal{T}_{b,\Delta t}\psi\|_{L^2} \leq CM_{k+1}(\psi)\Delta x^{k+1}. \quad (25)$$

Remark 2.5. Some assumptions can be weakened, for instance (i) and (ii) are still valid using that $b^{(2k+1)}$ is in L^∞ , then in the error bounds (21) and (22) the $\Delta t\Delta x^2$ term should be replaced by $\Delta t\Delta x$. However these bounds will be used in Section 3 and the form (21) and (22) is preferred. Also, it is possible to prove that the error term $O(M_{k+1}(\psi)\Delta x^{k+1})$ in (ii), (iii) and (iv) can be improved to $O(M_{2k+1}(\psi)\Delta x^{k+2})$ provided that $\psi \in \mathcal{C}^{2k+1}$.

Proof of Proposition 2.1

Notice that the estimates of (i) and (iii) are a consequence of (ii) (either by choosing $\psi \equiv 0$ to obtain (i), or by choosing $\Delta t \equiv 0$ and $u \equiv 0$ to obtain (iii)). Then (iv) is deduced from (iii) when applied to the regular function $\psi_1(x) := \psi(y_x(-\Delta t))$.

The plan is first to prove (i), and then to generalize to (ii). Precise estimates for the $2k+2$ derivative of $x \rightarrow u(y_x(-\Delta t))$ will be needed in order to estimate the error when using Gaussian quadrature formula. In the following, we first bound the derivatives of $x \rightarrow y_x(-t)$.

Lemma 2.2. Assume that $b \in \mathcal{C}^k$, for some $k \geq 1$, and 1-periodic. Let $L := \|b'\|_{L^\infty}$ and let $t \in \mathbb{R}$. Then $x \rightarrow y \equiv y_x(-t)$ is of class \mathcal{C}^k , 1-periodic, and

$$\begin{cases} \|y\|_{L^\infty(0,1)} \leq 1 + \|b\|_{L^\infty}|t|, \\ \|\frac{\partial}{\partial x}y\|_{L^\infty(0,1)} \leq e^{L|t|}, \\ \text{and, if } k \geq 2, \quad \|\frac{\partial^q}{\partial x^q}y\|_{L^\infty(0,1)} \leq C|t|^{q-1}e^{L|t|}, \quad \forall q \in [2, \dots, k], \end{cases} \quad (26)$$

for some constant $C \geq 0$. In particular, all the previous derivatives are bounded on a fixed time interval $t \in [0, T]$.

Proof. We consider y as a function of the time t and of x . We can assume that $x \in [0, 1]$ since we have $y_{k+x}(t) = k + y_x(t)$ for all $k \in \mathbb{Z}$ and $t, x \in \mathbb{R}$. We denote $y^{(k)} \equiv \frac{\partial^k}{\partial x^k}y$ the k -th derivative of y with respect to x .

Firstly, $y(t, x) = x + \int_0^t b(y(s, x))ds$ and therefore, for $x \in (0, 1)$, $|y(t, x)| \leq 1 + \|b\|_{L^\infty}|t|$.

For $k = 1$ and $b \in \mathcal{C}^1$, we have $\frac{\partial}{\partial t}\frac{\partial}{\partial x}y = b'(y)\frac{\partial}{\partial x}y$ and $\frac{\partial}{\partial x}y(0) = 1$, therefore $|\frac{\partial}{\partial x}y(t)| = \exp(\int_0^t b'(y(s))ds) \leq e^{L|t|}$.

For $k \geq 2$, we have

$$\begin{aligned} \frac{\partial}{\partial t}y^{(k)} &= (b'(y)y^{(1)})^{(k-1)} \\ &= b'(y)y^{(k)} + \sum_{\ell=1}^{k-1} C_{k-1}^\ell (b'(y))^{(\ell)} y^{(k-\ell)}. \end{aligned}$$

Then we use a recursion argument for $\ell = 1, \dots, k$. Let us assume that the spatial derivatives $y^{(\ell)}$ are bounded for $1 \leq \ell \leq k-1$, with $\|y^{(\ell)}\|_{L^\infty(0,1)} \leq C_\ell |t|^{\ell-1} e^{L|t|}$. Then for $k \geq 2$, the function $f := \sum_{\ell=1}^{k-1} C_{k-1}^\ell (b'(y))^{(\ell)} y^{(k-\ell)}$ is bounded, with a bound of the form $\|f(\cdot, t)\|_{L^\infty(0,1)} \leq C|t|^{k-2} e^{L|t|}$, for some constant C . By using

the formula, for a given and fixed x ,

$$y^{(k)}(t) = e^{\int_0^t b'(y(s))ds} y^{(k)}(0) + \int_0^t e^{\int_s^t b'(y(\theta))d\theta} f(s) ds,$$

the fact that $y^{(k)}(0) = 0$ for $k \geq 2$ and for $s \in [0, t]$ (or $s \in [t, 0]$ if $t \leq 0$):

$$\begin{aligned} |e^{\int_s^t b'(y(\theta))d\theta} f(s)| &\leq C e^{L|t-s|} |s|^{k-2} e^{L|s|} \\ &\leq C |t|^{k-2} e^{L|t|} \end{aligned}$$

we conclude to $|y^{(k)}(t)| \leq C |t|^{k-1} e^{L|t|}$. \square

Lemma 2.3. *Assume $q \geq k+1$, and $u \in V_k$. On any interval J where u is regular,*

$$\left\| \frac{d^q}{dx^q}(u(y)) \right\|_{L^\infty(J)} \leq C \Delta t \sum_{p=1}^k \|u^{(p)}\|_{L^\infty(y(J))}.$$

Proof. We first recall an expression for the q -th derivative of the composite function $u(y)$, also known as "Faà di Bruno's formula" [10]:

$$\frac{1}{q!} \frac{d^q}{dx^q}(u(y(x))) = \sum_{p=1}^k u^{(p)}(y(x)) \left(\sum_{(\alpha_j), \sum_j \alpha_j = p, \sum_j j \alpha_j = q} \frac{(y^{(1)}/1!)^{\alpha_1} \dots (y^{(q)}/q!)^{\alpha_q}}{\alpha_1! \dots \alpha_q!} \right). \quad (27)$$

Here the sum is limited to $p \leq k$ (instead of $p \leq q$) since $u \in V_k$.

Therefore, together with Lemma 2.2, we obtain the bound

$$\left\| \frac{d^q}{dx^q}(u(y)) \right\|_{L^\infty(J)} \leq C \sum_{p=1}^k \|u^{(p)}\|_{L^\infty(y(J))} \left(\sum_{(\alpha_j), \sum_{j=1}^q \alpha_j = p, \sum_{j=1}^q j \alpha_j = q} \Delta t^{\alpha_2 + \dots + \alpha_q} \right).$$

The case when $\alpha_2 = \dots = \alpha_q = 0$ happens only if $\alpha_1 = p = q$. Since $q \geq k+1$, and $p \leq k$, this case never occurs. Therefore the power of Δt is at least 1, which concludes the proof. \square

Proof of Proposition 2.1(i): Let ε be the error term, defined by

$$\varepsilon := \int_0^1 u(y_x(-\Delta t)) \varphi(x) dx - \sum_{i=0}^{M-1} \sum_{q=0}^{p_i} \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i u(y_{\tilde{x}_{q,\alpha}^i}(-\Delta t)) \varphi(\tilde{x}_{q,\alpha}^i).$$

We have $\varepsilon = \sum_i \sum_{q=0}^{p_i} \varepsilon_{i,q}$ where

$$\varepsilon_{i,q} := \int_{J_{i,q}} u(y_x(-\Delta t)) \varphi(x) dx - \sum_{\alpha=0}^k \tilde{w}_{q,\alpha}^i u(y_{\tilde{x}_{q,\alpha}^i}(-\Delta t)) \varphi(\tilde{x}_{q,\alpha}^i) \quad (28)$$

and with $J_{i,q} := (x_{i,q}, x_{i,q+1})$.

Let $u(y)$ be the function $x \rightarrow u(y_x(-\Delta t))$. Since $u(y)$ is C^{2k+2} regular on $J_{i,q}$ for each fixed $i, q \in [0, \dots, p_i]$, and that the R.H.S. of (28) corresponds to the Gaussian quadrature rule on $J_{i,q}$, we have in particular

$$|\varepsilon_{i,q}| \leq C \Delta x_{i,q}^{2k+3} \|[u(y)\varphi]^{(2k+2)}\|_{L^\infty(J_{i,q})},$$

where $\Delta x_{i,q} := x_{i,q+1} - x_{i,q}$.

On the other hand, since $\varphi \in V_k$,

$$\|[u(y)\varphi]^{(2k+2)}\|_{L^\infty(J_{i,q})} \leq C \sum_{r=0}^k \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|[u(y)]^{(2k+2-r)}\|_{L^\infty(J_{i,q})}.$$

For all $r \in [0, \dots, k]$ we have $2k+2-r \geq k+2 \geq k+1$, hence we can use Lemma 2.3 and obtain the bound

$$\| [u(y)\varphi]^{(2k+2)} \|_{L^\infty(J_{i,q})} \leq C \left(\sum_{r=0}^k \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \right) \Delta t \left(\sum_{p=1}^k \|u^{(p)}\|_{L^\infty(y(J_{i,q}))} \right).$$

In particular,

$$\sum_{i,q} |\varepsilon_{i,q}| \leq C \sum_{r=0}^k \sum_{p=1}^k \sum_i \sum_{q=0}^{p_i} \Delta t \Delta x_{i,q}^{2k+3} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|u^{(p)}\|_{L^\infty(y(J_{i,q}))}$$

By a scaling argument [4, 18], and using that $\varphi \in V_k$ for fixed k , we have, $\forall 0 \leq r \leq k$,

$$\|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \leq \frac{C}{\Delta x_{i,q}^{r+1/2}} \|\varphi\|_{L^2(J_{i,q})} \leq \frac{C}{\Delta x_{i,q}^{k+1/2}} \|\varphi\|_{L^2(J_{i,q})}, \quad (29)$$

for some constant C , assuming also $\Delta x_{i,q} \leq 1$ (the idea is to use the fact that for polynomials of degree k , by using norm equivalences, $\|\varphi^{(r)}\|_{L^\infty(0,1)} \leq C\|\varphi\|_{L^2(0,1)}$ for some constant C independant of φ , and then to use a scaling argument from $(0,1)$ to $J_{i,q}$ to obtain the desired inequality).

Denoting $|J|$ the length of any interval J , we have also

$$|J_{i,q}| e^{-L\Delta t} \leq |y(J_{i,q})| \leq |J_{i,q}| e^{L\Delta t}, \quad L := \|b'\|_{L^\infty},$$

where $|J_{i,q}| = \Delta x_{i,q}$. Hence, for $r \leq k$ and $p \leq k$,

$$\begin{aligned} \Delta x_{i,q}^{2k+3} \sum_{i,q} \|\varphi^{(r)}\|_{L^\infty(J_{i,q})} \|u^{(p)}\|_{L^\infty(y(J_{i,q}))} &\leq C \Delta x_{i,q}^{2k+3} \sum_{i,q} \frac{\|\varphi\|_{L^2(J_{i,q})} \|u\|_{L^2(y(J_{i,q}))}}{\Delta x_{i,q}^{r+1/2} |y(J_{i,q})|^{p+1/2}} \\ &\leq C \Delta x_{i,q}^2 \sum_{i,q} \|\varphi\|_{L^2(J_{i,q})} \|u\|_{L^2(y(J_{i,q}))}. \end{aligned}$$

Finally, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \sum_{i,q} \|\varphi\|_{L^2(J_{i,q})} \|u\|_{L^2(y(J_{i,q}))} &\leq \left(\sum_{i,q} \|\varphi\|_{L^2(J_{i,q})}^2 \right)^{1/2} \left(\sum_{i,q} \|u\|_{L^2(y(J_{i,q}))}^2 \right)^{1/2} \\ &\leq \|\varphi\|_{L^2} \|u\|_{L^2}. \end{aligned}$$

since $\bigcup_{i,q} J_{i,q}$ is a covering of $[0, 1]$. Hence we obtain

$$\sum_{i,q} |\varepsilon_{i,q}| \leq C \Delta t \Delta x^2 \|\varphi\|_{L^2} \|u\|_{L^2},$$

which concludes the proof of (i).

Proof of Proposition 2.1(ii): Let us write $\psi = P + R$ where $P \in V_k$ is defined as the Taylor expansion of ψ on each $J_{i,q} = (x_{i,q}, x_{i,q+1})$, around $x_{i,q}$. We consider the decomposition

$$u(y(-\Delta t)) - \psi(y(-\Delta t)) \equiv (u - P)(y(-\Delta t)) - R(y(-\Delta t)) \quad (30)$$

Then by Proposition 2.1(i), for any $\varphi \in V_k$,

$$|((u - P)(y(-\Delta t)), \varphi)_G - ((u - P)(y(-\Delta t)), \varphi)| \leq C \Delta t \Delta x^2 \|u - P\|_{L^2} \|\varphi\|_{L^2}.$$

Using the fact that $\|R\|_{L^2} \leq C\|R\|_{L^\infty} \leq CM_{k+1}(\psi)\Delta x^{k+1}$, we obtain the bound

$$\begin{aligned} & |((u - P)(y(-\Delta t)), \varphi)_G - ((u - P)(y(-\Delta t)), \varphi)| \\ & \leq C\Delta t\Delta x^2\|u - \psi\|_{L^2}\|\varphi\|_{L^2} + CM_{k+1}(\psi)\Delta t\Delta x^{k+3}\|\varphi\|_{L^2}. \end{aligned} \quad (31)$$

There remains to bound the error

$$(R(y(-\Delta t)), \varphi)_G - (R(y(-\Delta t)), \varphi).$$

This is easily bounded by $C\|R\|_{L^\infty}\|\varphi\|_{L^2} = O(\Delta x^{k+1}\|\varphi\|_{L^2})$. Combined with (30) and (31), we obtain the desired bound. \square

2.5. Non-constant b : stability and error analysis. We now turn on the stability and convergence analysis. The following result shows the *unconditional* stability of the scheme, for any $k \geq 1$.

Proposition 2.2 (Stability). *Let $k \geq 0$ and let b be Lipschitz continuous and 1-periodic.*

(i) *For any $u \in L^2$,*

$$\|x \rightarrow u(y_x(-t))\|_{L^2} \leq e^{\frac{1}{2}L|t|} \|u\|_{L^2}, \quad \text{where } L := \|b'\|_{L^\infty}. \quad (32)$$

(ii) *If furthermore b is of class \mathcal{C}^{2k+2} , there exists a constant $C_1 \geq 0$ such that, $\forall u \in V_k$,*

$$\|\tilde{\mathcal{T}}_{b,\Delta t}u\|_{L^2} \leq e^{C_1\Delta t}\|u\|_{L^2} \quad \forall u \in V_k.$$

(iii) *In particular for the scheme $u^{n+1} = \tilde{\mathcal{T}}_{b,\Delta t}u^n$,*

$$\|u^n\|_{L^2} \leq e^{C_1t_n}\|u^0\|_{L^2}, \quad \forall n \geq 0,$$

where $t_n = n\Delta t$.

Proof. (i) We make use of the change of variable $x \rightarrow z := y_x(-t)$, with periodic boundary conditions for the integrands. Therefore we have $x = y_z(t)$ and

$$\frac{\partial x}{\partial z}(t) = \exp\left(\int_0^t b'(y_z(s))ds\right) \leq e^{L|t|}.$$

We then obtain

$$\int_{\Omega} |u(y_x(-t))|^2 dx = \int_{\Omega} |u(z)|^2 \left| \frac{\partial x}{\partial z}(t) \right| dz \leq e^{L|t|} \int_{\Omega} |u(z)|^2 dz.$$

(ii) By using (21), we have

$$\|\tilde{\mathcal{T}}_{b,\Delta t}u\|_{L^2} \leq \|u(y(-\Delta t))\|_{L^2} + C\Delta t\Delta x^2\|u\|_{L^2}. \quad (33)$$

Together with (32) we get a stability constant

$$e^{\frac{L}{2}\Delta t} + C\Delta t\Delta x^2 \leq e^{\frac{L}{2}\Delta t}(1 + C\Delta t\Delta x^2) \leq e^{\frac{L}{2}\Delta t}e^{C\Delta t\Delta x^2},$$

hence the desired result for any $C_1 \geq 0$ such that $C_1 \geq \frac{1}{2}L + C\Delta x^2$. \square

We now state a first convergence result. It generalizes the error estimate of Theorem 2.1 established in the case when b is constant, to the non-constant case.

Theorem 2.2 (Convergence). *Let $k \geq 0$. Assume the initial condition v_0 is 1-periodic and of class \mathcal{C}^{k+1} . Let b be 1-periodic and of class \mathcal{C}^{2k+2} . There exists constants $C_1 \geq 0$, $C \geq 0$ such that*

$$\|u^n - v^n\|_{L^2} \leq e^{C_1T} \left(\|v^0 - u^0\|_{L^2} + CT \frac{\Delta x^{k+1}}{\Delta t} \right), \quad \forall n \leq N. \quad (34)$$

Proof of Theorem 2.2. By using the regularity of v^{n+1} and Proposition 2.1(iv) we have

$$\Pi v^{n+1} = \mathcal{T}_{b,\Delta t} v^n = \tilde{\mathcal{T}}_{b,\Delta t} v^n + O(\Delta x^{k+1}). \quad (35)$$

Because of the projection error $\|v^{n+1} - \Pi v^{n+1}\| = O(\Delta x^{k+1})$ we obtain the following consistency estimate:

$$v^{n+1} = \tilde{\mathcal{T}}_{b,\Delta t} v^n + O(\Delta x^{k+1}). \quad (36)$$

Therefore

$$u^{n+1} - v^{n+1} = \tilde{\mathcal{T}}_{b,\Delta t}(u^n - v^n) + O(\Delta x^{k+1}). \quad (37)$$

By the stability bound of Proposition 2.2(ii),

$$\|u^{n+1} - v^{n+1}\|_{L^2} \leq e^{C_1 \Delta t} \|u^n - v^n\|_{L^2} + C \Delta x^{k+1}.$$

We conclude by induction. \square

2.6. Stability to perturbations. We conclude by a stability result with respect to the error of the position of the characteristics. Let $a(x) = y_x(-t)$ and let $b(x) = \bar{y}_x(-t)$ be some approximation of $y_x(t)$.

Proposition 2.3. *Then for all $u, \varphi \in V_k$, it holds*

$$\left| \int_0^1 u(b(x)) \varphi(x) dx - \int_0^1 u(a(x)) \varphi(x) dx \right| \leq C \frac{\|b - a\|_{L^\infty}}{\Delta x} \|u\|_{L^2} \|\varphi\|_{L^2}. \quad (38)$$

for some constant $C \geq 0$.

Proof. For each interval I ,

$$\begin{aligned} \|u(b) - u(a)\|_{L^2(I)} &\leq \|u'\|_{L^\infty(I)} \|b - a\|_{L^\infty} \Delta x^{1/2} \\ &\leq C \frac{\|u\|_{L^2}}{\Delta x^{3/2}} \|b - a\|_{L^\infty} \Delta x^{1/2} \leq C \|u\|_{L^2(I)} \frac{\|b - a\|_{L^\infty}}{\Delta x} \end{aligned}$$

for some constant $C > 0$ (we have used a scaling argument as before). The result follows by summation. \square

If at each time step an error of order $\varepsilon = \|a - b\|_{L^\infty}$ is made on the computation of the characteristics, after $N = \frac{T}{\Delta t}$ time steps the error on the computations of the integrals will be of order $O(\frac{\varepsilon}{\Delta x \Delta t})$. As a consequence, the computation of the characteristics should be precise enough such that $\frac{\varepsilon}{\Delta x \Delta t}$ be not greater than the expected numerical error of the scheme.

3. SECOND ORDER PDES

This section deals with SLDG schemes for second order PDEs. We will first deal with a simple diffusion problem with constant coefficients, for which specific schemes can be obtained, and then we consider the more general case of advection - diffusion problems with variable coefficients.

3.1. Case of a diffusion equation with constant coefficient. We first consider a diffusion equation with a constant coefficient $\sigma \in \mathbb{R}$:

$$v_t - \frac{\sigma^2}{2}v_{xx} = 0, \quad x \in \Omega, \quad t \in (0, T), \quad (39)$$

$$v(0, x) = v_0(x), \quad x \in \Omega. \quad (40)$$

and aim to construct simple schemes in this particular setting. Following Kushner and Dupuis [17], a first scheme, in semi-discrete form, is

$$u^{n+1}(x) = \frac{1}{2} \left(u^n(x - \sigma\sqrt{\Delta t}) + u^n(x + \sigma\sqrt{\Delta t}) \right) \equiv S_{\Delta t}^0 u^n(x). \quad (41)$$

It is easy to see that, taking $v^n(x) := v(t_n, x)$ where v is solution of (39) and is assumed sufficiently regular, the following consistency error estimate holds:

$$\left\| \frac{v^{n+1} - S_{\Delta t}^0 v^n}{\Delta t} \right\|_{L^2} = O(\Delta t).$$

The basic SLDG scheme (also called hereafter SLDG-1) is based on the weak formulation of (41).

SLDG-1 scheme: Define recursively u^{n+1} in V_k such that

$$\int u^{n+1}(x)\varphi(x)dx = \int \frac{1}{2} \left(u^n(x - \sigma\sqrt{\Delta t}) + u^n(x + \sigma\sqrt{\Delta t}) \right) \varphi(x) dx, \quad \forall \varphi \in V_k.$$

(The initialization of u^0 is done as before). The scheme will be also written in abstract form as follows:

$$u^{n+1} = \mathcal{S}_{\Delta t}(u^n),$$

where

$$\mathcal{S}_{\Delta t} := \Pi S_{\Delta t}^0 \equiv \frac{1}{2} \left(\mathcal{T}_{-\sigma\sqrt{\Delta t}} + \mathcal{T}_{\sigma\sqrt{\Delta t}} \right).$$

Before doing the numerical analysis, our aim is first to improve the accuracy with respect to the time discretization. The technic proposed here is to use convex combination of u , $S_{\Delta t}$, $S_{\Delta t}S_{\Delta t}$, \dots . It will work only for the constant coefficient case (σ constant).

Using Taylor expansions, for u sufficiently regular, we have, for Δt small,

$$S_{\Delta t}^0 u = u + \frac{\sigma^2}{2}u_{xx}\Delta t + \frac{\sigma^4}{24}u_x^{(4)}\Delta t^2 + O(\Delta t^3), \quad (42)$$

$$S_{\Delta t}^0 S_{\Delta t}^0 u = u + \sigma^2 u_{xx}\Delta t + \frac{\sigma^4}{3}u_x^{(4)}\Delta t^2 + O(\Delta t^3), \quad (43)$$

where $u_x^{(q)}$ denotes the q -th derivative of u w.r.t. x .

On the other hand, if $v^n = v(t_n, x)$ where v is the exact solution of $v_t = \frac{\sigma^2}{2}v_{xx}$, we have

$$v^{n+1} = v^n + v_t \Delta t + \frac{1}{2}v_{tt}\Delta t^2 + O(\Delta t^3) \quad (44)$$

$$= v^n + \frac{\sigma^2}{2}v_{xx}^n \Delta t + \frac{\sigma^4}{8}v^{n,(4)}\Delta t^2 + O(\Delta t^3) \quad (45)$$

Now, looking for coefficients a, b, c such that $av^n + bS_{\Delta t}^0 v^n + cS_{\Delta t}^0 S_{\Delta t}^0 v^n$ be equal to v^{n+1} up to $O(\Delta t^3)$, using (42) and (43), we obtain the system

$$\begin{cases} a + b + c = 1 \\ \frac{b}{2} + c = \frac{1}{2} \\ \frac{b}{24} + \frac{c}{3} = \frac{1}{8} \end{cases} \quad (46)$$

and we find that $a = b = c = \frac{1}{3}$. Therefore, a second order scheme (for constant coefficient) is now given by

SLDG-2 scheme:

$$u^{n+1} = S_{\Delta t}^2 u^n := \frac{1}{3}(u^n + S_{\Delta t} u^n + S_{\Delta t} S_{\Delta t} u^n). \quad (47)$$

Remark 3.1. A variant of this scheme can be

$$u^{n+1} = \Pi \frac{1}{3} \left(u^n + S_{\Delta t}^0 u^n + S_{\Delta t}^0 S_{\Delta t}^0 u^n \right). \quad (48)$$

This is in general slightly different from (47) because $S_{\Delta t} S_{\Delta t} = \Pi S_{\Delta t}^0 \Pi S_{\Delta t}^0$ may differ from $\Pi S_{\Delta t}^0 S_{\Delta t}^0$. Nevertheless, the difference between the two will be of the order of the projection error $O(\Delta x^{k+1})$ when applied to a regular data.

In order to obtain a third order scheme, we can proceed in a similar way. First, we obtain the following expansions:

$$\begin{aligned} S_{\Delta t}^0 u &= u + \frac{\sigma^2}{2} u_{xx} \Delta t + \frac{\sigma^4}{24} u_x^{(4)} \Delta t^2 + \frac{\sigma^6}{6!} u_x^{(6)} \Delta t^3 + O(\Delta t^4), \\ S_{\Delta t}^0 S_{\Delta t}^0 u &= u + \sigma^2 u_{xx} \Delta t + \frac{\sigma^4}{3} u_x^{(4)} \Delta t^2 + \frac{2}{45} \sigma^6 u_x^{(6)} \Delta t^3 + O(\Delta t^4), \\ S_{\Delta t}^0 S_{\Delta t}^0 S_{\Delta t}^0 u &= u + \frac{3}{2} \sigma^2 u_{xx} \Delta t + \frac{7}{8} \sigma^4 u_x^{(4)} \Delta t^2 + \frac{61}{240} \sigma^6 u_x^{(6)} \Delta t^3 + O(\Delta t^4). \end{aligned}$$

Looking for coefficients a, b, c, d such that $av^n + S_{\Delta t}^0 v^n + S_{\Delta t}^0 S_{\Delta t}^0 v^n + S_{\Delta t}^0 S_{\Delta t}^0 S_{\Delta t}^0 v^n$ be equal to v^{n+1} up to $O(\Delta t^4)$, we find the system

$$\begin{cases} a + b + c + d = 1 \\ \frac{b}{2} + c + \frac{3}{2}d = \frac{1}{2} \\ \frac{b}{24} + \frac{c}{3} + \frac{7}{8}d = \frac{1}{8} \\ \frac{b}{6!} + \frac{2}{45}c + \frac{61}{240}d = \frac{1}{48} \end{cases} \quad (49)$$

and its solution

$$(a, b, c, d) := \frac{1}{45}(13, 21, 9, 2).$$

Thus, the following scheme is of 3rd order in time:

SLDG-3 scheme:

$$u^{n+1} = S_{\Delta t}^3 u^n := \frac{13}{45} u^n + \frac{7}{15} S_{\Delta t} u^n + \frac{1}{5} S_{\Delta t} S_{\Delta t} u^n + \frac{2}{45} S_{\Delta t} S_{\Delta t} S_{\Delta t} u^n.$$

As in Remark 3.1, a variant of the scheme can be

$$u^{n+1} = \Pi \left(\frac{13}{45} u^n + \frac{7}{15} S_{\Delta t}^0 u^n + \frac{1}{5} S_{\Delta t}^0 S_{\Delta t}^0 u^n + \frac{2}{45} S_{\Delta t}^0 S_{\Delta t}^0 S_{\Delta t}^0 u^n \right). \quad (50)$$

Since we are using a convex combination of stable schemes ($S_{\Delta t}$, $S_{\Delta t} S_{\Delta t}$ or $S_{\Delta t} S_{\Delta t} S_{\Delta t}$), the schemes SLDG-1, SLDG-2 and SLDG-3 are all stable in the L^2 norm.

Remark 3.2. Up to 5th order schemes - in time - can also be obtained (see [1]), using convex combinations of the form $u^{n+1} = \sum_{i=0}^p a_i (S_{\Delta t}^0)^i u^n$.

We now state a convergence result for (39).

Theorem 3.1. Let $k \geq 0$ and let σ be a constant, and assume that the exact solution v of (39) has bounded derivative $\frac{\partial^q v}{\partial x^q}$ for $q = \max(k+2, 2p+2)$. We consider the SLDG- p schemes with $p = 1, 2$ or 3 . Then

$$\|v^n - u^n\|_{L^2} \leq \|v^0 - u^0\|_{L^2} + CT \left(\frac{\Delta x^{k+1}}{\Delta t} + \Delta t^p \right), \quad \forall n \leq N. \quad (51)$$

Furthermore the same results hold for the variants (48),(50) for $p = 2, 3$.

Proof. We will consider the proof in the case of the SLDG-2 scheme, with $p = 2$, the other cases being similar. By using the regularity of the exact solution ($\frac{\partial^3 v}{\partial t^3}$ and $v_x^{n,(6)}$ bounded), we have the following consistency estimate:

$$v^{n+1} = a_0 v^n + a_1 S_{\Delta t}^0 v^n + a_2 S_{\Delta t}^0 S_{\Delta t}^0 v^n + O(\Delta t^3), \quad (52)$$

where $a_0 = a_1 = a_2 = \frac{1}{3}$, and the bound $O()$ is in the norm $\|\cdot\|_{L^2}$. Since $\Pi S_{\Delta t}^0 \psi = \Pi S_{\Delta t}^0 \Pi \psi + O(\Delta x^{k+1})$ for regular data ψ , we have also $S_{\Delta t}^2 v^n = \Pi (S_{\Delta t}^0)^2 v^n + O(\Delta x^{k+1})$, and thus

$$v^{n+1} = a_0 v^n + a_1 S_{\Delta t} v^n + a_2 S_{\Delta t} S_{\Delta t} v^n + O(\Delta t^3) + O(\Delta x^{k+1}). \quad (53)$$

By the scheme definition we have

$$u^{n+1} = \sum_{i=0}^2 a_i (S_{\Delta t})^i u^n. \quad (54)$$

We deduce, using the consistency estimate (52),

$$\begin{aligned} \|u^{n+1} - v^{n+1}\|_{L^2} &\leq \left\| \sum_{i \leq 2} a_i (S_{\Delta t})^i (u^n - v^n) \right\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1} \\ &\leq \sum_{i \leq 2} a_i \|(S_{\Delta t})^i (u^n - v^n)\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1} \\ &\leq \|u^n - v^n\|_{L^2} + C\Delta t^3 + C\Delta x^{k+1}, \end{aligned}$$

(since $a_i \geq 0$ and $\sum_i a_i = 1$). The result follows by induction. \square

3.2. Advection-diffusion with variable coefficients. We recall that for the following PDE:

$$-v_t - \frac{\sigma(t, x)^2}{2} v_{xx} - b(t, x) v_x + r(t, x) v = f(t, x), \quad x \in \Omega, \quad t \in (0, T),$$

with $\Omega = \mathbb{R}$ and terminal condition $w(T, x)$, introducing a probability space $(\mathcal{Q}, \mathbb{F}, \mathbb{P})$ with a filtration $\{\mathbb{F}_t\}_{t \geq 0}$, and a one-dimensional Brownian motion $(W_t)_{t \geq 0}$, and the solution $X_s = X_s^{t, x}$ of the stochastic differential equation

$$\begin{aligned} dX_s &= b(s, X_s) ds + \sigma(s, X_s) dW_s, \quad s \geq t, \\ X_t &= x, \end{aligned}$$

and if v is a $C^{1,2}$ regular solution of (55) on (t, T) , then the following equivalent expectation, or "Feynman-Kac" formula, holds:

$$v(t, x) = \mathbb{E} \left[e^{-\int_t^T r(\theta, X_\theta) d\theta} w(T, X_T^{t,x}) + \int_t^T e^{-\int_t^s r(\theta, X_\theta) d\theta} f(s, X_s^{t,x}) ds \mid \mathcal{F}_t \right]. \quad (55)$$

To simplify, we shall focus here on the case when b and σ does not depend of time, and r is constant. We consider the forward PDE:

$$u_t - \frac{\sigma(x)^2}{2} u_{xx} - b(x)u_x + ru = f(t, x), \quad x \in \Omega, \quad t \in (0, T). \quad (56)$$

In that case the Feynman-Kac formula gives, with $h = \Delta t$, $T = t + h$ and $u^n(x) := u(t_n, x)$:

$$u^{n+1}(x) = \mathbb{E} \left[e^{-rh} u^n(X_h^{0,x}) \mid \mathcal{F}_t \right] + w(h, x) \quad (57)$$

with

$$w(h, x) := \mathbb{E} \left[\int_0^h e^{-rs} f(t_n + h - s, X_s^{0,x}) ds \mid \mathcal{F}_t \right]. \quad (58)$$

Let $\mathcal{A}w := \frac{\sigma(x)^2}{2} w_{xx} + b(x)w_x - rw$. The term $w(h, x)$ is also the solution at time $s = h$ of the linear problem $w_t(s, x) = (\mathcal{A}w)(s, x) + \bar{f}(s, x)$ with initial condition $w(0, x) = 0$, and with $\bar{f}(s, x) := f(t_n + s, x)$. Assuming that the source term f is regular and that we can use its derivatives, we can approximate it with an error $O(h^{q+1})$ by using a Taylor expansion: $w(h, x) \simeq \sum_{j=1}^q \frac{h^j}{j!} w_{jt}(0, x)$ (where w_{jt} denotes the j -th derivative with respect to time). In particular, $w_t(0, x) = \bar{f}(0, x) = f(t_n, x)$, and $w_{tt} = (\mathcal{A}w + \bar{f})_t = \mathcal{A}w_t + \bar{f}_t = \mathcal{A}(\mathcal{A}w + \bar{f}) + \bar{f}_t$, so $w_{tt}(0, x) = (\mathcal{A}f)(t_n, x) + f_t(t_n, x)$. Hence in order to devise a second order scheme we approximate (58) by

$$w(h, x) = hf(t_n, x) + \frac{h^2}{2} (\mathcal{A}f + f_t)(t_n, x) + O(h^3). \quad (59)$$

The modification of the scheme is obtained, therefore, by adding at each time step the following correction term at gauss quadrature points

$$hf(t_n, x) + \frac{h^2}{2} (\mathcal{A}f + f_t)(t_n, x). \quad (60)$$

For the approximation of the expectation in (57), we aim to use a higher order semi-discrete approximation also called "weak Taylor approximations" in the stochastic setting, see in particular Kloeden and Platen [15, Chapter 15]. General semi-discrete (and fully-discrete) approximations can be found in [17].

We will focus on first and second order weak Taylor approximations. Some of these approximation may use the derivatives of b and σ (Milstein [20], Talay [32], Pardoux and Talay [23]). In our case we shall use a derivative-free formula of Platen [25] (explicit second and third order derivative-free formula can be found in Kloeden and Platen [15], as well as multidimensional extentions).

Let us denote $b = b(x)$, $\sigma = \sigma(x)$ as well as $\gamma_h^q = \gamma_h^q(x)$:

$$\gamma_h^q(x) := x + b(x)h + q\sigma(x)\sqrt{h}. \quad (61)$$

Our SLDG-1 scheme, corresponding to a first order (weak Euler scheme), is defined by

$$u^{n+1} \equiv S_{\Delta t}^{(1)} u^n := \Pi \left(\sum_{q=\pm 1} \alpha_q u^n(y_{\Delta t}^q(\cdot)) \right) \quad (62)$$

with weights $\alpha_{-1} = \alpha_1 = \frac{1}{2}$ and characteristics $y_h^q = \gamma_h^q$.

Our SLDG-2 scheme, corresponding to the second order Platen's scheme, is defined by

$$u^{n+1} \equiv S_{\Delta t}^{(2)} u^n := \Pi \left(\sum_{-1 \leq q \leq 1} \alpha_q u^n(y_{\Delta t}^q(\cdot)) \right) \quad (63)$$

with weights $\alpha_{-1} = \alpha_1 = \frac{1}{6}$ and $\alpha_0 = \frac{2}{3}$ and characteristics $y_h^q = y_h^q(x)$ defined by:

$$\begin{aligned} y_h^q(x) &= x + \frac{1}{2}(b(\gamma_h^{\sqrt{3}q}) + b)h \\ &+ \frac{1}{4} \left[(\sigma(\gamma_h^1) + \sigma(\gamma_h^{-1}) + 2\sigma)\sqrt{3}q + (\sigma(\gamma_h^1) - \sigma(\gamma_h^{-1}))(3q^2 - 1) \right] \sqrt{h}. \end{aligned} \quad (64)$$

Remark 3.3. *In the constant coefficient case $\sigma(x) \equiv \sigma$, the scheme becomes*

$$u^{n+1} \equiv S_{\Delta t}^{(2)} u^n := \Pi \left(\frac{1}{6} u^n(x - \sigma\sqrt{3\Delta t}) + \frac{2}{3} u^n(x) + \frac{1}{6} u^n(x + \sigma\sqrt{3\Delta t}) \right) \quad (65)$$

Remark 3.4. *Higher-order weak Taylor schemes can be founded in [15] and could be used with DG to devise fully discrete schemes in the same way.*

The above SLDG-1/2 schemes are no more exactly implementable because $b(x)$ and $\sigma(x)$ are not constant. So, as in the advection case, we consider the use of the Gaussian quadrature rule on each interval of regularity of the data.

Remark 3.5. *Notice that if h is small enough such that*

$$\|hb' + \sqrt{h}\sigma'\|_{L^\infty} < 1, \quad (66)$$

then for each $q = \pm 1$ the function $x \rightarrow \gamma_h^q(x)$ is a one-to-one and onto function. Furthermore, its inverse can be easily and rapidly computed by using a fixed point method or Newton's algorithm. Details are left to the reader.

In the same way, for h small enough such that, for instance,

$$h\|b'\|_{L^\infty} + 3\sqrt{h}\|\sigma'\|_{L^\infty} < 1, \quad (67)$$

then $x \rightarrow y_h^q(x)$ as defined in (64) is one-to-one and onto function.

SLDG-1 scheme (fully discrete): For each given $\eta \in \{\pm 1\}$, we consider a partition of I_i into intervals $J_{i,q}^\eta$ such that all $y^\eta(J_{i,q}^\eta)$ be subintervals of some I_j . We then define Gauss points $\tilde{x}_{q,\alpha}^{i,\eta}$ and the bilinear product $(a, b)_{G^\eta}$ in a similar way as in (17), that is, using the Gaussian quadrature rule on each $J_{i,q}^\eta$. Hence we define $\tilde{S}_{\Delta t}^{(1)} u^n$ in V_k such that

$$(\tilde{S}_{\Delta t}^{(1)} u^n, \varphi) = \frac{1}{2} \sum_{\eta=\pm} (u^n(y^\eta), \varphi)_{G^\eta}, \quad \forall \varphi \in V_k. \quad (68)$$

Formula (68) involves two different quadrature rules, because the discontinuity points of $u^n(y^+(x))$ and $u^n(y^-(x))$ are not the same. It differs from the definition of $S_{\Delta t}^{(1)}u$, which satisfies

$$(S_{\Delta t}^{(1)}u, \varphi) = \frac{1}{2} \sum_{\eta=\pm} (u(y^\eta), \varphi), \quad \forall \varphi \in V_k. \quad (69)$$

SLDG-2 scheme (fully discrete): In a similar way, we define $\tilde{S}_{\Delta t}^{(2)}u^n$ in V_k by:

$$(\tilde{S}_{\Delta t}^{(2)}u^n, \varphi) = \sum_{-1 \leq \eta \leq 1} \alpha_\eta (u^n(y_{\Delta t}^\eta), \varphi)_{G^\eta}, \quad \forall \varphi \in V_k. \quad (70)$$

3.3. Stability and convergence. We first state some useful estimates for the operators $\tilde{S}_{\Delta t} \in \{\tilde{S}_{\Delta t}^{(1)}, \tilde{S}_{\Delta t}^{(2)}\}$. The proof is similar to the one of Proposition 2.1.

Proposition 3.1. *Let $k \geq 0$ and let σ be of class \mathcal{C}^{2k+2} and 1-periodic.*

(i) *There exists a constant $C \geq 0$, for all $u \in V_k$,*

$$\left| \sum_q (u(y_{\Delta t}^q), \varphi)_{G^q} - \sum_q (u(y_{\Delta t}^q), \varphi) \right| \leq C \sqrt{\Delta t \Delta x^2} \|u\|_{L^2} \|\varphi\|_{L^2}. \quad \forall \varphi \in V_k \quad (71)$$

In particular, for any $u \in V_k$,

$$\tilde{S}_{\Delta t}u = S_{\Delta t}u + O(\sqrt{\Delta t \Delta x^2} \|u\|_{L^2}). \quad (72)$$

(ii) *For all $u \in V_k$, for any ψ in \mathcal{C}^{k+1} , 1-periodic,*

$$\tilde{S}_{\Delta t}(u - \psi) = S_{\Delta t}(u - \psi) + O(\sqrt{\Delta t \Delta x^2} \|u - \psi\|_{L^2}) + O(M_{k+1}(\psi) \Delta x^{k+1}), \quad (73)$$

where $C \geq 0$ is a constant.

(iii) *For any regular $\psi \in \mathcal{C}^{k+1}$, 1-periodic, we have in the L^2 norm*

$$\tilde{S}_{\Delta t}\psi = S_{\Delta t}\psi + O(M_{k+1}(\psi) \Delta x^{k+1}) \quad (74)$$

We now establish stability properties.

Proposition 3.2. *Let $k \geq 0$, and assume that h is small enough in order that (66) (resp. (67)) holds.*

(i) *(Stability with exact integration as in (62).) For any $u \in V_k$,*

$$\|S_{\Delta t}u\|_{L^2} \leq (1 + C\Delta t) \|u\|_{L^2},$$

where $C \geq 0$ is a constant.

(ii) *(Stability with Gaussian quadrature rule as in (68).) For any $u \in V_k$,*

$$\|\tilde{S}_{\Delta t}u\|_{L^2} \leq (1 + C\Delta t + C\sqrt{\Delta t \Delta x^2}) \|u\|_{L^2}.$$

(iii) *In particular the fully discrete schemes SLDG-1 and -2 are L^2 stable under the "weak" CFL condition*

$$\Delta x^4 \leq \lambda \Delta t, \quad \text{for some } \lambda > 0. \quad (75)$$

Proof. (i) By making use of the convexity of $x \rightarrow x^2$, the change of variable formula $x \rightarrow y_h^q(x)$ (and denoting also $z \rightarrow x_h^q(z)$ the inverse function of y_h^q , with $h = \Delta t$), we have

$$\begin{aligned} \|S_h u\|_{L^2}^2 &= \int \left| \sum_q \alpha_q u(x + b(x)h + q\sigma(x)\sqrt{h}) \right|^2 dx \\ &\leq \int \sum_q \alpha_q \left| u(x + b(x)h + q\sigma(x)\sqrt{h}) \right|^2 dx \\ &= \int \sum_q \frac{\alpha_q}{1 + b'(x^q(z))h + q\sigma'(x^q(z))\sqrt{h}} |u(z)|^2 dz. \end{aligned}$$

Then we remark that $x_h^q(z) = x + O(\sqrt{h})$, so $1 + b'(x^q(z))h + q\sigma'(x^q(z))\sqrt{h} = 1 + q\sigma'(x)\sqrt{h} + O(h)$, and for h small enough $0 \leq (1 + b'(x^q(z))h + q\sigma'(x^q(z))\sqrt{h})^{-1} \leq 1 - q\sigma'(x)\sqrt{h} + Ch$ for some constant $C \geq 0$. Hence

$$\begin{aligned} \|S_h u\|_{L^2}^2 &\leq \int \sum_q \alpha_q (1 - q\sigma'(x)\sqrt{h} + Ch) |u(z)|^2 dz \\ &\leq (1 + Ch) \int |u(z)|^2 dz \end{aligned}$$

where we have used that $\sum \alpha_q = 1$, and $\sum_q q\alpha_q = 0$. The desired result follows.

(ii) This is a consequence of (i) and of the bound (72) of Proposition 3.1. \square

The convergence result for the approximation of (56) is the following.

Theorem 3.2. *Let $k \geq 0$ and let σ be a 1-periodic function, of class \mathcal{C}^{2k+2} . We consider the schemes SLDG- p for $p = 1, 2$ (implementable version).*

Assume the exact solution v has a bounded derivative $\frac{\partial^q v}{\partial x^q}$ for $q = \max(2p+2, k+1)$, and that the weak CFL condition (75) is satisfied, then

$$\|u^n - v^n\|_{L^2} \leq e^{L_1 T} \left(\|u^0 - v^0\|_2 + CT \left(\frac{\Delta x^{k+1}}{\Delta t} + \Delta t^p \right) \right), \quad \forall n \leq N, \quad (76)$$

for some constant $L_1 \geq 0$.

In particular for $\Delta t = \lambda \Delta x$ for any $\lambda > 0$, and $k = p \in \{1, 2\}$, the SLDG- p schemes are fully discrete schemes and of order $O(\Delta x^p)$.

Proof of Theorem 3.2. We first consider the SLDG-1 scheme $u^{n+1} = \tilde{S}_{\Delta t} u^n$. By making use of the consistency error estimate, it holds

$$v^{n+1} = \Pi S_{\Delta t}^0 v^n + O(\Delta t^2) + O(\Delta x^{k+1}) = S_{\Delta t} v^n + O(\Delta t^2) + O(\Delta x^{k+1}). \quad (77)$$

Furthermore, by proposition 3.1(iii),

$$\|\tilde{S}_{\Delta t} v^n - S_{\Delta t} v^n\|_{L^2} \leq CM_{k+1}(v^n) \Delta x^{k+1}. \quad (78)$$

Hence

$$v^{n+1} = \tilde{S}_{\Delta t} v^n + O(\Delta t^2) + O(\Delta x^{k+1}), \quad (79)$$

and by difference with the scheme $u^{n+1} = \tilde{S}_{\Delta t} u^n$:

$$\|u^{n+1} - v^{n+1}\| = \|\tilde{S}_{\Delta t} u^n - \tilde{S}_{\Delta t} v^n\|_{L^2} + C(\Delta t^2 + \Delta x^{k+1}) \quad (80)$$

$$\leq e^{C\Delta t} \|u^n - v^n\|_{L^2} + C(\Delta t^2 + \Delta x^{k+1}), \quad (81)$$

for some constant $C \geq 0$, where we have made use of the stability estimate for $\tilde{S}_{\Delta t}$. Therefore we obtain the desired error bound.

For the SLDG-2 scheme, the estimates are similar, using the fact Platen's scheme is second order to get the consistency estimate $v^{n+1} = S_{\Delta t}^{(2)}v^n + O(\Delta t^3) + O(\Delta x^{k+1})$. The conclusion follows. \square

4. EXTENSION TO TWO-DIMENSIONAL PDES AND SPLITTING STRATEGIES

4.1. First order PDEs - two dimensional case. We aim to extend the previous scheme to treat two-dimensional PDEs, by using splitting strategies and one-dimensional solvers of the previous section for advection in axis-directions.

Let Ω be a square box domain $\Omega = [x_{1,min}, x_{1,max}] \times [x_{2,min}, x_{2,max}]$ with periodic boundary conditions. Let us consider a spatial discretization of Ω into cells $I_{i,j} := I_i \times J_j$ where I_i (resp. J_j) is a cell discretization of $[x_{1,min}, x_{1,max}]$ (resp. $[x_{2,min}, x_{2,max}]$) as in the one-dimensional case using M_1 (resp. M_2) points. We define the corresponding space of 2d discontinuous Galerkin element by using the Q_k basis ($v \in Q_k$ if $v(x) = \sum_{i,j \leq k} v_{ij} x_1^i x_2^j$):

$$V_k^{(2)} := \left\{ v \in L^2(\Omega, \mathbb{R}), v|_{I_{i,j}} \in Q_k, \forall (i, j) \right\} \quad (82)$$

We consider the case of

$$u_t + b_1(x_1, x_2)u_{x_1} + b_2(x_1, x_2)u_{x_2} = 0, \quad (x_1, x_2) \in \Omega. \quad (83)$$

The idea, already proposed in [29] or [7] is to split the equation into

$$u_t + b_1(x_1, x_2)u_{x_1} = 0, \quad (x_1, x_2) \in \Omega \quad (84)$$

and

$$u_t + b_2(x_1, x_2)u_{x_2} = 0, \quad (x_1, x_2) \in \Omega. \quad (85)$$

Let the corresponding characteristics $X_{(x_1, x_2)}^q(t)$ be defined by :

- for $q = 1$: $X_{(x_1, x_2)}^1(t) = (y_1(t), x_2)$ where $y_1(t)$ is the solution of $\dot{y}_1(t) = b_1(y_1(t), x_2)$ with $y_1(0) = x_1$,
- for $q = 2$: $X_{(x_1, x_2)}^2(t) = (x_1, y_2(t))$ where $y_2(t)$ is the solution of $\dot{y}_2(t) = b_2(x_1, y_2(t))$ with $y_2(0) = x_2$.

Let \mathcal{E}_t^q be the corresponding exact evolution operator in the direction of x_q . The exact solution of (84), with $q = 1$ (resp. (85), with $q = 2$) satisfies

$$v^{n+1}(x_1, x_2) = v^n(X_{(x_1, x_2)}^q(-\Delta t)) = \mathcal{E}_{\Delta t}^q(v^n)(x_1, x_2).$$

We define the discrete evolution operator for (84), denoted $\tilde{\mathcal{T}}_{b_1, \Delta t}^1$, so that for each fixed gauss points $x_2 = x_\alpha^i$ the one-dimensional scheme is used for the evolution in the direction x_1 . We define in the same way the operator $\tilde{\mathcal{T}}_{b_2, \Delta t}^2$ for the approximation of (85).

Remark 4.1. *In the case of (84) we do not try to compute precisely the 2d integrals*

$$\int_{I_i \times J_j} u^n(X_{(x_1, x_2)}^1(-\Delta t)) \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2, \quad (86)$$

where φ_1 and φ_2 are polynomial basis functions. The discontinuities of the integrand are no more well localized and it would not be possible to obtain easily an

accurate approximation for (86). Rather, the discrete scheme computes a high order approximation of the following integrals on a full band $[0, 1] \times J_j$

$$\int_{[0,1] \times J_j} u^n(X_{(x_1, x_2)}^1(-\Delta t)) \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2, \quad (87)$$

and this is all what is needed.

Now, the results of Section 2, in particular Propositions 2.1 and 2.2, can be extended to the operators $\tilde{\mathcal{T}}_{b_q, \Delta t}^q$, $q = 1, 2$. The difference is now that the consistency estimates are typically as follows, for $q = 1, 2$:

$$\|\mathcal{E}_{\Delta t}^q \varphi - \tilde{\mathcal{T}}_{b_q, \Delta t}^q \varphi\|_{L^2} \leq C \Delta t^2 \Delta x_q^{k+1} \|\varphi\|_{L^2}, \quad \forall \varphi \in V_k^{(2)},$$

and

$$\|\mathcal{E}_{\Delta t}^q \psi - \tilde{\mathcal{T}}_{b_q, \Delta t}^q \psi\|_{L^2} \leq C(\psi) \Delta x_q^{k+1}, \quad \forall \psi \in C^{k+1}.$$

Let furthermore \mathcal{E}_t be the evolution operator for the initial advection problem (83). In the case when $b = (b_1, b_2)$ is constant we have

$$\mathcal{E}_{\Delta t} = \mathcal{E}_{\Delta t}^2 \mathcal{E}_{\Delta t}^1$$

and we can therefore approximate the exact evolution $\mathcal{E}_{\Delta t} v^n$ by $\mathcal{T}_{b_2, \Delta t}^2 \mathcal{T}_{b_1, \Delta t}^1 u^n$ with no error coming from the splitting.

In the following, when there is no ambiguity, we furthermore denote

$$\mathcal{T}_{b_q, \Delta t}^q = \mathcal{T}_{\Delta t}^2.$$

In the case when $b = (b_1, b_2)$ is non-constant, we recall the following approximations of the exponential $e^{(A+B)\Delta t}$ for A and B matrices and for small Δt :

$$e^{(A+B)\Delta t} = e^{B\Delta t} e^{A\Delta t} + O(\Delta t^2) \quad (\text{Trotter spitting}), \quad (88)$$

$$e^{(A+B)\Delta t} = e^{B\frac{\Delta t}{2}} e^{A\Delta t} e^{B\frac{\Delta t}{2}} + O(\Delta t^3) \quad (\text{Strang's spitting}). \quad (89)$$

leading to consider the following splitting approximations

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}_{\Delta t}^2 \mathcal{T}_{\Delta t}^1 \quad (\text{Trotter}) \quad (90)$$

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}_{\frac{\Delta t}{2}}^1 \mathcal{T}_{\Delta t}^2 \mathcal{T}_{\frac{\Delta t}{2}}^1 \quad (\text{Strang}) \quad (91)$$

of expected consistency error $O(\Delta t)$ and $O(\Delta t^2)$ respectively.¹ These last two splitting schemes are similar to the ones used in [26].

Following [29], we shall also consider a 3rd-order splitting scheme of Ruth [30], a 4th-order splitting scheme of Forest [13] (see also Forest and Ruth [14]), as well as a 6th-order splitting of Yoshida [36]).

Ruth's 3th order splitting:

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}_{c_1 \Delta t}^1 \mathcal{T}_{d_1 \Delta t}^2 \mathcal{T}_{c_2 \Delta t}^1 \mathcal{T}_{d_2 \Delta t}^2 \mathcal{T}_{c_3 \Delta t}^1 \mathcal{T}_{d_3 \Delta t}^2, \quad (92)$$

with

$$c_1 = 7/24, \quad c_2 = 3/4, \quad c_3 = -1/24 \quad \text{and} \quad d_1 = 2/3, \quad d_2 = -2/3, \quad d_3 = 1.$$

Forest's 4th order splitting:

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}_{\gamma_1 \frac{\Delta t}{2}}^1 \mathcal{T}_{\gamma_2 \Delta t}^2 \mathcal{T}_{(\gamma_1 + \gamma_2) \frac{\Delta t}{2}}^1 \mathcal{T}_{\gamma_2 \Delta t}^2 \mathcal{T}_{(\gamma_1 + \gamma_2) \frac{\Delta t}{2}}^1 \mathcal{T}_{\gamma_2 \Delta t}^2 \mathcal{T}_{\gamma_1 \frac{\Delta t}{2}}^1, \quad (93)$$

¹Denoting $\tau = T/N$ for $N \geq 1$, and $q \geq 0$, if linear operators A_τ and B_τ on a normed vector space satisfy $A_\tau = B_\tau + O(\tau^{q+1})$, with $\|A_\tau^n\|, \|B_\tau^n\| \leq C$ for all $0 \leq n \leq N$, then $A_\tau^N = B_\tau^N + O(\tau^q)$.

with

$$\gamma_1 := \frac{1}{2 - 2^{1/3}} \quad \text{and} \quad \gamma_2 = -\frac{2^{1/3}}{2 - 2^{1/3}}.$$

Yoshida's 6th order splitting:

$$\mathcal{T}_{b\Delta t} \simeq \mathcal{T}_{y_1\Delta t}^{4th} \mathcal{T}_{y_2\Delta t}^{4th} \mathcal{T}_{y_1\Delta t}^{4th}, \quad (94)$$

where $\mathcal{T}_{\Delta t}^{4th}$ denotes the previous Forest's 4-th order approximation method,

$$y_1 := \frac{1}{2 - 2^{1/5}} \quad \text{and} \quad y_2 := -\frac{2^{1/5}}{2 - 2^{1/5}}.$$

Remark 4.2. *Stability in the L^2 -norm is then easily obtained. Indeed, we have the L^2 -stability of the one-directional advection operators $\mathcal{T}_{\Delta t}^k$, that is, for variable coefficients*

$$\|\mathcal{T}_{\Delta t}^k u\|_{L^2} \leq e^{c\Delta t} \|u\|_{L^2} \quad (95)$$

for some constant c . Then, for instance for the Trotter splitting, it holds $\|\mathcal{T}_{\Delta t}^1 \mathcal{T}_{\Delta t}^2 u\|_{L^2} \leq e^{2c\Delta t} \|u\|_{L^2}$, which gives the L^2 stability result

$$\|(\mathcal{T}_{\Delta t}^1 \mathcal{T}_{\Delta t}^2)^n u\|_{L^2} \leq e^{2c\Delta t n} \|u\|_{L^2}. \quad (96)$$

In the same way any finite product of operators of the form of $\mathcal{T}_{\alpha_k \Delta t}^k$ (or any convex combination of such products) would lead to stable schemes.

Hence the results of Section 2 can be extended: for $\alpha = 1, 2, 3, 4$ and 6 corresponding to the splittings (90), (91), (92), (93) and (94) respectively, for regular solutions, the one time step error will be of order

$$O(\Delta t^{\alpha+1}) + O(\Delta x^{k+1}), \quad (97)$$

and the convergence error bound after N time steps will be of order

$$O(\Delta t^\alpha) + O\left(\frac{\Delta x^{k+1}}{\Delta t}\right). \quad (98)$$

4.2. Second order PDEs - two dimensional case. We consider the case of

$$u_t - \frac{1}{2} \text{Tr}(\sigma(x)\sigma(x)^T D^2 u) + b(x) \cdot \nabla u = f(t, x), \quad x \in \Omega, \quad t \in (0, T) \quad (99)$$

(with initial condition $u(0, x) = u_0(x)$), where $\sigma(x) \in \mathbb{R}^{2 \times 2}$ and $\text{Tr}(A)$ denotes the trace of the matrix A .

We introduce the following decomposition into the direction of diffusions represented by the column vectors of the matrix σ (similar decompositions have been used by Kushner and Dupuis [17], Menaldi [19], Camilli and Falcone [3], Debrabant and Jakobsen [9], etc.):

$$\sigma\sigma^T = \sum_{q=1}^2 \sigma_q \sigma_q^T, \quad \text{where } \sigma_q := \begin{pmatrix} \sigma_{1,q} \\ \sigma_{2,q} \end{pmatrix}.$$

Setting $B_1 = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, we write (99) as follows:

$$u_t + \sum_{q=1,2} \left(-\frac{1}{2} \text{Tr}(\sigma_q \sigma_q^T D^2 u) + B_q \cdot \nabla u \right) = f(t, x). \quad (100)$$

Let us first consider the one-directional problem (one direction of diffusion):

$$u_t - \frac{1}{2}Tr(\sigma_q \sigma_q^T D^2 u) + B_q \cdot \nabla u = 0. \quad (101)$$

For this subproblem we consider weak Taylor schemes exactly as for the one-dimensional SLDG-1 and SLDG-2 schemes (61)-(62) and (64)-(63). Indeed these approximations are known to be also of order 1 and 2 in time for (101) in any dimension [15].

It remains to give a scheme definition, of sufficient order, for the approximation in two dimensions for

$$\Pi(u^n(y_{\Delta t}^q(\cdot))) \quad (102)$$

where Π is the projection on $V_k^{(2)}$ and $y_{\Delta t}^q(x)$ is now a vector of \mathbb{R}^2 .

Remark 4.3. *The typical problem is to compute accurately the projection on $V_k^{(2)}$ of a function of the form*

$$(x_1, x_2) \rightarrow u^n(f_1(h, x_1, x_2), f_2(h, x_1, x_2)), \quad (103)$$

for small h of the order $\sqrt{\Delta t}$, where the functions f_1 and f_2 are such that $f_1(0, x_1, x_2) = x_1$ and $f_2(0, x_1, x_2) = x_2$ and are regular functions of h . A high-order approximation of the term (102), or (103) in the general case could be obtained by using an interpretation of $h \rightarrow (f_1(h, x_1, x_2), f_2(h, x_1, x_2))$ as a characteristic of a corresponding advection problem. This is not treated in the present work.

In the present work we will consider only examples involving terms of the form $\Pi u^n(f_1(h, x_1, x_2), x_2)$ or $\Pi u^n(x_1, f_2(h, x_1, x_2))$ (i.e. $f_2(h, x_1, x_2) \equiv x_2$, or $f_1(h, x_1, x_2) \equiv x_1$), or of the form $\Pi u^n(f_1(h, x_1), f_2(h, x_2))$, for which the one-dimensional discretization can be extended to two dimensions by straightforward splitting.

Finally, for the general case of (99), we define the scheme by using Strang's splitting of the one time-step evolution operators for (101) and by adding the correction (60) for the source term.

5. NUMERICAL EXAMPLES

The first four examples are devoted to advection problems, while the other examples concern second order equations.

Example 1. We consider an advection equation with non-constant advection term

$$v_t + b(x)v_x = 0, \quad x \in (0, 1), \quad t \in (0, T), \quad (104)$$

$$v(0, x) = \sin(2\pi x), \quad x \in (0, 1), \quad (105)$$

and

$$b(x) := C_0 + C_1 \sin(2\pi x), \quad \text{with } C_0 = 1 \text{ and } C_1 := 0.8 \quad (106)$$

together with periodic boundary conditions on $(0, 1)$. The exact solution is given by $v(t, x) = \sin(2\pi y_x(-t))$, where

$$y_x(-t) = \frac{1}{\pi} \operatorname{atan} \left(-r + \tan \left(\operatorname{atan} \left(\frac{\tan(\pi x) + r}{a} \right) - C_0 \pi a t \right) \right)$$

with $r := \frac{C_1}{C_0}$ and $a := \sqrt{1 - r^2}$.

Results are given in Table 1 for $\Delta t \equiv \Delta x$ with fixed CFL= 1.8 and terminal time $T = 1.3$. (Here the CFL corresponds to $\|b\|_\infty \frac{\Delta t}{\Delta x}$.) The numerical error behaves

approximately one order better than the expected one when $\Delta t = \lambda \Delta x$, that is of the order of $O(\frac{\Delta x^{k+1}}{\Delta t}) \equiv O(\Delta x^k)$. Superconvergence results can be explained in some cases for other DG methods [34].

L^2 error		k=1		k=2		k=3		k=4	
M	N	error	order	error	order	error	order	error	order
10	10	1.95E-01	-	3.45E-02	-	1.45E-02	-	7.83E-03	-
20	20	2.67E-02	1.93	6.06E-03	2.50	1.38E-03	3.39	2.33E-04	5.07
40	40	7.80E-03	1.77	6.39E-04	3.24	3.22E-05	5.42	4.31E-06	5.75
80	80	1.47E-03	2.40	3.62E-05	4.13	1.52E-06	4.40	7.74E-08	5.80
160	160	2.27E-04	2.69	3.31E-06	3.45	7.13E-08	4.41	2.48E-09	4.96
320	320	3.92E-05	2.53	4.03E-07	3.04	3.92E-09	4.18	8.03E-11	4.95

TABLE 1. (*Example 1*) non-constant advection, $\Delta t \equiv \Delta x$ and CFL= 1.8, $T = 1.3$.

Example 2 (2D advection) We consider a two-dimensional advection PDE

$$v_t + b_1 v_x + b_2 v_y = 0 \quad (x, y) \in \Omega, \quad t \in (0, T), \quad (107)$$

$$v(0, x, y) = \sin(2\pi(x + y)) \quad (x, y) \in \Omega \quad (108)$$

with $\Omega = (0, 1)^2$ and periodic boundary conditions, and with the constant dynamics $b_1 = b_2 = 1$.

Results are shown in Table 2, for $T = 1$. Here the scheme is based on a Trotter splitting; cpu times are given in sec. for Forest's splitting. Superconvergence of order $k + 1$ is also observed here.

L^2 error		k=1		k=2		k=3		k=4		
M	N	error	order	error	order	error	order	error	order	cpu(s)
10	3	6.50E-03	-	2.08E-04	-	1.08E-05	-	4.51E-07	-	0.02
20	6	1.59E-03	2.03	3.52E-05	2.57	6.65E-07	4.02	1.51E-08	4.90	0.07
40	12	3.96E-04	2.01	4.90E-06	2.84	4.17E-08	4.00	4.76E-10	4.99	0.32
80	24	9.90E-05	2.00	6.18E-07	2.99	2.66E-09	3.97	1.49E-11	5.00	1.46

TABLE 2. (*Example 2*) 2D advection, $u_0(x, y) = \sin(2\pi(x + y))$, terminal time $T = 1$, and CFL=3.33.

Example 3 (2D advection with non-constant coefficients). We consider the following rotation example of a "bump":

$$u_t + 2\pi(-x_2, x_1) \cdot \nabla u = 0, \quad x = (x_1, x_2) \in \Omega, \quad t \in (0, T),$$

$$u(0, x) = 1 - e^{-20((x_1-1)^2 + x_2^2 - r_0^2)},$$

with $\Omega := (-2, 2)^2$, $r_0 = 0.25$ and terminal time $T = 0.9$. Since $b(x_1, x_2) = 2\pi(-x_2, x_1)$ is non-constant, Trotter's splitting is no longer exact.

In Table 3, we test and compare the splitting algorithms as described in subsection 2.3, from order 2 to 6 (Strang's splitting, Ruth's 3rd order, Forest's 4th and

Yoshida's 6th order splittings, tested with $k = 2, 3, 4$, and $k = 6$ respectively). Trotter's splitting error, not represented in Table (3), is of order 1. We have avoided taking the particular case of $T = 1$ (full turn) because it gives better numerical results but prevents to well analyze the order of the method. The cpu times are given in the case of Yoshida's splitting.

In this example, the initial data is sufficiently close to 1 outside a ball of radius 1.5, so that the error coming from the boundary treatment is neglectible.

L^2 error	Strang ($k = 2$)		Ruth ($k = 3$)		Forest ($k = 4$)		Yoshida ($k = 6$)		
$M = N$	error	order	error	order	error	order	error	order	cpu(s)
10	2.91E-01	-	1.85E-02	-	1.66E-01	-	1.81E-02	-	0.07
20	6.62E-02	2.13	1.76E-03	3.40	1.01E-02	4.04	2.45E-04	6.21	0.26
40	1.60E-02	2.05	2.11E-04	3.06	6.24E-04	4.01	3.64E-06	6.07	1.65
80	3.99E-03	2.01	2.61E-05	3.02	3.89E-05	4.00	5.61E-08	6.02	15.06
160	9.96E-04	2.00	3.25E-06	3.01	2.43E-06	4.00	1.03E-09	5.77	120.98

TABLE 3. (Example 3), 2D rotation, L^2 errors at time $T = 0.9$.

Example 4 (2D deformation with non-constant coefficients) In this example, close to the one in for instance Qiu and Shu [26, Example 5], the advection term is non-constant

$$u_t - \left(g(t) \cos\left(\frac{x^2}{2}\right) \sin(y) \right) u_x + \left(g(t) \cos\left(\frac{y^2}{2}\right) \sin(x) \right) u_y = 0,$$

$$(x, y) \in \Omega, t \in (0, T),$$

with $\Omega := (-2, 2)^2$, $T = 1$ and same initial data as in Example 4. Here we furthermore consider $g(t) := 1$ for $t \in [0, \frac{T}{2}]$ and then $g(t) := -1$ for $t \in]\frac{T}{2}, T]$, so that the exact solution after time T is $u(T, x, y) = u_0(x, y)$.

In Table 4, we test and the splitting algorithms of orders 2, 4 and 6 (Strang's, Forest's and Yoshida's splittings, with polynomials of degree $k = 2, 4$ and 6 respectively). The cpu times are also given in seconds.

L^2 error		Strang (with $k = 2$)			Forest (with $k = 4$)			Yoshida (with $k = 6$)		
M	N	error	order	cpu(s)	error	order	cpu(s)	error	order	cpu(s)
10	10	1.28E-01	-	0.005	7.82E-03	-	0.08	7.70E-04	-	0.85
20	20	1.45E-02	3.14	0.034	2.78E-04	4.81	0.36	6.60E-06	6.87	3.65
40	40	1.44E-03	3.33	0.104	9.06E-06	4.94	1.58	3.32E-08	7.64	16.20
80	80	1.66E-04	3.12	0.620	3.30E-07	4.78	7.73	2.71E-10	6.94	140.41

TABLE 4. (Example 4) 2D deformation, L^2 errors at time $T = 1$ with splittings of order 2, 4 and 6.

Example 5 (1D convection diffusion). Now, we consider a diffusion equation

$$v_t - \frac{1}{2}\sigma^2 v_{xx} + bv_x = 0, \quad \forall x \in \Omega, t \in (0, T) \quad (109)$$

$$v(0, x) = \cos(2\pi x) + \frac{1}{2} \cos(4\pi x), \quad x \in \Omega \quad (110)$$

together with periodic boundary conditions on $\Omega = (0, 1)$, with constants $\sigma = 0.1$, $b = 0.3$, and $T = 0.2$. The exact solution is given by

$$v(t, x) = \sum_{k=1,2} c_k \exp(-2\sigma^2 k^2 \pi^2 t) \cos(2k\pi(x - bt)),$$

with $c_1 = 1$ and $c_2 = \frac{1}{2}$.

Since the operators $\frac{1}{2}\sigma^2\partial_x^2$ and $b\partial_x$ commute, we use the simple scheme

$$u^{n+1} = S_{\Delta t}^\sigma \mathcal{T}_{b\Delta t} u^n.$$

In Table 5 we study the orders of the SLDG-RK p schemes when $\Delta t \equiv \Delta x$ and $p \in \{1, 2, 3\}$. The orders are as expected. We also give in Table 6 the errors when taking larger time steps ($\Delta t \gg \Delta x$), still showing good behavior.

We have numerically also tested the case when $b = 0$ (pure diffusion), the numerical results are very close to the present case.

L^2 error		SLDG-RK1 (P_1)		SLDG-RK2 (P_2)		SLDG-RK3 (P_3)	
M	N	error	order	error	order	error	order
10	10	9.94E-03	-	1.37E-03	-	8.66E-05	-
20	20	1.39E-03	2.84	1.08E-04	3.67	3.70E-06	4.55
40	40	2.93E-04	2.25	3.63E-06	4.90	1.03E-07	5.17
80	80	8.02E-05	1.87	6.28E-07	2.53	9.81E-09	3.39
160	160	2.35E-05	1.77	9.72E-08	2.69	7.00E-10	3.81
320	320	8.22E-06	1.52	2.60E-08	1.90	5.79E-11	3.60
640	640	4.06E-06	1.02	6.17E-09	2.08	5.81E-12	3.32

TABLE 5. *Example 5 (1D diffusion)*, SLDG-RK p schemes with $\Delta t \equiv \Delta x$, $p \in \{1, 2, 3\}$.

M	N	$\frac{\Delta t}{\Delta x}$	SLDG-RK1 (P_1)	SLDG-RK2 (P_2)	SLDG-RK3 (P_3)
20	10	0.40	1.37E-03	4.34E-05	1.79E-06
40	15	0.53	5.13E-04	6.87E-06	1.41E-07
80	20	0.80	1.39E-04	1.40E-06	1.11E-08
160	25	1.28	1.05E-04	1.83E-07	5.20E-10
320	30	2.13	8.49E-05	6.14E-08	3.09E-11
640	35	3.66	7.26E-05	4.35E-08	1.15E-11
1280	40	6.40	6.35E-05	3.31E-08	7.02E-12

TABLE 6. *Example 5 (1D diffusion)*, SLDG-RK p with large time steps $\Delta t \gg \Delta x$, and $p \in \{1, 2, 3\}$.

Example 6 (1D Black and Scholes and boundary conditions) This example deals with the one-dimensional Black-Scholes (B&S) PDE for the pricing of a European put option with one asset [33]. After a change of variable in logarithmic coordinates,² the equation for the European put option becomes on $\Omega := (x_{min}, x_{max})$:

$$\begin{cases} u_t - \frac{1}{2}\sigma^2 u_{xx} + bu_x + ru = 0, & x \in \Omega, t \in (0, T), \\ u(0, x) = u_0(x) = K \max(1 - e^x, 0) & x \in \Omega, \\ u(t, x) = u_\ell(t) \equiv Ke^{-rt} - Ke^x & t \in (0, T), x \leq x_{min}, \\ u(t, x) = u_r(t) \equiv 0 & t \in (0, T), x \geq x_{max}, \end{cases} \quad (111)$$

with $b := -(r - \frac{1}{2}\sigma^2)$ and where $x_{min} < 0$ and $x_{max} > 0$, and we have imposed boundary conditions outside of Ω . Numerically, the initial data presents a singular behavior at $x = 0$ (as it is only Lipschitz regular).

For this PDE the scheme reads

$$u^{n+1} = e^{-r\Delta t} S_{\Delta t}^\sigma \mathcal{T}_{\Delta t}^b u^n.$$

The following financial parameters are used: $K = 100$ (strike price), $r = 0.10$ (interest rate), $\sigma = 0.2$ (volatility), and $T = 0.25$ (maturity). Since the interesting part of the solution lies in a neighborhood of $x = 0$ (notice that φ has a singularity at $x = 0$), for the computational domain we consider

$$\Omega = (x_{min}, x_{max}) := (-2, 2).$$

Results are reported in Table 7 for the L^1 , L^2 and L^∞ errors, where Δt is chosen of the same order as Δx , and the SLDG-RK3 scheme is used ($k = 3$). We numerically observe an order between 4 and 5 (at least for the L^1 or L^2 errors), while the theoretical order is at least 3.

Remark 5.1 (Boundary treatment). *For semi-Lagrangian schemes, the knowledge of $u(t, x)$ for $x \leq x_{min}$ or $x \geq x_{max}$ can be used rather than only the boundary values $u(t, x_{min})$ or $u(t, x_{max})$. Here, out-of-bound values are needed for computing $S^0 v^n$, $S^0 S^0 v^n$ and $S^0 S^0 S^0 v^n$ for $v^n = \mathcal{T}_{\Delta t}^b u^n$. In particular, the values $u^n(x + k\sigma\sqrt{\Delta t} - b\Delta t)$ for $|k| \leq 3$ are used when $y := x + k\sigma\sqrt{\Delta t} - b\Delta t$ lies outside of (x_{min}, x_{max}) . In that case, we simply directly use the "boundary" value $u_\ell(t_n, y)$ when $y \leq x_{min}$ or $u_r(t_n, y)$ when $y \geq x_{max}$.*

Example 7 (1D diffusion with non-constant $\sigma(x)$). Now, we consider the following diffusion equation

$$v_t - \frac{1}{2}\sigma^2(x)v_{xx} = f(t, x), \quad x \in (0, 1), t \in (0, T) \quad (112)$$

$$v(0, x) = 0 \quad x \in (0, 1), \quad (113)$$

with periodic boundary conditions,

$$\sigma(x) := \sin(2\pi x),$$

and, for testing purposes, $f(t, x) := \bar{v}_t(t, x) - \frac{1}{2}\sigma^2(x)\bar{v}_{xx}(t, x)$ where $\bar{v}(t, x) := \sin(2\pi t) \cos(2\pi(x - t))$, which is therefore the exact solution ($v \equiv \bar{v}$).

²The classical B&S PDE for the put option reads

$$v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} - bsv_s + rv = 0, \quad s \in (0, \infty), t \in (0, T),$$

(where $b = r - \frac{1}{2}\sigma^2$), with initial condition $v(0, s) = \varphi(s) \equiv \max(K - s, 0)$. Then using the change of variable $x = \log(s/K)$ and $u(t, x) := v(t, s)$, we obtain the PDE (111) on $x \in \mathbb{R}$.

M	N	L^1 -Error	Order	L^2 -Error	Order	L^∞ -Error	Order
10	10	2.99E-02	-	3.49E-02	-	6.22E-02	-
20	20	9.50E-04	4.98	1.50E-03	4.55	3.25E-03	4.26
40	40	5.04E-05	4.24	8.20E-05	4.19	2.82E-04	3.53
80	80	1.66E-06	4.92	2.75E-06	4.90	1.12E-05	4.65
160	160	4.79E-08	5.12	8.84E-08	4.96	4.49E-07	4.64

TABLE 7. *Example 6, 1D Black and Scholes PDE.* Error table with $\Delta t \sim \Delta x$, using SLDG-RK3 and P_3 polynomials.

In this case, in order to get higher than first-order accuracy in time, we use the SLDG-2 scheme corresponding to a Platen's weak Taylor scheme. The correction for the source term $f(t, x)$ is treated by adding the term (60) at gauss quadrature points, at each time step.

In Table 8 we first check the accuracy with respect to time discretization, with fixed spatial mesh size so that only the time discretization error appears.

Then, in Table 9 the errors are given for varying mesh sizes such that $\Delta t \equiv \Delta x$ and with P_1 or P_2 elements ($k = 1$ or $k = 2$). We find the expected orders for the schemes SLDG-1/2.

Remark 5.2. *Notice that there is no need of a non-vanishing assumption on the diffusion coefficient in the proposed method.*

L^2 error	SLDG-1		SLDG-2		
	N	error	order	error	order
	100	1.19E-03	-	1.89E-04	2.05
	200	5.95E-04	1.01	4.57E-05	1.97
	400	2.96E-04	1.01	1.16E-05	1.93
	800	1.48E-04	1.00	3.07E-06	1.91
	1600	7.40E-05	1.00	8.17E-07	1.92

TABLE 8. *Example 7 (1D diffusion with variable coefficient),* with fixed spatial mesh ($M = 100$ and P_4 polynomials) and varying time steps.

Example 8 (2D diffusion) We consider the following two-dimensional diffusion equation:

$$u_t - \frac{1}{2}(5u_{xx} - 4u_{xy} + u_{yy}) = 0, \quad x \in \Omega, \quad t \in (0, T), \quad (114)$$

$$u(0, x) = u_0(x), \quad x \in \Omega \quad (115)$$

set on $\Omega = (0, 1)^2$ with periodic boundary conditions, and $T = 0.2$. The initial data is given by $u_0(x) = u_{01}(x + 2y) + u_{02}(-y)$ and $u_{0i}(\xi) := \sum_{q=1,2} c_q^i \cos(2\pi q \xi)$ with the constant $c_q^i = \frac{1}{i+q}$. The exact solution is known.³

³Making the change of variable $\xi = (\xi_1, \xi_2)$ such that $\xi_1 = x + 2y$ and $\xi_2 = -y$ we find that $v(t, \xi) = u(t, x)$ satisfies $v_t - \frac{1}{2}(v_{\xi_1 \xi_1} + v_{\xi_2 \xi_2}) = 0$ and $v(0, \xi) = u_{01}(\xi_1) + u_{02}(\xi_2)$ and

L^2 error		SLDG-1 (with P_1)		SLDG-2 (with P_2)	
M	N	error	order	error	order
10	10	8.60E-02	-	4.13E-02	-
20	20	3.52E-02	1.29	7.30E-03	2.50
40	40	1.59E-02	1.15	1.39E-03	2.39
80	80	7.54E-03	1.08	3.03E-04	2.20
160	160	3.67E-03	1.04	7.17E-05	2.08
320	320	1.81E-03	1.02	1.80E-05	1.99

TABLE 9. *Example 7 (1D diffusion with variable coefficient), with $\Delta t \equiv \Delta x$;*

In order to define the numerical scheme, we use the fact that

$$A := \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \sum_{k=1,2} \Sigma_k \Sigma_k^T, \quad \text{with } \Sigma_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \Sigma_2 := \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Results are given in Table 10, where we consider variable time steps and mesh steps $\Delta t \sim \Delta x$, $p = k$, and expect a global error of order $O(\Delta t^p) + O(\frac{\Delta x^{k+1}}{\Delta t}) \equiv O(\Delta x^k)$.

L^2 error		SLDG-RK1 (P_1)		SLDG-RK2 (P_2)		SLDG-RK3 (P_3)	
M	N	error	order	error	order	error	order
10	10	6.66E-03	-	1.86E-04	-	2.20E-06	-
20	20	3.26E-03	1.02	4.52E-05	2.04	3.10E-07	2.83
40	40	1.61E-03	1.01	1.08E-05	2.06	3.20E-08	3.27
80	80	8.04E-04	1.00	2.69E-06	2.01	4.34E-09	2.88
160	160	4.01E-04	1.00	6.66E-07	2.01	4.90E-10	3.14

TABLE 10. *Example 8(2D diffusion equation), error table with $\Delta t \sim \Delta x$.*

Example 9 (2D diffusion with non-constant coefficients) We consider the following two-dimensional diffusion equation:

$$u_t - \frac{1}{2} \text{Tr}(\sigma \sigma^T D^2 u) = f(t, x), \quad x \in \Omega, \quad t \in (0, T), \quad (116)$$

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega \quad (117)$$

set on $\Omega = (-\pi, \pi)^2$ with periodic boundary conditions, $T = 1.0$. The diffusion matrix $A = \sigma \sigma^T$ is defined by

$$\sigma(x, y) := \begin{pmatrix} \cos(x) & \cos(2x) \\ 0 & \sin(y) \end{pmatrix}.$$

In this test we have chosen $u(t, x, y) := \cos(t) \sin(2x) \sin(x+y)$ and the source term $f(t, x)$ such that (116) holds. (The initial data is therefore $u_0(x, y) = u(0, x, y)$)

therefore the exact solution is given by $u(t, x) = v(t, \xi) = u_1(t, \xi_1) + u_2(t, \xi_2)$ where $u_i(t, \xi) = \sum_{q=1,2} c_q^i e^{-(2\pi q)^2 t/2} \cos(2\pi q \xi)$.

The scheme is defined here by using Strang’s splitting and a second order correction (59) for the treatment of the source term as explained in Section 4.2.

Results are given in Table 11, where we consider variable time steps and mesh steps $\Delta t \sim \Delta x$ and polynomials of degree $k = 2$. The scheme is numerically roughly of order 2.

As in remark 5.2, there is not need of a strict positivity assumption on the diffusion matrix in this approach.

		L^1 error		L^2 error		L^∞ error		
M	N	error	order	error	order	error	order	cpu(s)
3	2	1.23E+01	-	2.47E+00	-	9.82E-01	-	0.01
6	4	2.29E+00	2.43	4.52E-01	2.45	1.65E-01	2.57	0.03
12	8	3.60E-01	2.67	7.60E-02	2.57	3.03E-02	2.45	0.09
24	16	7.24E-02	2.32	1.55E-02	2.29	5.96E-03	2.35	0.39
48	32	1.69E-02	2.10	3.54E-03	2.14	1.19E-03	2.32	2.08
96	64	4.61E-03	1.87	9.52E-04	1.89	3.25E-04	1.87	12.28

TABLE 11. *Example 9* (2D diffusion equation with variable coefficients), error table with $\Delta t \sim \Delta x$.

APPENDIX A. UNSTABILITY OF THE DIRECT SCHEME

Here we consider the ”direct scheme”, which defines naively at each time iteration a new piecewise polynomial $u^{n+1} \in V_k$ such that,

$$u_\alpha^{n+1,i} := u^n(x_\alpha^i - b\Delta t), \quad \text{for all Gauss points } x_\alpha^i.$$

In Figure 2, we consider again $v_t + v_x = 0$ with periodic boundary conditions on $(0, 1)$, and with the initial data $v_0(x) = \sin(2\pi x)$. We have represented two graphs with different choices of parameter N . In each graph is plotted the result of the direct scheme (green line) and of the SLDG scheme (red line) at time $T = 1$, with piecewise P_1 elements ($k = 1$) and fixed spatial mesh using $M = 46$ mesh steps. In the left graph, $N = 80$ time steps and both curves are confounded; in the right graph, $N = 320$, and the direct scheme becomes unstable. (We have found that the error behaves as $c^N \Delta x^{k+1}$ where $c > 1$, when using V_k elements.)

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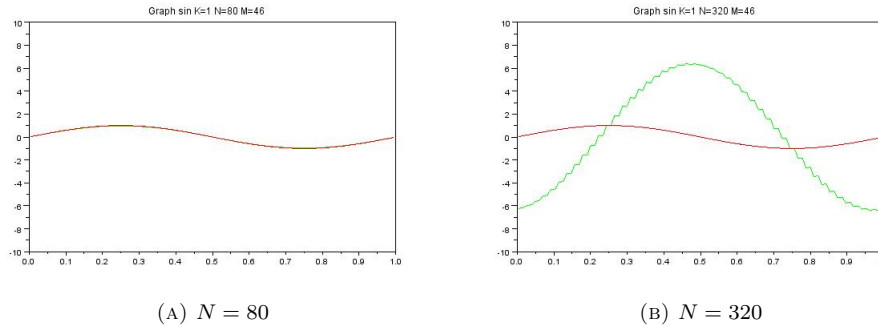


FIGURE 2. Results for $N = 80$ (left) and $N = 320$ (right), using P_1 elements with $M = 46$ in both cases. Instability appears on the right figure.

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