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Law of the iterated logarithm for the periodogram

Christophe Cuny^a, Florence Merlevède^b and Magda Peligrad^{c1}

^a Laboratoire MAS, Ecole Centrale de Paris, Grande Voie des Vignes, 92295 Chatenay-Malabry cedex, FRANCE. E-mail: christophe.cuny@ecp.fr

^b Université Paris-Est, LAMA (UMR 8050), UPEMLV, CNRS, UPEC, F-77454 Marne-La-Vallée, FRANCE. E-mail florence.merlevede@univ-mlv.fr

^c Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025, USA. E-mail: peligrm@ucmail.uc.edu

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Abstract

We consider the almost sure asymptotic behavior of the periodogram of stationary and ergodic sequences. Under mild conditions we establish that the limsup of the periodogram properly normalized identifies almost surely the spectral density function associated with the stationary process. Results for a specified frequency are also given. Our results also lead to the law of the iterated logarithm for the real and imaginary part of the discrete Fourier transform. The proofs rely on martingale approximations combined with results from harmonic analysis and technics from ergodic theory. Several applications to linear processes and their functionals, iterated random functions, mixing structures and Markov chains are also presented.

1 Introduction

The periodogram, introduced as a tool by Schuster in 1898, plays an essential role in the estimation of the spectral density of a stationary time series $(X_j)_{j \in \mathbb{Z}}$ of centered random variables with finite second moment. The finite Fourier transform is defined as

$$S_n(t) = \sum_{k=1}^n e^{ikt} X_k, \quad (1)$$

where $i = \sqrt{-1}$ is the imaginary unit, and the periodogram as

$$I_n(t) = \frac{1}{2\pi n} |S_n(t)|^2 \quad t \in [0, 2\pi]. \quad (2)$$

It is well-known since Wiener and Wintner [30] that for any stationary sequence $(X_j)_{j \in \mathbb{Z}}$ in \mathbb{L}^1 (namely $\mathbb{E}|X_0| < \infty$) there is a set Ω' of probability one such that for any $t \in [0, 2\pi]$ and any $\omega \in \Omega'$, $S_n(t)/n$ converges. To provide the speed of this convergence many authors (see Peligrad and Wu [21] and the references therein) established a central limit theorem for the real and imaginary parts of $S_n(t)/\sqrt{n}$ under various assumptions. Recently, Peligrad and Wu [21] showed that, under a very mild regularity condition and finite second moment, $\frac{1}{\sqrt{n}}[\mathcal{R}e(S_n(t)), \mathcal{I}m(S_n(t))]$ are asymptotically independent normal random variables with mean 0 and variance $\pi f(t)$ for almost all t (here f is the spectral density of $(X_j)_{j \in \mathbb{Z}}$). The central limit theorem implies that $I_n(t)/\log \log n$ converges

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to 0 in probability. An interesting and natural problem, that apparently has never been studied in depth before, is the law of the iterated logarithm, namely to identify in the almost sure sense, $\limsup_{n \rightarrow \infty} I_n(t) / \log \log n$ for almost all t , or for a t fixed. In this paper, we study both these problems. We provide mild sufficient conditions on the stationary sequence that are sufficient to have $\limsup_{n \rightarrow \infty} I_n(t) / \log \log n = f(t)$ almost surely. These results shed additional light on the importance of the periodogram in approximating the spectral density $f(t)$ of a stationary process. The techniques are based on martingale approximation, rooted in Gordin [15] and Rootzén [26] and developed by Gordin and Lifshitz [16] and Woodroffe [31], combined with tools from ergodic theory and harmonic analysis. Various applications are presented to linear processes and their functionals, iterated random functions, mixing structures and Markov chains.

We would like to point out that our results are formulated under the assumption that the underlying stationary sequence is assumed to be adapted to an increasing (stationary) filtration. Results in the non adapted case could also be obtained. We shall also assume that our stationary sequence is constructed via a measure-preserving transformation that is invertible. Since our proofs are based on martingale approximation, we could also obtain similar results when the measure-preserving transformation is assumed to be non invertible. In this situation, the conditions should be expressed with the help of the Perron-Frobenius operator associated to the transformation (see for instance [9]). In our paper we shall not pursue these last two cases.

Our paper is organized as follows. Section 2 contains the presentation of the results. Section 3 is devoted to the proofs. Applications are presented in Section 4.

2 Main results

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Assume, without loss of generality, that \mathcal{A} is a countably generated σ -field, and let $\theta : \Omega \rightarrow \Omega$ be a bijective bi-measurable transformation preserving \mathbb{P} . Let \mathcal{F}_0 be a σ -algebra such that $\mathcal{F}_0 \subseteq \theta^{-1}(\mathcal{F}_0)$. Let $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ be the non-decreasing filtration given by $\mathcal{F}_n = \theta^{-n}(\mathcal{F}_0)$, and let $\mathcal{F}_{-\infty} = \bigcap_{k \in \mathbb{Z}} \mathcal{F}_k$. All along the paper X_0 is a centered real random variable in \mathbb{L}^2 which is \mathcal{F}_0 -measurable. We then define a stationary sequence $(X_n, n \in \mathbb{Z})$ by $X_n = X_0 \circ \theta^n$. We denote $\mathbb{E}_k(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_k)$ and $P_k(\cdot) = \mathbb{E}_k(\cdot) - \mathbb{E}_{k-1}(\cdot)$. Throughout the paper, we say that a complex number z is an eigenvalue of θ if there exists $h \neq 0$ in $\mathbb{L}^2(\mathbb{P})$ such that $h \circ \theta = zh$ almost everywhere. We say that $A \in \mathcal{A}$ is invariant if $\theta^{-1}(A) = A$. If for any invariant set A , $\mathbb{P}(A) = 0$ or 1 , we say that θ is ergodic with respect to \mathbb{P} , or equivalently that the stationary sequence is ergodic.

Relevant to our results is the notion of spectral distribution function induced by the covariances. By Herglotz's Theorem (see e.g. Brockwell and Davis [3]), there exists a non-decreasing function G (the spectral distribution function) on $[0, 2\pi]$ such that, for all $j \in \mathbb{Z}$,

$$\text{cov}(X_0, X_j) = \int_0^{2\pi} \exp(ij\theta) dG(\theta), \quad j \in \mathbb{Z}.$$

If G is absolutely continuous with respect to the normalized Lebesgue measure λ on $[0, 2\pi]$, then the Radon-Nikodym derivative f of G with respect to the Lebesgue measure is called the spectral density and we have

$$\text{cov}(X_0, X_j) = \int_0^{2\pi} \exp(ij\theta) f(\theta) d\theta, \quad j \in \mathbb{Z}.$$

Our first theorem points out a projective condition which assures the law of the iterated logarithm for almost all frequencies.

All along the paper, denote

$$Y_k(t) = (\cos(kt)X_k, \sin(kt)X_k)',$$

where u' stands for the transposed vector of u .

Theorem 1 Assume that θ is ergodic and that

$$\sum_{k \geq 2} \frac{(\log k)}{k} \|\mathbb{E}_0(X_k)\|_2^2 < \infty. \quad (3)$$

Then the spectral density, say f , of $(X_k, k \in \mathbb{Z})$ exists and for almost all $t \in [0, 2\pi)$, the sequence $\{\sum_{k=1}^n Y_k(t)/\sqrt{2n \log \log n}, n \geq 3\}$ is \mathbb{P} -a.s. bounded and has the ball $\{x \in \mathbb{R}^2 : x'x \leq \pi f(t)\}$ as its set of limit points. In particular, for almost all $t \in [0, 2\pi)$, the following law of the iterated logarithm holds

$$\limsup_{n \rightarrow \infty} \frac{I_n(t)}{\log \log n} = f(t) \quad \mathbb{P}\text{-a.s.}$$

Note that condition (3) is satisfied by martingale differences. It is a very mild condition involving only a logarithmic rate of convergence to 0 of $\|\mathbb{E}_0(X_k)\|_2$.

If we assume a more restrictive moment condition, (3) can be weakened. Define the function $L(x) = \log(e + |x|)$.

Theorem 2 Assume that θ is ergodic. Assume in addition that

$$\mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty,$$

and that

$$\sum_{k \geq 3} \frac{\|\mathbb{E}_0(X_k)\|_2^2}{k(\log \log k)} < \infty. \quad (4)$$

Then the conclusions of Theorem 1 hold.

Note that condition (3), as well as condition (4), implies the following regularity condition:

$$\mathbb{E}(X_0 | \mathcal{F}_{-\infty}) = 0 \quad \mathbb{P}\text{-a.s.} \quad (5)$$

We point out that this regularity condition implies that the process $(X_k)_{k \in \mathbb{Z}}$ is purely non deterministic. Hence by a result of Szegő (see for instance [1, Theorem 3]) if (5) holds, the spectral density f of $(X_k)_{k \in \mathbb{Z}}$ exists and if X_0 is non degenerate,

$$\int_0^{2\pi} \log f(t) dt > -\infty;$$

in particular, f cannot vanish on a set of positive measure. We mention also that under (5), Peligrad and Wu [21] established that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}|S_n(t)|^2}{n} = 2\pi f(t) \quad \text{for almost all } t \in [0, 2\pi) \quad (6)$$

(see their Lemma 4.2).

Both theorems above hold for almost all frequencies. It is possible that on a set of measure 0 the behavior be quite different. This fact is suggested by a result of Rosenblatt [28] who established, on a set of measure 0, non-normal attraction for the Fourier transform under a different normalization than \sqrt{n} .

We give next conditions imposed to the stationary sequence which help to identify the frequencies for which the LIL holds. As we shall see, the next result is well adapted for linear processes generated by iid (independent identically distributed) sequences.

Theorem 3 Assume that (5) holds and that

$$\sum_{n \geq 0} \|P_0(X_n) - P_0(X_{n+1})\|_2 < \infty. \quad (7)$$

Then the spectral density $f(t)$ of $(X_k, k \in \mathbb{Z})$ is continuous on $(0, 2\pi)$, and the convergence (6) holds for all $t \in (0, 2\pi)$. Moreover if θ is ergodic, the conclusions of Theorem 1 hold for all $t \in (0, 2\pi) \setminus \{\pi\}$ such that e^{-2it} is not an eigenvalue of θ .

Remark 4 The conditions of this theorem do not imply that the spectral density is continuous at 0. This is easy to see by considering the time series $X_k = \sum_{j \geq 0} j^{-3/4} \varepsilon_{k-j}$ where $(\varepsilon_k)_{k \in \mathbb{Z}}$ is a sequence of iid centered real random variables in \mathbb{L}^2 . For this case all the conditions of Theorem 3 are satisfied (see Section 4.1) and $\text{var}(\sum_{k=1}^n X_k)/n$ converges to ∞ . This shows that the spectral density is not continuous at 0 since otherwise we would have $\text{Var}(\sum_{k=1}^n X_k)/n \rightarrow 2\pi f(0)$, which is not the case.

We would like to mention that Condition (7) above was used by Wu [32] in the context of the CLT. We also infer from our proof and Remark 10, that if (5) and (7) hold, and θ is ergodic, then $\limsup_{n \rightarrow \infty} I_n(\pi)/\log \log n = 2f(\pi)$ \mathbb{P} -a.s. Moreover, it follows from a recent result of Cuny [8] that if θ is ergodic, (5) holds and (7) is reinforced to $\sum_{n \geq 0} \|P_0(X_n)\|_2 < \infty$, then $\limsup_{n \rightarrow \infty} I_n(0)/\log \log n = 2f(0)$ \mathbb{P} -a.s.

We say that θ is *weakly-mixing*, if for all $A, B \in \mathcal{A}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mathbb{P}(\theta^{-k} A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| = 0.$$

It is well-known (see e.g. [22, Theorem 6.1]) that saying that θ is weakly mixing is equivalent to saying that θ is ergodic and its only eigenvalue is 1. Let us also mention that when $\mathcal{F}_{-\infty}$ is trivial then θ is weakly mixing (see section 2 of [22]).

As an immediate corollary to Theorem 3 we obtain the following LIL for all frequencies.

Corollary 5 Assume that θ is weakly mixing and that (5) and (7) hold. Then the conclusion of Theorem 3 holds for all $t \in (0, 2\pi) \setminus \{\pi\}$.

Next theorem involves a projective condition in the spirit of Rootzén [26]. It is very useful in order to treat several classes of Markov chains including reversible Markov chains.

Theorem 6 Assume that θ is ergodic. Let $t \in (0, 2\pi) \setminus \{\pi\}$ be such that e^{-2it} is not an eigenvalue of θ . Assume in addition that

$$\sup_n \|\mathbb{E}_0(S_n(t))\|_2 < \infty. \quad (8)$$

Then

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}|S_n(t)|^2}{n} = \sigma_t^2 \quad (\text{say}) \quad (9)$$

and $\{\sum_{k=1}^n Y_k(t)/\sqrt{2n \log \log n}, n \geq 3\}$ is \mathbb{P} -a.s. bounded and has the ball $\{x \in \mathbb{R}^2 : x'x \leq \sigma_t^2/2\}$ as its set of limit points. In particular,

$$\limsup_{n \rightarrow \infty} \frac{I_n(t)}{\log \log n} = \frac{\sigma_t^2}{2\pi} \quad \mathbb{P}\text{-a.s.}$$

Remark 7 Note that in Theorem 6 we do not require the sequence to be regular, i.e. it may happen that $\mathbb{E}(X_0|\mathcal{F}_{-n})$ does not converge to 0 in \mathbb{L}^2 . The spectral density might not exist.

3 Proofs

Proof of Theorem 1. The proof is based on martingale approximation. By Lemma 4.1 in Peligrad and Wu [21], since $X_0 \in \mathbb{L}^2$ and (5) holds under (3), we know that for almost all $t \in [0, 2\pi)$, the following limit exists in $\mathbb{L}^2(\mathbb{P})$ and \mathbb{P} -a.s.

$$D_0(t) = \lim_{n \rightarrow \infty} \sum_{k=0}^n e^{ikt} P_0(X_k). \quad (10)$$

Hence setting for all $\ell \in \mathbb{Z}$,

$$D_\ell(t) = e^{i\ell t} D_0(t) \circ \theta^\ell, \quad (11)$$

we get that for almost all $t \in (0, 2\pi)$, $(D_\ell(t))_{\ell \in \mathbb{Z}}$ forms a sequence of martingale differences in $\mathbb{L}^2(\mathbb{P})$ with respect to $(\mathcal{F}_\ell)_{\ell \in \mathbb{Z}}$. As we shall see, the conclusion of the theorem will then follow from Propositions 8 and 11 below.

Proposition 8 *Assume that θ is ergodic. Let $t \in (0, 2\pi) \setminus \{\pi\}$ and assume that e^{-2it} is not an eigenvalue of θ . Let D be a square integrable complex-valued random variable adapted to \mathcal{F}_0 and such that $\mathbb{E}_{-1}(D) = 0$ a.s. For any $k \in \mathbb{Z}$, let $d_k(t) = (\mathcal{R}e(e^{ikt} D \circ \theta^k), \mathcal{I}m(e^{ikt} D \circ \theta^k))'$. Then the sequence $\{\sum_{k=1}^n d_k(t) / \sqrt{2n \log \log n}, n \geq 3\}$ is \mathbb{P} -a.s. bounded and has the ball $\{x \in \mathbb{R}^2 : x' \cdot x \leq \mathbb{E}(|D|^2)/2\}$ as its set of limit points.*

Remark 9 *Since we assume \mathcal{A} to be countably generated, then $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is separable and (see Lemma 32) θ can admit at most countably many eigenvalues. Hence, Proposition 8 applies to almost all $t \in [0, 2\pi)$.*

Remark 10 *Let $t = 0$ or $t = \pi$, and assume that θ is ergodic. Then if D is a square integrable real-valued random variable adapted to \mathcal{F}_0 and such that $\mathbb{E}_{-1}(D) = 0$ a.s., the following result holds: $\limsup_{n \rightarrow \infty} |\sum_{k=1}^n \cos(kt) D \circ \theta^k|^2 / (2n \log \log n) = \mathbb{E}(D^2)$ a.s. For $t = 0$, it is the usual law of the iterated logarithm for stationary ergodic martingale differences sequences. For $t = \pi$, it follows from a direct application of [17, Theorem 1].*

Proposition 11 *Assume that condition (3) holds. Then, for almost all $t \in [0, 2\pi)$,*

$$\frac{|S_n(t) - M_n(t)|}{\sqrt{n \log \log n}} \rightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (12)$$

where $M_n(t) = \sum_{k=1}^n D_k(t)$ and $D_k(t)$ is defined by (11).

To end the proof of Theorem 1, we proceed as follows. By Proposition 11, it suffices to prove that the conclusion of Theorem 1 holds replacing $Y_k(t)$ with $d_k(t) = (\mathcal{R}e(D_k(t)), \mathcal{I}m(D_k(t)))'$. With this aim, it suffices to apply Proposition 8 together with Remark 9 and to notice the following fact: according to Lemma 4.2 in Peligrad and Wu [21], for almost all $t \in [0, 2\pi)$,

$$\frac{\mathbb{E}(|D_0(t)|^2)}{2} = \pi f(t).$$

It remains to prove the above propositions.

It is convenient to work on the product space. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = ([0, 2\pi] \times \Omega, \mathcal{B} \otimes \mathcal{A}, \lambda \otimes \mathbb{P})$ where λ is the normalized Lebesgue measure on $[0, 2\pi]$, and \mathcal{B} be the Borel σ -algebra on $[0, 2\pi]$. Let $t \in [0, 2\pi)$

be a real number, fixed for the moment. Clearly, the transformation $\tilde{\theta} = \tilde{\theta}_t$ (we omit the dependence with respect to t when t is fixed) given by

$$\tilde{\theta} : (u, \omega) \mapsto (u + t \text{ modulo } 2\pi, \theta(\omega)), \quad (13)$$

is invertible, bi-measurable and preserves $\tilde{\mathbb{P}}$.

Consider also the filtration $(\tilde{\mathcal{F}}_n)_{n \in \mathbb{Z}}$ given by $\tilde{\mathcal{F}}_n := \mathcal{B} \otimes \mathcal{F}_n$.

Define a random variable \tilde{X}_0 on $\tilde{\Omega}$ by $\tilde{X}_0(u, \omega) = e^{iu} X_0(\omega)$ for every $(u, \omega) \in \tilde{\Omega}$, and for any $n \in \mathbb{Z}$, $\tilde{X}_n = \tilde{X}_0 \circ \tilde{\theta}^n$. Notice that $(\tilde{X}_n)_{n \in \mathbb{Z}}$ is a stationary sequence of complex random variables adapted to the non-decreasing filtration $(\tilde{\mathcal{F}}_n)$. Moreover $e^{iu} e^{int} X_n(\omega) = \tilde{X}_n(u, \omega)$.

Proof of Proposition 8. Let $t \in [0, 2\pi)$ be fixed. Let $\tilde{D}(u) = e^{iu} D$ and $\tilde{D}_k = \tilde{D} \circ \tilde{\theta}^k$.

Let $\tilde{d}_k = (\mathcal{R}e(\tilde{D}_k), \mathcal{I}m(\tilde{D}_k))'$. Then

$$\tilde{d}_k(u) = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} d_k(t).$$

Since the unit ball is invariant under rotations, the result will follow if we prove that for λ -a.e. $u \in [0, 2\pi]$, the sequence $\{\sum_{k=1}^n \tilde{d}_k(u) / \sqrt{2n \log \log n}, n \geq 3\}$ has \mathbb{P} -a.s. the ball $\{y \in \mathbb{R}^2 : y' \cdot y \leq \|D_0\|_2^2 / 2\}$ as its set of limit points, or equivalently (by Fubini's Theorem), if the sequence $\{\sum_{k=1}^n \tilde{d}_k / \sqrt{2n \log \log n}, n \geq 3\}$ has $\tilde{\mathbb{P}}$ -a.s. the ball $\{y \in \mathbb{R}^2 : y' \cdot y \leq \|D\|_2^2 / 2\}$ as its set of limit points.

According to the almost sure analogue of the Cramér-Wold device (see Sections 5.1 and 5.2 in Philipp [23]), this will happen if we can prove that for any $x \in \mathbb{R}^2$ such that $x' \cdot x = 1$,

$$\limsup_{n \rightarrow \infty} \frac{\sum_{k=1}^n x' \cdot \tilde{d}_k}{\sqrt{2n \log \log n}} = \frac{\|D\|_2^2}{2} \quad \tilde{\mathbb{P}} - \text{a.s.} \quad (14)$$

To prove it we shall apply Corollary 2 in Heyde and Scott [17] to the stationary martingale differences $x' \cdot \tilde{d} \circ \tilde{\theta}_t^k$. We have to verify

$$\frac{1}{n} \sum_{k=1}^n (x' \cdot \tilde{d}_k)^2 \rightarrow \frac{\|D\|_2^2}{2} \quad \tilde{\mathbb{P}} - \text{a.s.} \quad (15)$$

In order to understand this convergence it is convenient to write

$$D = A + iB.$$

Therefore if $x = (a, b)'$,

$$\begin{aligned} (x' \cdot \tilde{d}_k)^2 &= (a \mathcal{R}e(\tilde{D}_k) + b \mathcal{I}m(\tilde{D}_k))^2 \\ &= (a \cos(u + kt) + b \sin(u + kt))^2 (A^2 \circ \theta^k) + (b \cos(u + kt) - a \sin(u + kt))^2 (B^2 \circ \theta^k) \\ &\quad + 2(a \cos(u + kt) + b \sin(u + kt))(b \cos(u + kt) - a \sin(u + kt))(A \circ \theta^k)(B \circ \theta^k). \end{aligned}$$

By using basic trigonometric formulas, it follows that if $x = (a, b)'$ is such that $a^2 + b^2 = 1$,

$$\begin{aligned} (x' \cdot \tilde{d}_k)^2 &= \frac{(A^2 + B^2) \circ \theta^k}{2} + \frac{(a^2 - b^2) \cos(2u + 2kt)}{2} (A^2 - B^2) \circ \theta^k \\ &\quad + ab \sin(2u + 2kt) (A^2 - B^2) \circ \theta^k + ab (\cos(2u + 2kt) + \sin(2u + 2kt)) (A \circ \theta^k)(B \circ \theta^k) \\ &\quad + (b^2 - a^2) \sin(2u + 2kt) (A \circ \theta^k)(B \circ \theta^k). \end{aligned}$$

By Lemma 32 applied with $t_0 = 2t$, we derive that, for any $t \in (0, 2\pi) \setminus \{\pi\}$ such that e^{-2it} is not an eigenvalue of θ then, for all u ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n ((x' \cdot \tilde{d}_k(u))^2 - \frac{a^2 + b^2}{2} (A^2 + B^2) \circ \theta^k) = 0 \quad \tilde{\mathbb{P}} - \text{a.s.} \quad (16)$$

Also by the ergodic theorem for θ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (A^2 + B^2) \circ \theta^k = \mathbb{E}(|A|^2 + |B|^2) = \|D\|_2^2 \quad \tilde{\mathbb{P}} - \text{a.s.} \quad (17)$$

Gathering (16) and (17), we get (15). This ends the proof of Proposition 8. \square

Proof of Proposition 11.

Let $\tilde{D}_0(u, \cdot) = e^{iu} \sum_{k \geq 0} P_0(e^{ikt} X_k) = \sum_{k \geq 0} \tilde{P}_0(\tilde{X}_k)$ which is defined for λ -a.e. $t \in [0, 2\pi)$. Write $\tilde{S}_n = \sum_{k=1}^n \tilde{X}_k$, $\tilde{M}_n = \sum_{k=1}^n \tilde{D}_0 \circ \tilde{\theta}^k$, and $\tilde{R}_n = \tilde{S}_n - \tilde{M}_n$.

Denote by $\tilde{\mathbb{E}}$ the expectation with respect to $\tilde{\mathbb{P}}$.

The next lemma follows from Corollary 4.2 in Cuny [7]. Notice that in [7], complex-valued variables are allowed.

Lemma 12 *Assume that*

$$\sum_{n \geq 1} \log n \frac{\tilde{\mathbb{E}}(|\tilde{R}_n|^2)}{n^2} < \infty. \quad (18)$$

Then

$$\tilde{R}_n = o(\sqrt{n \log \log n}) \quad \tilde{\mathbb{P}}\text{-a.s.} \quad (19)$$

To prove that (12) holds, it suffices to prove that for λ -a.e. $t \in [0, 2\pi)$, (19) holds. According to Lemma 12 it suffices then to prove that (18) is satisfied for λ -a.e. $t \in [0, 2\pi)$. To this end, we first prove that

$$\int_0^{2\pi} \tilde{\mathbb{E}}(|\tilde{R}_n|^2) dt = 2 \sum_{k=1}^n \|\mathbb{E}_0(X_k)\|_2^2. \quad (20)$$

Indeed, for almost all $t \in [0, 2\pi)$,

$$\tilde{R}_n(u, \omega) = e^{iu} \sum_{k=1}^n e^{ikt} \mathbb{E}_0(X_k)(\omega) + e^{iu} \sum_{k \geq n+1} e^{ikt} (\mathbb{E}_0(X_k)(\omega) - \mathbb{E}_n(X_k)(\omega)). \quad (21)$$

Whenever the R.H.S. below converges, the following identity holds:

$$\int_0^{2\pi} |\tilde{R}_n(u, \omega)|^2 dt = \sum_{k=1}^n (\mathbb{E}_0(X_k))^2(\omega) + \sum_{k \geq n+1} (\mathbb{E}_0(X_k)(\omega) - \mathbb{E}_n(X_k)(\omega))^2.$$

Then, using that $\mathbb{E}((\mathbb{E}_0(X_k) - \mathbb{E}_n(X_k))^2) = \|\mathbb{E}_0(X_{k-n})\|_2^2 - \|\mathbb{E}_0(X_k)\|_2^2$, and the fact that under (3), (5) holds, we obtain (20). Using (20), we see that under (3), for λ -a.e. $t \in [0, 2\pi)$, condition (18) holds. This ends the proof of (12) and then of the proposition. \square

Proof of Theorem 2. According to the proof of Theorem 1, it suffices to prove that under the conditions of Theorem 2, the almost sure convergence (12) holds for almost all $t \in [0, 2\pi)$. With this aim, we shall use truncation arguments. Given $\gamma > 0$ and $r \geq 0$, we set for any integer ℓ ,

$$\overline{X}_{\ell, r} := X_\ell \mathbf{1}_{\{|X_\ell| \leq 2\gamma r\}} - \mathbb{E}(X_\ell \mathbf{1}_{\{|X_\ell| \leq 2\gamma r\}})$$

and

$$\overline{D}_{\ell,r}(t) := e^{i\ell t} \sum_{k \geq 0} e^{ikt} (P_0(\overline{X}_{k,r})) \circ \theta^\ell.$$

We know that for almost all $t \in [0, 2\pi)$, $\overline{D}_{\ell,r}(t)$ is defined \mathbb{P} -a.s. and in $\mathbb{L}^2(\mathbb{P})$.

We define non stationary sequences $(\overline{X}_\ell)_{\ell \geq 1}$ and $(\overline{D}_\ell(t))_{\ell \geq 1}$ as follows: for every $r \in \mathbb{N}$ and every $\ell \in \{2^r, \dots, 2^{r+1} - 1\}$,

$$\overline{X}_\ell := \overline{X}_{\ell,r}, \quad \overline{D}_\ell(t) := \overline{D}_{\ell,r}(t). \quad (22)$$

Let also

$$X_\ell^* = X_\ell - \overline{X}_\ell \quad \text{and} \quad D_\ell^*(t) = D_\ell(t) - \overline{D}_\ell(t). \quad (23)$$

Lemma 13 *Assume that $\mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty$. Then, for a.e. $t \in [0, 2\pi)$,*

$$\sum_{n \geq 3} \frac{X_n^*}{\sqrt{n \log \log n}} e^{int} \quad \text{converges } \mathbb{P}\text{-a.s.} \quad (24)$$

In particular, by Kronecker's lemma, for a.e. $t \in [0, 2\pi)$, $\frac{\sum_{k=1}^n e^{ikt} X_k^}{\sqrt{n \log \log n}} \rightarrow 0$ \mathbb{P} -a.s.*

Proof. By Carleson's theorem [5], in order to establish (24) it suffices to prove that

$$\sum_{n \geq 3} \frac{(X_n^*)^2}{n \log \log n} < \infty \quad \mathbb{P}\text{-a.s.}$$

This is true because

$$\begin{aligned} \sum_{n \geq 4} \frac{\mathbb{E}((X_n^*)^2)}{n \log \log n} &= \sum_{r \geq 2} \sum_{\ell=2^r}^{2^{r+1}-1} \frac{\mathbb{E}((X_\ell - \overline{X}_{\ell,r})^2)}{\ell \log \log \ell} \leq 4 \sum_{r \geq 2} \mathbb{E}(X_0^2 \mathbf{1}_{\{|X_0| > 2^{r^r}\}}) \sum_{\ell=2^r}^{2^{r+1}-1} \frac{1}{\ell \log \log \ell} \\ &\ll \sum_{r \geq 2} \frac{1}{\log r} \mathbb{E}(X_0^2 \mathbf{1}_{\{|X_0| > 2^{r^r}\}}) \ll \mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty, \end{aligned} \quad (25)$$

where we used Fubini in the last step and the notation $a \ll b$ means there is a universal constant $C > 0$ such that $a < Cb$. \square

Lemma 14 *Assume that $\mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty$. Then, for a.e. $t \in [0, 2\pi)$,*

$$\frac{\sum_{k=1}^n D_k^*(t)}{\sqrt{n \log \log n}} \rightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

Proof. For almost all $t \in [0, 2\pi)$, $(D_\ell^*(t))_{\ell \geq 1}$ is a sequence of martingale differences in $\mathbb{L}^2(\mathbb{P})$. Hence using the Doob-Kolmogorov maximal inequality, we infer that the lemma will be established provided that

$$\sum_{k \geq 3} \int_0^{2\pi} \frac{\|D_k^*(t)\|_2^2}{k \log \log k} dt < \infty. \quad (26)$$

To prove it we use simple algebra and the projection's orthogonality, as follows:

$$\begin{aligned}
\sum_{k \geq 4} \int_0^{2\pi} \frac{\|D_k^*(t)\|_2^2}{k \log \log k} dt &= \sum_{r \geq 2} \sum_{\ell=2^r}^{2^{r+1}-1} \int_0^{2\pi} \frac{\|D_\ell(t) - \overline{D}_{\ell,r}(t)\|_2^2}{\ell \log \log \ell} dt \\
&\leq \sum_{r \geq 2} \frac{1}{2^r \log r} \sum_{\ell=2^r}^{2^{r+1}-1} \int_0^{2\pi} \frac{\|D_\ell(t) - \overline{D}_\ell(t)\|_2^2}{\ell \log \log \ell} dt \\
&\leq 2\pi \sum_{r \geq 2} \frac{1}{2^r \log r} \sum_{\ell=2^r}^{2^{r+1}-1} \sum_{k \geq 0} \|(P_0(X_k - \overline{X}_{k,r})) \circ \theta^\ell\|_2^2 \\
&= 2\pi \sum_{r \geq 2} \frac{1}{2^r \log r} \sum_{\ell=2^r}^{2^{r+1}-1} \sum_{k \geq 0} \|(P_{-k}(X_0 \mathbf{1}_{\{|X_0| > 2^{7r}\}})) \circ \theta^{\ell+k}\|_2^2 \\
&= 2\pi \sum_{r \geq 2} \frac{1}{\log r} \mathbb{E}(X_0 \mathbf{1}_{\{|X_0| > 2^{7r}\}} - \mathbb{E}(X_0 \mathbf{1}_{\{|X_0| > 2^{7r}\}} | \mathcal{F}_{-\infty}))^2 \leq 2\pi \sum_{r \geq 2} \frac{1}{\log r} \|X_0 \mathbf{1}_{\{|X_0| > 2^{7r}\}}\|_2^2.
\end{aligned}$$

Next, using Fubini's theorem as done in (25), (26) follows. \square

From Lemmas 13 and 14, we then deduce that if $\mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty$, then, for a.e. $t \in [0, 2\pi)$,

$$\frac{\sum_{k=1}^n (e^{ikt} X_k^* - D_k^*(t))}{\sqrt{n \log \log n}} \rightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

Therefore, to prove that the almost sure convergence (12) holds for almost all $t \in [0, 2\pi)$ (and then the theorem) it suffices to prove that for almost all $t \in [0, 2\pi)$,

$$\frac{|\overline{S}_n(t) - \overline{M}_n(t)|}{\sqrt{n \log \log n}} \rightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (27)$$

where $\overline{S}_n(t) = \sum_{j=1}^n e^{ijt} \overline{X}_j$ and $\overline{M}_n(t) = \sum_{j=1}^n \overline{D}_j(t)$ where the \overline{X}_j 's and $\overline{D}_j(t)$'s are defined in (22). Let

$$\overline{R}_n(t) = \overline{S}_n(t) - \overline{M}_n(t),$$

and for any $r \in \mathbb{N}$, let

$$A_r(t) := \sup_{0 \leq k \leq 2^r - 1} |\overline{R}_{k+2^r}(t) - \overline{R}_{2^r-1}(t)|.$$

Let $N \in \mathbb{N}^*$ and let $k \in [1, 2^N]$. We first notice that $A_r(t) \geq |\sum_{\ell=2^r}^{2^{r+1}-1} e^{i\ell t} (\overline{X}_\ell - \overline{D}_\ell(t))|$, so if K is the integer such that $2^{K-1} \leq k \leq 2^K - 1$, then

$$|\overline{R}_k(t)| \leq \sum_{r=0}^{K-1} A_r(t).$$

Consequently since $K \leq N$,

$$\sup_{1 \leq k \leq 2^N} |\overline{R}_k(t)| \leq \sum_{r=0}^{N-1} A_r(t).$$

Therefore, (27) will follow if we can prove that for almost all $t \in [0, 2\pi)$,

$$\sup_{0 \leq k \leq 2^r - 1} |\overline{R}_{k+2^r}(t) - \overline{R}_{2^r-1}(t)| = o(2^{r/2} \cdot (\log r)^{1/2}) \quad \mathbb{P}\text{-a.s.},$$

which will be true if we can prove that

$$\sum_{r \geq 0} \frac{1}{2^r \log r} \int_0^{2\pi} \mathbb{E} \left[\max_{2^r \leq k \leq 2^{r+1} - 1} |\bar{R}_k(t) - \bar{R}_{2^r-1}(t)|^2 \right] dt < \infty. \quad (28)$$

Notice that for any integer k in $[2^r, 2^{r+1} - 1]$,

$$\bar{R}_k(t) - \bar{R}_{2^r-1}(t) = e^{i(2^r-1)t} \left(\sum_{\ell=1}^{k-2^r+1} e^{i\ell t} (\bar{X}_{0,r} - \bar{D}_{0,r}(t)) \circ \theta^\ell \right) \circ \theta^{2^r-1}.$$

Therefore, by stationarity proving (28) amounts to prove that

$$\sum_{r \geq 0} \frac{1}{2^r \log r} \int_0^{2\pi} \mathbb{E} \left(\max_{1 \leq k \leq 2^r} \left| \sum_{\ell=1}^k (\bar{X}_{\ell,r} - \bar{D}_{\ell,r}(t)) \right|^2 \right) dt < \infty, \quad (29)$$

where $\bar{X}_{\ell,r}(t) = e^{i\ell t} \bar{X}_{\ell,r}$. Using Lemma 33 given in the Appendix with $M = 2^{\gamma r}$, we get that for any integer $s > 1$,

$$\begin{aligned} (2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \left(\max_{1 \leq k \leq 2^r} \left| \sum_{\ell=1}^k (\bar{X}_{\ell,r} - \bar{D}_{\ell,r}(t)) \right|^2 \right) dt \\ \leq 24 \times 2^r \|\mathbb{E}_{-s}(X_0)\|^2 + 24 \times 2^r \|X_0 \mathbf{1}_{|X_0| > 2^{\gamma r}}\|^2 + 12s^2 2^{2\gamma r}. \end{aligned}$$

To prove (29) and then to end the proof of the theorem, we select $\gamma < 1/4$ and use the above inequality with $s = [2^{\gamma r}] + 1$. Using Fubini's theorem as done in (25), we infer that (29) will be established provided that

$$\sum_{r \geq 2} \frac{1}{\log r} \|\mathbb{E}_{-[2^{\gamma r}]}(X_0)\|_2^2 < \infty. \quad (30)$$

This convergence follows from condition (4) by using the fact $(\|\mathbb{E}_{-n}(X_0)\|_2^2)_{n \geq 1}$ is decreasing and by noticing that by the usual comparison between the series and the integrals, for any non-increasing and positive function h on \mathbb{R}^+ and any positive γ ,

$$\sum_{n \geq 1} n^{-1} h(n^\gamma) < \infty \text{ if and only if } \sum_{n \geq 1} n^{-1} h(n) < \infty.$$

and that (30) is equivalent to $\sum_{n \geq 3} \frac{1}{n(\log \log n)} \|\mathbb{E}_{-[n^\gamma]}(X_0)\|_2^2 < \infty$. \square

Proof of Theorem 3. We divide the proof of this theorem in two parts.

1. **Proof of the continuity of f and of relation (6).** Let $(c_n)_{n \in \mathbb{Z}}$ denote the Fourier coefficients of f , i.e. $c_n := \mathbb{E}(X_0 X_n)$. Then, the Fourier coefficients of $(1 - e^{it})f(t)$ are $(c_n - c_{n+1})_{n \in \mathbb{Z}}$ and the Fourier coefficients of $h(t) := |1 - e^{it}|^2 f(t)$ are $(b_n)_{n \in \mathbb{Z}}$ with $b_n = 2c_n - c_{n+1} - c_{n-1}$, $n \in \mathbb{Z}$.

One can easily see that h is the spectral density associated with the stationary process $(Z_n)_{n \in \mathbb{Z}} := (X_n - X_{n-1})_{n \in \mathbb{Z}}$, i.e. $b_n = \mathbb{E}(Z_0 Z_n)$. Hence for $n \geq 0$,

$$|b_n| = |\mathbb{E}(Z_0 Z_n)| = \left| \sum_{k \geq 0} \mathbb{E}(P_{-k}(Z_0) P_{-k}(Z_n)) \right| \leq \sum_{k \geq 0} \|P_{-k}(Z_0)\|_2 \|P_{-k-n}(Z_0)\|_2.$$

Therefore,

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} |b_n| = 2 \sum_{n \geq 1} |b_n| \leq 2 \sum_{n \geq 0} (\|P_{-n}(Z_0)\|_2)^2.$$

By (7) it follows that $(b_n)_{n \in \mathbb{Z}}$ is absolutely summable. Therefore, by well known results on spectral density, (see for instance Bradley [2], Ch 8 and 9) h must be continuous and bounded on $[0, 2\pi]$, which in turn implies that f is continuous on $(0, 2\pi)$.

We prove now that (6) holds for every $t \in (0, 2\pi)$. With this aim, it suffices to show that, for every $t \in (0, 2\pi)$, $|1 - e^{it}|^2 \mathbb{E}(|S_n(t)|^2)/n \rightarrow h(t)$.

Define $T_n(t) := \sum_{k=1}^n e^{ikt} Z_k$. It is easy to see, using the fact that $c_n \rightarrow 0$ as $n \rightarrow \pm\infty$, that

$$\frac{\mathbb{E}(|T_n(t)|^2)}{n} = \frac{|1 - e^{it}|^2 \mathbb{E}(|S_n(t)|^2)}{n} + o(1),$$

(the little o is uniform in $t \in [0, 2\pi]$). Now, by (4.6) of [21], $(\mathbb{E}(|T_n(t)|^2)/n)_{n \geq 0}$ is nothing else but the Cesàro averages of the partial sums of the Fourier series associated with h , hence it converges to $h(t)$ by Fejer's theorem.

2. End of the proof.

By assumption (7), we have

$$\sum_{k \geq 0} |P_0(X_k - X_{k+1})| \quad \text{converges in } \mathbb{L}^2. \quad (31)$$

Let $t \in (0, 2\pi)$ be *fixed*. Using that $P_0(X_{-1}) = 0$, we obtain

$$\begin{aligned} \sum_{m=0}^k e^{imt} P_0(X_m - X_{m-1}) &= \sum_{m=0}^k e^{imt} P_0(X_m) - \sum_{m=0}^{k-1} e^{i(m+1)t} P_0(X_m) \\ &= (1 - e^{it}) \sum_{m=0}^k e^{imt} P_0(X_m) + e^{i(k+1)t} P_0(X_k). \end{aligned}$$

Since $\|P_0(X_k)\|_2 \rightarrow 0$, by (31), we see that the series $\sum_{m=0}^k e^{imt} P_0(X_m)$ converges in \mathbb{L}^2 as $k \rightarrow \infty$. Hence defining $D_0(t)$ by (10), it follows that $(D_0(t) \circ \theta^\ell, \ell \in \mathbb{Z})$ is a stationary sequence of martingale differences in \mathbb{L}^2 adapted to $(\mathcal{F}_\ell, \ell \in \mathbb{Z})$. Hence the theorem will follow by Proposition 8, if we can prove that $|S_n(t) - M_n(t)|/\sqrt{n \log \log n} \rightarrow 0$ \mathbb{P} -a.s. where $M_n(t) = \sum_{k=1}^n e^{ikt} D_0(t) \circ \theta^k$. With this aim, we first notice that

$$(1 - e^{it})D_0(t) = F_0(t) \quad \text{where} \quad F_0(t) = \sum_{m \geq 0} e^{imt} P_0(X_m - X_{m-1}).$$

Hence, writing $F_k(t) = F_0(t) \circ \theta^k$, we obtain the representation

$$(1 - e^{it})(S_n(t) - M_n(t)) = \sum_{k=0}^{n-1} e^{ikt} (Z_k - F_k(t)),$$

where $Z_k = X_k - X_{k-1}$. Therefore, the proof of the theorem will be complete if we can show that

$$\left| \sum_{k=0}^{n-1} e^{ikt} (Z_k - F_k(t)) \right| / \sqrt{n \log \log n} \rightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (32)$$

To prove this almost sure convergence, we shall work on the product space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ introduced in the proof of Theorem 1. Recall that $\tilde{\theta}$ has been defined in (13), $\tilde{\mathcal{F}}_n := \mathcal{B}([0, 2\pi]) \otimes \mathcal{F}_n$ and $\tilde{\mathbb{E}}$ stands for the expectation under $\tilde{\mathbb{P}}$.

Define $\tilde{Z}_0(u, \omega) := e^{iu} Z_0(\omega)$ and $\tilde{Z}_k := \tilde{Z}_0 \circ \tilde{\theta}^k$. Similarly, define $\tilde{F}_0(u, \omega) = e^{iu} F_0(t)(\omega)$ and $\tilde{F}_k = \tilde{F}_0 \circ \tilde{\theta}^k$. Let $\tilde{P}_0(\cdot) = \tilde{\mathbb{E}}(\cdot | \tilde{\mathcal{F}}_0) - \tilde{\mathbb{E}}(\cdot | \tilde{\mathcal{F}}_{-1})$. Note that $\tilde{F}_0 = \sum_{k \geq 0} \tilde{P}_0(\tilde{Z}_k) = e^{iu} \sum_{k \geq 0} e^{ikt} P_0(X_k - X_{k-1})$. By assumption (7), we have

$$\sum_{n \geq 0} \|\tilde{P}_0(\tilde{Z}_n)\|_{2, \tilde{\mathbb{P}}} < \infty,$$

where $\|\cdot\|_{2, \tilde{\mathbb{P}}}$ is the \mathbb{L}^2 norm with respect to $\tilde{\mathbb{P}}$.

Therefore, by Theorem 2.7 of Cuny [8] (see (21) of [8]), identifying \mathbb{C} with \mathbb{R}^2 , we obtain that

$$\left| \sum_{k=0}^{n-1} (\tilde{Z}_k - \tilde{F}_k) \right| / \sqrt{n \log \log n} \rightarrow 0 \quad \tilde{\mathbb{P}}\text{-a.s.}$$

Now, $\tilde{Z}_k(u, \cdot) = e^{iu} e^{ikt} Z_k$ and $\tilde{F}_k(u, \cdot) = e^{iu} e^{ikt} F_k$, hence (32) follows. \square

Proof of Theorem 6. Define an operator R_t on $\mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P})$ by $R_t(Y) := e^{it} \mathbb{E}_0(Y \circ \theta)$. Note that for every $n \geq 0$, $R_t^n(Y) = e^{int} \mathbb{E}_0(Y \circ \theta^n)$. Hence by assumption $\sup_{n \geq 1} \|\sum_{k=0}^n R_t^k(X_0)\|_2 < \infty$. By Browder [4, Lemma 5], there exists $Z_0 = Z_0(t) \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P})$ such that

$$X_0 = Z_0 - R_t(Z_0) = Z_0 - e^{it} \mathbb{E}_0(Z_1). \quad (33)$$

Now we denote $Z_k = Z_0 \circ \theta^k$. Note that $(Z_k)_k$ is a stationary sequence, $R_t(Z_0) = e^{it} \mathbb{E}_0(Z_1)$ and we have the decomposition:

$$X_0 = Z_0 - \mathbb{E}_{-1}(Z_0) + \mathbb{E}_{-1}(Z_0) - e^{it} \mathbb{E}_0(Z_1).$$

Denote the martingale difference $D_0(t) = Z_0 - \mathbb{E}_{-1}(Z_0) = P_0(Z_0)$. So,

$$\begin{aligned} S_n(t) &= \sum_{k=1}^n e^{itk} D_0(t) \circ \theta^k + \sum_{k=1}^n (e^{itk} \mathbb{E}_{k-1}(Z_k) - e^{it(k+1)} \mathbb{E}_k(Z_{k+1})) \\ &= \sum_{k=1}^n e^{itk} D_0(t) \circ \theta^k + e^{it} \mathbb{E}_0(Z_1) - e^{it(n+1)} \mathbb{E}_n(Z_{n+1}). \end{aligned}$$

By the Borel-Cantelli lemma,

$$|\mathbb{E}_n(Z_{n+1})| / \sqrt{n} = |\mathbb{E}_{-1}(Z_t)| \circ \theta^{n+1} / \sqrt{n} \rightarrow 0 \quad \mathbb{P}\text{-a.s.}$$

Therefore we have the following martingale approximation:

$$\frac{1}{n} (S_n(t) - \sum_{k=1}^n e^{itk} D_0(t) \circ \theta^k) \rightarrow 0 \text{ a.s. and in } \mathbb{L}^2.$$

Hence, since e^{-2it} is not an eigenvalue of θ , the proposition follows from Proposition 8 with $E(|D_0|^2) = E(|Z_0 - \mathbb{E}_{-1}(Z_0)|^2) = \sigma_t^2$.

It is convenient to express σ_t in terms of the original variables. With this aim notice that by equation (33) we obtain

$$\begin{aligned} \sum_{k=0}^n e^{itk} P_0(X_k) &= \sum_{k=0}^n e^{itk} P_0(Z_k) - \sum_{k=0}^n e^{it(k+1)} P_0(Z_{k+1}) \\ &= P_0(Z_0) - e^{it(n+1)} P_0(Z_{n+1}) = D_0(t) - e^{it(n+1)} P_0(Z_{n+1}). \end{aligned}$$

Since

$$\|P_0(Z_{n+1})\|_2^2 = \|\mathbb{E}_0(Z_{n+1})\|_2^2 - \|\mathbb{E}_{-1}(Z_{n+1})\|_2^2 = \|\mathbb{E}_{-n-1}(Z_0)\|_2^2 - \|\mathbb{E}_{-n-2}(Z_0)\|_2^2 \rightarrow 0,$$

we obtain

$$\sum_{k=0}^n e^{itk} P_0(X_k) \rightarrow D_0(t) \text{ in } \mathbb{L}^2.$$

This shows that, for this case, the representation (10) holds for all $t \in [0, 2\pi]$ such that (8) is satisfied. \square

4 Examples

4.1 Linear processes.

Let us consider the following linear process $(X_k)_{k \in \mathbb{Z}}$ defined by $X_k = \sum_{j \geq 0} a_j \varepsilon_{k-j}$ where $(\varepsilon_k)_{k \in \mathbb{Z}}$ is a sequence of iid real random variables in \mathbb{L}^2 and $(a_k)_{k \in \mathbb{Z}}$ is a sequence of reals in ℓ^2 . Taking $\mathcal{F}_0 = \sigma(\varepsilon_k, k \leq 0)$, it follows that $P_0(X_i) = a_i \varepsilon_0$. Therefore (7) is reduced to

$$\sum_{n \geq 3} |a_n - a_{n+1}| < \infty. \quad (34)$$

Hence, because $\mathcal{F}_{-\infty}$ is trivial, we conclude, by Corollary 5, that the conclusions of Theorem 3 hold for all $t \in (0, \pi) \cup (\pi, 2\pi)$ as soon as (34) is satisfied. Let us mention that when a_n is decreasing (34) is always satisfied.

4.2 Functions of linear processes

In this section, we shall focus on functions of real-valued linear processes. Define

$$X_k = h\left(\sum_{i \geq 0} a_i \varepsilon_{k-i}\right) - \mathbb{E}\left(h\left(\sum_{i \geq 0} a_i \varepsilon_{k-i}\right)\right), \quad (35)$$

where $(a_i)_{i \in \mathbb{Z}}$ be a sequence of real numbers in ℓ^2 and $(\varepsilon_i)_{i \in \mathbb{Z}}$ is a sequence of iid random variables in \mathbb{L}^2 . We shall give sufficient conditions in terms of the regularity of the function h , for $I_n(t)$ to satisfy a law of the iterated logarithm as described in Theorem 1.

Denote by $w_h(\cdot, M)$ the modulus of continuity of the function h on the interval $[-M, M]$, that is

$$w_h(u, M) = \sup\{|h(x) - h(y)|, |x - y| \leq u, |x| \leq M, |y| \leq M\}.$$

Corollary 15 *Assume that h is γ -Hölder on any compact set, with $w_h(u, M) \leq Cu^\gamma M^\alpha$, for some $C > 0$, $\gamma \in]0, 1]$ and $\alpha \geq 0$. Assume that*

$$\sum_{k \geq 2} (\log k)^2 |a_k|^{2\gamma} < \infty \text{ and } \mathbb{E}(|\varepsilon_0|^{2\nu(2\alpha+2\gamma)}) < \infty. \quad (36)$$

Then the conclusions of Theorem 1 hold with $(X_k)_{k \in \mathbb{Z}}$ defined by (35).

Proof. We shall apply Theorem 1 by taking $\mathcal{F}_k = \sigma(\varepsilon_\ell, \ell \leq k)$. Since X_0 is regular, $\|\mathbb{E}_0(X_k)\|_2^2 = \sum_{\ell \geq k} \|P_{-\ell}(X_0)\|_2^2$. Therefore (3) is equivalent to

$$\sum_{\ell \geq 2} (\log \ell)^2 \|P_0(X_\ell)\|_2^2 < \infty. \quad (37)$$

Let ε' be an independent copy of ε , and denote by $\mathbb{E}_\varepsilon(\cdot)$ the conditional expectation with respect to ε . Clearly

$$P_0(X_k) = \mathbb{E}_\varepsilon \left(h \left(\sum_{i=0}^{k-1} a_i \varepsilon'_{k-i} + a_k \varepsilon_0 + \sum_{i>k} a_i \varepsilon_{k-i} \right) - h \left(\sum_{i=0}^{k-1} a_i \varepsilon'_{k-i} + a_k \varepsilon'_0 + \sum_{i>k} a_i \varepsilon_{k-i} \right) \right).$$

Since $w_h(u_1 + u_2, M) \leq w_h(u_1, M) + w_h(u_2, M)$, it follows that

$$|P_0(X_k)| \leq \mathbb{E}_\varepsilon \left(2 \|X_0\|_\infty \wedge \left(w_h(|a_k| |\varepsilon_0|, |Y_1| \vee |Y_2|) + w_h(|a_k| |\varepsilon'_0|, |Y_1| \vee |Y_2|) \right) \right), \quad (38)$$

where $Y_1 = \sum_{i=0}^k a_i \varepsilon'_{k-i} + \sum_{i>k} a_i \varepsilon_{k-i}$ and $Y_2 = \sum_{i=0}^{k-1} a_i \varepsilon'_{k-i} + \sum_{i \geq k} a_i \varepsilon_{k-i}$. Noting that $(\varepsilon_0, |Y_1| \vee |Y_2|)$ and $(\varepsilon'_0, |Y_1| \vee |Y_2|)$ are both distributed as (ε_0, M_k) , where $M_k = \max \left\{ \left| \sum_{i \geq 0} a_i \varepsilon'_i \right|, \left| a_k \varepsilon_0 + \sum_{i \neq k} a_i \varepsilon'_i \right| \right\}$, and taking the L^2 -norm in (38), it follows that (37) is satisfied as soon as (36) is \square

4.3 Autoregressive Lipschitz models.

In this section, we give an example of iterative Lipschitz model, which fails to be irreducible, to which our results apply. For the sake of simplicity, we do not analyze the iterative Lipschitz models in their full generality, as defined in Diaconis and Freedman [12] and Dufflo [13].

For δ in $[0, 1[$ and C in $]0, 1]$, let $\mathcal{L}(C, \delta)$ be the class of 1-Lipschitz functions h which satisfy

$$h(0) = 0 \quad \text{and} \quad |h'(t)| \leq 1 - C(1 + |t|)^{-\delta} \quad \text{almost everywhere.}$$

Let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be a sequence of iid real-valued random variables. For $S \geq 1$, let $AR\mathcal{L}(C, \delta, S)$ be the class of Markov chains on \mathbb{R} defined by

$$Y_n = h(Y_{n-1}) + \varepsilon_n \quad \text{with} \quad h \in \mathcal{L}(C, \delta) \quad \text{and} \quad \mathbb{E} \left(\frac{|\varepsilon_0|^S L(\varepsilon_0)}{L(L(\varepsilon_0))} \right) < \infty. \quad (39)$$

(Recall that $L(x) = \log(e + |x|)$). For this model, there exists a unique invariant probability measure μ (see Proposition 2 of Dedecker and Rio [11]). Moreover we have

Proposition 16 *Assume that $(Y_i)_{i \in \mathbb{Z}}$ belongs to $AR\mathcal{L}(C, \delta, S)$. Then there exists a unique invariant probability measure that satisfies*

$$\int |x|^{S-\delta} \frac{L(x)}{L(L(x))} \mu(dx) < \infty.$$

Applying Theorem 2 we derive the following result.

Corollary 17 *Assume that $(Y_i)_{i \in \mathbb{Z}}$ is a stationary Markov chain belonging to $AR\mathcal{L}(C, \delta, S)$ for some $S \geq 2 + \delta$. Then, for any Lipschitz function g , the conclusions of Theorem 1 hold for $(g(Y_i) - \mu(g))_{i \in \mathbb{Z}}$.*

Remark 18 *The proof of this result reveals that an application of Theorem 1 would require the following moment condition on μ : $\int |x|^2 L(x) \mu(dx) < \infty$ which according to the proof of Proposition 16 is satisfied provided that $\mathbb{E}(|\varepsilon_0|^S L(\varepsilon_0)) < \infty$ for some $S \geq 2 + \delta$.*

Proof of Proposition 16. To prove Proposition 16, we shall modify the proof of Proposition 2 of Dedecker and Rio [11] as follows. Let K be the transition kernel of the stationary Markov chain

$(Y_i)_{i \in \mathbb{Z}}$ belonging to $ARL(C, \delta, \eta)$. For $n > 0$, we write $K^n g$ for the function $\int g(y) K^n(x, dy)$. Let $V(x) = |x|^S \frac{L(x)}{L(L(x))}$. Notice that

$$KV(x) = \mathbb{E}(V(Y_{n+1}) | Y_n = x) = \mathbb{E}(V(h(x) + \varepsilon_0)) \leq \mathbb{E}(V(|h(x)| + |\varepsilon_0|)).$$

By assumption on h , there exists $R_1 \geq 1$ and some $c \in]0, 1/2[$ such that for every x , with $|x| > R_1$, $|h(x)| \leq |x| - c|x|^{1-\delta} := g(x) \leq |x|$. Therefore, using the fact that for any positive reals a and b , $\log(e + a + b) = \log(e + a) + \log(1 + b/(e + a))$, we get for any $|x| > R_1$ (using that for $u \geq 0$, $\log(1 + u) \leq u$),

$$\begin{aligned} V(|h(x)| + |\varepsilon_0|) &\leq (g(x) + |\varepsilon_0|)^S \frac{L(x)}{L(L(|x| + |\varepsilon_0|))} + (|x| + |\varepsilon_0|)^S \frac{L(\varepsilon_0/(1 + |x|))}{L(L(|x| + |\varepsilon_0|))} \\ &\leq (g(x) + |\varepsilon_0|)^S \frac{L(x)}{L(L(x))} + \frac{2^S |x|^{S-1} |\varepsilon_0|}{L(L(R_1))} + \frac{2^S |\varepsilon_0|^S L(\varepsilon_0)}{L(L(\varepsilon_0))}. \end{aligned} \quad (40)$$

To deal now with the first term in the right hand side of the above inequality, we shall use inequality (54), in the Appendix, with $a = g(x)$ and $b = |\varepsilon_0|$. We get that there exist positive constants c and R_2 such that for any $|x| > R_2$,

$$\begin{aligned} V(|h(x)| + |\varepsilon_0|) &\leq (g(x))^S \frac{L(x)}{L(L(x))} + 2^{S+1} |\varepsilon_0| (g(x))^{S-1} \frac{L(x)}{L(L(x))} \\ &\quad + 2^S |\varepsilon_0|^S \frac{L(x)}{L(L(x))} + 2^S |x|^{S-1} |\varepsilon_0| + \frac{2^S |\varepsilon_0|^S L(\varepsilon_0)}{L(L(\varepsilon_0))} \\ &\leq |x|^S \frac{L(x)}{L(L(x))} - c|x|^{S-\delta} \frac{L(x)}{L(L(x))} + 3 \times 2^S |\varepsilon_0| |x|^{S-1} L(x) + 2^S |\varepsilon_0|^S \frac{L(x)}{L(L(x))} + \frac{2^S |\varepsilon_0|^S L(\varepsilon_0)}{L(L(\varepsilon_0))}. \end{aligned}$$

Taking the expectation, considering the moment assumption on ε_0 and using the fact that $\delta \in [0, 1[$ and $S \geq 1$, it follows that there exist positive constants d and R such that for any $|x| > R$,

$$KV(x) \leq V(x) - d|x|^{S-\delta} \frac{L(x)}{L(L(x))}.$$

So overall it follows that there exists a positive constant b such that

$$KV(x) \leq V(x) - d|x|^{S-\delta} \frac{L(x)}{L(L(x))} + b\mathbf{1}_{[-R, R]}(x). \quad (41)$$

This inequality allows to use the arguments given at the end of the proof of Proposition 2 in Dedecker and Rio [11]. Indeed, iterating n times the inequality (41), we get

$$\frac{d}{n} \sum_{k=1}^n \int |y|^{S-\delta} \frac{L(y)}{L(L(y))} K^k(x, dy) \leq \frac{1}{n} KV(x) + \frac{b}{n} \sum_{k=1}^n K^k([-R, R])(x),$$

and letting n tend to infinity, it follows that

$$d \int |x|^{S-\delta} \frac{L(x)}{L(L(x))} \mu(dx) \leq b\mu([-R, R]) < \infty.$$

□

Proof of Corollary 17. Let $X_i = g(Y_i) - \mu(g)$. According to Proposition 16, $\mathbb{E}\left(\frac{X_0^2 L(X_0)}{L(L(X_0))}\right) < \infty$. Hence Corollary 17 will follow from Theorem 2 if we can prove that the condition (4) is satisfied which will clearly hold if

$$\sum_{n>0} n^{-1} |K^n g(x) - \mu(g)|^2 \mu(dx) < \infty. \quad (42)$$

According to the inequality (5.7) in Dedecker and Rio [11], there exists a positive constant A such that

$$|K^n g(x) - \mu(g)| \leq An^{1-S/\delta} \int |x-y|\mu(dy) + A(1-B_n(x))^n \int |x-y|\mu(dy) \\ + An^{1-(S-1)/\delta} \int |x-y||y|^{S-\delta-1}\mu(dy), \quad (43)$$

where $B_n(x) = C[4(1+|x|) + (n-1)\mathbb{E}|\varepsilon_0|]^{-\delta}$. Noticing that $\sum_{n>0} n^{1-2(S-1)/\delta} < \infty$ as soon as $S > 1 + \delta$ and that according to Proposition 16, x^2 is μ -integrable as soon as $S \geq 2 + \delta$, we infer from (43) that (42) will be satisfied if we can prove that

$$\sum_{n>0} n^{-1} \int x^2 (1-B_n(x))^{2n} \mu(dx) < \infty. \quad (44)$$

Notice that that $(1-B_n(x))^{2n} \leq \exp(-2nB_n(x))$. If $|x| \leq 1$,

$$\exp(-2nB_n(x)) \leq \exp(-2Cn(8+4n)^{-\delta}),$$

implying that

$$\int_{-1}^1 x^2 \sum_{n>0} n^{-1} \exp(-2nB_n(x)) \mu(dx) < \infty.$$

Now if $|x| > 1$,

$$\sum_{n \geq 2} n^{-1} \exp(-2nB_n(x)) \leq \int_1^\infty u^{-1} \exp(-2Cu[4(1+|x|) + u\mathbb{E}|\varepsilon_0|]^{-\delta}) du \\ \leq \int_1^\infty u^{-1} \exp(-2Cu|x|^{-\delta}[8 + u|x|^{-\delta}\mathbb{E}|\varepsilon_0|]^{-\delta}) du \\ \leq \int_1^\infty z^{-1} \exp(-2Cz[8 + z\mathbb{E}|\varepsilon_0|]^{-\delta}) dz.$$

Hence there exists a positive constant M such that

$$\int_{|x|>1} x^2 \sum_{n>0} n^{-1} \exp(-2nB_n(x)) \mu(dx) \leq M \int_{|x|>1} x^2 \mu(dx), \quad (45)$$

which according to Proposition 16 is finite as soon as $S \geq 2 + \delta$. All the above computations then show that (44) (and then (42)) holds provided that $S \geq 2 + \delta$. \square

4.4 Application to weakly dependent sequences

Theorems 1 and 2 can be successively applied to large classes of weakly dependent sequences. In this section, we give an application to α -dependent sequences. With this aim, we first need some definitions.

Definition 19 For a sequence $\mathbf{Y} = (Y_i)_{i \in \mathbb{Z}}$, where $Y_i = Y_0 \circ \theta^i$ and Y_0 is an \mathcal{F}_0 -measurable and real-valued random variable, let for any $k \in \mathbb{N}$,

$$\alpha_{\mathbf{Y}}(k) = \sup_{u \in \mathbb{R}} \|\mathbb{E}(\mathbf{1}_{Y_k \leq u} | \mathcal{F}_0) - \mathbb{E}(\mathbf{1}_{Y_k \leq u})\|_1.$$

Remark 20 Recall that the strong mixing coefficient of Rosenblatt [27] between two σ -algebras \mathcal{F} and \mathcal{G} is defined by

$$\alpha(\mathcal{F}, \mathcal{G}) = \sup_{A \in \mathcal{F}, B \in \mathcal{G}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

For a strictly stationary sequence $(Y_i)_{i \in \mathbb{Z}}$ of real valued random variables, and the σ -algebra $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$, define then

$$\alpha(0) = 1 \text{ and } \alpha(k) = 2\alpha(\mathcal{F}_0, \sigma(Y_k)) \text{ for } k > 0. \quad (46)$$

Between the two above coefficients, the following relation holds: for any positive k , $\alpha_{\mathbf{Y}}(k) \leq \alpha(k)$. In addition, the α -dependent coefficient as defined in Definition 19 may be computed for instance for many Markov chains associated to dynamical systems that fail to be strongly mixing in the sense of Rosenblatt [27].

Definition 21 A quantile function Q is a function from $]0, 1]$ to \mathbb{R}_+ , which is left-continuous and non increasing. For any nonnegative random variable Z , we define the quantile function Q_Z of Z by $Q_Z(u) = \inf\{t \geq 0 : \mathbb{P}(|Z| > t) \leq u\}$.

Definition 22 Let μ be the probability distribution of a random variable X . If Q is an integrable quantile function (see Definition 21), let $\text{Mon}(Q, \mu)$ be the set of functions g which are monotonic on some open interval of \mathbb{R} and null elsewhere and such that $Q_{|g(X)|} \leq Q$. Let $\text{Mon}^c(Q, \mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and f_ℓ belongs to $\text{Mon}(Q, \mu)$.

Applying Theorem 2, we get

Corollary 23 Let Y_0 be a real-valued random variable with law P_{Y_0} , and $Y_i = Y_0 \circ \theta^i$. Let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$ where f belongs to $\text{Mon}^c(Q, P_{Y_0})$ with $Q^2 L(Q)/L(L(Q))$ integrable. Assume in addition that

$$\sum_{k \geq 3} \frac{1}{k(\log \log k)} \int_0^{\alpha_{\mathbf{Y}}(k)} Q^2(u) du < \infty. \quad (47)$$

Then (4) is satisfied and consequently, the conclusions of Theorem 1 hold for $(X_k)_{k \in \mathbb{Z}}$.

To prove that (47) implies (4), it suffices to notice that

$$\|\mathbb{E}_0(X_k)\|_2^2 = \mathbb{E}(X_k \mathbb{E}_0(X_k)) \leq \int_0^{\alpha_{\mathbf{Y}}(k)} Q^2(u) du$$

(see the proof of (4.17) in Merlevède and Rio [20] for the last inequality).

The definition 22 describes spaces similar to weak \mathbb{L}^p where we require a monotonicity condition plus a uniform bound on the tails of the functions. Let us introduce in the same spirit \mathbb{L}^p -like spaces.

Definition 24 If μ is a probability measure on \mathbb{R} and $p \in [2, \infty)$, $M \in (0, \infty)$, let $\text{Mon}_p(M, \mu)$ denote the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are monotonic on some interval and null elsewhere and such that $\mu(|f|^p) \leq M^p$. Let $\text{Mon}_p^c(M, \mu)$ be the closure in $\mathbb{L}^p(\mu)$ of the set of functions which can be written as $\sum_{\ell=1}^L a_\ell f_\ell$, where $\sum_{\ell=1}^L |a_\ell| \leq 1$ and $f_\ell \in \text{Mon}_p(M, \mu)$.

Let $X_i = f(Y_i) - \mathbb{E}(f(Y_i))$, where f belongs to $\text{Mon}_{2+\delta}^c(M, P_{Y_0})$ for some $\delta > 0$. If

$$\sum_{k \geq 3} \frac{(\alpha_{\mathbf{Y}}(k))^{\delta/(2+\delta)}}{k(\log \log k)} < \infty,$$

then Corollary 23 applies.

Application to functions of Markov chains associated to intermittent maps.

For γ in $]0, 1[$, we consider the intermittent map T_γ from $[0, 1]$ to $[0, 1]$, which is a modification of the Pomeau-Manneville map [24]:

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

Recall that T_γ is ergodic and that there exists a unique T_γ -invariant probability measure ν_γ on $[0, 1]$, which is absolutely continuous with respect to the Lebesgue measure. We denote by L_γ the Perron-Frobenius operator of T_γ with respect to ν_γ . Recall that for any bounded measurable functions f and g ,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(L_\gamma(f)g).$$

Let $(Y_i)_{i \geq 0}$ be a Markov chain with transition Kernel L_γ and invariant measure ν_γ .

Corollary 25 *Let $\gamma \in (0, 1)$ and $(Y_i)_{i \geq 1}$ be a stationary Markov chain with transition kernel L_γ and invariant measure ν_γ . Let Q be a quantile function such that*

$$\int_0^1 \frac{L(Q(u))}{L(L(Q(u)))} Q^2(u) du < \infty. \tag{48}$$

Let $X_i = f(Y_i) - \nu_\gamma(f)$ where f belongs to $\text{Mon}^c(Q, \nu_\gamma)$. Then (4) is satisfied and the conclusions of Theorem 1 hold for $(X_k)_{k \in \mathbb{Z}}$.

Remark 26 *Notice that, by standard arguments on quantile functions, (48) is equivalent to the following condition:*

$$\int_0^\infty \frac{xL(x)}{L(L(x))} Q^{-1}(x) dx < \infty,$$

where Q^{-1} is the generalized inverse of Q .

Proof. To prove this corollary, it suffices to see that (48) implies (47). For this purpose, we first notice that (47) can be rewritten in the following equivalent way (see Rio [25]):

$$\int_0^1 \frac{L(\alpha_{\mathbf{Y}}^{-1}(u))}{L(L(\alpha_{\mathbf{Y}}^{-1}(u)))} Q^2(u) du < \infty,$$

where $\alpha_{\mathbf{Y}}^{-1}(x) = \min\{q \in \mathbb{N} : \alpha_{\mathbf{Y}}(q) \leq x\}$. Now, according to Proposition 1.17 in Dedecker *et al.* [10], there exists a positive constant C such that $\alpha_{\mathbf{Y}}^{-1}(u) \leq Cu^{-\gamma/(1-\gamma)}$. Therefore, for any $\eta \in]0, 1/2[$, there exists a constant c depending on γ , C and η such that

$$\begin{aligned} \int_0^1 \frac{L(\alpha_{\mathbf{Y}}^{-1}(u))}{L(L(\alpha_{\mathbf{Y}}^{-1}(u)))} Q^2(u) du &\leq c \int_0^1 \frac{L(u^{-\eta})}{L(L(u^{-\eta}))} Q^2(u) du \\ &\leq c \int_0^1 \frac{L(Q(u))}{L(L(Q(u)))} Q^2(u) du + c \int_0^1 \frac{L(u^{-\eta})}{L(L(u^{-\eta}))} u^{-2\eta} du, \end{aligned}$$

which is finite under (47). \square

In particular, if f is with bounded variation and $\gamma < 1$, we infer from Corollary 25 that the conclusions of Theorem 1 hold for $(X_k)_{k \in \mathbb{Z}}$. Note also that (48) is satisfied if Q is such that $Q(u) \leq$

$Cu^{-1/2}(\log(u^{-1}))^{-1}(\log \log(u^{-1}))^{-b}$ for u small enough and $b > 1/2$. Therefore, since the density h_{ν_γ} of ν_γ is such that $h_{\nu_\gamma}(x) \leq Cx^{-\gamma}$ on $(0, 1]$, one can easily prove that if f is positive and non increasing on $(0, 1)$, with

$$f(x) \leq \frac{C}{x^{(1-\gamma)/2} |\log(x)| (\log |\log(x)|)^b} \quad \text{near 0 for some } b > 1/2,$$

then (48) holds.

4.5 Application to a class of Markov chains

We shall point out next some consequences of Theorem 6 in terms of stationary Markov chains characteristics. Let T be a regular transition probability on the measurable space $(\mathbb{S}, \mathcal{S})$, leaving invariant a probability μ on $(\mathbb{S}, \mathcal{S})$. We also denote by T the Markov operator induced on $\mathbb{L}^2(\mu)$ via

$$Tg(x) = \int_{\mathbb{S}} g(y)T(x, dy)$$

and we assume that it is ergodic (i.e. $Tf = f$ μ a.e. for $f \geq 0$ implies f is constant μ a.e.). Let $(\xi_n)_{n \in \mathbb{Z}}$ be the (stationary) canonical Markov chain with state space $(\mathbb{S}, \mathcal{S})$ associated with T , defined on the canonical space $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{S}^{\mathbb{Z}}, \mathcal{S}^{\otimes \mathbb{Z}}, \mathbb{P})$ (the law of ξ_0 under \mathbb{P} is μ). Denote by θ the shift on Ω . Denote by $\mathcal{F}_n = \sigma(\xi_j; j \leq n)$.

Corollary 27 *Let $(\xi_n)_{n \in \mathbb{Z}}$ be a stationary Markov chain. Let $h \in \mathbb{L}^2(\mathbb{S}, \mu)$ centered and define $X_k = h(\xi_k)$. Let $t \in (0, 2\pi) \setminus \{\pi\}$ be such that e^{-it} is not in the spectrum of T , and e^{-2it} is not an eigenvalue of T . Then the conclusions of Theorem 6 hold for $(X_k)_{k \in \mathbb{Z}}$.*

Remark 28 *In Corollary 27 we do not assume the regularity condition $\|T^n h\|_{2, \mu} \rightarrow 0$. The spectral density might not exist.*

Proof of Corollary 27. By assumption, there exists $g \in \mathbb{L}^2(\mathbb{S}, \mu)$ such that $h = g - e^{it}Tg$. Therefore,

$$\mathbb{E}_0(S_n(t)) = \sum_{k=1}^n \mathbb{E}_0(e^{itk}g(\xi_k) - e^{it(k+1)}g(\xi_{k+1})) = e^{it}\mathbb{E}_0(g(\xi_1)) - e^{it(n+1)}\mathbb{E}_0(g(\xi_{n+1})),$$

showing that condition (8) is satisfied. Hence, for $t \in (0, 2\pi) \setminus \{\pi\}$ such that e^{-2it} is not an eigenvalue of θ (the shift on Ω), the proposition will follow from Theorem 6. To end the proof we notice that if e^{-2it} is an eigenvalue for θ , it is an eigenvalue for T , see e.g. Proposition 2.3 of Cuny [6] (notice that the proof there extends easily to \mathbb{L}^2) \square

We give now a consequence of Corollary 27 for reversible Markov chains. This follows from the fact that the spectrum of T is real and lies in $[-1, 1]$.

Corollary 29 *Let $(\xi_n)_{n \in \mathbb{Z}}$ be a stationary and reversible Markov chain ($T = T^*$). Let $h \in \mathbb{L}^2(\mathbb{S}, \mu)$ centered and define $X_k = h(\xi_k)$. Then the conclusion of Theorem 6 holds for every $t \in (0, 2\pi) \setminus \{\pi\}$.*

The (independent) Metropolis Hastings Algorithm leads to a Markov chain with transition function

$$T(x, A) = p(x)\delta_x(A) + (1 - p(x))\nu(A),$$

where δ_x denotes the Dirac measure at point x , ν is a probability measure on \mathcal{S} and $p : \mathbb{S} \rightarrow [0, 1]$ is a measurable function for which $\theta = \int_{\mathbb{S}} \frac{1}{1-p(x)}\nu(dx) < \infty$. Then there is a unique invariant distribution

$$\mu(dx) = \frac{1}{\theta(1-p(x))}\nu(dx)$$

and the associated stationary Markov chain $(\xi_i)_i$ is reversible and ergodic. Hence Corollary 29 applies to this example.

5 Appendix

5.1 Facts from ergodic theory

We first recall the following consequence of the Dunford-Schwartz ergodic theorem, see sections VIII.5 and VIII.6 of [14].

Proposition 30 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and θ be a measure-preserving transformation on Ω . Let $s \in \mathbb{R}$. For every $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, there exists $\pi_s(X) \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$ such that*

$$\frac{1}{n} \sum_{k=0}^{n-1} e^{iks} X \circ \theta^k \xrightarrow[n \rightarrow \infty]{} \pi_s(X) \quad \mathbb{P}\text{-a.s.} \quad (49)$$

and in $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Moreover, $\pi_s(X) \circ \theta = e^{-is} \pi_s(X)$ \mathbb{P} -a.s.

Remark 31 *It follows from the Wiener-Wintner theorem that the set of measure 1 in (49) may be chosen independently of s , but we shall not need that refinement.*

Proof. Define an operator V_s on $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, by $V_s(X) = e^{is} X \circ \theta$. Then, V_s is a contraction of \mathbb{L}^1 which also contracts the \mathbb{L}^∞ norm. Hence we may apply [14, Theorem 6 p. 675], to obtain the almost sure convergence. The \mathbb{L}^1 convergence follows from [14, Corollary 5 p. 664] (see also the proof of the next lemma). \square

We also give the following lemma, that should be well-known. We give a proof for completeness.

Lemma 32 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and θ be a measure-preserving transformation on Ω . Let $t_0 \in \mathbb{R}$ be fixed. If there is no non trivial $Y \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, such that $Y \circ \theta = e^{-it_0} Y$ \mathbb{P} -a.s., then, for every $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ $\pi_{t_0}(X) = 0$ \mathbb{P} -a.s. Furthermore, when $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is separable, there exists a countable (at most) set $\mathbf{S} \subset \mathbb{R}$ such that for every $t \in \mathbb{R} \setminus \mathbf{S}$ and every $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, $\pi_t(X) = 0$ \mathbb{P} -a.s.*

Proof. Define V_{t_0} as above. Since $\sup_{n \geq 1} \frac{1}{n} \left\| \sum_{k=0}^{n-1} V_{t_0}^k \right\|_{\mathbb{L}^1 \rightarrow \mathbb{L}^1} < \infty$, by the Banach-principle, the set $\mathcal{Y} := \{X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P}) : \left\| \frac{1}{n} \sum_{k=0}^{n-1} e^{ikt_0} X \circ \theta^k \right\|_1 \rightarrow 0\}$ is closed in \mathbb{L}^1 . Now, by von Neumann's mean ergodic theorem

$$\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P}) = \overline{(I - V_{t_0})\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})} \oplus \text{Fix } V_{t_0},$$

where $\text{Fix } V_{t_0}$ stands for the fixed points of V_{t_0} in $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ and the closure is in norm $\|\cdot\|_2$. By assumption, V_{t_0} has no non trivial fixed point. Obviously, \mathcal{Y} contains $(I - V_{t_0})\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, hence $\mathcal{Y} = \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$.

Assume now that $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ is separable. Define an operator on $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ by $UX = X \circ \theta$. It is well known that the eigenspaces of U corresponding to different eigenvalues are orthogonal. By separability there are at most countably many eigenvalues for U , hence the result. \square

5.2 Technical approximation results

Lemma 33 *Assume that X_0 is almost surely bounded by M . For any integer $s \geq 1$*

$$(2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \left(\max_{1 \leq \ell \leq m} |S_\ell(t) - M_\ell(t)|^2 \right) dt \leq 12(m \|\mathbb{E}_{-s}(X_0)\|^2 + s^2 M^2),$$

where $M_n(t) = \sum_{k=1}^n D_k(t)$ and $D_k(t)$ is defined by (11).

The proof of this lemma follows from the following result (which is of independent interest) by selecting $a_j = 1$ for $0 \leq j \leq s-1$ and $a_j = 0$ for any $j \geq s$.

Lemma 34 *Assume that X_0 is almost surely bounded by M . Let (a_n) be a sequence of positive numbers nonincreasing smaller than 1 with $\sum_{j=1}^{\infty} a_j < \infty$ and $a_0 = 1$. Then*

$$(2\pi)^{-1} \int_0^{2\pi} \mathbb{E}(\max_{1 \leq \ell \leq m} |S_\ell(t) - M_\ell(t)|^2) dt \leq 12 \left(m \sum_{j=1}^{\infty} (a_{j-1} - a_j) \|\mathbb{E}_{-j}(X_0)\|^2 + M^2 \left(\sum_{j=0}^{\infty} a_j \right)^2 \right).$$

Proof of Lemma 34.

Step 1: Martingale decomposition.

We start with a traditional martingale decomposition (see for instance Section 4.1 in Merlevède, Peligrad and Utev [19]). Let $t \in [0, 2\pi)$ and $X_k(t) = e^{ikt} X_k$.

$$\theta_k(t) = X_k(t) + \sum_{j=1}^{\infty} a_j \mathbb{E}_k(X_{k+j}(t)); \quad \theta'_k(t) = \sum_{j=1}^{\infty} a_j \mathbb{E}_k(X_{k+j}(t))$$

and

$$\mathbb{E}_k(\theta_{k+1}(t)) - \theta_k(t) = -X_k(t) + \sum_{j=1}^{\infty} (a_{j-1} - a_j) \mathbb{E}_k(X_{k+j}(t)).$$

Finally, denote by

$$D'_{k+1}(t) = \theta_{k+1}(t) - \mathbb{E}_k(\theta_{k+1}(t)) = \sum_{j=0}^{\infty} a_j P_{k+1}(X_{k+j+1}(t)) \quad ; \quad M'_n(t) = \sum_{k=1}^n D'_k(t).$$

Then, $(D'_k(t))_{k \in \mathbb{Z}}$ is a sequence of martingale differences with respect to the stationary filtration $(\mathcal{F}_j)_{j \in \mathbb{Z}}$. Note

$$X_k(t) = D'_{k+1}(t) + \theta_k(t) - \theta_{k+1}(t) + \sum_{j=1}^{\infty} (a_{j-1} - a_j) \mathbb{E}_k(X_{k+j}(t)).$$

Taking into account the definition of $\theta'_k(t)$ we can also write

$$X_k(t) = D'_k(t) + \theta'_{k-1}(t) - \theta'_k(t) + \sum_{j=1}^{\infty} (a_{j-1} - a_j) \mathbb{E}_{k-1}(X_{k+j-1}(t)).$$

It follows that for almost all $t \in [0, 2\pi)$,

$$\begin{aligned} S_\ell(t) - M_\ell(t) &= \sum_{k=1}^{\ell} \sum_{j=0}^{\infty} (a_j - 1) P_k(X_{k+j}(t)) + \sum_{k=0}^{\ell-1} \sum_{j=1}^{\infty} (a_{j-1} - a_j) \mathbb{E}_k(X_{k+j}(t)) + \theta'_0(t) - \theta'_\ell(t) \\ &= I + II + \theta'_0(t) - \theta'_\ell(t). \end{aligned} \tag{50}$$

Step 2: The estimation of $\int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |S_\ell(t) - M_\ell(t)|^2 dt$.

We shall estimate separately this maximum for all the terms in the decomposition (50).

By the Doob-Kolmogorov martingale maximal inequality, stationarity, Fubini theorem and orthogonality of e^{itk} the first term will be (remind $a_0 = 1$) dominated by

$$\begin{aligned}
(2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |I|^2 dt &\leq (2\pi)^{-1} m \int_0^{2\pi} \mathbb{E} |D'_0(t) - D_0(t)|^2 dt \\
&= (2\pi)^{-1} m \int_0^{2\pi} \mathbb{E} \left| \sum_{j=0}^{\infty} (a_j - 1) P_0(X_j(t)) \right|^2 dt \\
&= m \sum_{j=1}^{\infty} (a_j - 1)^2 \|\mathbb{P}_{-j} X_0\|^2.
\end{aligned}$$

By simple computations,

$$\begin{aligned}
\sum_{j=1}^{\infty} (a_j - 1)^2 \|\mathbb{P}_{-j} X_0\|^2 &= \sum_{j=1}^{\infty} (a_j - 1)^2 (\|\mathbb{E}_{-j}(X_0)\|^2 - \|\mathbb{E}_{-j-1}(X_0)\|^2) \\
&= (a_1 - 1)^2 \|\mathbb{E}_{-1}(X_0)\|^2 + \sum_{j=2}^{\infty} [(a_j - 1)^2 - (a_{j-1} - 1)^2] \|\mathbb{E}_{-j}(X_0)\|^2 \\
&= \sum_{j=1}^{\infty} [(a_j - 1)^2 - (a_{j-1} - 1)^2] \|\mathbb{E}_{-j}(X_0)\|^2 \leq 2 \sum_{j=1}^{\infty} (a_{j-1} - a_j) \|\mathbb{E}_{-j}(X_0)\|^2.
\end{aligned}$$

So

$$(2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |I|^2 dt \leq 2m \sum_{j=1}^{\infty} (a_{j-1} - a_j) \|\mathbb{E}_{-j}(X_0)\|^2. \quad (51)$$

By Cauchy Schwarz inequality the second term is estimated as follows.

$$\int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |II|^2 dt \leq \sum_{\ell=1}^{\infty} (a_{\ell-1} - a_{\ell}) \sum_{j=1}^{\infty} (a_{j-1} - a_j) \int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} \left| \sum_{k=0}^{\ell-1} e^{i(k+j)t} \mathbb{E}_k(X_{k+j}) \right|^2 dt.$$

By Hunt and Young maximal inequality [18],

$$\begin{aligned}
(2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |II|^2 dt &\leq \sum_{j=1}^{\infty} (a_{j-1} - a_j) \sum_{k=0}^{m-1} \|\mathbb{E}_0(X_j)\|^2 \\
&= m \sum_{j=1}^{\infty} (a_{j-1} - a_j) \|\mathbb{E}_0(X_j)\|^2.
\end{aligned} \quad (52)$$

The last terms are estimated in a trivial way as follows:

$$(2\pi)^{-1} \int_0^{2\pi} \mathbb{E} \max_{1 \leq \ell \leq m} |\theta'_0(t) - \theta'_k(t)|^2 dt = 4M^2 \left(\sum_{j=0}^{\infty} a_j \right)^2. \quad (53)$$

Gathering (51), (52) and (53), the lemma follows. \square

5.3 An algebraic inequality

Lemma 35 For any positive reals a and b and any real $S \geq 1$,

$$(a + b)^S \leq 2^S b^S + a^S (1 + 2^{S+1} b/a). \quad (54)$$

Proof. To prove the above inequality we first notice that if $a \leq b$, the inequality is trivial. Let then assume that $b < a$. The Newton binomial formula gives

$$(a + b)^S \leq a^S(1 + b/a)^{[S]+1} \leq a^S(1 + b/a) \sum_{k=1}^{[S]+1} C_{[S]+1}^k (b/a)^{k-1} \leq a^S(1 + 2^{S+1}b/a).$$

□

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