

# A Simplified 2-Dimensional Model for some Elastic Masonries

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## Abstract

By homogenization, we propose a simplified 2-dimensional model for effective behaviors of structures made of some thin flat masonries where the mortar occupied a rather thin domain and is far softer than the bricks. Because a steady-state problem considered here may be formulated in terms of minimization, we use the method of variational convergence. Our model is simpler than the genuine one because it involves a homogeneous body and is accurate enough due to our convergence result. This work also shows what a reasonable candidate for effective energy density of the assembly of bricks and mortar could be.

*Keywords:* homogenization, variational convergence, elastic masonries.

## 1 Introduction

We do not make any difference between the physical Euclidean space of  $\mathbb{R}^3$ , if  $x := (x_1, x_2, x_3) \in \mathbb{R}^3$  then  $\hat{x}$  stands for  $(x_1, x_2)$ . A model of static behavior of a masonry can be described as follows. Let  $a$  a positive real number,  $H \in a\mathbb{N}$ , and  $h$  a small positive number. The domain occupied by the thin masonry under considerations here is  $\Omega_h := \omega \times (-h, h)$  where  $\omega := (0, L) \times (0, H)$  (see Figure 1). This masonry is made of a periodic distribution of bricks linked to each other by adhesive mortar occupying a very

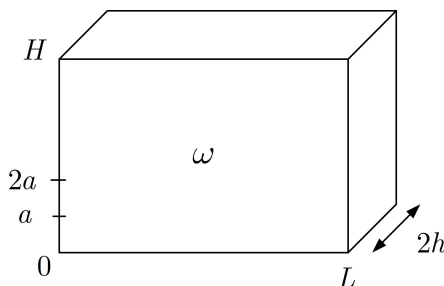


Figure 1: The thin masonry  $\Omega_h$

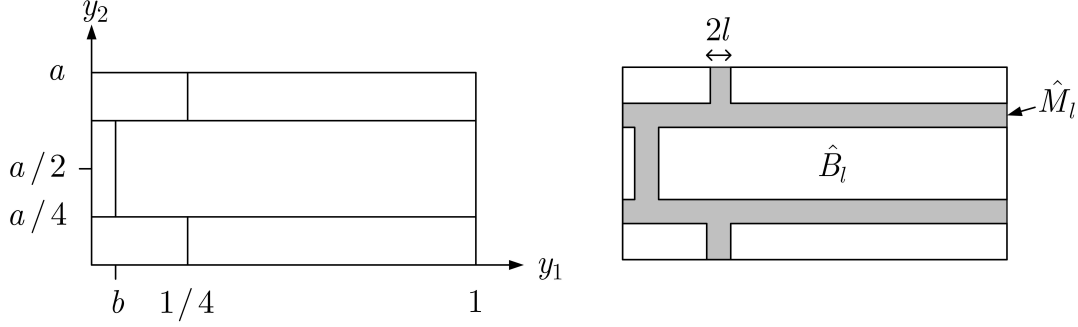


Figure 2: The unit cell  $\hat{Y}$ ,  $S$ ,  $\hat{M}_l$  and  $\hat{B}_l$

thin domain. More precisely, if  $\hat{Y} := (0, 1) \times (0, a)$ ,

$$S := \{ \hat{y} \in \hat{Y} \mid y_2 = a/4 \text{ or } 3a/4; \\ y_1 = 1/4, \text{ and } 0 < y_2 < a/4 \text{ or } 3a/4 < y_2 < a; \\ y_1 = b \text{ and } a/4 < y_2 < 3a/4, \text{ where } b \in (0, 3/4] \},$$

$$\hat{M}_l := \{ \hat{x} \in \hat{Y} \mid \text{dist}(\hat{x}, S) < l \}, \quad l < \min\{b, a/4\},$$

$$\hat{B}_l := \hat{Y} \setminus \hat{M}_l,$$

$$\varepsilon = L/n, \quad n \in \mathbb{N},$$

$$\hat{Y}_\varepsilon^i := \varepsilon i + \varepsilon \hat{Y}, \quad i \in I_\varepsilon := \{ i \in \mathbb{Z}^2 \mid \hat{Y}_\varepsilon^i \subset \omega \},$$

$$\hat{M}_{l\varepsilon}^i = \varepsilon i + \varepsilon \hat{M}_l, \quad \hat{M}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{M}_{l\varepsilon}^i,$$

$$\hat{B}_{l\varepsilon}^i = \varepsilon i + \varepsilon \hat{B}_l, \quad \hat{B}_{l\varepsilon} = \cup_{i \in I_\varepsilon} \hat{B}_{l\varepsilon}^i,$$

then  $B_{l\varepsilon h} := \hat{B}_{l\varepsilon} \times (-h, h)$  and  $M_{l\varepsilon h} := \hat{M}_{l\varepsilon} \times (-h, h)$  are the domains occupied by the bricks and the mortar, respectively (see Figure 2). The bricks are made of a homogeneous linearly elastic material whose bulk energy density denoted by  $W$  is a strictly convex quadratic function satisfying

$$\exists \alpha, \beta > 0; \quad \alpha |e|^2 \leq W(e) \leq \beta |e|^2 \quad \forall e \in \mathbb{S}^3,$$

the space of symmetric  $3 \times 3$  matrices. The mortar is assumed to be made of a homogeneous, isotropic linearly elastic material of density  $W_{\lambda\mu}$  whose Lamé coefficients  $\lambda$  and  $\mu$  are supposed to be far smaller than  $\alpha$  and  $\beta$ . Eventually, the wall is assumed to be clamped on a part  $\Gamma_{0h} = \gamma_0 \times (-h, h)$  of  $\partial\Omega_h$ ,  $\gamma_0$  being a part of  $\partial\omega$  with positive length, and subjected to body forces of density  $f^h$ .

To find the equilibrium configurations of the wall, we are led to the problem :

$$(\mathcal{P}) \quad \min \left\{ \int_{B_{l\varepsilon h}} W(e(u)(x)) dx + \int_{M_{l\varepsilon h}} W_{\lambda\mu}(e(u)(x)) dx - \int_{\Omega_h} f^h(x) \cdot u(x) dx \right. \\ \left. \mid u \in H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) \right\}$$

where

$$\begin{cases} H_{\Gamma_{0h}}^1(\Omega_h; \mathbb{R}^3) := \{ v \in H^1(\Omega_h; \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma_{0h} \text{ in the sense of the traces} \}, \\ e(v) := \frac{1}{2}(\nabla v + \nabla v^T). \end{cases}$$

Assuming  $f^h$  in  $L^2(\Omega_h; \mathbb{R}^3)$ , this problem clearly has a unique solution, but, due to the low values of  $(\varepsilon, l, \lambda, \mu, h)$ , obtaining numerical approximation may be difficult. Thus, it is of interest to propose a simplified but accurate enough model. A first attempt [4] is to replace  $(\mathcal{P})$  by a suitable 2-dimensional problem set in the cross section  $\omega$  of the wall which may read as :

$$(\hat{\mathcal{P}}_s) \quad \min \left\{ \int_{\hat{B}_{l\varepsilon}} \hat{W}(e(u)(\hat{x})) d\hat{x} + \int_{\hat{M}_{l\varepsilon}} \hat{W}_{\lambda,\mu}(e(u)(\hat{x})) d\hat{x} - \int_{\omega} \hat{f}(\hat{x}) \cdot u(\hat{x}) d\hat{x} \right. \\ \left. \mid u \in H_{\gamma_0}^1(\omega; \mathbb{R}^2) \right\},$$

where

- we will still denote the symmetric part of the gradient in the sense of distribution of  $\hat{x} \in \omega \mapsto u(\hat{x}) \in \mathbb{R}^2$  by  $e(u)$  which is then a distribution on  $\omega$  with value in  $\mathbb{S}^2$  the space of  $2 \times 2$  symmetric matrices,
- $\hat{W}$ , deduced from  $W$ , is a strictly convex quadratic function satisfying
$$\exists \hat{\alpha}, \hat{\beta} > 0; \quad \hat{\alpha}|e|^2 \leq \hat{W}(e) \leq \hat{\beta}|e|^2 \quad \forall e \in \mathbb{S}^2,$$
- $\hat{W}_{\lambda'\mu}(e) = \frac{\lambda'}{2}(e_{11} + e_{22})^2 + \mu|e|^2$ ,  $\lambda' = \frac{2\lambda\mu}{\lambda+2\mu}$ , which corresponds to  $W_{\lambda\mu}$  in the plane stress assumption.
- $\hat{f}$  is deduced from  $f^h$ ,
- $\hat{s} := (\varepsilon, l, \lambda', \mu)$ .

We will use the following notations :

$$\hat{W}_{\hat{s}}(\hat{y}, e) := \begin{cases} \hat{W}(e) & \text{if } \hat{y} \in \hat{B}_l \\ \hat{W}_{\lambda'\mu}(e) & \text{if } \hat{y} \in \hat{M}_l \end{cases}$$

which is extended into  $\mathbb{R}^2 \times \mathbb{S}^2$  by  $\hat{Y}$ -periodicity, and

$$F_{\hat{s}}(u) = \int_{\omega} \hat{W}_{\hat{s}}(\hat{x}/\varepsilon, e(u)(\hat{x})) d\hat{x}.$$

Assuming again that  $\hat{f} \in L^2(\omega; \mathbb{R}^2)$ , problem  $(\hat{\mathcal{P}}_{\hat{s}})$  has a unique solution  $\hat{u}_{\hat{s}}$  but, due to the low values of  $\hat{s}$ , getting numerical approximations remains difficult. Thus, taking into account this low values, we will propose a simplified but accurate enough model by studying the asymptotic behavior of  $(\hat{\mathcal{P}}_{\hat{s}})$  when  $\hat{s}$  goes to zero. Since we will only consider 2-dimensional problems in the sequel, to shorten notation we will denote a current point of  $\mathbb{R}^2$  by  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$  and  $s$  not by  $\hat{x}$ ,  $\hat{y}$  and  $\hat{s}$ , respectively.

Actually,  $(\hat{\mathcal{P}}_{\hat{s}})$  looks like a problem of periodic homogenization since the geometry and the mechanical properties of the microstructure are  $\varepsilon\hat{Y}$ -periodic, but the geometry involves an additional parameter  $l$  while the bulk energy of the mortar involves two other ones  $\lambda'$ ,  $\mu$ . If  $(l, \lambda', \mu)$  were fixed, the asymptotic effective bulk energy of the heterogeneous structure will be given by

$$W_{l\lambda'\mu}^{\text{eff}}(E) := \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{B}_l} \hat{W}(E + e(v)(y)) dy + \int_{\hat{M}_l} \hat{W}_{\lambda'\mu}(E + e(v)(y)) dy \right) \mid v \in H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) \right\}$$

where

$$H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2) := \{ v \in H^1(\hat{Y}; \mathbb{R}^2) \mid \text{the traces of } v \text{ on the opposite sides of } \hat{Y} \text{ are equal} \}.$$

The asymptotic behavior of  $W_{l\lambda'\mu}^{\text{eff}}$  when  $(l, \lambda', \mu) \rightarrow 0$  is a problem of modelling of soft elastic junctions. By using the arguments of [7] in a very general setting or of [1] in a setting close to the present one, it may be shown that

$$\lim_{(l, \lambda', \mu) \rightarrow 0} W_{l\lambda'\mu}^{\text{eff}}(E) = \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\lambda'\mu}([v](y)) dl \right) \mid v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right\},$$

where

$$H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) := \{ v \in H^1(\hat{Y} \setminus S; \mathbb{R}^2) \mid \text{the traces of } v \text{ on the opposite sides of } \hat{Y} \text{ are equal} \},$$

we made the additional assumption

$$\exists \bar{\lambda}', \bar{\mu} \in (0, +\infty); \quad \lambda' \sim 2\bar{\lambda}'l, \quad \mu \sim 2\bar{\mu}l, \quad (1)$$

which leads to the most interesting case from the mechanical point of view,

$$\begin{cases} \bar{W}_{\bar{\lambda}'\bar{\mu}}(v) := \hat{W}_{\bar{\lambda}'\bar{\mu}}(v \otimes_s n) \quad \forall v \in \mathbb{R}^2 \\ a \otimes_s b := \frac{1}{2}(a \otimes b + b \otimes a), \end{cases}$$

$n$  is a chosen unit normal to  $S$ ,  $[v]$  is the difference, taken in the direction of  $n$ , of the traces of  $v$  on  $S$ .

In fact, in the sequel we prove that this previous limit is *the effective bulk energy of the wall* by studying the asymptotic behavior of the strain energy functional  $F_s$  involved by  $(\hat{P}_s)$  when  $s$  goes to zero with the sole condition (1). This will be done in the setting of variational convergence by establishing that the strain energy functional converges in some sense to an integral functional on  $\omega$  whose density is :

$$W^{\text{eff}}(E) := \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right\}.$$

## 2 The Asymptotic Model

### 2.1 An Auxiliary Problem

We have just emphasized the role played, for all  $E$  in  $\mathbb{S}^2$ , by the problem

$$(\hat{P}_E) \quad \min \left\{ \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v](y)) dl \right) \right. \\ \left. \mid v \in H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2) \right\}$$

which has a unique (up to a constant element of  $\mathbb{R}^2$ ) solution  $v_E$  and we have

$$W^{\text{eff}}(E) = \frac{1}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v_E)(y)) dy + \int_S \bar{W}_{\bar{\lambda}'\bar{\mu}}([v_E](y)) dl \right)$$

and

$$\exists \gamma, \beta > 0; \quad \gamma|E|^2 \leq W^{\text{eff}}(E) \leq \hat{W}(E) \leq \beta|E|^2 \quad \forall E \in \mathbb{S}^2. \quad (2)$$

Let us still denote the extension of  $v_E$  into  $\mathbb{R}^2$  by  $\hat{Y}$ -periodicity by  $v_E$  and let  $\sigma_E = \hat{W}'(E + e(v_E))$ . It is easy to check that  $\text{div } \sigma_E = 0$  in the sense of the distributions on  $\mathbb{R}^2$ . Thus,  $v_E$ , being solution of an elastostatic problem in quadrants  $Q_i$  with boundary conditions like  $\sigma_E n \in H^{1/2}(\partial Q_i; \mathbb{R}^2)$ , belongs to  $H^2(\hat{Y} \setminus S; \mathbb{R}^2)$  [5], [6]. The Sobolev embedding implies that  $v_E$  on each connected components of  $\hat{Y} \setminus S$  is the restriction of Hölder continuous functions on  $\mathbb{R}^2$  and that  $e(v_E)$  and consequently  $\sigma_E$  belongs to  $L^r(\hat{Y}; \mathbb{S}^2) \forall r \in [1, +\infty)$ . This regularity property of  $\sigma_E$  is crucial to prove the lower bound : the function  $x \in \omega \mapsto \sigma_E(x/\varepsilon) \in \mathbb{S}^2$  weakly converges in  $L^r(\omega; \mathbb{S}^2)$  toward  $\int_{\hat{Y}} \sigma_E(y) dy \forall r \in [1, +\infty)$ !

The field  $v_E$  does not belong to  $H^1(\hat{Y}; \mathbb{R}^2)$ , thus in order to use this field for building suitable test functions we proceed to a regularization as in [4] and [7]. For all  $v$  in  $H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2)$  we define  $R_i v$ ,  $i = 1, 2, 3, 4$ , by :

$$\begin{aligned} R_1 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_1 - \frac{1}{4} \right| \right\} \left( v(y) - v \left( \frac{1}{2} - y_1, y_2 \right) \right) + v(y) + v \left( \frac{1}{2} - y_1, y_2 \right) \right] \\ R_2 v(y) &:= \frac{1}{2} [\min\{1, |y_1 - b|/l\} (v(y) - v(2b - y_1, y_2)) + v(y) + v(2b - y_1, y_2)] \\ R_3 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_2 - \frac{a}{4} \right| \right\} \left( v(y) - v \left( y_1, \frac{a}{2} - y_2 \right) \right) + v(y) + v \left( y_1, \frac{a}{2} - y_2 \right) \right] \\ R_4 v(y) &:= \frac{1}{2} \left[ \min \left\{ 1, \frac{1}{l} \left| y_2 - \frac{3a}{4} \right| \right\} \left( v(y) - v \left( y_1, \frac{3a}{2} - y_2 \right) \right) + v(y) + v \left( y_1, \frac{3a}{2} - y_2 \right) \right] \end{aligned}$$

and let  $R_l := R_4 \circ R_3 \circ R_2 \circ R_1$ .

Clearly for all  $v$  in  $H_{\text{per}}^1(\hat{Y} \setminus S; \mathbb{R}^2)$ ,  $R_l v$  belongs to  $H_{\text{per}}^1(\hat{Y}; \mathbb{R}^2)$  and  $R_l v(y) = v(y)$   $\forall y \in \hat{B}_l$ . It is straight forward to check that

$$\lim_{l \rightarrow 0} |R_l v_E - v_E|_{L^r(\hat{Y}; \mathbb{R}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (3)$$

$$\lim_{s \rightarrow 0} |\hat{W}'_{\lambda', \mu}(e(R_l v_E))|_{L^r(\hat{M}_l; \mathbb{S}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (4)$$

$$\lim_{l \rightarrow 0} |e(v_E)|_{L^r(\hat{M}_l; \mathbb{S}^2)} = 0 \quad \forall r \in [1, +\infty) \quad (5)$$

As for  $v_E$  we still denote the extension of  $R_l v_E$  into  $\mathbb{R}^2$  by  $\hat{Y}$ -periodicity by  $R_l v_E$ .

## 2.2 The Convergence Result

We start by establishing a compactness property for sequences with bounded energy. Let  $\mathcal{M}_b(\omega; \mathbb{S}^2)$  the space of bounded  $\mathbb{S}^2$ -valued measures on  $\omega$  and  $BD(\omega) := \{u \in L^1(\omega; \mathbb{R}^2) \mid e(u) \in \mathcal{M}_b(\omega; \mathbb{S}^2)\}$ .

**Proposition 1 (compactness property)** *Let  $(u_s)$  a sequence in  $H_{\gamma_0}^1(\omega; \mathbb{R}^2)$  such that  $F_s(u_s) \leq C$ , then there exist  $u$  in  $BD(\omega)$  and a not relabelled subsequence such that  $u_s$  weak\* converges in  $BD(\omega)$  toward  $u$  and consequently strongly in  $L^q(\omega; \mathbb{R}^2)$ ,  $q$  arbitrary in  $[1, 2)$  and weakly in  $L^2(\omega; \mathbb{R}^2)$ .*

*Proof of Proposition 1 :* We have

$$\begin{aligned} C &\geq \hat{\alpha} \int_{\hat{B}_{l\varepsilon}} |e(u_s)|^2 dx + \mu \int_{\hat{M}_{l\varepsilon}} |e(u_s)|^2 dx \\ &\geq \frac{\hat{\alpha}}{a} \left( \int_{\hat{B}_{l\varepsilon}} |e(u_s)| dx \right)^2 + \frac{\mu}{|\hat{M}_{l\varepsilon}|} \left( \int_{\hat{M}_{l\varepsilon}} |e(u_s)| dx \right)^2 \\ &\geq \frac{\hat{\alpha}}{a} \left( \int_{\hat{B}_{l\varepsilon}} |e(u_s)| dx \right)^2 + \frac{C\mu}{l} \left( \int_{\hat{M}_{l\varepsilon}} |e(u_s)| dx \right)^2 \end{aligned}$$

Hence (1) and the boundary condition  $u_s = 0$  on  $\gamma_0$  imply that  $u_s$  is bounded in  $LD(\omega; \mathbb{R}^2) := \{u \in L^1(\omega; \mathbb{R}^2) \mid e(u) \in L^1(\omega; \mathbb{S}^2)\}$  which gives the desired assertion [8].  $\square$

Now, we are in a position to state our main convergence result.

### Theorem 1 (upper and lower bound)

- *Upper bound :* for all  $u$  in  $H^1(\omega; \mathbb{R}^2)$  there exists a sequence  $(u_s)$  in  $H^1(\omega; \mathbb{R}^2)$  such that  $u_s$  weak\* converges in  $BD(\omega)$  toward  $u$  and

$$F^{\text{eff}}(u) := \int_{\omega} W^{\text{eff}}(e(u)) dx = \lim_{s \rightarrow 0} F_s(u_s).$$

- *Lower bound :* for all  $u$  in  $H^1(\omega; \mathbb{R}^2)$  and all sequences  $(u_s)$  in  $H^1(\omega; \mathbb{R}^2)$  which weak\* converges in  $BD(\omega)$  toward  $u$ , we have :

$$F^{\text{eff}}(u) \leq \liminf_{s \rightarrow 0} F_s(u_s).$$

*Proof of the upper bound :* We add some ingredients of the mathematical theory of bonding joints [7] and [1] to the classical proof by [2] for homogenization of elliptic operators.

First we assume that  $u$  is affine  $u(x) = Ex + d$ ,  $E \in \mathbb{S}^2$ ,  $d \in \mathbb{R}^2$ . Let  $w_{E_s}$  such that  $w_{E_s}(x) = \varepsilon(R_l v_E)(x/\varepsilon)$ , (3) and (5) imply :

$$\begin{aligned} \lim_{s \rightarrow 0} \int_{\omega} |w_{E_s}(x) - \varepsilon v_E(x/\varepsilon)|^r dx &= 0 \quad \forall r \in [1, 2] \\ \lim_{s \rightarrow 0} \int_{\omega} |\varepsilon v_E(x/\varepsilon)|^r dx &= 0 \end{aligned}$$

so that the field  $u_s = u + w_{E_s}$  belongs to  $H^1(\omega; \mathbb{R}^2)$  and strongly converges in  $L^q(\omega; \mathbb{R}^2)$ , and in  $L^2(\omega; \mathbb{R}^2)$ , toward  $u$ . Moreover

$$\begin{aligned} F_s(u_s) &= \int_{\hat{B}_{l\varepsilon}} \hat{W}(E + e(v_E)(x/\varepsilon)) dx + \int_{\hat{M}_{l\varepsilon}} \hat{W}_{\lambda', \mu}(E + e(R_l v_E)(x/\varepsilon)) dx \\ &= \frac{|\omega|}{|\hat{Y}|} \left( \int_{\hat{B}_l} \hat{W}(E + e(v_E)(y)) dy + \int_{\hat{M}_l} \hat{W}_{\lambda', \mu}(E + e(R_l v_E)(y)) dy \right). \end{aligned}$$

Hence, a simple computation (see [7] and [1]) gives

$$\begin{aligned} \lim_{s \rightarrow 0} F_s(u_s) &= \frac{|\omega|}{|\hat{Y}|} \left( \int_{\hat{Y} \setminus S} \hat{W}(E + e(v_E)(y)) dy + \int_S \bar{W}_{\lambda', \mu}([v_E](y)) dl \right) \\ &= \int_{\omega} W^{\text{eff}}(e(u)(x)) dx. \end{aligned}$$

Next, we take  $u$  as a piecewise continuous affine function :  $u(x) = E^i x + d^i$  on  $\omega^i$ ,  $i \in I$  finite, where the  $\omega^i$  form a partition by polyhedral sets. Like in first step, we define  $u_s$  by  $u_s^i(x) = u(x) + w_{E^i s}$  on each  $\omega^i$ . But by due account to the possible discontinuities on the interface  $\sigma_{jk}$  between  $\omega^j$  and  $\omega^k$  we need to introduce  $\phi_\delta$  in  $W^{1,\infty}(\omega)$ ,  $0 \leq \phi_\delta \leq 1$ ,  $\phi_\delta = 1$  on  $\sigma_{jk\delta} = \{x \in \omega \mid \text{dist}(x, \sigma_{jk\delta}) < \delta\}$ ,  $\delta > 0$ ,  $\phi_\delta = 0$  on  $\omega \setminus \sigma_{jk2\delta}$  and

$$u_{\delta s} = \phi_\delta u + (1 - \phi_\delta) u_s^i \quad \text{on } \omega^i.$$

Hence, we can repeat the end of the proof by [2] p.47–48 because  $u_s^i$  converges strongly toward  $u$  not only in  $L^q(\omega^i; \mathbb{R}^2)$  but also in  $L^2(\omega^i; \mathbb{R}^2)$  while  $\hat{W}_{\lambda', \mu}$  is convex and satisfies  $\hat{W}_{\lambda', \mu}(e) \leq C|e|^2$ .

Eventually, the proof is complete by a diagonalization and density argument.  $\square$

*Proof of the lower bound :* Once more we proceed by introducing a continuous piecewise affine function  $v(x) = E^i x + d^i$  as *approximation in  $H^1$*  of  $u$  on  $\omega^i$ . For each  $\omega^i$ , let us introduce  $\phi^i \in \mathcal{D}(\omega^i)$  such that  $0 \leq \phi^i \leq 1$ . The subdifferential inequality yields

$$\begin{aligned} F_s(u_s) &\geq \sum_{i \in I} \left( \int_{\omega^i} \phi^i(x) \hat{W}_s(x/\varepsilon, E^i + e(w_{E^i s})(x)) dx \right. \\ &\quad \left. + \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(w_{E^i s})(x)) \cdot e(u_s - v - w_{E^i s}) dx \right). \end{aligned}$$

A slight and obvious modification of the argument used in establishing the first step of the proof of the upper bound gives :

$$\lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}_s(x/\varepsilon, E^i + e(w_{E^i s})(x)) dx = \int_{\omega^i} \phi^i(x) W^{\text{eff}}(e(v)(x)) dx.$$

Moreover, (3), (4) and (5) imply :

$$\begin{aligned} &\lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(w_{E^i s})(x)) \cdot e(u_s - v - w_{E^i s}) dx \\ &= \lim_{s \rightarrow 0} \int_{\omega^i} \phi^i(x) \hat{W}'_s(x/\varepsilon, E^i + e(v_{E^i})(x/\varepsilon)) \cdot e(u_s - v - w_{E^i s}) dx \\ &= \lim_{s \rightarrow 0} \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot e(\phi^i(u_s - v - w_{E^i s})) dx \\ &\quad - \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot \nabla \phi^i \otimes_s (u_s - v - w_{E^i s}) dx \\ &= - \int_{\omega^i} \sigma_{E^i}(x/\varepsilon) \cdot \nabla \phi^i \otimes_s (u_s - v - w_{E^i s}) dx \quad (\text{div } \sigma_{E^i} = 0) \\ &= - \int_{\omega^i} \left( \int_Y \sigma_{E^i}(y) dy \right) \cdot \nabla \phi^i \otimes_s (u - v) dx \end{aligned}$$

because  $\sigma_{E^i}(\cdot/\varepsilon)$  weakly converges in  $L^q(\omega; \mathbb{S}^2)$  toward  $\int_{\hat{\gamma}} \sigma_{E^i}(y) dy$  and  $(u_s - v - w_{E^i s})$  converges strongly in  $L^q(\omega; \mathbb{R}^2)$  to  $u - v$ . Hence,

$$\liminf_{s \rightarrow 0} F_s(u_s) \geq \sum_i \int_{\omega_i} \phi_i W^{\text{eff}}(e(v)) dx + \sum_i \int_{\omega_i} \phi_i (W^{\text{eff}})'(e(v)) \cdot e(u - v) dx$$

And we conclude as in [2] by letting  $\phi_i$  converge increasingly to one on  $\omega_i$  for the first term and using (2) and the density of the piecewise affine continuous functions in  $H^1(\omega; \mathbb{R}^2)$  for the second term.  $\square$

### 3 Mechanical Interpretation, a Proposal of Model

Indeed, we did not succeed in proving that cluster points of sequence with uniformly bounded finite strain energy belongs to  $H^1(\omega; \mathbb{R}^2)$  and satisfies an homogeneous Dirichlet boundary condition on  $\gamma_0$ . Thus we cannot (see [2], [3]) assert that the unique solution  $\hat{u}_s$  of  $(\hat{\mathcal{P}}_s)$  weak\* converges in  $BD(\omega)$  toward the obviously unique solution  $\bar{u}$  of  $(\mathcal{P}^{\text{eff}})$

$$(\mathcal{P}^{\text{eff}}) \quad \min \left\{ \int_{\omega} W^{\text{eff}}(e(v)) dx - \int_{\omega} \hat{f} \cdot v dx \mid v \in H_{\gamma_0}^1(\omega; \mathbb{R}^2) \right\}.$$

Problem  $(\mathcal{P}^{\text{eff}})$  describes the equilibrium of a homogeneous elastic flat body occupying  $\omega$  as reference configuration with bulk energy  $W^{\text{eff}}$ , clamped on  $\gamma_0$  and subjected to body forces of density  $\hat{f}$ . Hence, our model is simpler than the genuine one because it involves a homogeneous body and accurate enough due to our convergence result. We only can claim that a reasonable candidate for effective energy bulk energy density of the assembly bricks mortar is  $W^{\text{eff}}$ . This was also derived in [4] through rather heuristical arguments.

Due to (2) the effective homogeneous material is weaker than the genuine material of the bricks. This is the price to pay due to the difficulty of making a homogeneous wall. The effective strength of the wall should be greater than the one of the bricks when stiff mortar is used. It should be interesting to consider the case when the magnitudes of  $\lambda', \mu$  is of order  $1/l$ . Hence our model is simpler than the genuine one because it involves a homogeneous body and is accurate enough due to our convergence result.

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