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CYCLE COMPLEX OVER \mathbb{P}^1 MINUS 3 POINTS : TOWARD MULTIPLE ZETA VALUES CYCLES.

ISMAEL SOUDÈRES

ABSTRACT. In this paper, we construct a family of algebraic cycles in Bloch's cycle complex over \mathbb{P}^1 minus three points which are expected to correspond to multiple polylogarithms in one variable. Elements in this family of weight p belong to the cubical cycle group of codimension p in $(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \times (\mathbb{P}^1 \setminus \{1\})^{2p-1}$ and in weight greater than or equal to 2, they naturally extend as equidimensional cycles over \mathbb{A}^1 .

Thus, we can consider their fibers at the point 1. This is one of the main differences with the work of Gangl, Goncharov and Levin. Considering the fiber of our cycles at 1 makes it possible to view these cycles as those corresponding to weight n multiple zeta values which are here thought as the value at 1 of multiple polylogarithms.

After the introduction, we recall some properties of Bloch's cycle complex, and explain the difficulties on a few examples. Then a large section is devoted to the combinatorial situation, essentially involving the combinatorics of trivalent trees in relation with the structure of the free Lie algebra on two generators. In the last section, two families of cycles are constructed as solutions to a "differential system" in Bloch's cycle complex. One of these families contains only cycles with empty fiber at 0; these correspond to multiple polylogarithms.

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1. INTRODUCTION

1.1. Multiple polylogarithms. Multiple polylogarithm functions are defined (cf. [Gon95]) by the power series

$$Li_{k_1, \dots, k_m}(z_1, \dots, z_m) = \sum_{n_1 > \dots > n_m > 0} \frac{z_1^{n_1}}{n_1^{k_1}} \frac{z_2^{n_2}}{n_2^{k_2}} \dots \frac{z_m^{n_m}}{n_m^{k_m}} \quad (z_i \in \mathbb{C}, |z_i| < 1).$$

where the k_i 's are strictly positive integers. They admit an analytic continuation to a Zariski open subset of \mathbb{C}^m . The case $m = 1$ is nothing but the classical *polylogarithm* functions. The case $z_1 = z$ and $z_2 = \dots = z_m = 1$ gives a one variable version of the multiple polylogarithm function

$$Li_{k_1, \dots, k_m}^{\mathbb{C}}(z) = Li_{k_1, \dots, k_m}(z, 1, \dots, 1) = \sum_{n_1 > \dots > n_m > 0} \frac{z^{n_1}}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}.$$

When k_1 is greater or equal to 2, the series converges as z tends to 1, where one recovers the multiple zeta value

$$\zeta(k_1, \dots, k_m) = Li_{k_1, \dots, k_m}^{\mathbb{C}}(1) = Li_{k_1, \dots, k_m}(1, \dots, 1) = \sum_{n_1 > \dots > n_m > 0} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}.$$

To the tuple of positive integers (k_1, \dots, k_m) of weight $n = \sum k_i$, we associate a tuple of 0's and 1's

$$(\varepsilon_n, \dots, \varepsilon_1) := (\underbrace{0, \dots, 0}_{k_1-1 \text{ times}}, 1, \dots, \underbrace{0, \dots, 0}_{k_m-1 \text{ times}}, 1).$$

This allows us to write multiple polylogarithms as iterated integrals ($z_i \neq 0$ for all i)

$$Li_{k_1, \dots, k_m}(z_1, \dots, z_m) = (-1)^m \int_{\Delta_\gamma} \frac{dt_1}{t_1 - \varepsilon_1 x_1} \wedge \dots \wedge \frac{dt_n}{t_n - \varepsilon_n x_n},$$

where γ is a path from 0 to 1 in $\mathbb{C} \setminus \{x_1, \dots, x_n\}$. The integration domain Δ_γ is the associated real simplex consisting of all m -tuples of points $(\gamma(t_1), \dots, \gamma(t_n))$ with $t_i < t_j$ for $i < j$, where we have set

$$(x_n, \dots, x_1) := (\underbrace{z_1^{-1}, \dots, z_1^{-1}}_{k_1 \text{ times}}, \underbrace{(z_1 z_2)^{-1}, \dots, (z_1 z_2)^{-1}}_{k_2 \text{ times}}, \dots, \underbrace{(z_1 \dots z_m)^{-1}, \dots, (z_1 \dots z_m)^{-1}}_{k_m \text{ times}}).$$

As shown in [Gon05a], iterated integrals have Hodge/motivic avatars living in a Hopf algebra equivalent to the Tannakian Hopf algebra of mixed \mathbb{Q} -Hodge-Tate

structures. Working with these motivic/Hodge iterated integrals reveals more structure – in particular the coproduct, which is not visible on the level of numbers – conjecturally without losing any information.

1.2. Multiple polylogarithms and algebraic cycles. The relations between motivic world and the higher Chow groups on the one hand (e.g. [Lev05, Voe02]), and the relations between multiple polylogarithms and regulators (e.g. [Zag91, Gon05b]) on the other hand, suggest the question of whether there exist avatars of the multiple polylogarithms in terms of algebraic cycles.

Given a number field \mathbb{K} , in [BK94], Bloch and Kriz used algebraic cycles to construct a graded Hopf algebra. They conjectured that this Hopf algebra was isomorphic to the Tannakian Hopf algebra of the category of mixed Tate motives over \mathbb{K} which was proved later by Spitzweck in [Spi] (as presented in [Lev05]). Moreover, Bloch and Kriz described a direct Hodge realization functor for these “cyclic motives”. For any integer n greater or equal to 2 and any point z in \mathbb{K} , they produced an algebraic cycle $Li_n^{cy}(z)$. This cycle $Li_n^{cy}(z)$ induces a motive. They showed in [BK94, Theorem 9.1] that the “bottom-left” coefficient of its period matrix in the Hodge realization is exactly $-Li_n(z)/(2i\pi)^n$.

More recently in [GGL09], Gangl, Goncharov and Levin, using a combinatorial approach, constructed algebraic cycles corresponding to the multiple polylogarithm values $Li_{k_1, \dots, k_m}(z_1, \dots, z_m)$ with parameters z_i in \mathbb{K}^* , under the condition that the corresponding x_i (as defined above) are all distinct. In particular, all the z_i except for z_1 must be different from 1. This implies that their method does not give algebraic cycles corresponding to multiple zeta values.

1.3. Algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The goal of this project is to develop a geometric construction for multiple polylogarithm cycles which removes the previous obstacle; thus we will obtain multiple zeta cycles.

The general idea underlying this project consists in viewing cycles fibered over a larger base, and not just point-wise cycles for some fixed parameter (z_1, \dots, z_m) . Levine, in [Lev11], shows that there exists a short exact sequence relating the Bloch-Kriz Hopf algebra over $\text{Spec}(\mathbb{K})$, its relative version over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the Hopf algebra associated to Goncharov and Deligne’s motivic fundamental group over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which contains the motivic iterated integrals associated to multiple polylogarithms in one variable.

This one variable version of multiple polylogarithms gives multiple zeta values for $z = 1$. Thus, in order to obtain motives corresponding to multiple zeta values in this algebraic cycles setting, it is natural to investigate first the Bloch-Kriz construction over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This should lead to algebraic cycles and motives corresponding to multiple polylogarithms in one variable. The multiple zeta values objects shall then arise as a limit motive or as the limit of a variation of mixed Hodge structure as z tends to 1.

However before computing any motives or any matrix periods, one first needs to obtain explicit algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ such that:

- their boundary (in Bloch-Kriz complex) is related to the differential (or derivative) of multiple polylogarithms functions;
- their Zariski closure over \mathbb{A}^1 have a well defined fiber at $z = 1$ as element of the Bloch-Kriz complex.

This last condition is a priori guided by the fact that, even if the periods matrix corresponding to $Li_n(z)$ is not well defined at $z = 1$ ($\log(1 - z)$ appears in some coefficients), its “bottom-left coefficient” $-Li_n(z)/(2i\pi)^n$ is well-defined at $z = 1$. Moreover we shall see later that this condition is also imposed by our geometric construction.

This paper gives such a class of cycles. In the final remarks, we provide some evidence that it is a good family by computing an integral in low weight.

1.4. Strategy and Main results. The Bloch-Kriz Hopf algebra and its relative version over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the H^0 of the bar construction over a commutative differential graded algebra (cdga) \mathcal{N}_X^\bullet constructed from algebraic cycles. We will use this construction in the case $\mathbb{K} = \mathbb{Q}$ and $X = \text{Spec}(\mathbb{Q})$ or $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ or $X = \mathbb{A}^1$. This cdga comes from the cubical construction of the higher Chow groups, and setting $\square^1 = \mathbb{P}^1 \setminus \{1\} \simeq \mathbb{A}^1$, we have:

$$\mathcal{N}_X^\bullet = \mathbb{Q} \oplus (\oplus_{p \geq 1} \mathcal{N}_X^\bullet(p)),$$

where the $\mathcal{N}_X^n(p)$ are generated by codimension p cycles in $X \times \square^{2p-n}$ which are admissible (cf. Definition 2.6 and Remark 2.7). The cohomology of the complex $\mathcal{N}_X^\bullet(p)$ recovers the higher Chow groups $\text{CH}^p(X, 2p - \bullet)$.

As the H^0 of the bar construction over $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$ is isomorphic to one of its 1-minimal model [BK94], our strategy is to follow the inductive construction of this 1-minimal model as presented in [DGMS75].

Let $S^{gr}(V) = \oplus_n S^{gr, n}(V)$ denote the graded symmetric algebra over a graded vector space V . Roughly speaking this inductive construction of the 1-minimal model of $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$ goes as follows. One begins with $V_1 = H^1(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet)$ (in degree 1) and a map φ_1 given by a choice of representative in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$ of a basis of V_1 . This map induces a map $S^{gr}(V_1) \rightarrow \mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$ and a map

$$H^k(S^{gr}(V_1)) \xrightarrow{\tilde{\varphi}_1} H^k(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet)$$

where the differential on $S^{gr}(V_1)$ is 0. The above map might not be injective on the H^2 which is one of the desired property of a 1-minimal model. Hence the first inductive step consists in killing the kernel of $\tilde{\varphi}_1$. In general at the i -th step of the induction (e.g. $i = 1$), one defines

$$V_{i+1} = V_i \oplus \ker(\tilde{\varphi}_i)$$

where the kernel is added in degree 1. One then extends φ_i to a map

$$\varphi_{i+1} : V_{i+1} \longrightarrow \mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^1$$

by defining it on $\ker(\tilde{\varphi}_i)$. One chooses a family (b_k) of degree 2 elements in $S^{gr}(V_i)$ inducing a basis $([b_k])$ of $\ker(\tilde{\varphi}_i)$. The image of $[b_k]$ under φ_{i+1} is defined as follows: by definition b_k is a linear combination of products of elements in V_i ,

$$b_k = \sum \alpha_{a,b}^k c_a \cdot c_b,$$

The element b_k is mapped by φ_i to $\sum \alpha_{a,b}^k \varphi_i(c_a) \cdot \varphi_i(c_b)$ in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^2$. As b_k gives a class in the H^2 , the differential of the previous sum is 0. And because $[b_k]$ is in the kernel of $\tilde{\varphi}_i$, the sum

$$\varphi_i(b_k) = \sum \alpha_{a,b}^k \varphi_i(c_a) \cdot \varphi_i(c_b)$$

is a boundary in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$. That is, there exists c_k in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^1$ such that

$$\varphi_i(b_k) = \partial(c_k)$$

where ∂ denotes the differential in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^\bullet$. One defines $\varphi_{i+1}([b_k])$ to be

$$\varphi_{i+1}([b_k]) = c_k.$$

The discussion above can be summarized in the following diagram

$$\begin{array}{ccc}
S^{gr,2}(V_i) & \xrightarrow{\varphi_i} & \mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^2 \\
b_k \longmapsto & \longrightarrow & \sum \alpha_{i,j}^k \varphi_i(c_a) \cdot \varphi_i(c_b) \longmapsto \partial \longrightarrow 0 \\
& & \begin{array}{c} \uparrow \hat{\partial}_i \\ \perp \\ \exists c_k \in \mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^1 \end{array}
\end{array}$$

If our construction does not follow exactly the above description, it is largely inspired by its mains aspect :

- Which linear combinations of products of degree 1 elements are boundary? Note that these linear combinations are degree 2 elements.
- How does one keep tract of the degree 1 elements c which are mapped by the differential onto these linear combinations of products?

Hence, we will inductively find linear combinations $\sum \alpha_{i,j} c_i \cdot c_j$ that have a zero differential, and see under what conditions they can be written as an explicit boundary, i.e. as $d(c)$ for some explicit cycle c in \mathcal{N}_X^1 .

In weight p , we will consider linear combinations constructed from elements obtained in lower weight; under some geometric conditions the cycle c can be constructed easily. It is the pull-back of $\sum \alpha_{i,j} c_i \cdot c_j$ induced by the multiplication map ($\square^1 \simeq \mathbb{A}^1$, $X = \mathbb{A}^1$):

$$X \times \mathbb{A}^1 \times \mathbb{A}^{2p-2} \longrightarrow X \times \mathbb{A}^{2p-2}$$

$$(t, s, x_1, \dots, x_{2p-2}) \longmapsto (ts, x_1, \dots, x_{2p-2}).$$

Even though it is not formalized in their paper, it is reasonable to believe that Bloch and Kriz used this idea to build their cycles $\text{Li}_n^{cy}(z)$. Thus, we naturally recover these cycles using the method described above. However, the cycles corresponding to multiple polylogarithms constructed using this method are different from the ones proposed by Gangl, Goncharov and Levin in [GGL09].

In particular, the geometric conditions on $\sum \alpha_{i,j} c_i \cdot c_j$ and the computation of the pull-back in the above construction oblige the constructed cycles on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to admit an extension to $\mathcal{N}_{\mathbb{A}^1}^\bullet$ which is dominant over \mathbb{A}^1 with pure relative dimension (cf. Definition 5.1) and to have an empty fiber at 0 or 1. Such cycles are called *equidimensional* (over \mathbb{A}^1).

A complete description of the inductive construction is based on a combinatorial setting using the coLie algebra dual to the free Lie algebra on two generators together with Ihara's action by special derivations. More precisely, for any Lyndon word W in the letters $\{0, 1\}$, we have considered linear combinations of decorated rooted trivalent trees, $T_{W^*}(x)$. These $T_{W^*}(x)$ are dual to the Lyndon brackets basis of the free Lie algebra, hence the subscript W^* in $T_{W^*}(x)$. In this notation, x denotes a parameter in \mathbb{A}^1 labeling the root. Dualizing, the action of the Lie algebra on itself by Ihara's special derivations, we obtain a twisted cobracket d_{cy} which can applied to $T_{W^*}(x)$. The main point of the combinatorial setting is the following result (Theorem 4.27).

Theorem. *For any Lyndon word W in the letters $\{0, 1\}$, the cobracket of $T_{W^*}(x)$ can be expressed as:*

$$(1) \quad d_{cy}(T_{W^*}(x)) = \sum \alpha_{i,j} T_{W_i^*}(x) \wedge T_{W_j^*}(x) + \sum \beta_{k,l} T_{W_k^*}(x) \wedge T_{W_l^*}(1)$$

where \wedge is the exterior product and W_i , W_j , W_k and W_l are Lyndon words of smaller length than W . The coefficients $\alpha_{i,j}$'s and $\beta_{k,l}$'s are integers.

Note that Gangl, Goncharov and Levin in [GGL09] defined a differential d_{ggl} on a commutative differential graded algebra built on a closely related type of trees. Gangl, Goncharov and Levine approach is essentially given by the universal enveloping algebra of the Lie algebra used in this paper (see Remark 4.29). The differential d_{ggl} was originally built to mimic the differential in \mathcal{N}_X^\bullet while the cobracket d_{cy} reveals the underlying structure of Ihara (co)action by special derivations.

This result gives us the combinatorial structure of the elements we want to build, and play a central role in constructing our explicit algebraic cycles in a general framework. Modifying the above “differential system” (a cobracket inducing a differential), we inductively construct cycles \mathcal{L}_W^0 corresponding to $T_{W^*}(x)$ (or in some sense $T_{W^*}(x) - T_{W^*}(0)$) and cycles \mathcal{L}_W^1 corresponding to the difference $T_{W^*}(x) - T_{W^*}(1)$. In this way, we obtain (cf. Theorem 5.8) algebraic cycles that are expected, when specialized at 1, to correspond to multiple zeta values under Bloch-Kriz Hodge realization functor (which is yet not completely computable in any explicit and coherent way).

Theorem. *Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For any Lyndon word W in the letters $\{0, 1\}$ of length $p \geq 2$, there exists a cycle \mathcal{L}_W^0 in $\mathcal{N}_X^1(p)$, i.e. a cycle of codimension p in $X \times \square^{2p-1}$, such that:*

- \mathcal{L}_W^0 has a decomposable boundary,
- \mathcal{L}_W^0 admits an equidimensional extension to \mathbb{A}^1 with empty fiber at 0.

A similar statement holds for 1 in place of 0.

In particular,

- $\mathcal{L}_W^0|_{\{x=1\}}$ gives an element of $\mathcal{N}_{\mathbb{Q}}^\bullet$ which is expected to correspond to a multiple zeta value in the same manner that, $\mathcal{L}_W^0|_{\{x=z\}}$ is expected to correspond to a multiple polylogarithm at z .
- The computation of the actual integral for $W = 011$ is done in the last section.

The paper is organized as follows:

- The next section (Section 2) is devoted to a general review of the Bloch-Kriz cycle complex. In particular, we detail the construction of the cycle complex and recall some of its main properties (relation to higher Chow groups, localization long exact sequence, etc.).
- Then, we apply in Section 3 our strategy to the nice examples of polylogarithms as described in [BK94]. Then through a weight 3 example we present the main difficulties and how to overcome them in general.
- Next, in Section 4, we deal with the combinatorial situation, beginning by presenting the trivalent trees attached to Lyndon words and their relations with the free Lie algebra on two generators $\text{Lie}(X_0, X_1)$. We review then Ihara's action by stable derivations. Then, we introduce linear combinations of trees T_{W^*} corresponding to the dual situation; that is in the Lie coalgebra graded dual to $\text{Lie}(X_0, X_1)$ and study the cobracket corresponding to Ihara's action. Note that we are not simply looking at Ihara Poisson bracket; we are also keeping track of the structure coefficients of the action.

This leads to Theorem 4.27 which proves that the image of T_{W^*} under d_{cy} is decomposable in terms of T_{U^*} (with U of smaller length).

- In Section 5 we prove our main Theorem. It begins by presenting some properties of equidimensional algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and \mathbb{A}^1 . Then, we study the relation between the two situations and explain how the pull-back by the multiplication (resp. a twisted multiplication) gives a homotopy between the identity and the fiber at 0 (resp. at 1) pulled-back to a cycle over \mathbb{A}^1 by $p : \mathbb{A}^1 \rightarrow \{pt\}$. Finally, the above work allows us to inductively construct the desired families of cycles \mathcal{L}_W^0 and \mathcal{L}_W^1 in Theorem 5.8.
- The last section is devoted to some concluding remarks. In particular, computing the integral attached to the cycle \mathcal{L}_{011}^0 , we show that its specialization at the point 1 is $-\zeta(2, 1)$.

2. CYCLE COMPLEX OVER $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

2.1. Commutative differential graded algebras. We recall here some definitions and properties of commutative differential graded algebras (cdga's) over \mathbb{Q} .

Definition 2.1 (cdga). A commutative differential graded algebra A is a commutative graded algebra (with unit) $A = \bigoplus_n A^n$ over \mathbb{Q} together with a graded homomorphism $d = \bigoplus d^n$, $d^n : A^n \rightarrow A^{n+1}$ such that

- $d^{n+1} \circ d^n = 0$
- d satisfies the Leibniz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b) \quad \text{for } a \in A^n, b \in A^m.$$

We recall that a graded algebra is *commutative* if and only if for any homogeneous elements a and b , we have

$$ab = (-1)^{\deg(a)\deg(b)}ba.$$

Definition 2.2. A cdga A is

- *connected* if $A^n = 0$ for all $n < 0$ and $A^0 = \mathbb{Q} \cdot 1$.
- *cohomologically connected* if $H^n(A) = 0$ for all $n < 0$ and $H^0(A) = \mathbb{Q} \cdot 1$.

In our context, the cdga involved are not necessarily connected, but come with an Adams grading.

Definition 2.3 (Adams grading). An *Adams graded* cdga is a cdga A together with a decomposition into subcomplexes $A = \bigoplus_{p \geq 0} A(p)$ such that

- $A(0) = \mathbb{Q}$ is the image of the algebra morphism $\mathbb{Q} \rightarrow A$.
- The Adams grading is compatible with the product of A , i.e.

$$A^k(p) \cdot A^l(q) \subset A^{k+l}(p+q).$$

However, no sign is introduced as a consequence of the Adams grading.

For an element $a \in A^k$, we call k the cohomological degree and denote it by $|a| := k$. In the case of an Adams graded cdga, for $a \in A^k(p)$, we call p its weight or Adams degree and denote it by $wt(a) := p$.

We assume all the commutative differential graded algebras to have an augmentation $\varepsilon : A \rightarrow \mathbb{Q}$. Note that an Adams graded cdga A has a canonical augmentation $A \rightarrow \mathbb{Q}$ with augmentation ideal $A^+ = \bigoplus_{p \geq 1} A(p)$.

2.2. General construction of Bloch-Kriz cycles complex. This subsection is devoted to the construction of the cycle complex as presented in [Blo86, Blo97, BK94, Lev94].

For simplicity we work over $\text{Spec}(\mathbb{Q})$. Let \square^n be the algebraic n -cube

$$\square^n = (\mathbb{P}^1 \setminus 1)^n.$$

Insertion morphisms $s_i^\varepsilon : \square^{n-1} \rightarrow \square^n$ are given by the identification

$$\square^{n-1} \simeq \square^{i-1} \times \{\varepsilon\} \times \square^{n-i}$$

for $\varepsilon = 0, \infty$. Similarly, for $I \subset \{1, \dots, n\}$ and $\varepsilon : I \rightarrow \{0, \infty\}$, we define $s_I^\varepsilon : \square^{n-|I|} \rightarrow \square^n$.

Definition 2.4. A *face* F of codimension p of \square^n is the image $s_I^\varepsilon(\square^{n-p})$ for some I and ε as above such that $|I| = p$.

In other word, a codimension p face of \square^n is given by the equation $x_{i_k} = \varepsilon_k$ for k in $\{1, \dots, p\}$ and ε_k in $\{0, \infty\}$ where x_1, \dots, x_n are the usual affine coordinates on \mathbb{P}^1 .

The permutation group \mathfrak{S}_n act on \square^n by permutation of the factor.

Remark 2.5.

- In some references, [Lev94, Lev11] for example, \square^n is defined to be the usual affine space \mathbb{A}^n and the faces by setting various coordinates equal to 0, or 1. This make the correspondence with the “usual” cube more natural. However, the above presentation, which agree with [BK94] or [GGL09], makes some comparisons and some formulas “nicer”. In particular, the relation between the construction in the setting $\square^1 = \mathbb{P}^1 \setminus \{1\}$ and the Chow group $\text{CH}^1(X)_{\mathbb{Q}}$ is simpler.
- Let **Cube** be the subcategory of the category of finite sets whose objects are $\underline{n} = \{0, 1\}^n$ and morphisms are generated by forgetting a factor, inserting 0 or 1 and permutation of the factors ; these morphisms being subject to natural relations. Similarly to the usual description of a simplicial object, \square^\bullet is a functor from **Cube** into the category of smooth \mathbb{Q} -varieties and the various \square^n are geometric equivalents of \underline{n} .

Now, let X be a smooth quasi-projective variety over \mathbb{Q} .

Definition 2.6. Let p and n be non negative integers. Let $\mathcal{Z}^p(X, n)$ be the free group generated closed irreducible sub-varieties of $X \times \square^n$ of codimension p which intersect all faces $X \times F$ properly (where F is a face of \square^n). That is:

$$\mathbb{Z} \left\langle W \subset X \times \square^n \text{ such that } \begin{cases} W \text{ is closed and irreducible} \\ \text{codim}_{X \times F}(W \cap X \times F) = p \\ \text{or } W \cap (X \times F) = \emptyset \end{cases} \right\rangle$$

Remark 2.7.

- A sub-variety W of $X \times \square^n$ as above is *admissible*.
- As $p_i : \square^n \rightarrow \square^{n-1}$ is smooth, we have the corresponding induced pull-back: $p_i^* : \mathcal{Z}^p(X, n-1) \rightarrow \mathcal{Z}^p(X, n)$.
- s_i^ε induces a regular closed embedding $X \times \square^{n-1} \rightarrow X \times \square^n$ which is of local complete intersection. As we are considering only admissible cycles, i.e. cycles in “good position” with respect to the faces, s_i^ε induces $s_i^{\varepsilon,*} : \mathcal{Z}^p(X, n) \rightarrow \mathcal{Z}^p(X, n-1)$.
- The morphism $\partial = \sum_{i=1}^n (-1)^{i-1} (s_i^{0,*} - s_i^{\infty,*})$ induces a differential

$$\mathcal{Z}^p(X, n) \longrightarrow \mathcal{Z}^p(X, n-1).$$

One can extend the action of \mathfrak{S}_n on \square^n to an action of the semi-direct product $G_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ where each $\mathbb{Z}/2\mathbb{Z}$ acts on \square^1 by sending the usual affine coordinate x to $1/x$. The sign representation of \mathfrak{S}_n extends to a sign representation $G_n \mapsto \{\pm 1\}$. Let $\text{Alt}_n \in \mathbb{Q}[G_n]$ be the corresponding projector.

Definition 2.8. Let p and k be integers with $p > 0$. Set

$$\mathcal{N}_X^k(p) = \text{Alt}_{2p-k}(\mathcal{Z}(X, 2p-k) \otimes \mathbb{Q}).$$

We will refer to k as the *cohomological degree*, and to p as the *weight*.

Remark 2.9. In this presentation, we have not dealt with degeneracies (images in $\mathcal{Z}(X, n)$ of p_i^*) because we use an alternative version with rational coefficients. For more details, see the first section of [Lev09] which presents the general setting of cubical objects. A similar remark was made in [BK94][after equation (4.1.3)].

Definition 2.10 (Cycle complex). For p and k as above, the pull-back

$$s_i^{\varepsilon^*} : \mathcal{Z}^p(X, 2p-k) \longrightarrow \mathcal{Z}^p(X, 2p-k-1)$$

induces a morphism $\partial_i^{\varepsilon} : \mathcal{N}_X^k(p) \longrightarrow \mathcal{N}_X^{k+1}(p)$. Thus, the differential ∂ on $\mathcal{Z}^p(X, 2p-k)$ extends to a differential

$$\partial = \sum_{i=1}^{2p-k} (-1)^{i-1} (\partial_i^0 - \partial_i^\infty) : \mathcal{N}_X^k(p) \xrightarrow{\partial} \mathcal{N}_X^{k+1}(p).$$

Let $\mathcal{N}_X^\bullet(p)$ be the complex

$$\mathcal{N}_X^\bullet(p) : \quad \cdots \longrightarrow \mathcal{N}_X^k(p) \xrightarrow{\partial} \mathcal{N}_X^{k+1}(p) \longrightarrow \cdots$$

Define the cycle complex as

$$\mathcal{N}_X^\bullet = \mathbb{Q} \oplus \bigoplus_{p \geq 1} \mathcal{N}_X^\bullet(p).$$

In [Lev94][§5] and in [Lev11][Example 4.3.2], Levine proved the following proposition.

Proposition 2.11. *Concatenation of the cube factors and pull-back by the diagonal*

$$X \times \square^n \times X \times \square^m \xrightarrow{\sim} X \times X \times \square^n \times \square^m \xrightarrow{\sim} X \times X \times \square^{n+m} \xleftarrow{\Delta_X} X \times \square^{n+m}$$

induces, after applying the Alt projector, a well-defined product:

$$\mathcal{N}_X^k(p) \otimes \mathcal{N}_X^l(q) \longrightarrow \mathcal{N}_X^{k+l}(p+q)$$

denoted by \cdot .

Remark 2.12. The smoothness hypothesis on X allows us to consider the pull-back by the diagonal $\Delta_X : X \longrightarrow X \times X$ which is, in this case, a local complete intersection.

We have the following theorem (also stated in [BK94, Blo97] for $X = \text{Spec}(\mathbb{Q})$).

Theorem 2.13 ([Lev94]). *The cycle complex \mathcal{N}_X^\bullet is a differential graded commutative algebra. In weight p , its cohomology groups are the p -th higher Chow group of X :*

$$H^k(\mathcal{N}_X^\bullet(p)) = \text{CH}^p(X, 2p-k)_{\mathbb{Q}},$$

where $\text{CH}^p(X, 2p-k)_{\mathbb{Q}}$ stands for $\text{CH}^p(X, 2p-k) \otimes \mathbb{Q}$.

Moreover, this construction is functorial in X with respect to flat pull-back and proper push-forward (up to the usual shifts in degree and weight). Using Levine's work [Lev94], we have a more general pull-back functoriality on the level of cohomology group; one could also use Bloch's moving Lemma [Blo94].

2.3. Some properties of Higher Chow groups. In this section, we present some well-known properties of the higher Chow groups and some applications that will be used later. Proof of the different statements can be found in [Blo86] or [Lev94].

2.3.1. Relation with higher K -theory. Higher Chow groups, in a simplicial version, were first introduced in [Blo86] in order to achieve a better understanding of the K -groups of higher K -theory. In [Lev94][Theorem 3.1], Levine gives a cubical version of the desired isomorphisms:

Theorem 2.14 ([Lev94]). *Let X be a smooth quasi-projective \mathbb{Q} -variety, and let p, k be two positive integers. Then*

$$\mathrm{CH}^p(X, 2p - k)_{\mathbb{Q}} \simeq \mathrm{Gr}^p K_{2p-k}(X) \otimes \mathbb{Q}$$

In particular, using the work of Borel [Bor74], computing the rank of K -groups of a number field, in the case $k = 2$ and $p \geq 2$ we find:

$$(2) \quad \mathrm{CH}^p(\mathbb{Q}, 2p - 2)_{\mathbb{Q}} \simeq \mathrm{Gr}^p K_{2p-2}(\mathbb{Q}) \otimes \mathbb{Q} = 0$$

2.3.2. A^1 -homotopy invariance. From Levine [Lev94][Theorem 4.5], one can deduce the following proposition.

Proposition 2.15 ([Lev94]). *Let X be as above and p_X be the projection $p_X : X \times \mathbb{A}^1 \rightarrow X$. Then the projection p_X induces (by flat pull-back) a quasi-isomorphism for any positive integer p*

$$p_X^* : \mathcal{N}_X^\bullet(p) \xrightarrow{q,i} \mathcal{N}_{X \times \mathbb{A}^1}^\bullet(p)$$

Moreover, an inverse to p_X^* on the cohomology is given by the pull-back by the zero section $i_0^* : \mathrm{H}^k(\mathcal{N}_{X \times \mathbb{A}^1}^\bullet(p)) \rightarrow \mathrm{H}^k(\mathcal{N}_X^\bullet(p))$.

Remark 2.16. The proof of Levine's theorem also tells how this quasi-isomorphism arises using the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. This leads to the proof of Proposition 5.7.

We now apply the above result in the case where $X = \mathrm{Spec}(\mathbb{Q})$, and use the relation with K -theory via Equation (2).

Corollary 2.17. *The second cohomology group of $\mathcal{N}_{\mathbb{A}^1}^\bullet$ vanishes:*

$$\forall p \geq 1 \quad \mathrm{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(p)) \simeq \mathrm{CH}^p(\mathbb{A}^1, 2p - 2)_{\mathbb{Q}} \simeq \mathrm{CH}^p(\mathbb{Q}, 2p - 2)_{\mathbb{Q}} = 0.$$

2.3.3. Localization sequence. Let W be a smooth closed subvariety of pure codimension d of a smooth quasi-projective variety X . Let U denote the open complement $U = X \setminus W$. A version adapted to our needs of Theorem 3.4 in [Lev94] gives the localization sequence for higher Chow groups.

Theorem 2.18 ([Lev94]). *Let p be a positive integer and l an integer. There is a long exact sequence*

$$(3) \quad \cdots \rightarrow \mathrm{CH}^p(U, l + 1)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{CH}^{p-d}(W, l)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{j^*} \mathrm{CH}^p(U, l)_{\mathbb{Q}} \xrightarrow{\delta} \cdots$$

where $i : W \rightarrow X$ denotes the closed immersion and $j : U \rightarrow X$ the open one.

Remark 2.19. Here, i_* and j^* denote the usual push-forward for proper morphisms and pull-back for flat ones.

In order to study the cycle complex over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we begin by applying the above theorem to the case where $X = \mathbb{A}^1$, $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $W = \{0, 1\}$.

Corollary 2.20. *We have the following description of $H^k(\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^\bullet)$:*

$$H^k(\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^\bullet) \simeq H^k(\mathcal{N}_{\mathbb{Q}}^\bullet) \oplus \left(H^{k-1}(\mathcal{N}_{\mathbb{Q}}^\bullet) \otimes \mathbb{Q}L_0 \right) \oplus \left(H^{k-1}(\mathcal{N}_{\mathbb{Q}}^\bullet) \otimes \mathbb{Q}L_1 \right),$$

where L_0 and L_1 are in cohomological degree 1 and weight 1 (i.e. of codimension 1).

Proof. The above long exact sequence gives

$$(4) \quad \cdots \longrightarrow \mathrm{CH}^{p-1}(\{0,1\}, l)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(\mathbb{A}^1, l)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\delta} \\ \mathrm{CH}^{p-1}(\{0,1\}, l-1)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(\mathbb{A}^1, l-1)_{\mathbb{Q}} \longrightarrow \cdots$$

The map i_* is induced by the inclusions i_0 and i_1 of 0 and 1 in \mathbb{A}^1 . The morphism $i_0^* : H^k(\mathcal{N}_{\mathbb{A}^1}^\bullet) \rightarrow H^k(\mathcal{N}_{\{0\}}^\bullet)$, and more generally i_x^* for any \mathbb{Q} point x of \mathbb{A}^1 , is an isomorphism inverse to $p_{\mathrm{Spec}(\mathbb{Q})}^* : \mathbb{A}^1 \rightarrow \mathrm{Spec}(\mathbb{Q})$. Hence the Cartesian diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{Spec}(\mathbb{Q}) \\ \downarrow & & \downarrow i_x \\ \mathrm{Spec}(\mathbb{Q}) & \xrightarrow{i_0} & \mathbb{A}^1 \end{array}$$

shows that $i_{0,*}$ (and respectively $i_{1,*}$) are 0 on cohomology.

In particular, the sequence (4) is a collection of short exact sequences

$$0 \longrightarrow \mathrm{CH}^p(\mathbb{A}^1, l)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{CH}^{p-1}(\{0,1\}, l-1)_{\mathbb{Q}} \longrightarrow 0.$$

Thus, for a fixed l , using the \mathbb{A}^1 -homotopy property and the fact that

$$\mathrm{CH}^p(\{0,1\}, l) \simeq \mathrm{CH}^p(\mathrm{Spec}(\mathbb{Q}), l) \oplus \mathrm{CH}^p(\mathrm{Spec}(\mathbb{Q}), l),$$

we obtain the following short exact sequence

$$0 \longrightarrow \mathrm{CH}^p(\mathrm{Spec}(\mathbb{Q}), l)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{CH}^{p-1}(\mathrm{Spec}(\mathbb{Q}), l-1)_{\mathbb{Q}}^{\oplus 2} \longrightarrow 0;$$

and an isomorphism

$$\mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}^p(\mathbb{A}^1, l)_{\mathbb{Q}} \oplus \mathrm{CH}^{p-1}(\mathrm{Spec}(\mathbb{Q}), l-1)_{\mathbb{Q}}^{\oplus 2}.$$

The relation between the cohomology groups of $\mathcal{N}_X^\bullet(p)$ and the higher Chow groups concludes the proof. \square

Remark 2.21. The above corollary for $k = 1$, gives us a description of $H^1(\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^\bullet)$ which is the key object of the first step of the 1-minimal construction mentioned in the introduction. Hence this corollary is the starting point of the construction of our algebraic cycle.

The generators L_0 and L_1 will be given explicitly in terms of cycles in $\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^\bullet$ at Subsection 3. This allows us to follow the 1-minimal model construction in order to build our algebraic cycles.

2.4. Cycle complex over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and mixed Tate motives. Levine in [Lev11] makes the link between the category of mixed Tate motives (in the sense of Levine [Lev05] or Voevodsky [Voe00]) over a base X and the cycle complex \mathcal{N}_X . The relation between mixed Tate motives and the cycle complex was developed earlier for $X = \mathrm{Spec}(\mathbb{Q})$, the spectrum of a number field, by Bloch and Kriz in [BK94].

In the following X still denotes a smooth, quasi-projective variety over \mathbb{Q} . We will work with coefficients in \mathbb{Q} .

Under more general conditions, Cisinski and Déglise [CD09] defined a triangulated category $\mathrm{DM}(X)$ of (effective) motives over a base with the expected properties. Levine's work [Lev93, Lev11] shows that when the motive of X is mixed Tate over $\mathrm{Spec}(\mathbb{Q})$ and satisfies the Beilinson-Soulé vanishing conjecture, there exists a Tannakian category $\mathrm{MTM}(X)$ of mixed Tate motives over X .

Together with defining an avatar of \mathcal{N}_X in $\mathrm{DM}(X)$, Levine [Lev11][Theorem 5.3.2 and beginning of the section 6.6] shows that when X satisfies the above conditions, one can identify the Tannakian group associated with $\mathrm{MTM}(X)$ with the spectrum of the H^0 of the bar construction over the cdga \mathcal{N}_X :

$$G_{\mathrm{MTM}(X)} \simeq \mathrm{Spec}(H^0(B(\mathcal{N}_X))).$$

Then, he uses a relative bar-construction in order to relate the natural morphisms

$$p^* : \mathrm{MTM}(\mathrm{Spec}(\mathbb{Q})) \longrightarrow \mathrm{MTM}(X), \quad x^* : \mathrm{MTM}(X) \longrightarrow \mathrm{MTM}(\mathrm{Spec}(\mathbb{K})),$$

induced by the structural morphism $p : X \rightarrow \mathrm{Spec}(\mathbb{Q})$ and a choice of a \mathbb{Q} -point x , to the motivic fundamental group of X at the base point x defined by Goncharov and Deligne, $\pi_1^{\mathrm{mot}}(X, x)$ (see [Del89] and [DG05]).

In particular, applying this to the the case $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, we have the following result.

Theorem 2.22 ([Lev11][Corollary 6.6.2]). *Let x be a \mathbb{Q} -point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $G_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}$ and $G_{\mathbb{Q}}$ denote the Spectrum of $H^0(B(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}))$ and $H^0(B(\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}))$ respectively. Then there is a split exact sequence:*

$$(5) \quad 1 \longrightarrow \pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x) \longrightarrow G_{\mathbb{P}^1 \setminus \{0, 1, \infty\}} \begin{array}{c} \xrightarrow{p^*} \\ \xleftarrow{x^*} \end{array} G_{\mathbb{Q}} \longrightarrow 1$$

where p is the structural morphism $p : \mathbb{P}^1 \setminus \{0, 1, \infty\} \rightarrow \mathrm{Spec}(\mathbb{Q})$.

Let $\mathrm{co}\mathcal{L}_{\mathbb{Q}}$ and $\mathrm{co}\mathcal{L}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}$ denote the Lie coalgebra given by the set of indecomposable elements of the H^0 of the bar construction of $\mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{\bullet}$ and $\mathcal{N}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^{\bullet}$ respectively. Then, taking the ring of functions and passing to the set of indecomposable elements, the short exact sequence (5) can be reformulated in terms of coLie algebras.

Thus there is a split exact sequence of coLie algebras:

$$0 \longrightarrow \mathrm{co}\mathcal{L}_{\mathbb{Q}} \longrightarrow \mathrm{co}\mathcal{L}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}} \longrightarrow \mathrm{co}\mathcal{L}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^{\mathrm{geom}} \longrightarrow 0$$

where $\mathrm{co}\mathcal{L}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^{\mathrm{geom}}$ is the graded dual of the Lie algebra associated to $\pi_1^{\mathrm{mot}}(X, x)$.

In particular $\mathrm{co}\mathcal{L}_{\mathbb{P}^1 \setminus \{0, 1, \infty\}}^{\mathrm{geom}}$ is the Lie coalgebra graded dual of the free Lie algebra on two generators $\mathrm{Lie}(X_0, X_1)$. This observation motivated the study of this Lie coalgebra in Section 4 in order to understand the combinatorial structure of our construction. This Lie coalgebra is presented in a ‘‘trivalent trees’’ version because of the combinatorial construction of Gangl, Goncharov and Levin [GGL09] which was based on linear combination of trivalent tress. Their approach seems to be related to the graded dual of the universal enveloping algebra of $\mathrm{Lie}(X_0, X_1)$.

3. LOW WEIGHT EXAMPLES AND CYCLES CORRESPONDING TO POLYLOGARITHMS

From this point until the end of the article, we let X denote $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

In this section, we give a first presentation of our strategy to construct general cycles in \mathcal{N}_X^{\bullet} corresponding to multiple polylogarithms by examining the simple case of the polylogarithms $\mathcal{L}i_n$. We will pay special attention to the Totaro cycle, which is known to correspond to the function $Li_2^{\mathbb{C}}(z)$, and then explain how its

construction can be generalized to obtain cycles already present in [BK94] and [GGL09] corresponding to the functions $Li_n^{\mathbb{C}}(z)$.

3.1. Two weight 1 examples of cycles generating the H^1 . We want to construct cycles in \mathcal{N}_X^\bullet in order to obtain the inductive construction of the 1-minimal model. This means that we will

- (1) find in \mathcal{N}_X^2 linear combinations of products of already constructed cycles that are boundaries, i.e. $d(c)$ for some c in \mathcal{N}_X^1 ;
- (2) explicitly construct the desired c .

But the first step begins with a basis of $H^1(\mathcal{N}_X^\bullet)$. However, as we only want a description of the geometric part of the 1 minimal model (that is relatively to the situation over $\text{Spec}(\mathbb{Q})$) and we do not need to consider a full basis of $H^1(\mathcal{N}_X^\bullet)$. We have seen that $H^1(\mathcal{N}_X^\bullet)$ (Corollary 2.20) is the direct sum of $H^1(\mathcal{N}_{\mathbb{Q}}^\bullet)$ and two copies of $H^0(\mathcal{N}_{\mathbb{Q}}^\bullet)$.

Lemma 3.1. *Let Γ_0 and Γ_1 be respectively the graph of $\rho_0 : X \rightarrow \square^1$ which sends x to x and the graph of $\rho_1 : X \rightarrow \square^1$ which sends x to $1 - x$. Then, Γ_0 and Γ_1 define admissible algebraic cycles in $X \times \square^1$, applying the projector Alt on the alternating elements gives two elements L_0 and L_1 in \mathcal{N}_X^1 , and we have*

$$H^1(\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^\bullet) \simeq H^1(\mathcal{N}_{\mathbb{Q}}^\bullet) \oplus (H^0(\mathcal{N}_{\mathbb{Q}}^\bullet) \otimes \mathbb{Q}L_0) \oplus (H^0(\mathcal{N}_{\mathbb{Q}}^\bullet) \otimes \mathbb{Q}L_1).$$

When we speak about parametrized cycles, we will usually omit the projector Alt and write

$$L_0 = [x; x] \quad \text{and} \quad L_1 = [x; 1 - x] \quad \subset X \times \square^1$$

where the notation $[x; f(x)]$ denotes the set

$$\{(x, f(x)) \text{ such that } x \in X\}.$$

Proof. First of all, note that L_0 and L_1 are codimension 1 cycles in $X \times \square^1 = X \times \square^{2*1-1}$. Moreover, as

$$L_0 \cap (X \times \{\varepsilon\}) = L_0 \cap (\mathbb{P}^1 \setminus \{0, 1, \infty\} \times \{\varepsilon\}) = \emptyset,$$

for $\varepsilon = 0, \infty$, L_0 is admissible (intersect each face in the right codimension or not at all) and gives an element of $\mathcal{N}_X^1(1)$. Furthermore, the above intersection tells us that $\partial(L_0) = 0$. Similarly, one can show that L_1 gives an element of $\mathcal{N}_X^1(1)$, and that $\partial(L_1) = 0$. Thus L_0 and L_1 yield well defined class in $H^1(\mathcal{N}_X^\bullet(1))$.

In order to show that they are non-trivial, one shows that in the localization sequence (3), their images under the boundary map

$$H^1(\mathcal{N}_X^\bullet(1)) \xrightarrow{\delta} H^0(\mathcal{N}_{\{0\}}^\bullet(0)) \oplus H^0(\mathcal{N}_{\{1\}}^\bullet(0))$$

are non-zero. It is enough to treat the case of L_0 . Let $\overline{L_0}$ be the closure of L_0 in $\mathbb{A}^1 \times \square^1$. Indeed, $\overline{L_0}$ is given by the parametrized cycle

$$\overline{L_0} = [x; x] \subset \mathbb{A}^1 \times \square^1$$

and the intersection with the face $u_1 = 0$ is of codimension 1 in $\mathbb{A}^1 \times \{0\}$ and the intersection with $u_1 = \infty$ is empty. Hence $\overline{L_0}$ is admissible.

Thus, considering the definition of δ , $\delta(L_0)$ is given by the intersection of the differential of $\overline{L_0}$ with $\{0\}$ and $\{1\}$ on the first and second factors respectively. The above discussion on the admissibility of $\overline{L_0}$ shows that $\delta(L_0)$ is non-zero on the factor $H^0(\mathcal{N}_{\{0\}}^\bullet(0))$ and 0 on the other factor, as the admissibility condition is trivial in $H^0(\mathcal{N}_{\{0\}}^\bullet(0))$ and the restriction of $\overline{L_0}$ to 1 is empty. The situation is reversed for L_1 . \square

Later we will consider cycles depending on many parameters, and denote by

$$[x; f_1(x, \mathbf{t}), f_2(x, \mathbf{t}), \dots, f_n(x, \mathbf{t})] \subset X \times \square^n$$

the (image under the projector \mathcal{Alt} of the) restriction to $X \times \square^n$ of the image of

$$\begin{array}{ccc} X \times (\mathbb{P}^1)^k & \longrightarrow & X \times (\mathbb{P}^1)^n \\ (x, \mathbf{t}) & \longmapsto & (x, f_1(x, \mathbf{t}), f_2(x, \mathbf{t}), \dots, f_n(x, \mathbf{t})). \end{array}$$

3.2. A weight 2 example: the Totaro cycle. Consider the linear combination

$$b = L_0 \cdot L_1 \in \mathcal{N}_X^2(2).$$

It is given as a parametrized cycle by

$$b = [x; x, 1 - x] \subset X \times \square^2$$

or in terms of defining equations by

$$X_1 V_1 - U_1 X_2 = 0 \quad \text{and} \quad U_1 V_2 + U_2 V_1 = V_1 V_2$$

where X_1 and X_2 denote the homogeneous coordinates on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and U_i, V_i the homogeneous coordinates on each factor $\square^1 = \mathbb{P}^1 \setminus \{1\}$ of \square^2 . The intersection of b with any faces (U_i or $V_i = 0$ for any i 's) is empty, because X_1 and X_2 are different from 0 in X and because U_i is different from V_i in \square^1 . This comment ensures that b is admissible.

Moreover, it implies that $\partial(b) = 0$. So b gives a class

$$[b] \in \mathbb{H}^2(\mathcal{N}_X^\bullet(2)).$$

Let us show that this class is trivial.

Let \bar{b} denote the algebraic closure of b in $\mathbb{A}^1 \times \square^2$. As before, the intersection with $\mathbb{A}^1 \times F$ for any face F of \square^2 is empty, and \bar{b} (after applying the projector \mathcal{Alt}) gives

$$\bar{b} \in \mathcal{N}_{\mathbb{A}^1}^2(2).$$

Write $\partial_{\mathbb{A}^1}$ for the differential in $\mathcal{N}_{\mathbb{A}^1}$. Then $\partial_{\mathbb{A}^1}(\bar{b}) = 0$ and \bar{b} defines a class

$$[b] \in \mathbb{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(2)).$$

As Corollary 2.17 ensures that $\mathbb{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(2)) = \text{CH}^2(\mathbb{A}^1, 2) = 0$, there exists $\bar{c} \in \mathcal{N}_{\mathbb{A}^1}^1(2)$ such that

$$\partial_{\mathbb{A}^1}(\bar{c}) = \bar{b}.$$

Moreover, note that $\bar{b}|_0 = \bar{b}|_1 = \emptyset$. The multiplication map

$$\mathbb{A}^1 \times \square^1 \times \square^2 \xrightarrow{\mu} \mathbb{A}^1 \times \square^2, \quad [x; u_1, u_2, u_3] \longmapsto \left[\frac{x}{1-u_1}; u_2, u_3 \right]$$

is flat. Hence, one can consider the pull-back by μ of the cycle \bar{b} . This pull-back is given explicitly (after reparametrization) by

$$\mu^*(\bar{b}) = \left[x; 1 - \frac{x}{t_1}, t_1, 1 - t_1 \right] \subset \mathbb{A}^1 \times \square^3.$$

This is nothing other than Totaro's cycle [Tot92], already described in [BK94, Blo91], and it gives a well-defined element in $\mathcal{N}_{\mathbb{A}^1}^1(2)$.

Definition 3.2. Let $L_{01} = \text{Li}_2^{cy}$ denote the cycle

$$L_{01} = \left[x; 1 - \frac{x}{t_1}, t_1, 1 - t_1 \right] \subset X \times \square^3$$

in $\mathcal{N}_X^1(2)$.

Remark 3.3. The cycle L_{01} corresponds to the function $x \mapsto \text{Li}_2(x)$, as shown in [BK94] or in [GGL09].

From the parametrized expression above, it follows that:

Lemma 3.4. *The cycle L_{01} satisfies the following properties*

- (1) $\partial(L_{01}) = b$.
- (2) L_{01} extends to \mathbb{A}^1 , i.e. its closure $\overline{L_{01}}$ in $\mathbb{A}^1 \times \square^3$ gives a well-defined element in $\mathcal{N}_{\mathbb{A}^1}^1(2)$.
- (3) $\overline{L_{01}}|_{x=0} = \emptyset$ and $\overline{L_{01}}|_{x=1}$ are well-defined.

3.3. Polylogarithm cycles. We construct the cycles $\text{Li}_n^{cy} = L_{0\dots 01}$ for $n > 2$ inductively. Define Li_1^{cy} to be equal to L_1 .

Lemma 3.5. *For any integer $n \geq 2$, there exist cycles Li_n^{cy} in $\mathcal{N}_X^1(n)$ satisfying*

- (1) $\partial(\text{Li}_n^{cy}) = L_0 \cdot \text{Li}_{n-1}^{cy}$
- (2) Li_n^{cy} extends to \mathbb{A}^1 , i.e. its closure $\overline{\text{Li}_n^{cy}}$ in $\mathbb{A}^1 \times \square^{2n-1}$ yields a well-defined element in $\mathcal{N}_{\mathbb{A}^1}^1(n)$.
- (3) $\overline{\text{Li}_n^{cy}}|_{x=0} = \emptyset$ and $\overline{\text{Li}_n^{cy}}|_{x=1}$ are well-defined.
- (4) Li_n^{cy} is explicitly given as a parametrized cycle by

$$\left[x; 1 - \frac{x}{t_{n-1}}, t_{n-1}, 1 - \frac{t_{n-1}}{t_{n-2}}, t_{n-2}, \dots, 1 - \frac{t_2}{t_1}, t_1, 1 - t_1 \right] \subset X \times \square^{2n-1}$$

Proof. For $n = 2$, we already defined $\text{Li}_2^{cy} = L_{01}$ satisfying the expected properties.

Assume now that we have constructed the cycles Li_k^{cy} for $2 \leq k \leq n$. As before, let b be the product

$$b = L_0 \cdot \text{Li}_{n-1}^{cy} = \left[x; x, 1 - \frac{x}{t_{n-2}}, t_{n-2}, 1 - \frac{t_{n-2}}{t_{n-3}}, t_{n-3}, \dots, 1 - \frac{t_2}{t_1}, t_1, 1 - t_1 \right].$$

and \bar{b} its algebraic closure in $\mathbb{A}^1 \times \square^{2n-2}$.

Computing the differential with the Leibniz rule, we obtain $\partial(b) = -L_0 \cdot L_0 \cdot \text{Li}_{n-2}^{cy} = 0$, and b gives a class in $H^2(\mathcal{N}_X^\bullet(n))$.

Using its expression as a parametrized cycle, we compute the differential of \bar{b} in $\mathcal{N}_{\mathbb{A}^1}^\bullet$

$$(6) \quad \partial_{\mathbb{A}^1}(\bar{b}) = \sum_{i=1}^{2n-2} (\partial_{\mathbb{A}^1, i}^0(\bar{b}) - \partial_{\mathbb{A}^1, i}^\infty(\bar{b})) = 0$$

as many terms are the empty cycle because intersecting with a face $u_i = 0, \infty$ on a factor \square^1 leads to a 1 appearing on another \square^1 while the other terms cancel after applying the projector \mathcal{Alt} .

Just as it does in the case of Li_2^{cy} , \bar{b} gives a class in $H^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(2)) = 0$ by Corollary 2.17, and there exists

$$\overline{\text{Li}_{n-1}^{cy}} = \bar{c} \in \mathcal{N}_{\mathbb{A}^1}^1$$

such that

$$\partial_{\mathbb{A}^1}(\bar{c}) = \bar{b}.$$

As $\overline{\text{Li}_{n-1}^{cy}}|_{x=0} = \emptyset$, $\bar{b}|_{x=0} = \emptyset$ and the element \bar{c} is given by the pull-back by the multiplication

$$\mathbb{A}^1 \times \square^1 \times \square^{2n-2} \xrightarrow{\mu} \mathbb{A}^1 \times \square^{2n-2},$$

given in coordinates by

$$\left[x; u_1, u_2, \dots, u_{2n-1} \right] \longmapsto \left[\frac{x}{1-u_1}; u_2, \dots, u_{2n-1} \right].$$

Reparametrizing the factor \mathbb{A}^1 and the first \square^1 factor, one writes $\bar{c} = \mu^*(\bar{b})$ explicitly as a parametrized cycle

$$\overline{\text{Li}}_n^{cy} = \bar{c} = [x; 1 - \frac{x}{t_{n-1}}, t_{n-1}, 1 - \frac{t_{n-1}}{t_{n-2}}, t_{n-2}, \dots, 1 - \frac{t_2}{t_1}, t_1, 1 - t_1] \subset \mathbb{A}^1 \times \square^{2n-1}.$$

Now, let Li_n^{cy} be the restriction of \bar{c} to $\mathcal{N}_X^1(n)$, i.e. the parametrized cycle

$$\text{Li}_n^{cy} = [x; 1 - \frac{x}{t_{n-1}}, t_{n-1}, 1 - \frac{t_{n-1}}{t_{n-2}}, t_{n-2}, \dots, 1 - \frac{t_2}{t_1}, t_1, 1 - t_1] \subset X \times \square^{2n-1}.$$

The different properties, $d(\text{Li}_n^{cy}) = L_0 \cdot \text{Li}_{n-1}^{cy}$, extension to \mathbb{A}^1 , $\text{Li}_n^{cy}|_{x=0} = \emptyset$, can now be derived easily either for the explicit parametric representation or using the properties of \bar{c} . \square

Remark 3.6. In Equation (6), the fact that $\partial_{\mathbb{A}^1, 1}^0(\bar{b}) = 0$ is related to the induction hypothesis $\overline{\text{Li}}_{n-1}^{cy}|_{x=0} = \emptyset$: in terms of cycles, the above part of the differential is given by

$$\bar{b} \cap (\mathbb{A}^1 \times \{0\} \times \square^{2n-3}) = \text{Li}_{n-1}^{cy}|_{x=0}.$$

The other terms in the differential are related to the equation satisfied by Li_{n-1}^{cy} in $\mathcal{N}_X^1(n-1)$, giving

$$\partial(b) = -L_0 \cdot L_0 \cdot \text{Li}_{n-2}^{cy} = 0.$$

Even if L_0 is not defined in $\mathcal{N}_{\mathbb{A}^1}^\bullet$, the fact that $\overline{\text{Li}}_{n-1}^{cy}|_{x=0} = 0$ ensures that the product really corresponds to an element in $\mathcal{N}_{\mathbb{A}^1}^\bullet$.

Remark 3.7. • The expression of Li_n^{cy} as a parametrized cycle was already given fiberwise in [BK94] and in [GGL09].

- Moreover, Li_n^{cy} corresponds to the function $z \mapsto Li_n^{\mathbb{C}}(z)$ as shown in [BK94].
- The construction is given in full detail for more general cycles in Section 5, and is nothing but a direct application of Theorem 5.8 to the word $0 \cdots 01$ (with $n-1$ zero).

The case of cycles Li_n^{cy} is, however, simple enough to be treated separately as the “good” case.

- It is a general fact that pulling back by the multiplication preserves the property of having empty fiber at $x=0$ as proved in Proposition 5.5.

3.4. Admissibility problem at $x=1$ in weight 3. It seems that the first attempt to define algebraic cycles corresponding not only to polylogarithms but also to multiple polylogarithms was made by Gangl, Goncharov and Levin in [GGL09]. In their work, they succeeded in constructing cycles corresponding to the value $\text{Li}_{n_1, \dots, n_k}(z_1, \dots, z_k)$ for fixed parameters z_i in a number field F , with the conditions $z_i \neq 1$ and $z_i \neq z_j$ for $i \neq j$. However, their cycles are not admissible if one removes the conditions on the z_i . In this section, we consider the first example where this problem appears.

In the example of the Li_n^{cy} cycles, we have kept on multiplying by the cycle L_0 :

$$\partial(\text{Li}_n^{cy}) = L_0 \cdot \text{Li}_{n-1}^{cy}.$$

Below we explain the first example where a multiplication by the cycle L_1 arises and where both geometric and combinatorial difficulties arise.

The cycle L_{01} was defined previously, so was the cycle $L_{001} = \text{Li}_3^{cy}$ by considering the product

$$b = L_0 \cdot L_{01}.$$

Now, we would also like to consider the product

$$b = L_{01} \cdot L_1 \in \mathcal{N}_X^2(3),$$

given as a parametrized cycle by

$$b = [x; 1 - \frac{x}{t_1}, t_1, 1 - t_1, 1 - x] \subset X \times \square^4.$$

From this expression, we see that $\partial(b) = 0$ because $x \in X$ cannot be equal to 1.

Let \bar{b} be the closure of the defining cycle of b in $\mathbb{A}^1 \times \square^4$, i.e.

$$\bar{b} = \left\{ (x, 1 - \frac{x}{t_1}, t_1, 1 - t_1, 1 - x) \text{ such that } x \in \mathbb{A}^1, t_1 \in \mathbb{P}^1 \right\}.$$

Let F be a face of \square^4 , and let u_i denote the coordinates on each factor \square^1 . Then $u_i \neq 1$. If F is contained in a hyperplane defined by $u_2 = \infty$ or $u_3 = \infty$, then, since $u_1 \neq 1$, we have

$$\bar{b} \cap \mathbb{A}^1 \times F = \emptyset.$$

Similarly, the intersection of \bar{b} with a face contained in $u_4 = \infty$ is empty, because $x \in \mathbb{A}^1$ is different from ∞ . This remark reduces the case where F is contained in $u_1 = \infty$ to the case where F is contained in $u_2 = 0$, which gives an empty intersection since $u_3 \neq 1$. By symmetry, the intersection with an F contained in $u_3 = 0$ is also empty.

In order to prove that \bar{b} is admissible and gives an element in $\mathcal{N}_{\mathbb{A}^1}^2$, it remains to check the (co)dimension condition on the three remaining faces: the face where $u_1 = 0$, the face where $u_4 = 0$ and the face $u_1 = u_4 = 0$. The intersection of \bar{b} with the face $u_1 = u_4 = 0$ is empty since $u_2 \neq 1$. The intersection \bar{b} with the face defined by $u_1 = 0$ or $u_4 = 0$ is 1 dimensional, so of codimension 3 in $\mathbb{A}^1 \times F$.

Remark 3.8. Let F_4^0 denote the face of \square^4 defined by $u_4 = 0$. The intersection of b with $X \times F_4^0$ is empty since $x \neq 1$ in $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

From the above discussion, we obtain a well-defined element in $\mathcal{N}_{\mathbb{A}^1}^2(3)$, which we again denote by \bar{b} . Since the intersection with $u_1 = 0$ is killed by the projector Alt , computing the differential in $\mathcal{N}_{\mathbb{A}^1}^\bullet$ gives

$$\partial_{\mathbb{A}^1}(\bar{b}) = -\overline{L_{01}}|_{x=1} \neq 0$$

and \bar{b} does not give a class in $\mathbb{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet)$.

In order to overcome this obstacle, we introduce the constant cycle $L_{01}(1)$ in $\mathcal{N}_X^1(2)$ defined by

$$L_{01}(1) = [x; 1 - \frac{1}{t_1}, t_1, 1 - t_1] \subset X \times \square^3.$$

The cycle $L_{01}(1)$ satisfies

$$\forall a \in X \quad L_{01}(1)|_{x=a} = L_{01}|_{x=1}.$$

and extends to a well-defined cycle in $\mathcal{N}_{\mathbb{A}^1}^1(2)$.

Instead of considering the product $L_{01} \cdot L_1$, we consider the linear combination

$$(7) \quad b = (L_{01} - L_{01}(1)) \cdot L_1 \in \mathcal{N}_X^2(3).$$

As above, we check that b extends to a well-defined element \bar{b} in $\mathcal{N}_{\mathbb{A}^1}^2(3)$. The correction by $-L_{01}(1) \cdot L_1$ ensures that

$$\partial(b) = 0, \quad \partial_{\mathbb{A}^1}(\bar{b}) = 0, \quad \bar{b}|_{x=0} = \emptyset.$$

Considering the previous computation of the pull-back by the multiplication $\mu : \mathbb{A}^1 \times \square^1 \rightarrow \mathbb{A}^1$, we define L_{011} in $\mathcal{N}_X^\bullet(3)$ as the parametrized cycle

$$(8) \quad L_{011} = [x; 1 - \frac{x}{t_2}, 1 - \frac{t_2}{t_1}, t_1, 1 - t_1, 1 - t_2] \\ + [x; 1 - \frac{x}{t_2}, 1 - t_2, 1 - \frac{1}{t_1}, t_1, 1 - t_1] \subset X \times \square^5$$

Since $x \neq 1$ in $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, it is easy to check that L_{011} is admissible on $X \times \square^5$ and gives a well-defined element in $\mathcal{N}_X^\bullet(3)$. An explicit computation also yields

$$(9) \quad \partial(L_{011}) = b = (L_{01} - L_{01}(1)) \cdot L_1.$$

Remark 3.9. However, we point out that

- The cycle L_{011} is defined as parametrized cycle; not using the pull-back by the multiplication which serves here as a support to “guess” the parametrized expression.
- The fiber at $x = 1$ of the algebraic closure $\overline{L_{011}}$ of L_{011} in $\mathbb{A}^1 \times \square^5$ is not admissible.
- This non-admissibility problem was also encountered by Gangl, Goncharov and Levin in [GGL09].

Section 5 will explain how to obtain general cycles admissible at $x = 1$, and the particular example of a cycle \mathcal{L}_{011}^0 related to L_{011} above will be detailed in the next Subsection 3.5.

Remark 3.10. Even if L_{011} is not admissible at $x = 1$, we *could* still go on looking for “good” linear combinations of products in \mathcal{N}_X^2 . In particular, in weight 4, we *could* consider

$$(10) \quad b = L_0 \cdot L_{011} + L_{001} \cdot L_1 - L_{001}(1) \cdot L_1 + L_{01} \cdot L_{01}(1)$$

and observe that

- The terms $L_0 \cdot L_{011} + L_{001} \cdot L_1$ correspond to the expression of the cobracket of the Lyndon word 0011 in the Lie coalgebra graded dual to the free Lie algebra $\text{Lie}(X_0, X_1)$ (cf. Equation (4.24)). This is explained in generality at Section 4.3.
- When computing the differential $\partial(b)$, the term $\partial(L_{01} \cdot L_{01}(1))$ gives

$$\partial(L_{01} \cdot L_{01}(1)) = L_0 \cdot L_1 \cdot L_{01}(1).$$

From the differential of $L_0 \cdot L_{011}$ arises a term in $L_0 \cdot L_{01}(1) \cdot L_1$. Hence the differential of $L_{01} \cdot L_{01}(1)$ cancels with part of the differential of $L_0 \cdot L_{011}$ and can be think as “a propagation” of the correction introduced for L_{011} .

- The term $-L_{001}(1) \cdot L_1$ is similar to the correction $-L_{01}(1) \cdot L_1$ introduced earlier for L_{011} and ensures that $\partial_{\mathbb{A}^1}(\bar{b}) = 0$.

The above remarks motivated the introduction of the cobracket d_{cy} on the Lie coalgebra dual to the Lie algebra representing the action of $\text{Lie}(X_0, X_1)$ by Ihara special derivations. In particular the similarity between Equation (10) above and Equation (21) leads the author to uncover the relation between Ihara’s action and the geometry of above cycles. The goal of section 4 is to understand the global combinatorics underlying the correction terms and to explicit their relation to Ihara’s special derivations.

Note however that this combinatorics can not be applied directly. The reason is that the lack of admissibility of the fiber at $x = 1$ of $\overline{L_{011}}$ “propagates” to higher weight, leading to closed subvarieties which are not admissible anymore. In the next section we explain how to overcome this issue.

3.5. Twisted multiplication and admissibility in weight 3. All previously given examples of cycles are the one obtained by the general construction *with the exception* of L_{011} . Namely

$$L_0 = \mathcal{L}_0^1, \quad L_1 = \mathcal{L}_1^0, \quad L_{01} = \mathcal{L}_{01}^0, \quad \text{Li}_n^{cy} = L_{W_n} = \mathcal{L}_{W_n}^0,$$

with $W_n = \underbrace{0 \cdots 0}_{n-1 \text{ times}} 1$.

In the previous section, the cycle L_{011} was built by considering the product

$$b = (L_{01} - L_{01}(1))L_1 \in \mathcal{N}_X^2(3).$$

However, the fiber at 1 of $\overline{L_{011}}$ is not an admissible cycle. This comes from the lack of admissibility of the fiber at 1 of the cycle $\overline{L_1}$; closure in $\mathbb{A}^1 \times \square^1$ of L_1 .

Building the cycles Li_n^{cy} the lack of admissibility of the fiber at $x = 0$ of $\overline{L_0}$ was balanced by an empty fiber at $x = 0$ of $\overline{\text{Li}_{n-1}^{cy}}$. This phenomenon allowed the induction to go through in the construction of the cycles Li_n^{cy} .

Hence, we want to replace the factor $L_{01} - L_{01}(1)$ in the expression of b by a cycle \mathcal{L}_{01}^1 whose closure $\overline{\mathcal{L}_{01}^1}$ in $\mathbb{A}^1 \times \square^3$ is admissible with an empty fiber at $x = 1$. We define a twisted multiplication map ν

$$\mathbb{A}^1 \times \square^1 \times \square^2 \xrightarrow{\nu} \mathbb{A}^1 \times \square^2, \quad [x; u_1, u_2, u_3] \longmapsto \left[\frac{x-u_1}{1-u_1}; u_2, u_3 \right]$$

which exchanges the role of 0 and 1 with respect to the multiplication μ . As in the case of L_{01} one obtains

$$\overline{\mathcal{L}_{01}^1} = \nu^*(\overline{L_0 \cdot L_1}) \subset \mathbb{A}^1 \times \square^3$$

and \mathcal{L}_{01}^1 is defined as its restriction to $X \times \square^3$. The cycle $\overline{\mathcal{L}_{01}^1}$ can also be described as a parametrized cycle by

$$\overline{\mathcal{L}_{01}^1} = \left[x; \frac{t_1 - x}{t_1 - 1}, t_1, 1 - t_1 \right] \subset \mathbb{A}^1 \times \square^3.$$

The cycle $\overline{\mathcal{L}_{01}^1}$ is a well-defined element of $\mathcal{N}_{\mathbb{A}^1}^1(2)$ and its fiber at x is empty. \mathcal{L}_{01}^1 is an element of $\mathcal{N}_X^1(2)$ and a direct computation shows that

$$\mathcal{L}_{01}^1 = L_{01} - L_{01}(1) + \partial \left(\left[x; s, \frac{s - \frac{t_1 - x}{t_1}}{s - \frac{t_1 - 1}{t_1}}, t_1, 1 - t_1 \right] \right)$$

Then the product

$$b' = \mathcal{L}_{01}^1 L_1 = \left[x; \frac{t_1 - x}{t_1 - 1}, t_1, 1 - t_1; 1 - x \right]$$

can be extended in an admissible cycle $\overline{b'}$ in $\mathbb{A}^1 \times \square^4$ with empty fiber at $x = 0$ and zero under $\partial_{\mathbb{A}^1}$. Its pull-back by the multiplication gives

$$\overline{\mathcal{L}_{011}^0} = \mu^*(\overline{b'}) = \left[x, 1 - \frac{x}{t_2}, \frac{t_1 - t_2}{t_1 - 1}, t_1, 1 - t_1, 1 - t_2 \right] \in \mathcal{N}_{\mathbb{A}^1}^1(3)$$

which has an empty fiber at 0. By proposition 5.7 (or direct computation) it satisfies:

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{011}^0}) = \overline{b'} = \overline{\mathcal{L}_{01}^1 L_1}.$$

By definition \mathcal{L}_{011}^0 is its restriction to $X \times \square^5$ and satisfies

$$\partial(\mathcal{L}_{011}^0) = \mathcal{L}_{01}^1 L_1.$$

This example shows that in order to build a well defined family of cycle whose extension to \mathbb{A}^1 admits a well-defined fiber at $x = 1$ (as elements of $\mathcal{N}_{\{1\}}^1$) and an empty fiber at 0, we need to build at the same time a family of cycle satisfying the same properties but with 0 and 1 exchange.

4. COMBINATORIAL SETTINGS

A *plane* or *planar* tree is a finite tree whose internal vertices are of valency ≥ 3 and on which a cyclic ordering is given on the edges coming out of each vertex. All other vertices are of valency 1; we call them *external vertices*.

A *rooted* tree is a planar tree as above with one distinguished external vertex of valency 1, called its root. In particular a rooted tree has at least one edge. The external vertices which are not the root are called *leaves*.

We will draw trees so that the root vertex is at the top and so that the cyclic order around the vertices is counterclockwise.

The following combinatorial section is organized as follows. Subsection 4.1 reviews some properties of the free Lie algebra on two generators $\text{Lie}(X_0, X_1)$. In particular it presents the basis of Lyndon brackets and a presentation of the brackets as trivalent trees. Subsection 4.2 introduces Ihara's special derivations [Iha90, Iha92] and their actions on $\text{Lie}(X_0, X_1)$. Using the tree representation for $\text{Lie}(X_0, X_1)$, we keep track of the part of Ihara bracket (or Poisson bracket) on $\text{Lie}(X_0, X_1)$ which comes from the special derivations. This is done by taking the semi-direct sum of $\text{Lie}(X_0, X_1)$ by itself.

Section 4.3 dualizes the above situation. Ihara coaction is then written down in terms of the basis dual to the Lyndon brackets. The structure coefficients of this coaction give us the "system of differential equations" after a change of basis.

As a last comment to this introduction to this combinatorial part, we remarks that the usages of trivalent tree is not necessary. However, the presentation with trees is somehow more visual. More importantly it allows to shade a new light on the relation between our construction and the one Gangl, Goncharov and Levin [GGL09].

4.1. Lyndon words and the free Lie algebra $\text{Lie}(X_0, X_1)$. The material developed in this section is detailed in full generality in [Reu93, Reu03] and recalls the basic definitions and some properties of the free Lie algebra on two generators and its relations to trivalent trees and Lyndon words.

4.1.1. Trees and free Lie algebra. Recall that a *Lie algebra* over \mathbb{Q} is a \mathbb{Q} vector space L , equipped with a bilinear mapping $[\cdot, \cdot] : L \otimes L \rightarrow L$, satisfying the two following properties for any x, y, z in L :

$$(11) \quad [x, x] = 0$$

$$\text{(Jacobi)} \quad [[x, y]z] + [[y, z], x] + [[z, x], y] = 0.$$

Remark 4.1. Note that applying the first relation to $[x + y, x + y]$ yields the *anti-symmetry* relation

$$[x, y] = -[y, x].$$

Thus, we may rewrite Jacobi identity as

$$[[x, y], z] = [x, [y, z,]] + [[x, z,], y].$$

Definition 4.2. Given a set S , a free Lie algebra on S over \mathbb{Q} is a Lie algebra L over \mathbb{Q} together with a mapping $i : S \rightarrow L$ with the following *universal property* :

For each Lie algebra K and each mapping $f : S \rightarrow K$, f factors uniquely through L .

In what follows, we will only consider S to be a set with two elements, either $S = \{0, 1\}$ or $S = \{X_0, X_1\}$.

It is usual to consider the free Lie algebra on $\{X_0, X_1\}$ as a subspace of $\mathbb{Q} \langle X_0, X_1 \rangle$ (its enveloping algebra), the space of polynomials in two non commuting variables X_0 and X_1 . Let $\text{Lie}(X_0, X_1)$ denote this free Lie algebra.

In order to show the existence of free Lie algebras, one generally uses a tree representation.

Definition 4.3. Let $\mathbb{Q}[\mathcal{T}^{tri}]$ denote the \mathbb{Q} vector space generated by the set \mathcal{T}^{tri} of rooted, planar, trivalent trees with leaves decorated (i.e. labeled) by 0's and 1's.

For two trees T_1, T_2 in \mathcal{T}^{tri} , define $T_1 \wedge T_2$ to be the tree obtained by joining the root (marked by a circle around the vertex) of T_1 and T_2 and adding a new root:

$$\begin{array}{c} \circ \\ | \\ T_1 \end{array} \wedge \begin{array}{c} \circ \\ | \\ T_2 \end{array} := \begin{array}{c} \circ \\ / \quad \backslash \\ T_1 \quad T_2 \end{array}$$

The internal law \wedge is standard in the the study of binary operations. It can also be seen as a binary version of the B^+ operator studied by Bergbauer and Kreimer [BK05]

\mathcal{T}^{tri} is isomorphic to the free magma on $\{0, 1\}$; a branch \wedge in a tree corresponds to a bracketing in a well-formed expression.

The composition law \wedge extends by bilinearity to $\mathbb{Q}[\mathcal{T}^{tri}]$. Let I_{Jac} denote the ideal of $\mathbb{Q}[\mathcal{T}^{tri}]$ generated by elements of the form

$$T \wedge T \quad \text{and} \quad (T_1 \wedge T_2) \wedge T_3 + (T_2 \wedge T_3) \wedge T_1 + (T_3 \wedge T_1) \wedge T_2.$$

The quotient $\mathbb{Q}[\mathcal{T}^{tri}]/I_{Jac}$ is a Lie algebra with bracket $[\cdot, \cdot]$ given by \wedge ; in fact it is a free Lie algebra on $\{0, 1\}$. Identifying $\{0, 1\}$ to $\{X_0, X_1\}$ by the obvious morphism and using the correspondence $\wedge \leftrightarrow [\cdot, \cdot]$, one obtains

Lemma 4.4. *The quotient $\mathcal{T}^{Lie} = \mathbb{Q}[\mathcal{T}^{tri}]/I_{Jac}$ is isomorphic to $\text{Lie}(X_0, X_1)$.*

For T in \mathcal{T}^{tri} let $[T]$ denote its image in \mathcal{T}^{Lie}

4.1.2. *Lyndon words.* We are interested here in a particular basis of the vector space \mathcal{T}^{Lie} , the one induced by the Lyndon words.

Let S be the set $\{0, 1\}$, and let S^* denote the set of finite words in the letters 0, 1. Let $<$ be the lexicographic order on S^* with $0 < 1$.

Definition 4.5 (Lyndon words). A *Lyndon word* W in S^* is a nonempty word which is smaller than all its nontrivial proper right factors, i.e. $W \neq \emptyset$ and

$$W = UV \text{ with } U, V \neq \emptyset \quad \Rightarrow \quad W < V.$$

Note that 0 and 1 are Lyndon words.

Example 4.6. The Lyndon words of length ≤ 4 are

$$0, 1, 01, 001, 011, 0001, 0011, 0111.$$

They are ordered by the lexicographic order, which gives

$$0 < 0001 < 001 < 0011 < 01 < 011 < 0111 < 1.$$

In order to associate a tree to a Lyndon word, we need the following definition.

Definition 4.7 (Standard factorization). Let W be a word in S^* of length ≥ 2 . The standard factorization of W is the decomposition

$$W = UV \text{ with } \begin{cases} U, V \in S^* \setminus \emptyset \\ \text{and } V \text{ is the smallest non-trivial proper right factor of } W. \end{cases}$$

To any Lyndon word W we associate a tree τ_W in \mathcal{T}^{tri} . If $W = 0$ or $W = 1$, set

$$\tau_0 = \begin{array}{c} \circ \\ | \\ 0 \end{array} \quad \tau_1 = \begin{array}{c} \circ \\ | \\ 1 \end{array}.$$

For a Lyndon word W of length ≥ 2 , let $W = UV$ be its standard factorization and set

$$\tau_W = \tau_U \wedge \tau_V.$$

Let H_L be the set $\{\tau_W\}$ where W runs through the Lyndon words in the letters $0, 1$.

Remark 4.8. The set H_L is endowed with the total order $<$ induced by the ordering of the Lyndon words W given by the lexicographic order on S^* .

Definition 4.9. Let Lyn be the set of the Lyndon words. For any Lyndon word W , let $[\tau_W]$, or simply $[W]$, be the image of τ_W in \mathcal{T}^{Lie} .

We say that τ_W is a *Lyndon tree* and that $[W]$ is a *Lyndon bracket*.

Theorem 4.10 ([Reu93][Theorem 5.1]). *The family $([W])_{W \in Lyn} = ([\tau_W])_{W \in Lyn}$ forms a basis of \mathcal{T}^{Lie} .*

Example 4.11. In length ≤ 3 , the Lyndon trees are given by:

$$\tau_0 = \begin{array}{c} \circ \\ | \\ 0 \end{array}, \quad \tau_1 = \begin{array}{c} \circ \\ | \\ 1 \end{array}, \quad \tau_{01} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad 1 \end{array}, \quad \tau_{001} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad \circ \\ | \quad / \quad \backslash \\ 0 \quad 0 \quad 1 \end{array}, \quad \tau_{011} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad \circ \\ | \quad / \quad \backslash \\ 0 \quad 1 \quad 1 \end{array},$$

and in length 4 by

$$\tau_{0001} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad \circ \\ | \quad / \quad \backslash \\ 0 \quad 0 \quad 0 \quad 1 \end{array}, \quad \tau_{0011} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad \circ \\ | \quad / \quad \backslash \\ 0 \quad 0 \quad 1 \quad 1 \end{array}, \quad \tau_{0111} = \begin{array}{c} \circ \\ / \quad \backslash \\ 0 \quad \circ \\ | \quad / \quad \backslash \\ 0 \quad 1 \quad 1 \quad 1 \end{array}.$$

Moreover, a basis of $\mathcal{T}^{Lie} \wedge \mathcal{T}^{Lie}$ is then given by the family $([U] \wedge [V])$ for U, V Lyndon words such that $U < V$. Writing the Lie bracket in this basis, one obtains the structure coefficients of $\mathcal{T}^{Lie} = \text{Lie}(X_0, X_1)$

Definition 4.12. The structure coefficients $\alpha_{U,V}^W$ of $\mathcal{T}^{Lie} = \text{Lie}(X_0, X_1)$ are given for any Lyndon words W and $U < V$ by the family of relations

$$(12) \quad [U] \wedge [V] \xrightarrow{[\cdot, \cdot]} [[U], [V]] = \sum_{W \in Lyn} \alpha_{U,V}^W [W].$$

The $\alpha_{U,V}^W$ are integers

4.2. Special derivation and Ihara cobracket and coaction. We review below Ihara bracket [Iha90, Iha92] providing $\text{Lie}(X_0, X_1)$ with another Lie structure denoted $\{, \}$. We also explained how Ihara's bracket is represented in terms of trivalent trees.

A derivation of $\text{Lie}(X_0, X_1)$ is a linear endomorphism D of $\text{Lie}(X_0, X_1)$ compatible with the bracket $[\cdot, \cdot]$ in the following way:

$$D([f, g]) = [D(f), g] + [f, D(g)] \quad \forall f, g \in \text{Lie}(X_0, X_1).$$

The commutator of two derivations D and D' , given by

$$[D, D']_{Der} = D \circ D' - D' \circ D,$$

provides a Lie algebra structure on the set $Der(\text{Lie}(X_0, X_1))$ of derivations.

Definition 4.13 (Special derivations). For any f in $\text{Lie}(X_0, X_1)$ we define a derivation D_f by

$$(13) \quad D_f(X_0) = 0, \quad D_f(X_1) = [X_1, f]$$

The map $\text{Lie}(X_0, X_1) \longrightarrow \text{Der}(\text{Lie}(X_0, X_1))$ sending $f \mapsto D_f$ is linear and its image is a subLie algebra of $\text{Der}(\text{Lie}(X_0, X_1))$ such that for any f and g in $\text{Lie}(X_0, X_1)$ one has

$$[D_f, D_g]_{\text{Der}} = D_h \quad \text{with } h = [f, g] + D_f(g) - D_g(f).$$

Definition 4.14 (Ihara bracket, [Iha90, Iha92]). The Lie bracket $\{ , \}$ on $\text{Lie}(X_0, X_1)$ is defined by

$$\{f, g\} = [f, g] + D_f(g) - D_g(f).$$

Remark 4.15. Note that even if Ihara bracket is defined consistently on $\text{Lie}(X_0, X_1)$, the derivation $D_{X_1} = 0$. Moreover D_{X_0} is the adjoint derivation

$$D_{X_0}(g) = \text{ad}_{X_0}(g) = [g, X_0];$$

In particular, $\{X_0, g\} = [D_{X_0}, D_g]_{\text{Der}} = 0$.

As we have just seen $\text{Lie}(X_0, X_1)$ acts on itself by the non-inner derivations D_f ; that is D_f is not an adjoint derivation (with the exception of D_{X_0} and D_{X_1}). Ihara bracket controls the relation between the usual bracket, the derivation bracket and the action. However it loses the description of the action. Using the trees representation for $\text{Lie}(X_0, X_1)$ we will keep track of the action. We begin by adding a root decoration to the trivalent trees. Trees corresponding to an element in the Lie algebra $\text{Lie}(X_0, X_1)$ have a root decorated by a generic parameter x . Trees corresponding to derivation have a root decorated by 1 (as reminder that they act on X_1).

More formally, let $\mathbb{Q}[\mathcal{T}_x^{\text{tri}}]$ (resp. $\mathbb{Q}[\mathcal{T}_1^{\text{tri}}]$) denote the \mathbb{Q} vector space generated by the set $\mathcal{T}_x^{\text{tri}}$ (resp. $\mathcal{T}_1^{\text{tri}}$) of rooted, planar, trivalent trees with leaves decorated (i.e. labeled) by 0's and 1's and a root decorated by x (resp. 1).

The internal law \blacktriangleleft is defined as previously on each set $\mathcal{T}_x^{\text{tri}}$ and $\mathcal{T}_1^{\text{tri}}$ that is it joins the two trees and add a new root redecorated by x and 1 respectively. It is then extended to the disjoint union

$$\{0\} \cup \mathcal{T}_1^{\text{tri}} \cup \mathcal{T}_x^{\text{tri}}$$

by 0 when ever the two trees do not have the same root decoration and extended by bilinearity to

$$\mathbb{Q}[\mathcal{T}_x^{\text{tri}}] \oplus \mathbb{Q}[\mathcal{T}_1^{\text{tri}}].$$

The ideal I_{Jac} is defined in the obvious way separately on $\mathbb{Q}[\mathcal{T}_x^{\text{tri}}]$ and on $\mathbb{Q}[\mathcal{T}_1^{\text{tri}}]$ and we defined the Lie algebras:

$$\mathcal{T}_x^{\text{Lie}} = \mathbb{Q}[\mathcal{T}_x^{\text{tri}}]/I_{\text{Jac}} \quad \text{and} \quad \mathcal{T}_1^{\text{Lie}} = \mathbb{Q}[\mathcal{T}_1^{\text{tri}}]/I_{\text{Jac}}$$

We identify $\mathcal{T}_x^{\text{Lie}}$ with the Lie algebra $\text{Lie } X_0, X_1$. Again, we will write $[T(a)]$ the image in $\mathcal{T}_a^{\text{Lie}}$ of tree T in $\mathcal{T}_a^{\text{tri}}$ for $a \in \{1, x\}$.

Similarly a generic element of \mathcal{T}^{Lie} is denoted by $[F]$ while its image in $\mathcal{T}_x^{\text{Lie}}$ (resp. $\mathcal{T}_1^{\text{Lie}}$); that is with its root decorated by x (resp. 1); is denoted by $[F(x)]$ (resp. $[F(1)]$).

To any element $[F(1)]$ in $\mathcal{T}_1^{\text{Lie}}$ we associate a derivation on the direct sum

$$\mathcal{T}_{1;x}^{\text{Lie}} = \mathcal{T}_1^{\text{Lie}} \oplus \mathcal{T}_x^{\text{Lie}}$$

through

$$D_{F(1)}\left(\begin{bmatrix} 1 & \circ \\ & \bullet \\ & 0 \end{bmatrix}\right) = D_{F(1)}\left(\begin{bmatrix} x & \circ \\ & \bullet \\ & 0 \end{bmatrix}\right) = 0, \quad D_{F(1)}\left(\begin{bmatrix} 1 & \circ \\ & \bullet \\ & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \circ \\ & \bullet \\ & 1 \end{bmatrix} \blacktriangleleft [F(1)]$$

$$\text{and } D_{F(1)}\left(\begin{bmatrix} x & \circ \\ & \bullet \\ & 1 \end{bmatrix}\right) = \begin{bmatrix} x & \circ \\ & \bullet \\ & 1 \end{bmatrix} \blacktriangleleft [F(x)] = \begin{bmatrix} x & \circ \\ & \bullet \\ 1 & \text{---} & F \end{bmatrix}$$

where the last equality means that both trees are joined by their root and a new root is added with root decoration x ; the operation being linear in F . An element $[F(x)]$ of \mathcal{T}_x^{Lie} acts on $\mathcal{T}_{1;x}^{Lie}$ by the derivation $D_{F(x)} = 0$.

Now, we give $\mathcal{T}_{1;x}^{Lie}$ the Lie algebra structure of the semi-direct sum (see [GOV97]) of $\mathcal{T}_x^{Lie} \simeq \text{Lie}(X_0, X_1)$ by $\mathcal{T}_1^{Lie} \simeq \text{Lie}(X_0, X_1)$ acting by the above derivations. More precisely, We define on $\mathcal{T}_{1;x}^{Lie}$ the bilinear map

$$\{[F(a)], [G(b)]\} = [F(a)] \wedge [G(b)] + D_{F(a)}([G(b)]) - D_{G(b)}([F(a)])$$

where $F(a)$ and $G(b)$ denote two generic elements of $\mathcal{T}_{1;x}^{Lie}$; that is a and b are in $\{1, x\}$.

Lemma 4.16. *The direct sum $\mathcal{T}_{1;x}^{Lie} = \mathcal{T}_1^{Lie} \oplus \mathcal{T}_x^{Lie}$ endows with $\{, \}$ is a Lie algebra.*

Proof. The proof takes place in $\mathcal{T}_{1;x}^{Lie}$ and its subspaces \mathcal{T}_x^{Lie} and \mathcal{T}_1^{Lie} . Hence we simply write $F(a)$ to denote the element $[F(a)]$ in $\mathcal{T}_{1;x}^{Lie}$. It is enough to check that $\{, \}$ satisfies Jacobi identity. The definition of

$$\{F(a), G(b)\}$$

insures that $\{, \}$ is the usual bracket \wedge on \mathcal{T}_x^{Lie} , when $a = b = x$. When $a = b = 1$, then $\{, \}$ is the Ihara bracket on $\mathcal{T}_1^{Lie} \simeq \text{Lie}(X_0, X_1)$. Hence Jacobi identity holds when the three terms are all in \mathcal{T}_x^{Lie} or all in \mathcal{T}_1^{Lie} . When $a = x$ and $b = 1$, the bracket $\{F(a), G(b)\}$ reduces to $-D_{G(1)}(F(x))$. We have now to prove that

$$\{F(a), \{G(b), H(c)\}\} + \{G(b), \{H(c), F(a)\}\} + \{H(c), \{F(a), G(b)\}\} = 0$$

when two out of the three elements $a, b, c \in \{1, x\}$ are equal and the other is different. We can assume that $a = b$. When $a = b = x$ Jacobi identity reduces to the fact that $H(1)$ act as a derivation on $\{F(x), G(x)\}$. When $a = b = 1$, Jacobi identity reduces to the definition of the bracket of two derivations. \square

Note that a basis of $\mathcal{T}_{1;x}^{Lie} \wedge \mathcal{T}_{1;x}^{Lie}$ is given by the union of the following families

$$\begin{aligned} & [\tau_U(x)] \wedge [\tau_V(x)] \text{ for any Lyndon word } U < V \\ & [\tau_U(x)] \wedge [\tau_V(1)] \text{ for any Lyndon word } U \neq V \\ & [\tau_U(1)] \wedge [\tau_V(1)] \text{ for any Lyndon word } U < V. \end{aligned}$$

Definition 4.17. The structure coefficients $\alpha_{U,V}^W$, $\beta_{U,V}^W$ and $\gamma_{U,V}^W$ of $\mathcal{T}_{1;x}^{Lie}$ are given for any Lyndon words W by the family of relations

$$(14) \quad \begin{aligned} \{[\tau_U(x)], [\tau_V(x)]\} &= \sum_{W \in \text{Lyn}} \alpha_{U,V}^W [\tau_W(x)]. \quad \text{for any Lyndon word } U < V \\ \{[\tau_U(x)], [\tau_V(1)]\} &= \sum_{W \in \text{Lyn}} \beta_{U,V}^W [\tau_W(x)] \quad \text{for any Lyndon word } U \neq V \\ \{[\tau_U(1)], [\tau_V(1)]\} &= \sum_{W \in \text{Lyn}} \gamma_{U,V}^W [\tau_W(1)] \quad \text{for any Lyndon word } U < V. \end{aligned}$$

Note that the $\alpha_{U,V}^W$ are the α 's of Equation (12) because $\{, \}$ restricted to \mathcal{T}_x^{Lie} is the usual Lie bracket. All coefficients above are integers.

We will need later for our geometric application the following property of the above coefficients.

Lemma 4.18. *Let W be a Lyndon word of length greater than or equal to 2. Then the following holds for any Lyndon words U, V :*

- $\beta_{0,V}^W = 0$,
- $\beta_{V,0}^W = \alpha_{0,V}^W$

- $\beta_{U,1} = 0$,
- $\beta_{1,U}^W = \alpha_{U,1}^W$.
- $\gamma_{U,V}^W = \alpha_{U,V}^W + \beta_{U,V}^W - \beta_{V,U}^W$.

In particular, $\beta_{0,0}^W = \beta_{1,1}^W = 0$. We also have

$$\alpha_{U,V}^\varepsilon = \beta_{U,V}^\varepsilon = \gamma_{U,V}^\varepsilon = 0$$

for $\varepsilon \in \{0, 1\}$.

Proof. The coefficient $\beta_{0,V}^W$ arises by decomposing on the basis the bracket

$$\{[\tau_0(x)], [\tau_V(1)]\} = -D_{\tau_V(1)}\left(\left[\begin{array}{c} x \\ \circ \\ \vdots \\ 0 \end{array} \right]\right) = 0.$$

Hence $\beta_{0,V}^W = 0$. Similarly, $\beta_{1,U}^W$ arises from

$$-D_{\tau_U(1)}\left(\left[\begin{array}{c} x \\ \circ \\ \vdots \\ 1 \end{array} \right]\right) = -\left[\begin{array}{c} x \\ \circ \\ \vdots \\ 1 \end{array} \right] \blacktriangleleft [\tau_U(x)] = [\tau_U(x)] \blacktriangleleft \left[\begin{array}{c} x \\ \circ \\ \vdots \\ 1 \end{array} \right]$$

which shows that $\beta_{1,U}^W = \alpha_{U,1}^W$. In the same way, $\beta_{U,1} = 0$ arises from $-D_{\tau_1(1)}([\tau_U(x)]) = 0$ because $D_{\tau_1(1)} = D_{X_1} = 0$ after identifying \mathcal{T}_1^{Lie} and $\text{Lie}(X_0, X_1)$. The same identification show that

$$-D_{\tau_0} = -ad_{\tau_0} : [F] \mapsto -[F] \blacktriangleleft [\tau_0]$$

which proves $\beta_{V,0}^W = \alpha_{0,V}^W$. The relation

$$\gamma_{U,V}^W = \alpha_{U,V}^W + \beta_{U,V}^W - \beta_{V,U}^W$$

comes from the relation between the commutator of two derivations, the action and the usual bracket. It is given in terms of the lie algebra $\text{Lie}(X_0, X_1)$ by:

$$\{D_f, D_g\} = D_h \quad \text{with } h = [f, g] + D_f(g) - D_g(f)$$

for any f, g in $\text{Lie}(X_0, X_1)$. □

4.3. Trivalent trees and duality. Here, we dualize Equation (14) by considering the vector space graded dual to $\mathcal{T}_{1;x}^{Lie}$. The dual d_{cy} of the bracket $\{, \}$ coming from the semi-direct sum is a cobracket; that is essentially a differential where the relation $d^2 = 0$ is dual to the Jacobi identity. Hence the coefficients α 's and β 's from Equation (14) give us a “differential system” (cf. Equation (18)) which will leads us, after a change of basis, to the differential system satisfied by our algebraic cycle.

We begin by making explicit the construction of the graded dual of $\mathcal{T}_{1;x}^{Lie}$.

Remark 4.19. One does not really need to explicitly construct the dual vector space of $\mathcal{T}_{1;x}^{Lie}$ and a basis dual to one above. However, this explicit construction allows us to work with concrete objects. Moreover, using this explicit construction, we can relate our work to the combinatorial construction of Gangl Goncharov and Levin in [GGL09](see Remark 4.29).

The construction of the vector spaces dual to \mathcal{T}_x^{Lie} and \mathcal{T}_1^{Lie} are parallel. Hence a will denote an element of $\{1, x\}$.

Let $\mathbb{Q}[\mathcal{T}_a^{tri, <}]$ be the quotient of $\mathbb{Q}[\mathcal{T}_a^{tri}]$ by the ideal (for \blacktriangleleft) I_s generated by

$$T_1 \blacktriangleleft T_2 + T_2 \blacktriangleleft T_1.$$

Let T be a tree in \mathcal{T}_a^{tri} with subtree $T_1 \blacktriangleleft T_2$, and let T' be the tree T in which $T_1 \blacktriangleleft T_2$ has been replaced by $T_2 \blacktriangleleft T_1$. The following relation holds in $\mathbb{Q}[\mathcal{T}_a^{tri, <}]$:

$$T = -T'.$$

The total order on H_L (Remark 4.8) induces a total order $<$ on \mathcal{T}_a^{tri} . Let $\mathfrak{B}_a^<$ be the set of trees T in \mathcal{T}_a^{tri} such that

$$T' = T_1 \wedge T_2 \text{ is subtree of } T \Rightarrow T_1 < T_2.$$

Writing $T(a) \in \mathfrak{B}_a^<$ also for the image of a tree $T(a)$ in $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$, we see that

Lemma 4.20. *The set $\mathfrak{B}_a^<$ induces a basis of $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$, also denoted by $\mathfrak{B}_a^<$.*

From now on we identify $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$ with its dual, via the basis $\mathfrak{B}_a^<$.

Let $I_{Jac,a}^<$ denote the image of the ideal I_{Jac} in $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$. The Lie algebra \mathcal{T}_a^{Lie} is then isomorphic to the quotient $\mathbb{Q}[\mathcal{T}_a^{tri,<}] / I_{Jac,a}^<$ and, using the identification between $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$ and its graded dual (the grading coming from the number of leaves), one can identify the graded dual of \mathcal{T}_a^{Lie} with a subspace of $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$.

Definition 4.21. Let $\mathcal{T}_a^{coL} \subset \mathbb{Q}[\mathcal{T}_a^{tri,<}]$ denote the subvector space of $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$ graded dual of $\mathcal{T}_a^{Lie} \simeq \text{Lie}(X_0, X_1)$.

Let $(T_{W^*}(a))_{W \in L_{yn}}$ in \mathcal{T}_a^{coL} denote the dual basis of the basis $([W])_{W \in L_{yn}}$ of the free Lie algebra \mathcal{T}_a^{Lie} .

The $T_{W^*}(a)$ are linear combinations of trees in $\mathfrak{B}_a^<$. Observe that any Lyndon tree $\tau_W(a)$ is in $\mathfrak{B}_a^<$ and that by definition its coefficient in T_{W^*} is 1.

Example 4.22. Up to length ≤ 3 , $T_{W^*}(a) = \tau_W(a)$, i.e.

$$T_{0^*}(a) = \begin{array}{c} a \odot \\ | \\ 0 \end{array}, \quad T_{1^*} = \begin{array}{c} a \odot \\ | \\ 1 \end{array}, \quad T_{01^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 1 \end{array}, \quad T_{001^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 0 \\ | \quad | \\ 0 \quad 1 \end{array}, \quad T_{011^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 1 \\ | \quad | \\ 0 \quad 1 \end{array}.$$

In length 4, the first linear combination appears:

$$T_{0001^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 0 \\ | \quad | \\ 0 \quad 1 \end{array}, \quad T_{0011^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 0 \\ | \quad | \\ 0 \quad 1 \end{array} + \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 0 \\ | \quad | \\ 1 \quad 1 \end{array}, \quad T_{0111^*} = \begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 1 \\ | \quad | \\ 1 \quad 1 \end{array}.$$

Remark 4.23. Using $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$ instead of \mathcal{T}_a^{tri} to built explicitly the graded dual of \mathcal{T}_a^{Lie} , makes it possible so shrink the size of the linear combinations involved in the basis dual to $([\tau_W])_{W \in L_{yn}}$. As an example, the linear combination of trees corresponding to τ_{01} in the dual basis working in \mathcal{T}_a^{tri} should be

$$\frac{1}{2} \left(\begin{array}{c} a \odot \\ / \quad \backslash \\ 0 \quad 1 \end{array} - \begin{array}{c} a \odot \\ / \quad \backslash \\ 1 \quad 0 \end{array} \right)$$

As the Lie bracket on \mathcal{T}_a^{Lie} (for the free Lie algebra structure) is induced by $\wedge : \mathcal{T}_a^{tri}[a] \wedge \mathcal{T}_a^{tri}[a] \rightarrow \mathcal{T}_a^{tri}[a]$; it is also induced by \wedge on $\mathbb{Q}[\mathcal{T}_a^{tri,<}]$. By duality, one obtains a cobracket

$$d_{Lie} : \mathcal{T}_a^{coL} \longrightarrow \mathcal{T}_a^{coL} \wedge \mathcal{T}_a^{coL}$$

dual to the Lie bracket and induced by the map $\mathbb{Q}[\mathcal{T}_a^{tri}][a] \longrightarrow \mathbb{Q}[\mathcal{T}_a^{tri}][a] \wedge \mathbb{Q}[\mathcal{T}_a^{tri}][a]$ also denoted by d_{Lie} :

$$(15) \quad d_{Lie} : \begin{array}{c} a \odot \\ / \quad \backslash \\ T_1 \quad T_2 \end{array} \longmapsto \begin{array}{c} a \odot \\ | \\ T_1 \end{array} \wedge \begin{array}{c} a \odot \\ | \\ T_2 \end{array}.$$

The property that $d_{Lie} \circ d_{Lie} = 0$ on \mathcal{T}_a^{coL} is dual to the Jacobi identity on \mathcal{T}_a^{Lie} .

Example 4.24. As example in weight 4 one has

$$d_{Lie}(T_{0011^*}(a)) = T_{0^*}(a) \wedge T_{011^*}(a) + T_{001^*}(a) \wedge T_{1^*}(a).$$

Proposition 4.25. *By duality, the following hold in \mathcal{T}_a^{coL} :*

- $T_{0^*}(a) = \begin{smallmatrix} a \circlearrowleft \\ \circ \\ 0 \end{smallmatrix}, T_{1^*}(a) = \begin{smallmatrix} a \circlearrowleft \\ \circ \\ 1 \end{smallmatrix};$
- $d_{Lie}(T_{0^*}(a)) = d_{Lie}(T_{1^*}(a)) = 0;$
- *for all Lyndon words W of length ≥ 2 ,*

$$(16) \quad d_{Lie}(T_{W^*}(a)) = \sum_{\substack{U < V \\ U, V \in L_{yn}}} \alpha_{U, V}^W T_{U^*}(a) \wedge T_{V^*}(a)$$

where the $\alpha_{U, V}^W$ are the structure coefficients of $\mathcal{T}^{Lie} \simeq \text{Lie}(X_0, X_1)$ defined by Equation (12).

Moreover one can construct the linear combinations $T_{W^*}(a)$ inductively by

$$(17) \quad T_{W^*}(a) = \sum_{\substack{U < V \\ U, V \in L_{yn}}} \alpha_{U, V}^W T_{U^*}(a) \lrcorner T_{V^*}(a)$$

for W of length greater than or equal to 2. Here \lrcorner denotes the bilinear map $\mathcal{T}_a^{coL} \otimes \mathcal{T}_a^{coL} \rightarrow \mathcal{T}_a^{coL}$ induced by \lrcorner .

Note that between Equation (12) and Equation (16) the summation is ‘‘reversed’’ due to the duality : Equation (16) computes the transpose of the matrix representation of the Lie bracket given in basis by Equation (12). Equation (17) provides an inductive constructions of trees $T_{W^*}(a)$.

Having built the graded dual vector spaces \mathcal{T}_1^{coL} and \mathcal{T}_x^{coL} above, we define:

Definition 4.26. Let $\mathcal{T}_{1;x}^{coL}$ be the graded dual of $\mathcal{T}_{1;x}^{Lie}$; as a vector space it is the direct sum

$$\mathcal{T}_{1;x}^{coL} = \mathcal{T}_1^{coL} \oplus \mathcal{T}_x^{coL}.$$

A basis of $\mathcal{T}_{1;x}^{coL}$ is given by the union of the two families:

$$\begin{aligned} &T_{W^*}(x) \text{ for any Lyndon word } W \\ &T_{W^*}(1) \text{ for any Lyndon word } W. \end{aligned}$$

Similarly a basis of $\mathcal{T}_{1;x}^{coL} \wedge \mathcal{T}_{1;x}^{coL}$ is given by the union of the following families:

$$\begin{aligned} &T_{U^*}(x) \wedge T_{V^*}(x) \text{ for any Lyndon word } U < V \\ &T_{U^*}(x) \wedge T_{V^*}(1) \text{ for any Lyndon word } U \neq V \\ &T_{U^*}(1) \wedge T_{V^*}(1) \text{ for any Lyndon word } U < V. \end{aligned}$$

Proposition 4.27. *The bracket $\{, \}$ on $\mathcal{T}_{1;x}^{Lie}$ induces a cobracket on $\mathcal{T}_{1;x}^{coL}$*

$$d_{cy} : \mathcal{T}_{1;x}^{coL} \rightarrow \mathcal{T}_{1;x}^{coL} \wedge \mathcal{T}_{1;x}^{coL}$$

which is written in terms of the above basis as

$$(18) \quad d_{cy}(T_{W^*}(x)) = \sum_{U < V} \alpha_{U, V}^W T_{U^*}(x) \wedge T_{V^*}(x) + \sum_{U, V} \beta_{U, V}^W T_{U^*}(x) \wedge T_{V^*}(1)$$

and

$$(19) \quad d_{cy}(T_{W^*}(1)) = \sum_{U < V} \gamma_{U, V}^W T_{U^*}(1) \wedge T_{V^*}(1)$$

where U and V are Lyndon words, the $\alpha_{U, V}^W$, $\beta_{U, V}^W$ and $\gamma_{U, V}^W$ are those defined in Equation (14).

In particular $d_{cy}^2 = 0$.

Proof. The proposition follows by duality. In particular equations (18) and (19) are just the transpose of (14). \square

We give below some examples of computation of $d_{cy}(T_{W^*}(x))$:

Example 4.28.

- For Lyndon words of length 1 and 2, one has

$$d_{cy}(T_0(x)) = d_{cy}(T_1(x)) = 0, \quad \text{and } d_{cy}(T_{01}(x)) = T_{0^*}(x) \wedge T_{1^*}(x) + T_{1^*}(x) \wedge T_{0^*}(1).$$

- For Lyndon words of weight 3, one has

$$d_{cy}(T_{001}(x)) = T_{0^*}(x) \wedge T_{01^*}(x) + T_{01^*}(x) \wedge T_{0^*}(1)$$

$$(20) \quad d_{cy}(T_{011}(x)) = T_{01^*}(x) \wedge T_{1^*}(x) + T_{1^*}(x) \wedge T_{01^*}(1)$$

- In weight 4, one finds

$$(21) \quad d_{cy}(T_{0011^*}(x)) = T_{0^*}(x) \wedge T_{011^*}(x) + T_{011^*}(x) \wedge T_{0^*}(1) \\ + T_{001^*}(x) \wedge T_{1^*}(x) + T_{1^*}(x) \wedge T_{001^*}(1) + T_{01^*} \wedge T_{01^*}(1)$$

Equations (20) and (21) should in particular be compared with equations (9) and (10).

Note that the only difference between (21) and (10) is the terms with a factor $T_{0^*}(1)$. This difference also appears when comparing $d_{cy}(T_{001^*})$ above to $\partial(L_{001}) = \partial(\text{Li}_3^{cy})$ presented at Lemma 3.5. One could simply kill these terms in the expression $d_{cy}(T_{W^*})$ by taking the appropriate quotient. However, Lemma 4.18 insures that terms of the form $T_{W^*}(x) \wedge T_{0^*}(1)$ can always be regroup with a unique term $T_{0^*}(x) \wedge T_{W^*}(x)$ giving a term in

$$(T_{0^*}(x) - T_{0^*}(1)) \wedge T_{W^*}(x).$$

Hence, terms with a factor $T_{0^*}(1)$ do not really change the combinatorial situation.

However, as presented in section 3.4, the geometric situation do not fit exactly the above combinatorial setting which need to be rewritten in a suitable way. Before doing so, we would like to comment the relation between Equation (18) and the combinatorial approach of Gangl Goncharov and Levin in [GGL09].

Remark 4.29. In [GGL09], Gangl Goncharov and Levin built parametrized algebraic cycles in $\mathcal{N}_{\text{Spec}(\mathbb{Q})}^1$ out of linear combinations of trivalent trees and a *forest cycling map*. They worked in cdga setting where:

- the product is induced by the disjoint union of trees (hence the name “forest”);
- the graded commutativity is induced by an ordering of the edges of the trees and of the forests and an alternating relation;
- the differential d_{ggl} consists in the appropriate alternating sums of the following operation : (a) contracting internal edges of trees and (b) contracting and splitting root and external edges as pictured below (figures 1 and 2 :

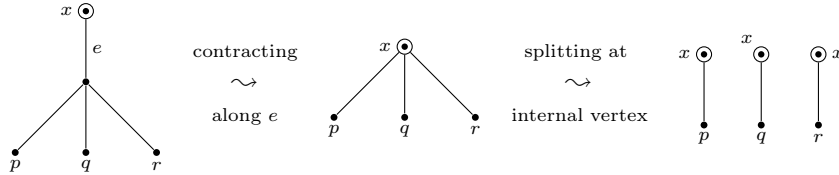


FIGURE 1. Contracting the root

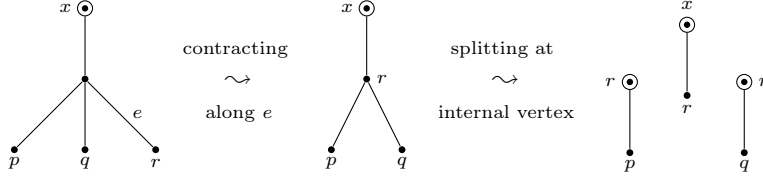


FIGURE 2. Contracting a leaf

Their linear combination of trees differ from ours by their decorations but also by their structures. More precisely, with our decoration (x decorates the root and 0 and 1 the leaves) their linear combinations of trees are dual to the standard basis of the universal enveloping algebra of $\text{Lie}(X_0, X_1)$ presented as a quotient of $\mathbb{Q}[\mathcal{T}_x^{tri}]$.

Moreover, the authors of [GGL09] are not very precise about the case where the leaves or the root are decorated by 0. Gangl Goncharov and Levin *forest cycling map* sends any tree with root decorated by 0 to the empty cycle. Hence, taking the quotient by the ideal they generate, we kill these trees with root decorated by 0.

We can endow the linear combinations of trees T_{W^*} with a canonical ordering of each tree (root edge is the first edge, then one goes down and left). Then one computes $d_{ggl}(T_{W^*}(x))$ in [GGL09] setting and observes that it satisfies exactly Equation (18) after killing trees with 0 as root decoration :

$$(22) \quad d_{ggl}(T_{W^*}(x)) = \sum_{U < V} \alpha_{U,V}^W T_{U^*}(x) \cdot T_{V^*}(x) + \sum_{U,V} \beta_{U,V}^W T_{U^*}(x) \cdot T_{V^*}(1)$$

We will not give a proof of the above claim which involves long computations on the decompositions of Lie brackets into the Lyndon bracket basis. However, we explain informally below why this is true:

- Because of the ordering of edges and its alternating relation, working in the vector space

$$\mathcal{T}_{1;x}^{coL} = \mathcal{T}_1^{coL} \oplus \mathcal{T}_x^{coL} \subset \mathbb{Q}[\mathcal{T}_1^{tri,<}] \oplus \mathbb{Q}[\mathcal{T}_x^{tri,<}]$$

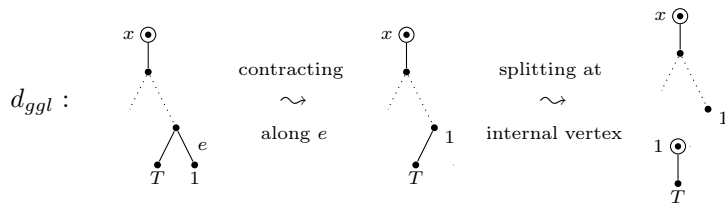
is possible. The graded commutativity in Gangl Goncharov Levin construction corresponds to the exterior product $\mathcal{T}_{1;x}^{coL} \wedge \mathcal{T}_{1;x}^{coL}$.

- Computing $d_{ggl}(T_{W^*}(x))$ the internal edges do not contribute. To see this, one proceeds by induction and uses the fact that $d_{Lie}^2 = 0$.
- The part of d_{ggl} corresponding to the root edge is nothing but d_{Lie} , hence the part

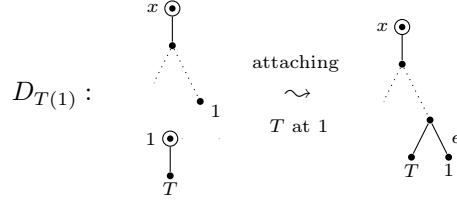
$$\sum_{U < V} \alpha_{U,V}^W T_{U^*}(x) \cdot T_{V^*}(x)$$

in $d_{ggl}(T_{W^*}(x))$.

- The part of d_{ggl} corresponding to external edges with 0 as decoration gives a factor with a tree having 0 as root decoration. Hence this part is 0 because we have killed these trees.
- The part of d_{ggl} corresponding to an external edge e with 1 as decoration (given by “contract and split”) is dual to the action by special derivations (given by “attach at 1”) as draw below:



where T denotes a subtree; and



4.4. A combinatorial statement. Because of the admissibility issue at 1 for algebraic cycles explained at Section 3.4, Equation (18) can not be used directly. The solution found at Section 3.5 leads us to express (18) using products as $T_{U^*}(x) \wedge (T_{V^*}(x) - T_{V^*}(1))$ instead of $T_{U^*}(x) \wedge T_{V^*}(1)$.

Definition 4.30. For any Lyndon word W , let $T_{W^*}^1$ be the difference

$$T_{W^*}^1 = (T_{W^*}(x) - T_{W^*}(1)).$$

In order to use a consistent notation we define $T_{W^*}^0 = T_{W^*}(x)$ which can be thought as $T_{W^*}^0 = T_{W^*}(x) - T_{W^*}(0)$ where $T_{W^*}(0) = 0$.

Because

$$T_{W^*}(1) = T_{W^*}^1 - T_{W^*}^0,$$

a basis of $\mathcal{T}_{1;x}^{coL}$ is given by the union of the two families:

$$\begin{aligned} & T_{W^*}^0 \text{ for any Lyndon word } W \\ & T_{W^*}^1 \text{ for any Lyndon word } W. \end{aligned}$$

Hence a basis of $\mathcal{T}_{1;x}^{coL} \wedge \mathcal{T}_{1;x}^{coL}$ is given by the union of the following families:

$$\begin{aligned} & T_{U^*}^0 \wedge T_{V^*}^0 \text{ for any Lyndon word } U < V \\ & T_{U^*}^1 \wedge T_{V^*}^0 \text{ for any Lyndon word } U \neq V \\ & T_{U^*}^1 \wedge T_{V^*}^1 \text{ for any Lyndon word } U < V. \end{aligned}$$

We can now rewrite the Equation (18) in terms of the above basis. We also write the cobracket $d_{cy}(T_{W^*}^1)$ in terms of this basis.

Definition 4.31. Let W be a Lyndon word. We define coefficients $a_{U,V}^W$, $a'_{U,V}$ for any Lyndon words $U < V$ and coefficient $b_{U,V}^W$ and $b'_{U,V}$ for any Lyndon words U, V through:

$$(ED-T^0) \quad d_{cy}(T_{W^*}^0) = \sum_{U < V} a_{U,V}^W T_{U^*}^0 \wedge T_{V^*}^0 + \sum_{U,V} b_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0,$$

and

$$(ED-T^1) \quad d_{cy}(T_{W^*}^1) = \sum_{U < V} a'_{U,V} T_{U^*}^1 \wedge T_{V^*}^1 + \sum_{U,V} b'_{U,V} T_{U^*}^1 \wedge T_{V^*}^0$$

Note that no term of the form $T_{U^*}^1 T_{V^*}^1$ appears in Equation (ED- T^0) because there is no term of the form $T_{U^*}(1) \wedge T_{V^*}(1)$ appearing in Equation (18). More precisely, one expresses the a 's, b 's, a' 's and b' 's in terms of the α 's and β 's of Equation (18).

Lemma 4.32. For any Lyndon words W , the following holds

$$(23) \quad \begin{aligned} a_{U,V}^W &= \alpha_{U,V}^W + \beta_{U,V}^W - \beta_{V,U}^W & \text{for } U < V \\ b_{U,V}^W &= \beta_{V,U}^W & \text{for any } U, V \end{aligned}$$

and

$$(24) \quad \begin{aligned} a_{U,V}^W &= -a_{U,V}^W && \text{for } U < V, \\ b_{U,V}^W &= a_{U,V}^W + b_{U,V}^W && \text{for } U < V, \\ b_{V,U}^W &= -a_{U,V}^W + b_{V,U}^W && \text{for } U < V, \\ b_{U,U}^W &= b_{U,U}^W && \text{for any } U. \end{aligned}$$

Note that coefficients a 's and b 's are defined in terms of a 's and b 's (from Equation (ED- T^0)) and not in term of coefficients α 's and β 's (from Equation (18)). It makes in particular easier the proof of the above Lemma and the comparison between $d_{cy}(T_{W^*}^0)$ and $d_{cy}(T_{W^*}^1)$:

$$(25) \quad d_{cy}(T_{W^*}^0 - T_{W^*}^1) = \sum_{U < V} a_{U,V}^W (T_{U^*}^0 \wedge T_{V^*}^0 + T_{U^*}^1 \wedge T_{V^*}^1 + T_{V^*}^1 \wedge T_{U^*}^0 - T_{U^*}^1 \wedge T_{V^*}^0)$$

Proof. Beginning with Equation (18) for a Lyndon word W

$$d_{cy}(T_{W^*}(x)) = \sum_{U < V} \alpha_{U,V}^W T_{U^*}(x) \wedge T_{V^*}(x) + \sum_{U,V} \beta_{U,V}^W T_{U^*}(x) \wedge T_{V^*}(1),$$

we write terms of the form $T_{U^*}(x) \wedge T_{V^*}(1)$ as

$$\begin{aligned} T_{U^*}(x) \wedge T_{V^*}(1) &= T_{U^*}(x) \wedge (-(T_{V^*}(x) - T_{V^*}(1)) + T_{V^*}(x)) \\ &= -T_{U^*}^0 \wedge T_{V^*}^1 + T_{U^*}^0 \wedge T_{V^*}^0 \\ &= T_{V^*}^1 \wedge T_{U^*}^0 + T_{U^*}^0 \wedge T_{V^*}^0. \end{aligned}$$

Then Equation (23) follows by reordering the terms of the sum. The second sum $\sum_{U,V} \beta_{U,V}^W T_{U^*}(x) \wedge T_{V^*}(1)$ is, in this reordering, cut in three pieces : when $U < V$, when $U > V$ and when $U = V$. Note that when $U = V$, the term $T_{U^*}^0 \wedge T_{V^*}^0$ vanishes leaving only the term $T_{V^*}^1 \wedge T_{U^*}^0$. The inversion of letters U and V in the equality

$$b_{U,V}^W = \beta_{V,U}^W$$

is induced by the choice of terms $T_{U^*}^1 \wedge T_{V^*}^0$ as ‘‘cross-terms’’ of the basis instead of $T_{U^*}^0 \wedge T_{V^*}^1$. This choice is motivated by the position of the tree $T_{U^*}^1$ in the products. The tree $T_{U^*}^1$ will correspond to the algebraic cycle \mathcal{L}_0^1 which usually arise as first term of a product.

In order to obtain Equation (24), one computes $d_{cy}(T_{W^*}^1) = d_{cy}(T_{W^*}(x)) - d_{cy}(T_{W^*}(1))$ as

$$(26) \quad \begin{aligned} d_{cy}(T_{W^*}^1) &= \sum_{U < V} a_{U,V}^W T_{U^*}^0 \wedge T_{V^*}^0 + \sum_U b_{U,U}^W T_{U^*}^1 \wedge T_{U^*}^0 \\ &\quad + \sum_{U < V} b_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0 + \sum_{V < U} b_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0 \\ &\quad - \sum_{U < V} \gamma_{U,V}^W T_{U^*}(1) \wedge T_{V^*}(1) \end{aligned}$$

where U, V are Lyndon words. One remarks that

$$T_{U^*}^0 \wedge T_{V^*}^0 = -T_{U^*}^1 \wedge T_{V^*}^1 + T_{U^*}^1 \wedge T_{V^*}^0 - T_{V^*}^1 \wedge T_{U^*}^0 + T_{U^*}(1) \wedge T_{V^*}(1).$$

Replacing this expression of $T_{U^*}^0 \wedge T_{V^*}^0$ in the above equation for $d_{cy}(T_{W^*}^1)$, one obtains

$$d_{cy}(T_{W^*}^1) = \sum_{U < V} -a_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^1 + \sum_{U < V} a_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0 + \sum_{U < V} -a_{U,V}^W T_{V^*}^1 \wedge T_{U^*}^0 + \sum_U b_{U,U}^W T_{U^*}^1 \wedge T_{U^*}^0 + \sum_{U < V} b_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0 + \sum_{V < U} b_{U,V}^W T_{U^*}^1 \wedge T_{V^*}^0$$

because by Lemma 4.18, one has

$$\gamma_{U,V}^W = \alpha_{U,V}^W + \beta_{U,V}^W - \beta_{V,U}^W = a_{U,V}^W.$$

Collecting terms in the last expression of $d_{cy}(T_{W^*}^1)$ gives Equation (24). \square

Lemma 4.18 gives us some extra information about coefficients $\alpha_{U,V}^W$ and $\beta_{U,V}^W$ when W, U or V is equal to the letter 0 or 1. This translates for coefficients a 's, b 's, a 's and b 's as:

Lemma 4.33. \bullet *If W is the Lyndon word 0 or 1, then :*

$$a_{U,V}^0 = b_{U,V}^0 = a'_{U,V}^0 = b'_{U,V}^0 = 0, \quad a_{U,V}^1 = b_{U,V}^1 = a'_{U,V}^1 = b'_{U,V}^1 = 0$$

for any Lyndon words U and V .

\bullet *For any Lyndon word W, U and V of length at least 2, one has*

$$a_{0,V}^W = a'_{0,V}^W = 0 \quad \text{and} \quad b_{U,0}^W = b'_{U,0}^W = 0.$$

which says that there is no term in $T_{0^}^0 T_{V^*}^0, T_{0^*}^1 T_{V^*}^1$ or $T_{U^*}^1 T_{0^*}^0$. Moreover, one has*

$$a_{U,1}^W = a'_{U,1}^W = 0, \quad \text{and} \quad b_{1,V}^W = b'_{1,V}^W = 0$$

which says that there is no term in $T_{U^}^0 T_{1^*}^0, T_{U^*}^1 T_{1^*}^1$ or $T_{1^*}^1 T_{V^*}^0$.*

One also notes that for W a Lyndon word,

$$a_{U,V}^W = b_{U,V}^W = a'_{U,V}^W = b'_{U,V}^W = 0$$

as soon as of U plus the length of V is not equal to the length of W .

In particular, Equation (ED- T^0) and Equation (ED- T^1) involve only Lyndon words of length smaller than the length of W .

Proof. This is a direct consequence of Lemma 4.18 and Lemma 4.32, \square

The two equations (ED- T^0) and (ED- T^1) gives us the combinatorial situation which will allow the construction of the algebraic cycles. However, we can not directly related the above equation to what happen in the cycle algebra $\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^{\bullet}$ because:

- \bullet the structures are not the same : $\mathcal{T}_{1;x}^{coL}$ is a coLie algebra while $\mathcal{N}_{\mathbb{A}^1}^{\bullet}$ is a commutative differential graded algebra;
- \bullet we can associated a cycle \mathcal{L}_W^0 to $T_{W^*}^0$ (resp. \mathcal{L}_W^1 to $T_{W^*}^1$) only after that the cycle is built.

We will apply later the following Proposition 4.34 to the complex $\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^{eq,\bullet}$ of equidimensional cycle over $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The result and the proof are algebraic and do not involved any geometry. The Proposition 4.34 concludes this combinatorial part. It relates the structure of Equations (ED- T^0) and (ED- T^1) to the ‘‘differential system’’ arising in the construction of the algebraic cycles in the next section.

Let $(\mathcal{A}^{\bullet}, \partial_{\mathcal{A}})$ be a cdga and p an integer ≥ 2 . We assume the following :

- \bullet There exist two degree 1 elements A_0^1 and A_1^0 in \mathcal{A}^1 such that

$$\partial_{\mathcal{A}}(A_0^1) = \partial_{\mathcal{A}}(A_1^0) = 0;$$

- For any Lyndon words W of length k with $2 \leq k \leq p-1$, there exist two degree 1 elements A_W^0 and A_W^1 in \mathcal{A}^1 satisfying (ED- T^0) and (ED- T^1) respectively:

$$\begin{aligned}\partial_{\mathcal{A}}(A_W^0) &= \sum_{U < V} a_{U,V}^W A_U^0 \wedge A_V^0 + \sum_{U,V} b_{U,V}^W A_U^1 \wedge A_V^0, \\ \partial_{\mathcal{A}}(A_W^1) &= \sum_{U < V} a'_{U,V}^W A_U^1 \wedge A_V^1 + \sum_{U,V} b'_{U,V}^W A_U^1 \wedge A_V^0\end{aligned}$$

Proposition 4.34. *Let W be a Lyndon word of length p . Let R_{A^0} and R_{A^1} be the degree 2 elements defined by*

$$R_{A^0} = \sum_{U < V} a_{U,V}^W A_U^0 A_V^0 + \sum_{U,V} b_{U,V}^W A_U^1 A_V^0,$$

and

$$R_{A^1} = \sum_{U < V} a'_{U,V}^W A_U^1 A_V^1 + \sum_{U,V} b'_{U,V}^W A_U^1 A_V^0,$$

where coefficient a 's, b 's a' 's, b' 's are, as above, the one defined by Equations (ED- T^0) and (ED- T^1).

Then

$$\partial_{\mathcal{A}}(R_{A^0}) = \partial_{\mathcal{A}}(R_{A^1}) = 0.$$

Proof. First we recall that the symmetric algebra $S^g(V^\bullet)$ over a graded vector space V is the tensor algebra modulo the ideal generated by

$$ab - (-1)^{\deg(a)\deg(b)}ba$$

when ever a and b are homogeneous.

Let $S_{\mathcal{T}}^\bullet = S^g(\mathcal{T}_{1;x}^{coL})$ be symmetric graded algebra over the vector space $\mathcal{T}_{1;x}^{coL}$ concentrated purely in degree 1. We shall use the same notation for an element in $\mathcal{T}_{1;x}^{coL}$ and its image in $S_{\mathcal{T}}^\bullet$. The cobracket d_{cy} and the Leibniz rule induces a differential on $S_{\mathcal{T}}^\bullet$ denote by ∂_{cy} . This makes $S_{\mathcal{T}}^\bullet$ into a cdga. In particular Equations (ED- T^0) and (ED- T^1) holds in $S_{\mathcal{T}}^\bullet$ after replacing d_{cy} by ∂_{cy} and the wedge product in $\mathcal{T}_{1;x}^{coL} \wedge \mathcal{T}_{1;x}^{coL}$ by the product in $S_{\mathcal{T}}^\bullet$.

Let $S_{\mathcal{T}, \leq p-1}^\bullet$ be the sub-algebra of $S_{\mathcal{T}}^\bullet$ generated by elements $T_{U^*}^0$ and $T_{V^*}^1$ for U and V Lyndon words of length $k \leq p-1$. It is a sub-cdga of $S_{\mathcal{T}}^\bullet$ because equations (ED- T^0) and (ED- T^1) involve only words of smaller length on the right hand side.

Let W be a Lyndon word of length p . Note that the degree 2 element $\partial_{cy}(T_{W^*}^0)$ and $\partial_{cy}(T_{W^*}^1)$ of $S_{\mathcal{T}}^\bullet$ are also in $S_{\mathcal{T}, \leq p-1}^\bullet$:

$$\partial_{cy}(T_{W^*}^0), \partial_{cy}(T_{W^*}^1) \in S_{\mathcal{T}, \leq p-1}^\bullet;$$

even if there are not boundary in $S_{\mathcal{T}, \leq p-1}^\bullet$.

We define an algebra morphism $\varphi : S_{\mathcal{T}, \leq p-1}^\bullet \rightarrow \mathcal{A}^\bullet$ on the degree one elements by

$$\varphi(T_{U^*}^0) = A_U^0, \quad \text{and} \quad \varphi(T_{U^*}^1) = A_U^1$$

for any Lyndon word U of length k with $2 \leq k \leq p-1$ and by $\varphi(T_{0^*}^1) = A_0^1$ and $\varphi(T_{1^*}^0) = A_1^0$. The morphism φ is a cdga morphism because of the assumption on $\partial_{\mathcal{A}}(A_U^0)$ and $\partial_{\mathcal{A}}(A_U^1)$ for Lyndon words U of length $\leq p-1$. Hence $R_{A^0} = \varphi(\partial_{cy}(T_{W^*}^0))$ and $R_{A^1} = \varphi(\partial_{cy}(T_{W^*}^1))$ satisfy

$$\partial_{\mathcal{A}}(R_{A^0}) = \varphi(\partial_{cy} \circ \partial_{cy}(T_{W^*}^0)) = 0, \quad \text{and} \quad \partial_{\mathcal{A}}(R_{A^1}) = \varphi(\partial_{cy} \circ \partial_{cy}(T_{W^*}^1)) = 0.$$

□

5. CONSTRUCTION OF ALGEBRAIC CYCLES

In this section we define two “differential systems” for algebraic cycles, one corresponding to cycles with empty fiber at $x = 0$ and another corresponding to cycles with empty fiber at $x = 1$. Then, we show that there exist two families of cycles in $\mathcal{N}_X^{eq,1}$ satisfying these systems induced by two families of cycles in $\mathcal{N}_{\mathbb{A}^1}^{eq,1}$.

5.1. Equidimensional cycles. We recall that the base field is \mathbb{Q} and that all varieties considered below are \mathbb{Q} -varieties. We also recall that $\square^1 = \mathbb{P}^1 \setminus \{1\}$, that $\square^n = (\square^1)^n$, and that $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Definition 5.1 (Equidimensionality). Let Y be an irreducible smooth variety

- Let $\mathcal{Z}_{eq}^p(Y, n)$ denote the free abelian group generated by irreducible closed subvarieties $Z \subset Y \times \square^n$ such that for any face F of \square^n , the intersection $Z \cap (Y \times F)$ is empty or the restriction of $p_1 : Y \times \square^n \rightarrow Y$ to

$$Z \cap (Y \times F) \rightarrow Y$$

is dominant and equidimensional of pure relative dimension $\dim(F) - p$; that is the non empty fibers have the same required dimension.

- We say that elements of $\mathcal{Z}_{eq}^p(Y, n)$ are *equidimensional over Y with respect to any face* or simply *equidimensional*.
- Following the definition of $\mathcal{N}_Y^k(p)$, let $\mathcal{N}_Y^{eq,k}(p)$ denote

$$\mathcal{N}_Y^{eq,k}(p) = \text{Alt}(\mathcal{Z}_{eq}^p(Y, 2p - k) \otimes \mathbb{Q}).$$

Definition 5.2. Let C be an element of \mathcal{N}_Y^\bullet decomposed in terms of cycles as

$$C = \sum_{i \in I} q_i Z_i, \quad q_i \in \mathbb{Q},$$

where I is a finite set and the Z_i are irreducible closed subvarieties of $Y \times \square^{n_i}$ intersecting all the faces of \square^{n_i} properly (i.e. in codimension p_i).

- The support of C is defined as $\text{Supp}(C) = \bigcup_i Z_i$.
- For C in $\mathcal{N}_Y^{eq,k}(p)$, we will say that C has empty fiber at a point y in Y if for any i in I the fiber of $Z_i \rightarrow Y$ at y is empty.

Proposition 5.3. *Let Y be an irreducible smooth variety.*

- (1) *The differential ∂_Y on \mathcal{N}_Y^\bullet induces a differential:*

$$\mathcal{N}_Y^{eq,k}(p) \xrightarrow{\partial_Y} \mathcal{N}_Y^{eq,k}(p)$$

which makes $\mathcal{N}_Y^{eq,\bullet}(p)$ into a sub-complex of $\mathcal{N}_Y^\bullet(p)$.

- (2) *$\mathcal{N}_Y^{eq,\bullet} = \bigoplus_{p \geq 0} \mathcal{N}_Y^{eq,\bullet}(p)$ is a subalgebra (sub-cdga) of \mathcal{N}_Y^\bullet .*
- (3) *Assume that Z or Z' has an empty fiber at a point y in Y . Then the fiber at y of $\overline{Z} \cdot \overline{Z}'$ is empty.*

Proof. As the generators of $\mathcal{Z}_{eq}^p(Y, 2p - k)$ are equidimensional over Y when intersected with any face, they stay equidimensional over Y with respect to any face when intersected with a codimension 1 face because a face intersected with a codimension 1 face is either another face or the intersection is empty. This gives the first point.

Let Z and Z' two generators of $\mathcal{Z}_{eq}^p(Y, 2p - k)$ and $\mathcal{Z}_{eq}^q(Y, 2q - l)$ respectively for p, q, k and l integers. By definition, for any face $F \subset \square^{2p-k}$, the restriction of the projection $p_1 : Y \times F \rightarrow Y$ to

$$p_1 : Z \cap (Y \times F) \rightarrow Y$$

is equidimensional of relative dimension $\dim(F) - p$, or the above intersections are empty. Similarly for Z' .

Let F and F' be two faces as above, and assume that none of the intersections $Z \cap (X \times F)$ and $Z' \cap (X \times F')$ is empty. Then

$$Z \times Z' \cap (Y \times Y \times F \times F') \subset Y \times Y \times \square^{2(p+q)-k-l}$$

is equidimensional over $Y \times Y$ of relative dimension $\dim(F) + \dim(F') - p - q$. For any point x in the image of the diagonal $\Delta : Y \rightarrow Y \times Y$, we denote with a subscript x the fiber over x . We have in particular

$$\begin{aligned} \dim \left(((Z \times Z') \cap (Y \times Y \times F \times F'))_x \right) &= \\ \dim (Z \times Z' \cap (\{x\} \times F \times F')) &= \\ \dim(\{x\}) + \dim(F) + \dim(F') - p - q & \end{aligned}$$

and $Z \times Z' \cap (\text{im}(\Delta) \times F \times F')$ is equidimensional over Y of relative dimension $\dim(F) + \dim(F') - p - q$ by either of the two projections $Y \times Y \rightarrow Y$. If either $Z \cap (Y \times F)$ or $Z' \cap (Y \times F')$ is empty, then the intersection

$$Z \times Z' \cap (Y \times Y \times F \times F')$$

is empty and so is $Z \times Z' \cap (\text{im}(\Delta) \times F \times F')$.

From this, we deduce that

$$(\Delta \times \text{id})^{-1}(Z \times Z') \simeq Z \times Z' \cap (\text{im}(\Delta) \times \square^{2(p+q)-k-l})$$

is equidimensional over Y with respect to any face. Hence,

$$Z \cdot Z' = \text{Alt}((\Delta \times \text{id})^{-1}(Z \times Z')) \in \mathcal{N}_Y^\bullet$$

and the product in \mathcal{N}_Y^\bullet induces a cdga structure on $\mathcal{N}_Y^{eq, \bullet}$ which makes it into a sub-cdga.

Moreover, from the above computation, we see that if the fiber of Z is empty at a point y , then, denoting with a subscript y the various fibers at y , we have

$$\left((\Delta \times \text{id})^{-1}(Z \times Z') \right)_y = Z \times Z' \cap (\{(y, y)\} \times \square^{2(p+q)-k-l}) = Z_y \times Z'_y = \emptyset.$$

The same holds if Z' is empty at y , which gives the last point of the proposition. \square

In order to compare the situation in \mathcal{N}_X^\bullet and in $\mathcal{N}_{\mathbb{A}^1}^\bullet$, we will use the following proposition.

Proposition 5.4. *Let Y_0 be an open dense subset of an irreducible smooth variety Y and let $j : Y_0 \rightarrow Y$ denote the inclusion.*

- (1) *The restriction of cycles from Y to Y_0 induces a morphism of cdga's*

$$j^* : \mathcal{N}_Y^{eq, \bullet} \rightarrow \mathcal{N}_{Y_0}^{eq, \bullet}.$$

- (2) *The morphism j^* is injective.*

- (3) *Let C be in $\mathcal{N}_{Y_0}^\bullet$ be decomposed in terms of cycles as*

$$C = \sum_{i \in I} q_i Z_i, \quad q_i \in \mathbb{Q}$$

where I is a finite set. Assume that for any i , the Zariski closure $\overline{Z_i}$ of Z_i is in $\mathcal{Z}_{eq}^{p_i}(Y, n_i)$. Define \overline{C} as

$$\overline{C} = \sum_{i \in I} q_i \overline{Z_i};$$

then

$$\overline{C} \in \mathcal{N}_Y^{eq, \bullet} \quad \text{and} \quad C = j^*(\overline{C}) \in \mathcal{N}_{Y_0}^{eq, \bullet}.$$

- (4) *In particular $\overline{j^*(D)} = D$ for any D in $\mathcal{N}_Y^{eq, \bullet}$.*

Proof. It is enough to prove the first part of proposition for generators of $\mathcal{N}_Y^{eq, \bullet}$.

Let Z be an irreducible, closed subvariety of codimension p of $Y \times \square^{2p-k}$ such that for any face F of \square^{2p-k} the intersection

$$Z \cap (Y \times F) \quad (\text{resp. } Z' \times (Y \times F'))$$

is dominant equidimensional over Y of relative dimension $\dim(F) - p$ or empty.

Let Z_0 be the intersections $Z \cap Y_0$. As for any face F of \square^{2p-k}

$$Z_0 \cap (Y_0 \times F) = (Z \cap (Y \times F)) \cap Y_0 \times \square^{2p-k}$$

Z_0 is equidimensional with respect to any face over Y_0 with relative dimension $\dim(F) - p$. This also shows that j^* commutes with the differential on $\mathcal{N}_Y^{eq, \bullet}$ and on $\mathcal{N}_{Y_0}^{eq, \bullet}$.

Moreover if $\overline{Z_0}$ denotes the Zariski closure of Z_0 in Y , one has $\overline{Z_0} = Z$ because Z is closed and irreducible. This gives part (4) of the proposition and the injectivity of j^* .

Let Z' be an irreducible, closed subvariety of codimension q of $Y \times \square^{2q-l}$ providing a generator of $\mathcal{Z}_{eq}^q(Y, 2q-l)$. In order to show that j^* commutes with the product structure, it suffices to remark that

$$Z_0 \times Z'_0 = (Z \times Z') \cap (Y_0 \times Y'_0 \times \square^{2(p+p')-k-k'}) \subset Y \times Y \times \square^{2(p+p')-k-k'}.$$

Let C and \bar{C} be as in the proposition. The fact that \bar{C} is in $\mathcal{N}_Y^{eq, \bullet}$ follows directly from the definition. To prove that

$$C = j^*(C') \in \mathcal{N}_{Y_0}^{eq, \bullet},$$

we can assume that I contains only one element and that $q_1 = 1$. Then it follows from the fact that $Z_1 = \overline{Z_1} \cap Y_0 \subset Y$. □

The main geometric tool of our construction comes from the usual multiplication on \mathbb{A}^1 which induces a homotopy on $\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$ between the identity and the constant cycle given by the fiber at 0.

Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the multiplication map sending (x, y) to xy , and let $\tau : \square^1 = \mathbb{P}^1 \setminus \{1\} \rightarrow \mathbb{A}^1$ be the isomorphism sending the affine coordinate u to $\frac{1}{1-u}$. The map τ sends ∞ to 0, 0 to 1 and extends as a map from \mathbb{P}^1 to \mathbb{P}^1 sending 1 to ∞ .

The maps m and τ are in particular flat and equidimensional of relative dimension 1 and 0 respectively.

Consider the following commutative diagram for a positive integer n ;

$$\begin{array}{ccc} \mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{(m \circ (\text{id}_{\mathbb{A}^1} \times \tau)) \times \text{id}_{\square^n}} & \mathbb{A}^1 \times \square^n \\ p_{\mathbb{A}^1 \times \square^1} \downarrow & & \downarrow p_{\mathbb{A}^1} \\ \mathbb{A}^1 \times \square^1 & \xrightarrow{m \circ (\text{id}_{\mathbb{A}^1} \times \tau)} & \mathbb{A}^1 \\ p_{\mathbb{A}^1} \downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

Proposition 5.5 (multiplication and equidimensionality). *In the following statement, p, k and n will denote positive integers subject to the relation $n = 2p - k$.*

- The composition $\tilde{m} = (m \circ (\text{id}_{\mathbb{A}^1} \times \tau)) \times \text{id}_{\square^n}$ induces a group morphism

$$\mathcal{Z}_{eq}^p(\mathbb{A}^1, n) \xrightarrow{\tilde{m}^*} \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$$

which extends into a morphism of complexes for any p ,

$$\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}(p) \xrightarrow{\tilde{m}^*} \mathcal{N}_{\mathbb{A}^1 \times \square^1}^{eq, \bullet}(p).$$

- Moreover, there is a natural morphism

$$h_{\mathbb{A}^1, n}^p : \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n) \longrightarrow \mathcal{Z}_{eq}^p(\mathbb{A}^1, n+1)$$

given by regrouping the \square 's factors.

- The composition $\mu^* = h_{\mathbb{A}^1, n}^p \circ \tilde{m}^*$ gives a morphism

$$\mu^* : \mathcal{N}_{\mathbb{A}^1}^{eq, k}(p) \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, k-1}(p)$$

sending equidimensional cycles with empty fiber at 0 to equidimensional cycles with empty fiber at 0.

- Let $\theta : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the involution sending the natural affine coordinate x to $1-x$. Twisting the multiplication \tilde{m} by θ via

$$\begin{array}{ccc} \mathbb{A}^1 \times \square^1 \times \square^n & \longrightarrow & \mathbb{A}^1 \times \square^n \\ \theta \times \text{id}_{\square^{n+1}} \downarrow & & \uparrow \theta \times \text{id}_{\square^n} \\ \mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{\tilde{m}} & \mathbb{A}^1 \times \square^n \end{array}$$

gives a morphism

$$\nu^* : \mathcal{N}_{\mathbb{A}^1}^{eq, k}(p) \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, k-1}(p)$$

sending equidimensional cycles with empty fiber at 1 to equidimensional cycles with empty fiber at 1.

Proof. It is enough to work with generators of $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$. Let Z be an irreducible subvariety of $\mathbb{A}^1 \times \square^n$ such that for any face F of \square^n , the first projection

$$p_{\mathbb{A}^1} : Z \cap (\mathbb{A}^1 \times F) \longrightarrow \mathbb{A}^1$$

is dominant and equidimensional of relative dimension $\dim(F) - p$ or empty. Let F be a face of \square^n . First we want to show that under the projection $\mathbb{A}^1 \times \square^1 \times \square^n \longrightarrow \mathbb{A}^1 \times \square^1$,

$$\tilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is dominant and equidimensional of relative dimension $\dim(F) - p$ or empty. This follows from the fact that $Z \cap (\mathbb{A}^1 \times F)$ is dominant equidimensional over \mathbb{A}^1 and m is flat and equidimensional of relative dimension 1 (hence are $m \circ (\text{id}_{\mathbb{A}^1} \times \tau)$ and \tilde{m}).

The map \tilde{m} is identity on the \square^n factor, thus for $Z \subset \mathbb{A}^1 \times \square^n$ as above and a codimension 1 face F of \square^n , $\tilde{m}^{-1}(Z)$ satisfies

$$\tilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) = \tilde{m}^{-1}(Z \cap (\mathbb{A}^1 \times F)).$$

This shows that \tilde{m}^* is a morphism of complexes.

Moreover, assuming that the fiber of Z at 0 is empty, the intersection

$$\tilde{m}^{-1}(Z) \cap (\{0\} \times \square^1 \times \square^n)$$

is empty because \tilde{m} restricted to

$$\{0\} \times \square^1 \times \square^n$$

factors through the inclusion $\{0\} \times \square^n \longrightarrow \mathbb{A}^1 \times \square^n$. Hence the fiber of $\tilde{m}^{-1}(Z)$ over $\{0\} \times \square^1$ (resp. over $\{0\}$) by $p_{\mathbb{A}^1 \times \square^1}$ (resp. $p_{\mathbb{A}^1} \circ p_{\mathbb{A}^1 \times \square^1}$) is empty.

Now, let Z be an irreducible subvariety of $\mathbb{A}^1 \times \square^1 \times \square^n$ such that for any face F of \square^n

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is dominant and equidimensional of relative dimension $\dim(F) - p$ when the intersection is not empty. Let F' be a face of

$$\square^{n+1} = \square^1 \times \square^n.$$

The face F' is either of the form $\square^1 \times F$ or of the form $\{\varepsilon\} \times F$ with F a face of \square^n and $\varepsilon \in \{0, \infty\}$. We can assume that $Z \cap (\mathbb{A}^1 \times \square^1 \times F)$ is not empty. When F' is of the first type, we observe that

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is dominant and equidimensional and that $\mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$ is equidimensional of relative dimension 1. Hence the projection

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p + 1 = \dim(F') - p.$$

When F' is of the second type, by symmetry of the role of 0 and ∞ , we can assume that $\varepsilon = 0$. Then, the intersection

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F)$$

is nothing but the fiber of $Z \cap (\mathbb{A}^1 \times \square^1 \times F)$ over $\mathbb{A}^1 \times \{0\}$. Hence, it has pure dimension $\dim(F) - p + 1$.

Moreover, denoting the fiber with a subscript, the composition

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F) = (Z \cap (\mathbb{A}^1 \times \square^1 \times F))_{\mathbb{A}^1 \times \{0\}} \longrightarrow \mathbb{A}^1 \times \{0\} \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p = \dim(F') - p.$$

This shows that $h_{\mathbb{A}^1, n}^p$ gives a well defined morphism and that it preserves the fiber at a point x in \mathbb{A}^1 ; in particular if Z has an empty fiber at 0, so does $h_{\mathbb{A}^1, n}^p(Z)$.

Finally, the last part of the proposition follows from the fact that θ exchanges the roles of 0 and 1. \square

Remark 5.6. We saw that \tilde{m} sends cycles with empty fiber at 0 to cycles with empty fiber at any point in $\{0\} \times \square^1$. Similarly, \tilde{m} sends cycles with empty fiber at 0 to cycles that also have an empty fiber at any point in $\mathbb{A}^1 \times \{\infty\}$.

From the proof of Proposition 4.2 in [Lev94], we deduce that μ^* gives a homotopy between $p_0^* \circ i_0^*$ and id where i_0 is the zero section $\{0\} \rightarrow \mathbb{A}^1$ and p_0 the projection onto the point $\{0\}$.

Proposition 5.7. *Let the notations be as in Proposition 5.5 above. For $\varepsilon = 0, 1$, let i_ε be the inclusion of ε into \mathbb{A}^1*

$$i_0 : \{\varepsilon\} \longrightarrow \mathbb{A}^1,$$

and let p_ε be the corresponding projections $p_\varepsilon : \mathbb{A}^1 \longrightarrow \{\varepsilon\}$.

Then μ^* provides a homotopy between

$$p_0^* \circ i_0^* \text{ and } \text{id} : \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet},$$

and similarly ν^* provides a homotopy between

$$p_1^* \circ i_1^* \text{ and } \text{id} : \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}.$$

In other words,

$$\partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1} = \text{id} - p_0^* \circ i_0^* \quad \text{and} \quad \partial_{\mathbb{A}^1} \circ \nu^* + \nu^* \circ \partial_{\mathbb{A}^1} = \text{id} - p_1^* \circ i_1^*$$

The proposition follows from commuting the different compositions involved, and from the relation between the differential on $\mathcal{N}_{\mathbb{A}^1 \times \square^1}^{eq, \bullet}$ and the one on $\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$ via the map $h_{\mathbb{A}^1, n}^p$.

Proof. Let $i_{0, \square}$ and $i_{\infty, \square}$ denote the zero section and the infinity section $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \times \square^1$. The action of θ only exchanges the role of 0 and 1 in \mathbb{A}^1 , hence it is enough to prove the statement for μ^* . As before, in order to obtain the proposition for $\mathcal{N}_{\mathbb{A}^1}^{eq, k}(p)$, it is enough to work with the generators of $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$ with $n = 2p - k$.

By the previous proposition 5.5, \tilde{m}^* commutes with the differential on $\mathcal{Z}_{eq}^p(\mathbb{A}^1, \bullet)$ and on $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, \bullet)$. As the morphism μ^* is defined by $\mu^* = h_{\mathbb{A}^1, n}^p \circ \tilde{m}^*$, the proof relies on computing $\partial_{\mathbb{A}^1} \circ h_{\mathbb{A}^1, n}^p$. Let Z be a generator of $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$. In particular,

$$Z \subset \mathbb{A}^1 \times \square^1 \times \square^n$$

and $h_{\mathbb{A}^1, n}^p(Z)$ is also given by Z but viewed in

$$\mathbb{A}^1 \times \square^{n+1}.$$

The differentials denoted by $\partial_{\mathbb{A}^1}^{n+1}$ on $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n+1)$ and $\partial_{\mathbb{A}^1 \times \square^1}^n$ on $\mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$ are both given by intersections with the codimension 1 faces, but the first \square^1 factor in \square^{n+1} gives two more faces and introduces a change of sign. Namely, using an extra subscript to indicate in which cycle groups the intersections take place, we have

$$\begin{aligned} \partial_{\mathbb{A}^1}^{n+1}(h_{\mathbb{A}^1, n}^p(Z)) &= \sum_{i=1}^{n+1} (-1)^{i-1} (\partial_{i, \mathbb{A}^1}^0(Z) - \partial_{i, \mathbb{A}^1}^\infty(Z)) \\ &= \partial_{1, \mathbb{A}^1}^0(Z) - \partial_{1, \mathbb{A}^1}^\infty(Z) - \sum_{i=2}^{n+1} (-1)^{i-2} (\partial_{i, \mathbb{A}^1}^0(Z) - \partial_{i, \mathbb{A}^1}^\infty(Z)) \\ &= i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) - \sum_{i=1}^n (-1)^{i-1} (\partial_{i+1, \mathbb{A}^1}^0(Z) - \partial_{i+1, \mathbb{A}^1}^\infty(Z)) \\ &= i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) \\ &\quad - \sum_{i=1}^n (-1)^{i-1} (h_{\mathbb{A}^1, n-1}^p \circ \partial_{i, \mathbb{A}^1 \times \square^1}^0(Z) - h_{\mathbb{A}^1, n-1}^p \circ \partial_{i, \mathbb{A}^1 \times \square^1}^\infty(Z)) \\ &= i_{0, \square}^*(Z) - i_{\infty, \square}^*(Z) - h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1 \times \square^1}^n(Z). \end{aligned}$$

Thus, we can compute $\partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1}$ on $\mathcal{Z}_{eq}^p(\mathbb{A}^1, n)$ as

$$\begin{aligned} \partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1} &= \partial_{\mathbb{A}^1} \circ h_{\mathbb{A}^1, n}^p \circ \tilde{m}^* + h_{\mathbb{A}^1, n-1}^p \circ \tilde{m}^* \circ \partial_{\mathbb{A}^1} \\ &= i_{0, \square}^* \circ \tilde{m}^* - i_{\infty, \square}^* \circ \tilde{m}^* - h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1} \circ \tilde{m}^* \\ &\quad + h_{\mathbb{A}^1, n-1}^p \circ \partial_{\mathbb{A}^1} \circ \tilde{m}^* \\ &= i_{0, \square}^* \circ \tilde{m}^* - i_{\infty, \square}^* \circ \tilde{m}^*. \end{aligned}$$

The morphism $i_{\infty, \square}^* \circ \tilde{m}^*$ is induced by

$$\begin{array}{ccccccc} \mathbb{A}^1 & \xrightarrow{i_{\infty, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ x & \longmapsto & (x, \infty) & \longmapsto & (x, 0) & \longmapsto & 0 \end{array}$$

which factors through

$$\begin{array}{ccccccc} \mathbb{A}^1 & \xrightarrow{i_{\infty, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ \downarrow p_0 & & & & & & \downarrow \text{id}_{\mathbb{A}^1} \\ \mathbb{A}^1 & \xrightarrow{i_0} & & & & & \mathbb{A}^1 \end{array}$$

Thus,

$$i_{\infty, \square}^* \circ \tilde{m}^* = (i_0 \circ p_0)^* = p_0^* \circ i_0^*.$$

Similarly $i_{0, \square}^* \circ \tilde{m}^*$ is induced by

$$\begin{array}{ccccccc} \mathbb{A}^1 & \xrightarrow{i_{\infty, \square}} & \mathbb{A}^1 \times \square^1 & \xrightarrow{\tau} & \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \\ x & \longmapsto & (x, 0) & \longmapsto & (x, 1) & \longmapsto & x \end{array}$$

which factors through $\text{id}_{\mathbb{A}^1} : \mathbb{A}^1 \rightarrow \mathbb{A}^1$, and we have

$$i_{0, \square}^* \circ \tilde{m}^* = \text{id}.$$

This concludes the proof of the proposition. \square

5.2. Cycles over $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ corresponding to multiple polylogarithms. Set $\mathcal{L}_0^1 = L_0$ and $\mathcal{L}_1^0 = L_1$, where L_0 and L_1 are the cycles in $\mathcal{N}_X^1(1)$ defined in Section 3 induced by the graph of $x \mapsto x$ and $x \mapsto 1 - x$ from $X \rightarrow \mathbb{P}^1$. Note that the superscript 1 in \mathcal{L}_0^1 refers to the fact that this cycle has an empty fiber at 1.

Consider the two following differential systems

$$(ED-\mathcal{L}^0) \quad \partial(\mathcal{L}_W^0) = \sum_{U < V} a_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^0 + \sum_{U,V} b_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^1$$

and

$$(ED-\mathcal{L}^1) \quad \partial(\mathcal{L}_W^1) = \sum_{U < V} a'_{U,V}^W \mathcal{L}_U^1 \mathcal{L}_V^1 + \sum_{U,V} b'_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^1$$

where the coefficients $a_{U,V}^W$, $b_{U,V}^W$, $a'_{U,V}^W$ and $b'_{U,V}^W$ are the ones defined in Definition 4.32.

These differential equations are exactly the differential system considered in section 4.4.

Theorem 5.8. *Let j be the inclusion $X \hookrightarrow \mathbb{A}^1$. For any Lyndon word W of length $p \geq 2$, there exist two cycles \mathcal{L}_W^0 and \mathcal{L}_W^1 in $\mathcal{N}_X^1(p)$ such that:*

- $\mathcal{L}_W^0, \mathcal{L}_W^1$ are elements of $\mathcal{N}_X^{eq, 1}(p)$.
- There exist cycles $\overline{\mathcal{L}}_W^0, \overline{\mathcal{L}}_W^1$ in $\mathcal{N}_{\mathbb{A}^1}^{eq, 1}(p)$ such that

$$\mathcal{L}_W^0 = j^*(\overline{\mathcal{L}}_W^0) \quad \text{and} \quad \mathcal{L}_W^1 = j^*(\overline{\mathcal{L}}_W^1).$$

- The restriction of $\overline{\mathcal{L}}_W^0$ (resp. $\overline{\mathcal{L}}_W^1$) to the fiber $x = 0$ (resp. $x = 1$) is empty.
- The cycle \mathcal{L}_W^0 (resp. \mathcal{L}_W^1) satisfies the equation (ED- \mathcal{L}^0) (resp. (ED- \mathcal{L}^1)) in \mathcal{N}_X^\bullet and the same holds for its extension $\overline{\mathcal{L}}_W^0$ (resp. $\overline{\mathcal{L}}_W^1$) to $\mathcal{N}_{\mathbb{A}^1}^{eq, \bullet}$.

The rest of the section is devoted to proving the above theorem. Let $A_{\mathcal{L}^0}$ and $A_{\mathcal{L}^1}$ denote the right-hand side of (ED- \mathcal{L}^0) and (ED- \mathcal{L}^1) respectively. The proof works by induction, and will be developed as follows:

- Reviewing the cycles \mathcal{L}_{01}^0 and \mathcal{L}_{01}^1 presented in subsection 3 in order to show that they give the desired cycles for $W = 01$.

- Proving that $A_{\mathcal{L}^0}$ and $A_{\mathcal{L}^1}$ have differential 0 in \mathcal{N}_X^\bullet . This was essentially proved in Proposition 4.34.
- Extending $A_{\mathcal{L}^0}$ and $A_{\mathcal{L}^1}$ to \mathbb{A}^1 and proving in Lemma 5.10 that the differential stays 0 in $\mathcal{N}_{\mathbb{A}^1}^\bullet$.
- Finally, constructing \mathcal{L}_W^0 and \mathcal{L}_W^1 by pull-back by the multiplication and pull-back by the twisted multiplication in Lemma 5.11.
- Proving that the pull-back by the (twisted) multiplication preserves the equidimensionality property and has empty fiber at $x = 0$ (resp. $x = 1$) is a direct consequence Proposition 5.5.
- Showing that \mathcal{L}_W^0 and \mathcal{L}_W^1 satisfy the expected differential equations follows from the homotopy property of the (twisted) multiplication given in Proposition 5.7.

Proof. We start the induction with the only Lyndon word of length 2: $W = 01$.

Example 5.9. In Section 3, we already considered the product

$$b = \mathcal{L}_0^1 \mathcal{L}_1^0 = [x; x, 1 - x] \subset X \times \square^2.$$

In other word, b is, up to projection onto the alternating elements, nothing but the graph of the function $X \rightarrow (\mathbb{P}^1)^2$ sending x to $(x, 1 - x)$. Its closure \bar{b} in $\mathbb{A}^1 \times \square^2$ is induced by the graph of $x \mapsto (x, 1 - x)$ viewed as a function from \mathbb{A}^1 to $(\mathbb{P}^1)^2$:

$$\bar{b} = [x; x, 1 - x] \subset \mathbb{A}^1 \times \square^2.$$

From this expression, we see that $\partial_{\mathbb{A}^1}(\bar{b}) = 0$.

Proposition 5.3 already ensures that b is equidimensional over X , as this is the case for both \mathcal{L}_0^1 and \mathcal{L}_1^0 . Then, in order to show that \bar{b} is equidimensional over \mathbb{A}^1 , it is enough to look at the fibers over 0 and 1. In both cases, the fiber is empty and \bar{b} is equidimensional over \mathbb{A}^1 . Now, set

$$\overline{\mathcal{L}_{01}^0} = \mu^*(\bar{b}) \quad \text{and} \quad \overline{\mathcal{L}_{01}^1} = \nu^*(\bar{b})$$

where μ^* and ν^* are defined in Proposition 5.5. The same proposition shows that $\overline{\mathcal{L}_{01}^0}$ and $\overline{\mathcal{L}_{01}^1}$ are equidimensional over \mathbb{A}^1 , and more precisely elements of $\mathcal{N}_{\mathbb{A}^1}^{eq, 1}(2)$. The same proposition shows that $\overline{\mathcal{L}_{01}^0}$ and $\overline{\mathcal{L}_{01}^1}$ have empty fiber at 0 and 1 respectively because \bar{b} has.

Since the fibers at 0 and 1 of \bar{b} are empty and $\partial_{\mathbb{A}^1}(\bar{b}) = 0$, we conclude from Proposition 5.7 that

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}^0}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}^1}) = \bar{b}.$$

Finally, we define

$$\mathcal{L}_{01}^0 = j^*(\overline{\mathcal{L}_{01}^0}) \quad \text{and} \quad \mathcal{L}_{01}^1 = j^*(\overline{\mathcal{L}_{01}^1})$$

where j is the inclusion $X \rightarrow \mathbb{A}^1$, and conclude using Proposition 5.4.

We can explicitly compute the two pull-backs, and obtain a parametric representation

$$\mathcal{L}_{01}^0 = [x; 1 - \frac{x}{t_1}, t_1, 1 - t_1], \quad \mathcal{L}_{01}^1 = [x; \frac{t_1 - x}{t_1 - 1}, t_1, 1 - t_1].$$

In order to compute the pull-back, observe that if $u = 1 - x/t_1$ then

$$\frac{x}{1 - u} = t_1.$$

Computing the pull-back by μ^* then comes down to simply rescaling the new \square^1 factor which arrives in first position. The case of ν^* is similar, but using the fact that for $u = \frac{t_1 - x}{t_1 - 1}$ we have

$$\frac{x - u}{1 - u} = t_1.$$

Let W be a Lyndon word of length p greater than or equal to 3. From now on, we assume that Theorem 5.8 holds for any Lyndon word of length strictly less than p . We set

$$A_{\mathcal{L}^0} = \sum_{U < V} a_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^0 + \sum_{U,V} b_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^1,$$

and

$$A_{\mathcal{L}^1} = \sum_{U < V} a'_{U,V} \mathcal{L}_U^1 \mathcal{L}_V^1 + \sum_{U,V} b'_{U,V} \mathcal{L}_U^0 \mathcal{L}_V^1,$$

Lemma 4.33 shows that $A_{\mathcal{L}^0}$ and $A_{\mathcal{L}^1}$ only involves Lyndon words U and V such that the sum of the length of U and the length of V is equal to the length of W . In particular the various coefficients are 0 as soon as U or V has length greater than or equal to W .

The induction hypothesis gives the existence of \mathcal{L}_U^0 and \mathcal{L}_V^1 for any U and V of smaller length, and by definition $\partial(\mathcal{L}_0^1) = \partial(\mathcal{L}_1^0) = 0$. So the combinatorial Proposition 4.34, with $A_U^0 = \mathcal{L}_U^0$ and $A_V^1 = \mathcal{L}_V^1$, shows that

$$(27) \quad \partial(A_{\mathcal{L}^0}) = \partial(A_{\mathcal{L}^1}) = 0.$$

Lemma 5.10 (extension to \mathbb{A}^1). *Let $\overline{A_{\mathcal{L}^0}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) denote the algebraic cycles in $\mathcal{Z}(\mathbb{A}^1 \times \square^{2p-2})$ obtained by taking the Zariski closure in $\mathbb{A}^1 \times \square^{2p-2}$ of each term in the formal sum defining $A_{\mathcal{L}^0}$ (resp. $A_{\mathcal{L}^1}$). Then*

- $\overline{A_{\mathcal{L}^0}}$ and $\overline{A_{\mathcal{L}^1}}$ are equidimensional over \mathbb{A}^1 with respect to any face of \square^{2p-2} ; i.e. $\overline{A_{\mathcal{L}^0}}$ and $\overline{A_{\mathcal{L}^1}}$ are in $\mathcal{N}_{\mathbb{A}^1}^{eq,2}(p)$.
- $A_{\mathcal{L}^0}$ has empty fiber at 0 and $A_{\mathcal{L}^1}$ has empty fiber at 1.
- $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) = \partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}}) = 0$

Proof. The cases of $A_{\mathcal{L}^0}$ and $A_{\mathcal{L}^1}$ are very similar, so we only discuss the case of $A_{\mathcal{L}^0}$.

Let U and V be Lyndon words different from 0 and 1, of respective length q and q' strictly smaller than the length p of W .

The induction hypothesis and Proposition 5.4 show that the equidimensional cycles over \mathbb{A}^1 $\overline{\mathcal{L}_U^0}$ (resp. $\overline{\mathcal{L}_V^1}$) and $\overline{\mathcal{L}_V^0}$ (resp. $\overline{\mathcal{L}_U^1}$) are the Zariski closure of \mathcal{L}_U^0 (resp. \mathcal{L}_U^1) and \mathcal{L}_V^0 (resp. \mathcal{L}_V^1) respectively.

Thus, Proposition 5.4 ensures that

$$\overline{\mathcal{L}_U^0} \cdot \overline{\mathcal{L}_V^0} = \overline{\mathcal{L}_U^0} \cdot \overline{\mathcal{L}_V^0} \in \mathcal{N}_{\mathbb{A}^1}^{eq,2}(p) \quad \text{and} \quad \overline{\mathcal{L}_U^0} \cdot \overline{\mathcal{L}_V^1} = \overline{\mathcal{L}_U^0} \cdot \overline{\mathcal{L}_V^1} \in \mathcal{N}_{\mathbb{A}^1}^{eq,2}(p).$$

and that the above products have empty fiber at 0, since it is the case for $\overline{\mathcal{L}_U^0}$ (Proposition 5.3).

In order to show that $A_{\mathcal{L}^0}$ extends in an equidimensional cycle over \mathbb{A}^1 , it is now enough to study the products $\mathcal{L}_0^1 \cdot \mathcal{L}_U^0$ and $\mathcal{L}_1^0 \cdot \mathcal{L}_V^1$ because Lemma 4.33 shows that those are the only types of product in $A_{\mathcal{L}^0}$ with a non zero coefficient involving \mathcal{L}_0^1 and \mathcal{L}_1^0 which are not equidimensional over \mathbb{A}^1 .

We show below that $\overline{\mathcal{L}_0^1 \mathcal{L}_U^0}$ is equidimensional over \mathbb{A}^1 and has empty fiber at 0.

We remark first that U has length $p-1$ and that \mathcal{L}_U^0 is in $\mathcal{N}_X^{eq,1}(p-1)$. Let Z be an irreducible component of $\text{Supp}(\mathcal{L}_U^0)$, and let \overline{Z} denote its Zariski closure

$$\mathbb{A}^1 \times \square^{2p-3}.$$

Then \overline{Z} is an equidimensional cycle over \mathbb{A}^1 because it is an irreducible component of $\text{Supp}(\overline{\mathcal{L}_U^0})$. Moreover, \overline{Z} has an empty fiber at 0.

Let Γ denote the graph of $\text{id} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Then we have

$$\mathcal{L}_0^1 = \text{Alt}(\Gamma \cap (X \times \square^1)) \quad \text{and} \quad \overline{\mathcal{L}_0^1} = \text{Alt}(\Gamma \cap (\mathbb{A}^1 \times \square^1)).$$

We simply write Γ_X and $\overline{\Gamma_X}$ for

$$\Gamma_X = \Gamma \cap (X \times \square^1) \quad \text{and} \quad \overline{\Gamma_X} = \Gamma \cap (\mathbb{A}^1 \times \square^1)$$

It is enough to show that $\overline{\Gamma_X} \cdot \overline{Z}$ is equidimensional over \mathbb{A}^1 (here \cdot denotes the product in $\mathcal{N}_X^{eq, \bullet}$). As the projection

$$\Gamma_X \times Z \longrightarrow X \times X$$

is equidimensional, one has

$$\overline{\Gamma_X} \cdot \overline{Z} \simeq (\overline{\Gamma_X} \times \overline{Z}) \cap \text{im}(\Delta_{\mathbb{A}^1})$$

where $\Delta_{\mathbb{A}^1} : \mathbb{A}^1 \times \square^{2p-2} \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1 \times \square^{2p-2}$.

Hence it is enough to show that for any face F of \square^{2p-2} the projection

$$(\overline{\Gamma_X} \times \overline{Z}) \cap (\text{im}(\Delta_{\mathbb{A}^1}) \cap \times F) \longrightarrow \text{im}(\Delta_{\mathbb{A}^1})$$

is either empty or dominant and equidimensional of relative dimension $\dim(F) - p$.

Restricting the above situation to $X \times X \subset \mathbb{A}^1 \times \mathbb{A}^1$, we see that it is enough to check that the fiber at $(0, 0)$ and $(1, 1)$ are empty because

$$(\Gamma_X \times Z) \cap (X \times X \times F)$$

is either empty or dominant and equidimensional of the right relative dimension over $X \times X$.

We write the face F as $F_1 \times F'$ with F^1 a face of \square^1 and F' a face of \square^{2p-3} . Using the fact that

$$\mathbb{A}^1 \times \mathbb{A}^1 \times \square^{2p-2} \simeq (\mathbb{A}^1 \times \square^1) \times (\mathbb{A}^1 \times \square^{2p-3}),$$

the fiber at $(1, 1)$ is given by

$$(\overline{\Gamma_X} \times \overline{Z}) \cap (\{(1, 1)\} \times F) \simeq (\overline{\Gamma_X} \cap \{1\} \times F_1) \times (\overline{Z} \cap \{1\} \times F') = \emptyset$$

because $\overline{\Gamma_X} \cap \{1\} \times F_1$ is empty in $\mathbb{A}^1 \times \square^1$ ($\overline{\Gamma_X}$ is the restriction of the graph of id).

Similarly, the fiber at $(0, 0)$ is given by

$$(\overline{\Gamma_X} \times \overline{Z}) \cap (\{(0, 0)\} \times F) \simeq (\overline{\Gamma_X} \cap \{0\} \times F_1) \times (\overline{Z} \cap \{0\} \times F') = \emptyset$$

because \overline{Z} has empty fiber at 0 (by induction hypothesis $\overline{\mathcal{L}_U^0}$ has empty fiber at 0).

Thus, $\overline{\Gamma_X} \overline{Z}$ is equidimensional over \mathbb{A}^1 with empty fiber at 0 (and also at 1) for any irreducible component Z of $\text{Supp}(\mathcal{L}_U^0)$. Hence $\overline{\mathcal{L}_0^1 \mathcal{L}_U^0}$ is equidimensional with respect to any face and has empty fiber at 0 (and at 1). Exchanging the role of 0 and 1, a similar argument shows that $\overline{\mathcal{L}_1^0 \mathcal{L}_V^1}$ is equidimensional over \mathbb{A}^1 and has empty fiber at 1 (and at 0).

The above discussion also shows that $\overline{A_{\mathcal{L}^0}}$ is equidimensional over \mathbb{A}^1 and has an empty fiber at 0.

Now, we need to show that

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) = 0.$$

One computes

$$j^*(\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}})) = \partial(j^*(\overline{A_{\mathcal{L}^0}})) = \partial(A_{\mathcal{L}^0}) = 0$$

as explained previously. The injectivity of j^* on equidimensional cycle (Proposition 5.4) insures that.

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) = 0.$$

□

The equality

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) = 0 \quad (\text{resp. } \partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}}) = 0)$$

shows that $\overline{A_{\mathcal{L}^0}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) gives a class in $H^2(\mathcal{N}_{\mathbb{A}^1}^\bullet)$. As Corollary 2 ensures that this cohomology group is 0, $\overline{A_{\mathcal{L}^0}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) is the boundary of some cycle c (resp. c') in $\mathcal{N}_{\mathbb{A}^1}^1$. Lemma 5.11 below gives this c (resp. c') explicitly and, after restriction to X , concludes the proof of Theorem 5.8. \square

Lemma 5.11. *Define $\overline{\mathcal{L}_W^0}$ and $\overline{\mathcal{L}_W^1}$ in $\mathcal{N}_{\mathbb{A}^1}^{eq,1}(p)$ by*

$$\overline{\mathcal{L}_W^0} = \mu^*(\overline{A_{\mathcal{L}^0}}) \quad \text{and} \quad \overline{\mathcal{L}_W^1} = \nu^*(\overline{A_{\mathcal{L}^1}})$$

where μ^* and ν^* are the morphisms defined in Proposition 5.5.

Let $j : X \rightarrow \mathbb{A}^1$ be the natural inclusion of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ into \mathbb{A}^1 and define \mathcal{L}_W^0 and \mathcal{L}_W^1 by

$$\mathcal{L}_W^0 = j^*(\overline{\mathcal{L}_W^0}) \quad \text{and} \quad \mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1}).$$

Then \mathcal{L}_W^0 and \mathcal{L}_W^1 satisfy the conditions of Theorem 5.8.

Proof. As in Proposition 5.7, let i_0 (resp. i_1) be the inclusion of 0 (resp. 1) in \mathbb{A}^1 :

$$i_0 : \{0\} \rightarrow \mathbb{A}^1 \quad i_1 : \{1\} \rightarrow \mathbb{A}^1,$$

and let p_0 and p_1 be the corresponding projection $p_\varepsilon : \mathbb{A}^1 \rightarrow \{\varepsilon\}$ for $\varepsilon = 0, 1$.

Proposition 5.5 ensures that $\overline{\mathcal{L}_W^0}$ (resp. $\overline{\mathcal{L}_W^1}$) is equidimensional over \mathbb{A}^1 with respect to any faces and has an empty fiber at $x = 0$ (resp. $x = 1$); in particular $i_0^*(\overline{A_{\mathcal{L}^0}}) = i_1^*(\overline{A_{\mathcal{L}^1}}) = 0$. Moreover, Proposition 5.7 enables us to compute $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W^0})$ as

$$\begin{aligned} \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W^0}) &= \partial_{\mathbb{A}^1} \circ \mu^*(\overline{A_{\mathcal{L}^0}}) \\ &= \text{id}(\overline{A_{\mathcal{L}^0}}) - p_0^* \circ i_0^*(\overline{A_{\mathcal{L}^0}}) - \mu^* \circ \partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) \\ &= \overline{A_{\mathcal{L}^0}} \end{aligned}$$

because $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^0}}) = 0$ and $i_0^*(\overline{A_{\mathcal{L}^0}}) = 0$.

Using Proposition 5.7 again, a similar computation gives

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W^1}) = \overline{A_{\mathcal{L}^1}}$$

because $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}}) = 0$ and $i_1^*(\overline{A_{\mathcal{L}^1}}) = 0$.

Now, as

$$\mathcal{L}_W^0 = j^*(\overline{\mathcal{L}_W^0}) \quad \text{and} \quad \mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1}),$$

\mathcal{L}_W^0 and \mathcal{L}_W^1 are equidimensional with respect to any faces over X by Proposition 5.4, and their closures in $\mathbb{A}^1 \times \square^{2p-1}$ are exactly $\overline{\mathcal{L}_W^0}$ and $\overline{\mathcal{L}_W^1}$. As j^* is a morphism of cdga's, \mathcal{L}_W^0 and \mathcal{L}_W^1 satisfy the expected differential equations, as do $\overline{\mathcal{L}_W^0}$ and $\overline{\mathcal{L}_W^1}$; that is

$$\partial(\mathcal{L}_W^0) = A_{\mathcal{L}^0} \quad \text{and} \quad \partial(\mathcal{L}_W^1) = A_{\mathcal{L}^1}.$$

This concludes the proof of the Lemma and of Theorem 5.8 \square

6. CONCLUDING REMARKS

6.1. Examples in weight 4 and 5. In weight 4, arises the first linear combination in the differential equation. In weight 5, arises the first case where the differential equation for \mathcal{L}_W^0 and \mathcal{L}_W^1 are not the same. There are actually two such examples in weight 5. We give below one of them.

There are three Lyndon word in weight 4: 0001, 0011 and 0111. The first linear combination arise from the word 0011. The image of

$$T_{0011^*}(x) = \begin{array}{c} x \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 0 \quad 0 \quad 1 \quad 1 \end{array} + \begin{array}{c} x \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 0 \quad 0 \quad 1 \quad 1 \end{array}$$

under d_{cy} , given at Equation 21, is

$$d_{cy}(T_{0011^*}(x)) = (T_{0^*}(x) - T_{0^*}(1)) \wedge T_{011^*}(x) + (T_{001^*}(x) - T_{001^*}(1)) \wedge T_{1^*}(x) + T_{01^*} \wedge T_{01^*}(1).$$

Hence the cycle $\overline{\mathcal{L}_{0011}^0}$ is defined as the pull-back by μ of

$$\overline{\mathcal{L}_0^1 \mathcal{L}_{011}^0} + \overline{\mathcal{L}_{001}^1 \mathcal{L}_1^0} + \overline{\mathcal{L}_{01}^1 \mathcal{L}_{01}^0}.$$

In weight 5 consider the Lyndon word 00101. Its corresponding tree $T_{00101^*}(x)$ is

$$\begin{array}{c} x \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 0 \quad 0 \quad 1 \quad 1 \end{array} - \begin{array}{c} x \odot \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 0 \quad 0 \quad 1 \quad 1 \end{array}$$

and computing $d_{cy}(T_{00101^*}(x))$ gives

$$d_{cy}(T_{00101^*}) = T_{001^*}(x) \wedge T_{01^*}(x) - T_{0001^*}(x) \wedge T_{1^*}(x) - T_{1^*}(x) \wedge T_{0001^*}(1).$$

Finally, \mathcal{L}_{00101}^0 and \mathcal{L}_{00101}^1 satisfy respectively

$$(28) \quad \partial(\mathcal{L}_{00101}^0) = \mathcal{L}_{001}^0 \mathcal{L}_{01}^0 - \mathcal{L}_{0001}^1 \mathcal{L}_1^0$$

and

$$(29) \quad \partial(\mathcal{L}_{00101}^1) = -\mathcal{L}_{001}^1 \mathcal{L}_{01}^1 - \mathcal{L}_{01}^1 \mathcal{L}_{001}^0 + \mathcal{L}_{001}^1 \mathcal{L}_{01}^0 - \mathcal{L}_{0001}^1 \mathcal{L}_1^0.$$

6.2. An integral associated to \mathcal{L}_{011}^0 . In this section, we sketch how to associate an integral to the cycle \mathcal{L}_{011}^0 . We directly follow the algorithm described in [GGL09][Section 9] and put in detailed practice in [GGL07]. There will be no general review of the direct Hodge realization from Bloch-Kriz motives [BK94][Section 8 and 9]. Gangl, Goncharov and Levine's construction seems to consist in setting particular choices of representatives in the intermediate Jacobians of \square^n in relation with their algebraic cycles.

We do not extend this description here, nor do we generalize the computations below. Relating the Bloch-Kriz approach to the explicit algorithms described by Gangl, Goncharov and Levine and the application to our particular family of cycles \mathcal{L}_W^0 will be the topic of a future paper, as it requires, in particular, a family $\mathcal{L}_W^{0,B}$ of elements in $H^0(B(\mathcal{N}_X^\bullet))$ not yet at our disposal.

Let us recall the expression of \mathcal{L}_{011}^0 as parametrized cycle:

$$\begin{aligned} \mathcal{L}_{011}^0 &= [x, 1 - \frac{x}{t_2}, \frac{t_1 - t_2}{t_1 - 1}, t_1, 1 - t_1, 1 - t_2] \\ &= -[x; 1 - \frac{x}{t_2}, 1 - t_2, \frac{t_1 - t_2}{t_1 - 1}, t_1, 1 - t_1]. \end{aligned}$$

where the second expression is due to alternating projector and seems to the author more suitable for the computations below.

One wants to bound \mathcal{L}_{011}^0 by an algebraic-topological cycle in a larger bar construction (not described here). This is done by introducing topological variables s_i in real simplices

$$\Delta_s^n = \{0 \leq s_1 \leq \dots \leq s_n \leq 1\}.$$

Let $d^s : \Delta_s^n \rightarrow \Delta_s^{n-1}$ denote the simplicial differential

$$d^s = \sum_{k=0}^n (-1)^k i_k^*$$

where $i_k : \Delta_s^{n-1} \rightarrow \Delta_s^n$ is given by the face $s_k = s_{k+1}$ in Δ_s^n with the usual conventions for $k = 0, n$.

Let us define

$$C_{011}^{s,1} = [x; 1 - \frac{s_3 x}{t_2}, 1 - t_2, \frac{t_1 - t_2}{t_1 - 1}, t_1, 1 - t_1]$$

for s_3 going from 0 to 1. Then, $d^s(C_{011}^{s,1}) = \mathcal{L}_{011}^0$ as $s_3 = 0$ implies that the first cubical coordinate is 1.

Now, the algebraic boundary ∂ of $C_{011}^{s,1}$ is given by the intersection with the codimension 1 faces of \square^5 ; giving

$$\partial(C_{011}^{s,1}) = [x; 1 - s_3 x, \frac{t_1 - s_3 x}{t_1 - 1}, t_1, 1 - t_1].$$

We can again bound this cycle by introducing a new simplicial variable $0 \leq s_2 \leq s_3$, and the cycle

$$C_{011}^{s,2} = [x; 1 - s_3 x, \frac{t_1 - s_2 x}{t_1 - s_2/s_3}, t_1, 1 - t_1].$$

The intersections with the faces of the simplex $\{0 \leq s_2 \leq s_3 \leq 1\}$ given by $s_2 = 0$ and $s_3 = 1$ lead to empty cycles (as at least one cubical coordinate equals 1) and a negligible cycle respectively. Thus, the simplicial boundary of $C_{011}^{s,2}$ satisfies (up to a negligible cycles)

$$d^s(C_{011}^{s,2}) = -\partial(C_{011}^{s,1}) = -[x; 1 - s_3 x, \frac{t_1 - s_3 x}{t_1 - 1}, t_1, 1 - t_1].$$

Its algebraic boundary is given by

$$\partial(C_{011}^{s,2}) = -[x; 1 - s_3 x, s_2 x, 1 - s_2 x] + [x; 1 - s_3 x, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}].$$

Finally, we introduce a last simplicial variable $0 \leq s_1 \leq s_2$ and a purely topological cycle

$$\widetilde{C}_{011}^{s,3} = -[x; 1 - s_3 x, s_2 x, 1 - s_1 x] + [x; 1 - s_3 x, \frac{s_2}{s_3}, 1 - \frac{s_1}{s_3}].$$

Its simplicial differential is (up to negligible terms) given in one hand by the face $s_1 = s_2$:

$$[x; 1 - s_3 x, s_2 x, 1 - s_2 x] - [x; 1 - s_3 x, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}]$$

which equals to $-\partial(C_{011}^{s,2})$ and in the other hand by the face $s_2 = s_3$:

$$-[x; 1 - s_3 x, s_3 x, 1 - s_1 x].$$

In order to cancel this extra boundary, we define

$$C_{011}^{s,3} = \widetilde{C}_{011}^{s,3} + [x; 1 - s_2 x, s_3 x, 1 - s_1 x]$$

whose algebraic boundary is 0.

Finally, we have

$$(d^s + \partial)(C_{011}^{s,1} + C_{011}^{s,2} + C_{011}^{s,3}) = \mathcal{L}_{011}^0$$

up to negligible terms.

Now, we fix the situation at the fiber x_0 and following Gangl, Goncharov and Levin, we associate to the algebraic cycle $\mathcal{L}_{011}^0|_{x=x_0}$ the integral $I_{011}(x_0)$ of the standard volume form

$$\frac{1}{(2i\pi)^3} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}$$

over the simplex given by $C_{011}^{s,3}$. That is:

$$\begin{aligned} I_{011}(x_0) = & -\frac{1}{(2i\pi)^3} \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{x_0 ds_3}{1-x_0s_3} \wedge \frac{ds_2}{s_2} \wedge \frac{x_0 ds_1}{1-x_0s_1} \\ & + \frac{1}{(2i\pi)^3} \int_{0 \leq s_3 \leq 1} \frac{x_0 ds_3}{1-x_0s_3} \int_{0 \leq s_1 \leq s_2 \leq 1} \frac{ds_2}{s_2} \wedge \frac{ds_1}{1-s_1} \\ & + \frac{1}{(2i\pi)^3} \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{x_0 ds_2}{1-x_0s_2} \wedge \frac{ds_3}{s_3} \wedge \frac{x_0 ds_1}{1-x_0s_1}. \end{aligned}$$

Taking care of the change of sign due to the numbering, the first term in the above sum is (for $x_0 \neq 0$ and up to the factor $(2i\pi)^{-3}$) equal to

$$Li_{1,2}^{\mathbb{C}}(x_0) = \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 1} \frac{ds_1}{x_0^{-1} - s_1} \wedge \frac{ds_2}{s_2} \wedge \frac{ds_3}{x_0^{-1} - s_3}$$

while the second term (up to the same multiplicative factor) equals

$$-Li_1^{\mathbb{C}}(x_0) Li_2^{\mathbb{C}}(1)$$

and the third factor (up to the inverse power of $2i\pi$) equals

$$Li_{2,1}^{\mathbb{C}}(x_0).$$

Globally the integral is well defined for $x_0 = 0$ and, more interestingly, also for $x_0 = 1$, as the divergencies when x_0 goes to 1 cancel each other in the above sums. A simple computation and the shuffle relation for $Li_1^{\mathbb{C}}(x_0) Li_2^{\mathbb{C}}(x_0)$ shows that the integral associated to the fiber of \mathcal{L}_{011}^0 at $x_0 = 1$ is

$$(2i\pi)^3 I_{011}(1) = -Li_{2,1}^{\mathbb{C}}(1) = -\zeta(2, 1).$$

6.3. Comments about quasi-finite cycles setting. In this paper we have decided to work with cycles in the original Bloch complex \mathcal{N}_X^\bullet having the extra property that their projection onto the base X is equidimensional. This allows us to compare easily our construction to previously built explicit cycles related to polylogarithms ([BK94]) or multiple polylogarithms (with conditions on the parameters in [GGL09]). Moreover, after computing the structures coefficients of $\mathcal{T}_{1;x}^{coL}$ (Definitions 14) up to some weight, our cycles can be written explicitly as parametrized cycles because the pull-back by the multiplication μ^* introduces a terms in $1 - \frac{x}{t_{p-1}}$ on the new \square^1 factor while the pull-back by the twisted multiplication ν^* introduces a terms in $\frac{t_{p-1}-x}{t_{p-1}-1}$.

However, in classical motivic constructions, one prefers to work with *quasi-finite* cycles (cf. Definition 6.1). Quasi-finite cycles behave better in particular when X is of dimension $d \geq 2$, with functoriality or with a sheaf theoretic approach. They are defined following Levine [Lev11] as follows.

Definition 6.1. Let Y be an irreducible smooth variety.

- Let $\mathcal{Z}_{q.f.}^p(Y, n)$ denote the free abelian group generated by irreducible closed subvarieties

$$Z \subset Y \times \square^n \times (\mathbb{P}^1 \setminus \{1\})^p$$

the restriction of the projection on $Y \times \square^n$,

$$p_1 : Z \longrightarrow Y \times \square^n,$$

is dominant and quasi finite (that is of pure relative dimension 0).

- We say that elements of $\mathcal{Z}_{q.f.}^p(Y, n)$ are *quasi-finite*.
- The symmetric group \mathfrak{S}_p acts on $\mathcal{Z}_{q.f.}^p(Y, n)$ by permutation of the factor in $(\mathbb{P}^1 \setminus \{1\})^p$. Let $Sym_{\mathbb{P}^1 \setminus \{1\}}^p$ denotes the projector corresponding to the *symmetric* representation.
- Following the definition of $\mathcal{N}_Y^k(p)$, let $\mathcal{N}_Y^{q.f. k}(p)$ denote

$$\mathcal{N}_Y^{q.f. k}(p) = Sym_{\mathbb{P}^1 \setminus \{1\}}^p \circ Alt_{2p-k} \left(\mathcal{Z}_{q.f.}^p(Y, 2p-k) \otimes \mathbb{Q} \right).$$

- As in the classical case, the intersection with codimension 1 faces of \square^{2p-k} induces a differential

$$\partial_Y = \sum_{i=1}^{2p-k} (-1)^{i-1} (\partial_i^0 - \partial_i^\infty)$$

of degree 1.

- We define the complex of quasi finite cycles as

$$\mathcal{N}_X^{q.f. \bullet} = \mathbb{Q} \oplus \bigoplus_{p \geq 1} \mathcal{N}_X^{q.f. \bullet}(p).$$

The product structure given by concatenation of factors and pull-back by the diagonal makes $\mathcal{N}_X^{q.f. \bullet}$ into a cdga. The cohomology of $\mathcal{N}_X^{q.f. \bullet}$ agrees with higher Chow groups by [Lev11, Lemma 4.2.1].

Remark 6.2. The condition on the quasi finite cycles is much more strict than the one for our equidimensional cycles as it requires equidimensionality (of dimension 0) over

$$Y \times \square^n$$

and not only over Y as in the case of equidimensional cycle.

However in the case of $\mathcal{N}_X^{q.f. \bullet}(p)$ the ambient space is much more larger due to the extra $(\mathbb{P}^1 \setminus \{1\})^p$ factors.

Because of this, propositions 5.3 – point (3) –, 5.4, 5.5 and 5.7 hold with $\mathcal{N}_-^{q.f. \bullet}$ instead of $\mathcal{N}_-^{eq. \bullet}$.

In order to obtain an equivalent of Theorem 5.8 in $\mathcal{N}_X^{q.f. \bullet}$, we need to have quasi-finite cycles $\mathcal{L}_{qf,0}$ and $\mathcal{L}_{qf,1}$ avatar of \mathcal{L}_0^1 and \mathcal{L}_1^0 in $\mathcal{N}_X^{q.f. 1}$. Once this is granted, Lemma 5.10 holds in $\mathcal{N}_X^{q.f. 2}$ by the same arguments using quasi finiteness over

$$X \times \square^n$$

and the empty fiber properties.

We define below these cycles $\mathcal{L}_{qf,0}$ and $\mathcal{L}_{qf,1}$ which (up to the two projectors) are each given by a single irreducible variety of

$$X \times \square^1 \times (\mathbb{P}^1 \setminus \{1\}).$$

Let x denote the standard affine coordinates on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$; let $[U : V]$ be the standard projective coordinate on \square^1 and $[A : B]$ the one on $\mathbb{P}^1 \setminus \{1\}$. Let Z_0 be defined by the following equation

$$Z_0 : (U - V)(A - B)(U - xV) + x(1 - x)UVB = 0.$$

Similarly, let Z_1 be defined by the following equation

$$Z_1 : (U - V)(A - B)(U - (1 - x)V) + x(1 - x)UVB = 0.$$

- Proposition 6.3.** (1) Let $\mathcal{L}_{qf,0}$ and $\mathcal{L}_{qf,1}$ the images under $Sym_{\mathbb{P}^1 \setminus \{1\}}^1 \circ Alt_1$ of Z_0 and Z_1 respectively. Then $\mathcal{L}_{qf,0}$ and $\mathcal{L}_{qf,1}$ are elements of $\mathcal{N}_X^{q.f. 1}$.
- (2) There Zariski closures $\overline{\mathcal{L}_{qf,0}}$ and $\overline{\mathcal{L}_{qf,1}}$ to \mathbb{A}^1 are in $\mathcal{N}_{\mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{1\})^2}^1(1)$ with empty fiber over $\{1\} \times (\mathbb{P}^1 \setminus \{1\})^2$ and $\{0\} \times (\mathbb{P}^1 \setminus \{1\})^2$ respectively.

- (3) *The Zariski closure of there product $\overline{\mathcal{L}_{qf,0}\mathcal{L}_{qf,1}}$ is an element of $\mathcal{N}_{\mathbb{A}^1}^{q.f.2}(2)$ with empty fiber at 0 and 1.*

Proof. Part (3) is a consequence of part (2). The case of $\mathcal{L}_{qf,0}$ and $\mathcal{L}_{qf,1}$ are symmetric with respect to the role of 0 and 1. Hence, we only treat the case of $\mathcal{L}_{qf,0}$.

Note that the projection $Z_0 \rightarrow X \times \square^1$ is dominant because the defining equation is homogeneous of degree 1 in A and B .

Note that $U - V \neq 0$ in \square^1 . In order to check the quasi-finiteness, one writes the equation as

$$A(U - V)(U - xV) = B((U - V)(U - xV) + x(1 - x)UV).$$

Remarks that for $x, 1 - x, U$ and V all non zero, the equation determines uniquely $[A : B]$. For $U = 0$ the equation becomes

$$Ax = Bx.$$

Hence $[A : B]$ should be uniquely determined provided that $x \neq 0$ (which holds in $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$). Recall that $[A : B]$ is the projective coordinate on $\mathbb{P}^1 \setminus \{1\}$. Thus $A - B$ is invertible and

$$Z_0 \cap (X \times \{U = 0\} \times (\mathbb{P}^1 \setminus \{1\})) = \emptyset.$$

For $V = 0$, the equation becomes

$$A = B$$

without restriction on x . Hence we have

$$Z_0 \cap (X \times \{V = 0\} \times (\mathbb{P}^1 \setminus \{1\})) = \emptyset.$$

The above discussion shows that Z_0 is quasi-finite over X . It also shows that its Zariski closure $\overline{Z_0}$ is not quasi-finite over \mathbb{A}^1 because of the fiber at $x = 0$. However one has

$$\overline{Z_0} \cap (\mathbb{A}^1 \times \{V = 0\} \times (\mathbb{P}^1 \setminus \{1\})) = \emptyset.$$

and

$$\overline{Z_0} \cap (\mathbb{A}^1 \times \{U = 0\} \times (\mathbb{P}^1 \setminus \{1\})) = \{0\} \times \{U = 0\} \times (\mathbb{P}^1 \setminus \{1\})$$

which is of codimension 1 in $\mathbb{A}^1 \times \{U = 0\} \times (\mathbb{P}^1 \setminus \{1\})$. This shows that $\overline{Z_0}$ is an admissible cycle over $\mathbb{A}^1 \times (\mathbb{P}^1 \setminus \{1\})$. One remarks that the above intersection is also only non empty the fiber of $\overline{Z_0}$ over $\{0\} \times \square^1$.

The fiber of $\overline{Z_0}$ over 1 is given by the equation

$$A(U - V)^2 = B(U - V)^2$$

and is empty in $\mathbb{A}^1 \times \square^1 \times (\mathbb{P}^1 \setminus \{1\})$ because there $U - V$ and $A - B$ are invertible. \square

From the above discussion we obtain

Theorem 6.4. *For any Lyndon word W of length $p \geq 2$, there exist two cycles $\mathcal{L}_{qf,W}^0$ and $\mathcal{L}_{qf,W}^1$ in $\mathcal{N}_X^{q.f.1}(p)$ satisfying conditions of Theorem 5.8 where quasi-finite cycles replace equidimensional ones.*

Remark 6.5. The proof of Proposition 6.3 also shows that

$$\mathbb{H}^1 \left(\mathcal{N}_{\mathbb{P}^1 \setminus \{0,1,\infty\}}^{q.f.1} \right) \simeq \mathbb{H}^1 \left(\mathcal{N}_{\mathbb{Q}}^{q.f.1} \right) \oplus \left(\mathbb{H}^0 \left(\mathcal{N}_{\mathbb{Q}}^{q.f.1} \right) \otimes \mathbb{Q}\mathcal{L}_{qf,0} \right) \oplus \left(\mathbb{H}^0 \left(\mathcal{N}_{\mathbb{Q}}^{q.f.1} \right) \otimes \mathbb{Q}\mathcal{L}_{qf,1} \right).$$

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