Cycle complex over the projective line minus three points : toward multiple zeta values cycles

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CYCLE COMPLEX OVER $\mathbb{P}^1$ MINUS 3 POINTS : TOWARD MULTIPLE ZETA VALUES CYCLES.

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Abstract. In this paper, the author constructs a family of algebraic cycles in Bloch's cycle complex over $\mathbb{P}^1$ minus three points which are expected to correspond to multiple polylogarithms in one variable. Elements in this family of weight p are in the cubical cycle group of codimension p in $(\mathbb{P}^1 \setminus \{0,1,\infty\}) \times (\mathbb{P}^1 \setminus \{1\})^{2p-1}$ and are, in weight greater or equal to 2, naturaly extended as equidimensional cycles over over $\mathbb{A}^1$.

This allows to consider their fibers at the point 1 and this is one of the main differences with the work of Gangl, Goncharov and Levin. Considering the fiber at 1 makes it possible to think of these cycles as corresponding to weight n multiple zeta values.

After the introduction, the author recalls some properties of Bloch's cycle complex and enlightens the difficulties on a few examples. Then a large section is devoted to the combinatorial situation involving the combinatoric of trivalent trees. In the last section, two families of cycles are constructed as solutions to a "differential system" in Bloch's cycle complex. One of this families contains only cycles with empty fiber at 0 which should correspond to multiple polylogarithms.

Contents

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1. Introduction

1.1. **Multiple polylogarithms.** The multiple polylogarithm functions were defined in [Gon95] by the power series

$$Li_{k_1,\dots,k_m}(z_1,\dots,z_m) = \sum_{n_1>\cdots>k_m} \frac{z_1^{n_1}}{n_1^{k_1}} \frac{z_2^{n_2}}{n_2^{k_2}} \cdots \frac{z_m^{n_m}}{n_m^{k_m}} \qquad (z_i \in \mathbb{C},\ |z_i| < 1).$$

They admit an analytic continuation to a Zariski open subset of $\mathbb{C}^m$. The case $m = 1$ is nothing but the classical *polylogarithm* functions. The case $z_1 = z$ and $z_2 = \cdots = z_m = 1$ gives a one variable version of multiple polylogarithm function

$$Li^{\mathbb{C}}_{k_1,\dots,k_m}(z) = Li_{k_1,\dots,k_m}(z_1, 1, \dots, 1) = \sum_{n_1>\cdots>k_m} \frac{z_1^{n_1}}{n_1^{k_1} n_2^{k_2} \cdots n_m^{k_m}}.$$

When k_1 is greater or equal to 2, the series converge as z goes to 1 and one recovers the multiple zeta value

$$\zeta(k_1,\dots,k_m) = Li^{\mathbb{C}}_{k_1,\dots,k_m}(1) = Li_{k_1,\dots,k_m}(1,\dots,1) = \sum_{n_1>\cdots>k_m} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_m^{k_m}}.$$

To the tuple of integer $(k_1,\dots,k_m)$ of weight $n = \sum k_i$, we can associate a tuple of 0 and 1

$$(\varepsilon_n,\dots,\varepsilon_1) := (\underbrace{0,\dots,0}_{k_1-1 \text{ times}},1,\dots,\underbrace{0,\dots,0}_{k_p-1 \text{ times}},1)$$

which allows to write multiple polylogarithms as iterated integrals ($z_i \neq 0$ for all i)

$$Li_{k_1,\dots,k_m}(z_1,\dots,z_m) = (-1)^m \int_{\Delta_\gamma} \frac{dt_1}{t_1 - \varepsilon_1 x_1} \wedge \frac{dt_m}{t_n - \varepsilon_n x_n}$$

where γ is a path from 0 to 1 in $\mathbb{C} \setminus \{x_1,\dots,x_n\}$, the integration domain Δ_γ is the associated real simplex consisting of all m-tuples of points $(\gamma(t_1),\dots,\gamma(t_n))$ with $t_i < t_j$ for $i < j$ and where we have set $x_n = z_1^{-1}$, $x_{n-i} = (z_1 \cdots z_l)^{-1}$ for $k_1 + \cdots + k_{l-1} + 1 \leqslant i < k_1 + \cdots + k_l$ and $x_1 = (z_1 \cdots z_m)^{-1}$.

As shown in [Gon05a], iterated integrals have Hodge/motivic avatars living in a Hopf algebra equivalent to the tannakian Hopf algebra of $\mathbb{Q}$ mixed Hodge-Tate structure. Working with these motivic/Hodge iterated integrals allows to see more structure, in particular the coproduct which is not visible on the level of numbers, conjecturaly without losing any information.

1.2. **Multiple polylogarithms and algebraic cycles.** Considering the relations between the motivic word and the higher Chow groups in one hand (e.g. [Lev05, Voe02]) and in the other hand the relations between multiple polylogarithm and regulators (e.g. [Zag91, Gon05b]), it is reasonable to ask whether there exists avatars of the multiple polylogarithms in terms of algebraic cycles.

Given a number field $\mathbb{K}$, in [BK94], Bloch and Kriz have constructed using algebraic cycles, a graded Hopf algebra, isomorphic to the tannakian Hopf algebra of the category of mixed Tate motive over $\mathbb{K}$ ([Spi01] and latter [Lev11]), together with a direct Hodge realization for this "cyclic motives". Moreover for any integer n greater or equal to 2 and any point z in $\mathbb{K}$ they have produced an algebraic cycle $\mathrm{Li}_n^{cy}(z)$. This cycle $\mathrm{Li}_n^{cy}(z)$ induces a motive. They have shown at Theorem 9.1 that the "bottom-left" coefficient of the periods matrix in the Hodge realization is exactly $-Li_n(z)/(2i\pi)^n$.

More recently, Gangl, Goncharov and Levin, using a combinatorial approach, have built algebraic cycles corresponding to the multiple polylogarithm values $Li_{k_1,\dots,k_m}(z_1,\dots,z_m)$ with parameters z_i in $\mathbb{K}^*$ under the condition that the corresponding x_i (as defined above) are all distinct. In particular, all the z_i but z_1 have to be different from 1 and their methods does not gives algebraic cycles corresponding to multiple zeta values.

1.3. **Algebraic cycles over $\mathbb{P}^1 \setminus \{0,1,\infty\}$.** The goal of my project is to develop a geometric construction for multiple polylogarithm cycles removing the previous obstruction which will allow to have multiple zeta cycles.

A general idea underlying this project consists on looking cycles fibered over a larger base and not just point-wise cycles for some fixed parameter $(z_1,...,z_m)$. Levine, in [Lev11], shows that there exists a short exact sequence relating the Bloch-Kriz Hopf algebra over $\mathrm{Spec}(\mathbb{K})$, its relative version over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and the Hopf algebra associated to Goncharov and Deligne's motivic fundamental group over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which contains the motivic iterated integrals associated to multiple polylogarithms in one variable.

As this one variable version of multiple polylogarithms gives multiple zeta values for $z = 1$, it is natural to investigate first the case of Bloch-Kriz construction over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ in order to obtain algebraic cycles corresponding to multiple polylogarithms in one variable with a "good specialization" at 1. However before computing any Hodge realization matrix periods, one needs first to obtain explicit algebraic cycles over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which can be specialized at 1 and have a chance to correspond to multiple zeta values. This paper gives such a class of cycles and final remarks gives some evidences that it is a good family by computing an integral in low weight.

1.4. **Strategy and Main results.** Bloch and Kriz Hopf algebra and its relative version over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is the H^0 of the bar construction over a commutative differential algebra (c.d.g.a) $\mathcal{N}_X^\bullet$ build out of algebraic cycles. We will use this construction in the case $\mathbb{K} = \mathbb{Q}$ and $X = \mathrm{Spec}(\mathbb{Q})$ or $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ or $X = \mathbb{A}^1$. This c.d.g.a comes form the cubical construction of the higher Chow groups and one has with $\square^1 = \mathbb{P}^1 \setminus \{1\} \simeq \mathbb{A}^1$:

$$\mathcal{N}_X^\bullet = \mathbb{Q} \oplus (\oplus_{p \geqslant 1} \mathcal{N}_k^\bullet(p))$$

where the $\mathcal{N}_X^n(p)$ are generated by codimension p cycles in $X \times \square^{2p-n}$ which are in good position. The cohomology of the complex $\mathcal{N}_X^\bullet(p)$ give back the higher Chow groups $\mathrm{CH}^p(\mathrm{Spec}(k), 2p - \bullet)$.

As the H^0 of the bar construction over $\mathcal{N}_{\mathbb{P}^1\setminus\{0,1,\infty\}}^\bullet$ is related to its 1-minimal model. The strategy to obtain our family of cycles is to follow the inductive construction of this 1-minimal model which gives a generalized nilpotent c.d.g.a

$M = \Lambda(V)$ together with a map $\phi : M \longrightarrow \mathcal{N}^{\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ inducing an isomorphism on the H^0 and the H^1 and an injection on the H^2.

More precisely this construction begins with the $V_1 = \mathrm{H}^1(\mathcal{N}^{\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}})$. Then, the inductive step of the 1-minimal model construction goes as follows, defining $V_{i+1} = V_i \oplus \ker(\mathrm{H}^2(\Lambda^{\bullet}(V_i)) \longrightarrow \mathrm{H}^2(\mathcal{N}^{\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$, one chooses specific representatives in $\mathcal{N}^{\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ of basis elements of the above kernel considering the diagram

$$\begin{array}{ccc} \Lambda^2(V_i) & \xrightarrow{\varphi_i} & \mathcal{N}^2_{\mathbb{P}^1\setminus\{0,1,\infty\}} \\ \\ b & \longmapsto & \sum \alpha_{i,j} c_i \cdot c_j \xmapsto{d} 0 \\ & & \hat{d} \uparrow \\ & & \exists c \in \mathcal{N}^1_{\mathbb{P}^1\setminus\{0,1,\infty\}} \end{array}$$

A particular choice of such a c corresponding to b induces the map $V_{i+1} \xrightarrow{\varphi_{i+1}} \mathcal{N}^{\bullet}_X$ and allows the inductive construction to go on.

Hence, one wants first to find inductively linear combinations $\sum \alpha_{i,j} c_i \cdot c_j$ that have a zero differential and under what conditions they can be written as an explicit boundary, that is as $d(c)$ for some explicit cycle c in $\mathcal{N}^1_X$.

In weight p, the considered linear combinations are built out of elements obtained in lower weight and under some geometric conditions (equidimensionality over $\mathbb{A}^1$ and empty fiber at 0), the cycle c can be constructed easily. It is the pull-back of $\sum \alpha_{i,j} c_i \cdot c_j$ induce by the multiplication map ($\square^1 \simeq \mathbb{A}^1$, $X = \mathbb{A}^1$):

$$\begin{array}{ccc} X \times \mathbb{A}^1 \times \mathbb{A}^{2p-2} & \longrightarrow & X \times \mathbb{A}^{2p-2} \\ \\ (t, s, x_1, \ldots, x_{2p-2}) & \longmapsto & (ts, x_1, \ldots, x_{2p-2}). \end{array}$$

Even though it is not formalized in their paper, it is reasonable to believe that Bloch and Kriz used this idea to build the cycles $\mathrm{Li}^{cy}_n(z)$. Thus, we naturally find back these cycles using the method described above. However, the cycles corresponding to multiple polylogarithms built using this method are different from the one proposed by Gangl, Goncharov and Levin.

In particular, the geometric conditions on $\sum \alpha_{i,j} c_i \cdot c_j$ and the computation of the pull-back in the above construction oblige the constructed cycles on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to admit an extension in $\mathcal{N}^{\bullet}_{\mathbb{A}^1}$ equidimensional over $\mathbb{A}^1$ and with empty fiber at 0 or 1.

A complete description of the inductive construction is based on a combinatorial setting using the coLie algebra dual to the defined a differential d^{cy} on trees which is closely related to the differential in $\mathcal{N}^{\bullet}_X$. The main point of the combinatorial setting is the following result (Theorem 3.44).

Theorem. *For any Lyndon word W in the letters $\{0, 1\}$, the element $T_W(t)$ is decomposable:*

$$d_{cy}(T_W(t)) = \sum \alpha_{i,j} T_{W_i}(t) \cdot T_{W_j}(t) + \sum \beta_{i,j} T_{W_k}(t) \cdot T_{W_l}(1) \tag{1}$$

where $\cdot$ denotes the disjoint union of trees and where W_i, W_j, W_k and W_l are Lyndon of smaller length than W.

This result gives us the combinatorial structure of the elements we want to built and allows us to construct two explicit families of cycles in a general framework. Modifying the above "differential system", one inductively constructs cycles $\mathcal{L}_W$ corresponding to $T_W(t)$ and cycles $\mathcal{L}^1_W$ corresponding to the difference $T_W(t) - T_W(1)$. In this way, one obtains, at Theorem 4.12, algebraic cycles that are expect to correspond to multiple zeta values when specialized at 1.

Theorem. *Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For any Lyndon word W in the letters $\{0, 1\}$ of length p greater or equal to 2, there exists cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ in $\mathcal{N}^1_X(p)$, that is cycles of codimension 1 in $X \times \square^{2p-1}$ such that*

- *$\mathcal{L}_W$ (resp. $\mathcal{L}^1_W$) admits an equidimensional extension to $\mathbb{A}^1$ with empty fiber at 0 (resp. 1).*
- *$\mathcal{L}_W$ (resp. $\mathcal{L}^1_W$), as its extensions to $\mathbb{A}^1$, satisfies*

$$d(\mathcal{L}_W) = \sum a_{i,j}\mathcal{L}_{W_i} \cdot \mathcal{L}_{W_j} + \sum b_{k,l}\mathcal{L}_{W_k} \cdot \mathcal{L}^1_{W_l} \tag{2}$$

$$\Big(\textit{resp.} \quad d(\mathcal{L}^1_W) = \sum a'_{i,j}\mathcal{L}^1_{W_i} \cdot \mathcal{L}^1_{W_j} + \sum b'_{k,l}\mathcal{L}_{W_k} \cdot \mathcal{L}^1_{W_l} + \sum c'_l\mathcal{L}_0 \cdot \mathcal{L}_{W_l} \quad \Big) \tag{3}$$

where coefficients $a_{i,j}$, $a'_{i,j}$, $b_{k,l}$, $b'_{k,l}$ and c'_l are derived from (1) (see Definition 4.1 and Proposition 4.3) and such that the above equations involved only words of smaller length than p

In particular,

- $\mathcal{L}_W|_{\{t=1\}}$ gives an element of $\mathcal{N}^\bullet_{\mathbb{Q}}$ which is expected to correspond to a multiple zeta value as $\mathcal{L}_W|_{\{t=z\}}$ is expected to correspond to a multiple polylogarithm at z. The computation of the actual integral for $W = 011$ is done in the last section.
- The two R.H.S in (2) and (3) admit equidimensional extensions over $\mathbb{A}^1$ and have empty fiber at 0 (resp. 1). Their pull-back by the multiplication (resp. a twisted multiplication) gives $\mathcal{L}_W$ (resp. $\mathcal{L}^1_W$).

The paper is organized as follows :

- The next section (Section 2) is devoted to a general review of Bloch-Kriz cycle complex. After recalling some basis definitions and properties of (Adams graded) c.d.g.a, their 1-minimal model and the bar construction, we detail the construction of the cycle complex. Then, we recall some of its main properties (relation to higher Chow groups, localization long exact sequence, etc.), some applications and the relations with mixed Tate motives. We conclude this section by applying our strategy to the nice example of polylogarithms as described in [BK94] and present the main difficulties trough a weight 3 example.
- Afterward in Section 3, we deal with the combinatorial situation first presenting the trivalent trees attached to Lyndon words and their relations with the free Lie algebra. Then, we present linear combinations of trees T_W corresponding to the dual situation and study some of their properties. From there, we review the construction of the differential graded algebra of R-deco forest introduced in [GGL09] and study the behavior the sums T_W under the differential d_{cy}. This leads to Theorem 3.44 and some relations satisfy by the coefficients appearing in this theorem (Cf. Equation (1) or Equation (26)).
- Section 4 proves our main Theorem. It begins with a purely combinatorial statement relating the trees situation of Equation (26) to its modified geometric version used at Theorem 4.12 (Cf. also above Equations (2) and (3)). Then, we present some properties of equidimensional algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\mathbb{A}^1$, study the relation between the two situation and

explain how the pull-back by the multiplication (resp. a twisted multiplication) gives a homotopy between the identity and the fiber at 0 (resp. at 1) pulled-back to a cycle over $\mathbb{A}^1$ by $p : \mathbb{A}^1 \to \{pt\}$. Finally, the above work allows us to construct inductively the desired families of cycle $\mathcal{L}_W$ and $\mathcal{L}^1_W$ at Theorem 4.12.

- The last section is devoted to some concluding remarks. In particular, it present a combinatorial description that make it possible to write explicitly cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ in terms of parametrized algebraic cycles. Then, computing the integral attached to the cycle $\mathcal{L}_{011}$, we show that it specialization at the point 1 is $-2\zeta(2,1)$.

2. Cycle complex over $\mathbb{P}^1 \setminus \{0,1,\infty\}$

2.1. **1-minimal model and bar construction.** We recall here some facts about the 1-minimal model of a commutative differential graded algebra. More details can be found in [Sul77], [DGMS75] or [BK87]. As explained in the introduction an underlying goal of this paper is to give an explicit description of the 1-minimal model of $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ in terms of explicit parametrized algebraic cycles over $\mathbb{P}^1 \setminus \{0,1,\infty\}$. An important idea in order to build the desired cycles is to follow step by step the inductive construction of the 1-minimal model reviewed below.

2.1.1. *Differential graded algebra.* We recall some definitions and properties of commutative differential graded algebra over $\mathbb{Q}$.

Definition 2.1 (cdga)**.** A commutative differential graded algebra A is a commutative graded algebra (with unit) $A = \oplus_n A^n$ over $\mathbb{Q}$ together with a graded homomorphism $d = \oplus d^n$, $d^n : A^n \longrightarrow A^{n+1}$ such that

- $d^{n+1} \circ d^n = 0$
- d satisfies the Leibniz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^n a \cdot d(b) \qquad \text{for } a \in A^n,\, b \in A^m.$$

We recall that a graded algebra is *commutative* if and only if for any homogeneous elements a and b one has

$$ab = (-1)^{\deg(a)\deg(b)} ba.$$

Definition 2.2. A cdga A is

- *connected* if $A^n = 0$ for all $n < 0$ and $A^0 = \mathbb{Q} \cdot 1$.
- *cohomologically connected* if $\mathrm{H}^n(A) = 0$ for all $n < 0$ and $\mathrm{H}^0(A) = \mathbb{Q} \cdot 1$.

In our context, the cdga involved are not necessarily connected but comes with an Adams grading.

Definition 2.3 (Adams grading)**.** An *Adams graded* cdga is a cdga A together with a decomposition into subcomplex $A = \oplus_{p \geqslant 0} A(p)$ such that

- $A(0) = \mathbb{Q}$ is the image of the algebra morphism $\mathbb{Q} \longrightarrow A$.
- The Adams grading is compatible with the product of A, that is

$$A^k(p) \cdot A^l(q) \subset A^{k+l}(p+q).$$

However, no sign is introduced as a consequence of the Adams grading.

For an element $a \in A^k$, we call k the cohomological degree and denote it by $|a| := k$. In the case of an Adams graded cdga, for $a \in A^k(p)$, we call p its weight or Adams degree and denote it by $\widetilde{(a)} := p$.

2.1.2. 1-*minimal model.* We assume all the commutative differential graded algebra to have an augmentation $\varepsilon A \longrightarrow \mathbb{Q}$. Note that an Adams graded cdga A has a canonical augmentation $A \longrightarrow \mathbb{Q}$ with augmentation ideal $A^+ = \oplus_{p \geqslant 1} A(p)$.

We recall that the free commutative algebra $\Lambda(E)$ over a graded vector space $E = E_{\text{odd}} \oplus E_{\text{even}}$ is the tensor product of the exterior algebra on the odd part E_{odd} and of the polynomial algebra on the even part E_{even},

$$\Lambda(E) = \Lambda^*(E_{\text{odd}}) \otimes \text{Sym}^*(E_{\text{even}}).$$

Definition 2.4 (Hirsch extension)**.** An *Hirsch extension* of a cdga (A, d) is a cdga (A', d') satisfying :

(1) There exists a 1 dimensional graded vector space $V = \mathbb{Q}v$ of some degree k such that
$$\mathbb{A}' = A \otimes \Lambda(V)$$
(2) the restriction of d' to A is d and
$$d(v) \in A^+ \cdot A^+.$$
where A^+ denotes the augmentation ideal.

Definition 2.5 (Generalized nilpotent cdga)**.** A cdga A is *generalized nilpotent* if there exists a sequence of sub-differential algebra

$$\mathbb{Q} \subset A_1 \subset \ldots \subset A_l \subset \ldots$$

such that

- $A = \bigcup A_l$
- $A_l = A_{l-1} \otimes \Lambda(V_l)$ is an Hirsch extension;

In particular, one has $A = \Lambda(E)$ for some graded vector space E. More precisely, one should remark the following.

Remark 2.6. Equivalently to the above definition, a cdga (A, d) over $\mathbb{Q}$ is *generalized nilpotent* if

(1) As a $\mathbb{Q}$ graded algebra $A = \Lambda(E)$ where $E = E_{\text{odd}} \oplus E_{\text{even}}$ is a graded vector space; that is
$$A = \Lambda(E_{\text{odd}}) \otimes \text{Sym}^*(E_{\text{even}}).$$
(2) For $n \geqslant 0$, let $A_{(n)} \subset A$ be the sub-algebra generated by elements of degree $\leqslant n$. Set $A_{(n+1,0)} = A_{(n)}$ and for $q \geqslant 0$ define inductively $A_{(n+1,q+1)}$ as the sub-algebra define by $A_{(n)}$ and the set
$$\{x \in A_{(n+1)} \,|\, d(x) \in A_{(n+1,q+1)}\}.$$
Then, for all $n \geqslant 0$, one has
$$A_{(n+1)} = \bigcup_{q \geqslant 0} A_{(n+1,q)}.$$

Definition 2.7. Let A be a cdga. A *n-minimal model* of A is a map of cdga

$$s : M_A\{n\} \longrightarrow A$$

with $M_A\{n\}$ generalized nilpotent and generated (as an algebra) in degree $\leqslant n$ such that s induces an isomorphism on H^k for $1 \leqslant k \leqslant n$ and an injection on H^{n+1}

Theorem 2.8 ([Sul77], see also [BK87])**.** *Let A be a cohomologically connected cdga. Then, for each $n = 1, 2, \ldots$ there exists an n-minimal model of A $s : M_A\{n\} \longrightarrow A$.*

Moreover such an n-minimal model is unique up to non-canonical *isomorphism.*

We will here be only interested in the case of the 1-minimal model. We would like to recall below the construction of the 1-minimal model as described in [DGMS75] as it explains our approach to build explicit cycles in $\mathcal{N}^{\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$. It also illustrate the non-canonicity of the 1-minimal model.

We recall below a possible construction for a 1-minimal model.

Inductive construction of a 1*-minimal model.*

Let A be a cohomologically connected cdga.

Initialization: Set $V_1 = \mathrm{H}^1(A)$ totally of degree 1. Let $(v_{1,n})_{1\leqslant n\leqslant \dim(V_1)}$ be a basis of V_1 and choose representative $a_{1,n}$ in A of each $v_{1,n}$. Now, define $d_1 : V_1 \longrightarrow V_1 \wedge V_1$ to be the 0 map and $s_1^{V_1} : V_1 \longrightarrow A$ by $s(v_{1,n}) = a_{1,n}$.

Finally, set $M_1 = \Lambda(V_1)$ and extend $s_1^{V_1}$ to a cdga morphism $s : M_1 \longrightarrow A$.

Inductive step: Assume that one has constructed the cdga $s_k : M_k = \Lambda(V_k) \longrightarrow A$ for $k \geqslant 1$. Define

$$V_{k+1} = V_k \oplus \ker\left(\mathrm{H}^2(M_k) \longrightarrow \mathrm{H}^2(A)\right)$$

where $\ker(\mathrm{H}^2(M_k) \to \mathrm{H}^2(A))$ is totally in degree 1. In order to define d_{k+1} and s_{k+1} one proceed as follow.

Let $(v_{k+1,n})$ be a basis of $\ker(\mathrm{H}^2(M_k) \to \mathrm{H}^2(A))$. For each $v_{k+1,n}$ choose a representative in $\Lambda^2(V_k)$, that is

$$v_{k+1,n} = \left[\sum_{i,j} a_{i,j} v_i \wedge v_j\right] \qquad \in \mathrm{H}^2(M_k) = \mathrm{H}^2(\Lambda(V_k))$$

for some v_i and v_j in V_k. Thus, the image of $v_{k+1,n}$ in $\mathrm{H}^2(A)$ is the class of

$$\sum_{i,j} a_{i,j} s_k(v_i) \cdot s_k(v_j) \in A^2$$

which has differential 0. Moreover, as $v_{k+1,n}$ is in $\ker(\mathrm{H}^2(M_k) \to \mathrm{H}^2(A))$, one has some $c_{k+1,n}$ in A^1 such that

$$d(c_{k+1,n}) = \sum_{i,j} a_{i,j} s_k(v_i) \cdot s_k(v_j).$$

Now, one defines $d_{k+1} : V_{k+1} \longrightarrow V_{k+1} \wedge V_{k+1}$ which extends d_k by sending $v_{k+1,n}$ to $\sum_{i,j} a_{i,j} v_i \wedge v_j$ and one defines $s_{k+1}^{V_{k+1}} : V_{k+1} \longrightarrow A$ which extends $s_k^{V_k}$ by sending $v_{k+1,n}$ to $c_{k+1,n}$. These definitions are summarized in the following diagram

(4)

$$\begin{array}{ccccccc}
 & & \Lambda^2(V_k) & \xrightarrow{\quad s_k \quad} & A^2 & \xrightarrow{\quad d \quad} & A^3 \\
 & & & & & & \\
v_{k+1,n} & \overset{d_{k+1}}{\longmapsto} & \sum_{i,j} a_{i,j} v_i \wedge v_j & \longmapsto & \sum_{i,j} a_{i,j} s_k(v_i) \cdot s_k(v_j) & \longmapsto & 0 \\
 & & & & \Big\uparrow{\scriptstyle d} & & \\
 & & & & \exists\, c_{k+1,n} & \in A^1 & \\
 & & & \overset{s_{k+1}}{\nearrow} & & &
\end{array}$$

which describe the fact that $v_{k+1,n}$ is in $\ker(\mathrm{H}^2(M_k) \to \mathrm{H}^2(A))$.

Finally, set $M_{k+1} = \Lambda(V_{k+1})$ with differential induced by d_{k+1} and $s_{k+1} : M_{k+1} \longrightarrow A$ to be the morphisms of cdga extending $s_{k+1}^{V_{k+1}}$.

One checks that $s : M = \cup M_k \longrightarrow A$ provides a 1-minimal model. This is insured by the fact that as each step creates some kernel in degree 2 which is killed at the next step in order to obtain the injection on the H^2. □

The main point of the above construction is to build the underlying vector space V_A of the 1-minimal model $M_A\{1\} = \Lambda(A)$. As explained in [BK94], V_A is endowed with a coLie algebra structure. It is a general fact for generalized nilpotent cdga.

Lemma 2.9 ([BK94][Lemma 2.29])**.** *Let M be a generalized nilpotent cdga and V a vector space freely generating M as cdga. The differential on M induces*

$$d : V \longrightarrow V \wedge V$$

giving V a coLie algebra structure, that is V is dual to a pro-finite dimensional Lie algebra with d dual to the bracket where the relation $d \circ d = 0$ is dual to Jacobi identity.

If one begins with an Adams graded cdga A, one will add an Adams grading to the above definition and properties. In particular, the construction of the 1-minimal model works similarly but one includes the induction into a first induction on the Adams degree.

2.1.3. *Bar construction.* The bar construction over a c.d.g.a has been used in various contexts and is reviewed in many places. However, as there do not seem to exist a global sign convention, the main definitions in the cohomological setting are recalled below following the (homological) description given in [VL12].

Let A be a c.d.g.a with augmentation $\varepsilon : A \longrightarrow \mathbb{Q}$, with product μ_A and let A^+ be the augmentation ideal $A^+ = \ker(\varepsilon)$. Define s to be a degree -1 generator and consider the degree 1 morphism

$$\mathbb{Q}s \otimes \mathbb{Q}s \longrightarrow \mathbb{Q}s.$$

Definition 2.10. The bar construction $B(A)$ over A is the tensor coalgebra over the suspension of $sA^+ := \mathbb{Q}s \otimes A^+$.

- In particular, as vector space $B(A)$ is given by :
$$B(A) = T(sA^+) = \bigoplus_{n \geqslant 0} (sA^+)^{\otimes n}.$$
- An homogeneous element $\mathbf{a}$ of tensor degree n is denoted using the *bar* notation (|), that is
$$\mathbf{a} = [sa_1|\dots|sa_n]$$
and its degree is
$$\deg_B(\mathbf{a}) = \sum_{i=1}^{n} \deg_{sA^+}(a_i) = \sum_{i=1}^{n} \deg_A(a_i) - 1.$$
- The coalgebra structure comes from the natural deconcatenation coproduct, that is
$$\Delta([sa_1|\dots|sa_n]) = \sum_{i=0}^{n} [sa_1|\dots|sa_i] \otimes [sa_{i+1}|\dots|sa_n].$$

Remark 2.11. This construction can be can be related (Cf. [BK94]) to the total complex associated to the simplicial complex

$$A^{\otimes \bullet} : \dots \to A^{\otimes n} \to A^{\otimes (n-1)} \to \dots$$

The augmentation makes it possible to use directly A^+ without referring to the tensor coalgebra over A and without the need of killing the degeneracies.

However this simplicial presentation usually hides the need of working with the shifted complex. Here, we use the extra -1 generator s which makes it easier to understand the signs convention using the Kozul rules.

We associate to any bar element $[sa_1|\dots|sa_n]$ the function $\eta(i)$ giving its "partial" degree

$$\eta(i) = \sum_{k=1}^{i} \deg_{sA}(sa_k) = \sum_{k=1}^{i} (\deg_A(a_k) - 1).$$

The original differential d_A induces a differential D_1 on $B(A)$ given by

$$D_1([sa_1|\dots|sa_n]) = -\sum_{i=1}^{n} (-1)^{\eta(i-1)} [sa_1|\dots|sd_A(a_i)|\dots|sa_n]$$

where the initial minus from comes from the fact the differential on the shifted complex sA is $-d_A$. Moreover, the multiplication on A induces another differential D_2 on $B(A)$ given by

$$D_2([sa_1|\dots|sa_n]) = -\sum_{i=1}^{n} (-1)^{\eta(i)} [sa_1|\dots|s\mu_A(a_i, a_{i+1})|\dots|sa_n]$$

where the signs are coming from Kozul commutation rules : the global sign in front of the i-th terms of the sum can be written as $(-1)^{\eta(i-1)}(-1)^{\deg(s)\deg_A(a_i)}$. One checks that the two differentials anticommute providing $B(A)$ with a total differential.

Definition 2.12. The total differential on $B(A)$ is defined by

$$d_{B(A)} = D_1 + D_2.$$

The last structure arising with the bar construction is the graded shuffle product

$$[sa_1|\dots|sa_n] \text{ ш } [sa_{n+1}|\dots|sa_{n+m}] = \sum_{\sigma \in sh(n,m)} (-1)^{\varepsilon_{gr}(\sigma)} [sa_{\sigma(1)}|\dots|sa_{\sigma(n+m)}]$$

where $sh(n,m)$ denotes the permutation of $\{1,\dots,n+m\}$ such that if $1 \leqslant i < j \leqslant n$ or $n+1 \leqslant i < j \leqslant n+m$ then $\sigma(i) < \sigma(j)$. The sign is the graded signature of the permutation (for the degree in sA^+) given by

$$\varepsilon_{gr}(\sigma) = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \deg_{sA}(sa_i)\deg_{sA}(sa_j) = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} (\deg_A(a_i)-1)(\deg_A(a_j)-1).$$

With this definitions, one can explicitly check the following.

Proposition 2.13. *Let A be a (Adams graded) c.d.g.a. The operations Δ, $d_{B(A)}$ and ш together with the obvious unit and counit give $B(A)$ a structure of (Adams graded) commutative graded differential Hopf algebra.*

In particular, these operations induces on $\mathrm{H}^0(B(A))$, and more generally on $\mathrm{H}^*(B(A))$, a (Adams graded) commutative Hopf algebra structure. This algebra is graded in the case of $\mathrm{H}^*(B(A))$ and graded concentrated in degree 0 in the case of $\mathrm{H}^0(B(A))$.

We recall that the set of indecomposable elements of an augmented c.d.g.a is defined as the augmentation ideal I modulo products, that is I/I^2. Applying a general fact about Hopf algebra, the coproduct structure on $\mathrm{H}^0(B(A))$ (resp. $\mathrm{H}^*(B(A))$) induces a coLie algebra structure on its set of indecomposables.

The bar construction is a quasi-isomorphism invariant and comparing a generalized nilpotent c.d.g.a. to its bar construction shows (see [BK94])

Proposition 2.14 ([BK94]). *Let A be a cohomologically connected c.d.g.a and let*

$$\varphi : M \longrightarrow A$$

be a minimal model of A.

Defining QM (resp. $Q\,\mathrm{H}^*(B(A))$) the set of indecomposable elements of M (resp. $\mathrm{H}^*(B(A))$), there is an isomorphism of coLie algebra

$$\varphi_Q : QM \otimes s\mathbb{Q} \xrightarrow{\sim} Q\,\mathrm{H}^*(B(A))$$

canonical after the choice of φ.

2.2. **General construction of Bloch-Kriz cycles complex.** This subsection is devoted to the construction of the cycle complex as presented in [Blo86, Blo97, BK94, Lev94].

Let $\mathbb{K}$ be a perfect field and let $\square^n_{\mathbb{K}}$ be the algebraic n-cube

$$\square^n_{\mathbb{K}} = (\mathbb{P}^1 \setminus 1)^n.$$

When $\mathbb{K} = \mathbb{Q}$, we will drop the subscript and simply write $\square^n$ for $\square^n_{\mathbb{Q}}$. Insertion morphisms $s^\varepsilon_i : \square^{n-1}_{\mathbb{K}} \longrightarrow \square^n_{\mathbb{K}}$ are given by the identification

$$\square^{n-1}_{\mathbb{K}} \simeq \square^{i-1}_{\mathbb{K}} \times \{\varepsilon\} \times \square^{n-i}_{\mathbb{K}}$$

for $\varepsilon = 0, \infty$. Similarly, for $I \subset \{1, \dots, n\}$ and $\varepsilon : I \to \{0, \infty\}$, one defines $s^\varepsilon_I : \square^{n-|I|}_{\mathbb{K}} \longrightarrow \square^n_{\mathbb{K}}$.

Definition 2.15. A *face* F of codimension p of $\square^n_{\mathbb{K}}$ is the image $s^\varepsilon_I(\square^{n-p})$ for some I and ε as above such that $|I| = p$.

In other word, a codimension p face of $\square^n_{\mathbb{K}}$ is given by the equation $x_{i_k} = \varepsilon_k$ for k in $\{1, \dots, p\}$ and ε_k in $\{0, \infty\}$ where $x_1, \dots, x_n$ are the usual affine coordinates on $\mathbb{P}^1$.

The permutation group $\mathfrak{S}_n$ act on $\square^n_{\mathbb{K}}$ by permutation of the factor.

Remark 2.16.

- In some references as [Lev94, Lev11] for example, $\square^n_{\mathbb{K}}$ is defined to be the usual affine space $\mathbb{A}^n$ and the faces by setting various coordinates equal to 0, or 1. This make the correspondence with the "usual" cube more natural. However, the above presentation, which agree with [BK94] or [GGL09], makes some comparisons and some formulas "nicer". In particular, the relation between the construction in the setting $\square^1_{\mathbb{K}} = \mathbb{P}^1 \setminus \{1\}$ and the Chow group $\mathrm{CH}^1(X)_{\mathbb{Q}}$ is simpler.
- Let Cube be the subcategory of the category of finite sets whom objects are $\underline{n} = \{0, 1\}^n$ and morphisms are generated by forgetting a factor, inserting 0 or 1 and permutation of the factors ; these morphisms being subject to natural relations. Similarly to the usual description of a simplicial object, $\square^\bullet_{\mathbb{K}}$ is a functor from Cube into the category of smooth $\mathbb{K}$-varieties and the various $\square^n_{\mathbb{K}}$ are geometric equivalents of $\underline{n}$.

Now, let X be a smooth quasi-projective variety over $\mathbb{K}$.

Definition 2.17. Let p and n be non negative integers. Let $\mathcal{Z}^p(X, n)$ be the free group generated closed irreducible sub-varieties of $X \times \square^n_{\mathbb{K}}$ of codimension p which intersect all faces $X \times F$ properly (where F is a face of $\square^n_{\mathbb{K}}$). That is:

$$\mathbb{Z}\left\langle W \subset X \times \square^n_{\mathbb{K}} \text{ such that } \begin{cases} W \text{ is smooth, closed and irreducible} \\ \mathrm{codim}_{X\times F}(W \cap X \times F) = p \\ \text{or } W \cap X \times F = \emptyset \end{cases} \right\rangle$$

Remark 2.18.

- A sub-variety W of $X \times \square^n_{\mathbb{K}}$ as above is *admissible*.

- As $p_i : \square^n_{\mathbb{K}} \to \square^{n-1}_{\mathbb{K}}$ is smooth, one has the corresponding induced pull-back: $p_i^* : \mathcal{Z}^p(X, n-1) \to \mathcal{Z}^p(X, n)$.
- s_i^ε induces a regular closed embedding $X \times \square^{n-1}_{\mathbb{K}} \to X \times \square^{n-1}_{\mathbb{K}}$ which is of local complete intersection. As we are considering only admissible cycles, that is cycles in "good position" with respect to the faces, s_i^ε induces $s_i^{\varepsilon\,*} : \mathcal{Z}^p(X, n) \to \mathcal{Z}^p(X, n-1)$.
- The morphism $\partial = \sum_{i=1}^n (-1)^{i-1}(s_i^{0,*} - s_i^{\infty,*})$ induces a differential
$$\mathcal{Z}^p(X, n) \longrightarrow \mathcal{Z}^p(X, n-1).$$

One extends the action of $\mathfrak{S}_n$ on $\square^n_{\mathbb{K}}$ to an action of the semi-direct product $G_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ where each $\mathbb{Z}/2\mathbb{Z}$ acts on $\square^1_{\mathbb{K}}$ by sending the usual affine coordinates x to $1/x$. The sign representation of $\mathfrak{S}_n$ extends to a sign representation $G_n \longmapsto \{\pm 1\}$. Let $\mathcal{A}lt_n \in \mathbb{Q}[G_n]$ be the corresponding projector.

Definition 2.19. Let p and k be integers with $p > 0$. One defines
$$\mathcal{N}^k_X(p) = \mathcal{A}lt_{2p-k}(\mathcal{Z}(X, 2p-k) \otimes \mathbb{Q}).$$
We will refer to k as the *cohomological degree* and to p as the *weight*.

Remark 2.20. In this presentation, we did not take care of degeneracies (images in $\mathcal{Z}(X, n)$ of p_i^*) because we use an alternating version with rational coefficients. For more details, one should see the first section of [Lev09] which presents the general setting of cubical objects. A similar remark was made in [BK94][after the equation (4.1.3)].

Definition 2.21 (Cycle complex)**.** for p and k as above, the pull-back
$$s_i^{\varepsilon\,*} : \mathcal{Z}^p(X, 2p-k) \longrightarrow \mathcal{Z}^p(X, 2p-k-1)$$
induces a morphism $\partial_i^\varepsilon : \mathcal{N}^k_X(p) \longrightarrow \mathcal{N}^{k+1}_X(p)$. Thus, the differential ∂ on $\mathcal{Z}^p(X, 2p-k)$ extends into a differential
$$\partial = \sum_{i=1}^{2p-k} (-1)^{i-1}(\partial_i^0 - \partial_i^\infty) : \mathcal{N}^k_X(p) \xrightarrow{\partial} \mathcal{N}^{k+1}_X(p).$$
Let $\mathcal{N}^\bullet_X(p)$ be the complex
$$\mathcal{N}^\bullet_X(p) : \qquad \cdots \longrightarrow \mathcal{N}^k_X(p) \xrightarrow{\partial} \mathcal{N}^{k+1}_X(p) \longrightarrow \cdots$$
One defines the cycle complex as
$$\mathcal{N}^\bullet_X = \mathbb{Q} \bigoplus_{p \geqslant 1} \mathcal{N}^\bullet_X(p).$$

Levine has shown in [Lev94][§5] or [Lev11][Example 4.3.2] the following proposition.

Proposition 2.22. *Concatenation of the cube factors and pull-back by the diagonal*
$$X \times \square^n_{\mathbb{K}} \times X \times \square^m_{\mathbb{K}} \xrightarrow{\sim} X \times X \times \square^n_{\mathbb{K}} \times \square^m_{\mathbb{K}} \xrightarrow{\sim} X \times X \times \square^{n+m}_{\mathbb{K}} \xleftarrow{\Delta_X} X \times \square^{n+m}_{\mathbb{K}}$$
induced, after applying the $\mathcal{A}lt$ projector, a well-defined product:
$$\mathcal{N}^k_X(p) \otimes \mathcal{N}^l_X(q) \longrightarrow \mathcal{N}^{k+l}_X(p+q)$$
denoted by $\cdot$

Remark 2.23. The smoothness hypothesis on X allows us to consider the pull-back by the diagonal $\Delta_X : X \longrightarrow X \times X$ which is in this case of local complete intersection.

One has the following theorem (also stated in [BK94, Blo97] for $X = \mathrm{Spec}(\mathbb{K})$).

Theorem 2.24 ([Lev94])**.** *The cycle complex $\mathcal{N}_X^\bullet$ is a differential graded commutative algebra. In weight p, its cohomology groups are the higher Chow group of X:*

$$\mathrm{H}^k(\mathcal{N}_X(p)) = \mathrm{CH}^p(X, 2p-k)_{\mathbb{Q}},$$

where $\mathrm{CH}^p(X, 2p-k)_{\mathbb{Q}}$ stand for $\mathrm{CH}^p(X, 2p-k) \otimes \mathbb{Q}$.

Moreover, one easily has flat pull-back and proper push-forward. Using Levine's work [Lev94], one has more general pull-back on the cohomology group; one could also use Bloch moving Lemma [Blo94].

2.3. **Some properties of Higher Chow groups.** In this section, we present some well-known properties of the higher Chow groups and some applications that will be used later. Proof of the different statements can be found in [Blo86] or [Lev94].

2.3.1. *Relation with higher K-theory.* Higher Chow groups, in a simplicial version, were first introduced in [Blo86] in order to understand better the K groups of higher K-theory. Levine in [Lev94][Theorem 3.1] gives a cubical version of the desired isomorphisms:

Theorem 2.25 ([Lev94])**.** *Let X be a smooth quasi-projective $\mathbb{K}$ variety and let p, k be two positive integers. One has :*

$$\mathrm{CH}^p(X, 2p-k)_{\mathbb{Q}} \simeq \mathrm{Gr}_\lambda^p K_{2p-k}(X) \otimes \mathbb{Q}$$

In particular, using the work of Borel [Bor74], computing the K groups of a number fields, one finds in the case $\mathbb{K} = \mathbb{Q}$ and $k = 2$:

$$\mathrm{CH}^p(\mathbb{Q}, 2p-2)_{\mathbb{Q}} \simeq \mathrm{Gr}_\lambda^p K_{2p-2}(\mathbb{Q}) \otimes \mathbb{Q} = 0 \tag{5}$$

2.3.2. *A^1-homotopy invariance.* From Levine [Lev94][Theorem 4.5], one deduces the following proposition.

Proposition 2.26 ([Lev94])**.** *Let X be as above and p be the projection $p : X \times \mathbb{A}^1 \longrightarrow X$. The projection p induced a quasi-isomorphism for any positive integer p*

$$p^* : \mathcal{N}_X^\bullet(p) \xrightarrow{q.i.} \mathcal{N}_{X\times\mathbb{A}^1}^\bullet(p)$$

Moreover, an inverse of the quasi-isomorphism is given by $i_0^* : \mathrm{H}^k(\mathcal{N}_{X\times\mathbb{A}^1}^\bullet(p)) \longrightarrow \mathrm{H}^k(\mathcal{N}_X^\bullet(p))$.

Remark 2.27. The proof of Levine's theorem also tells us how this quasi-isomorphism arise using the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ and this leads us to the proof of Proposition 4.11.

We now apply the above result in the case of $\mathbb{K} = \mathbb{Q}$ and $X = \mathrm{Spec}(\mathbb{Q})$, and use the relation the K-theory via Equation (5).

Corollary 2.28. *In the case of $\mathbb{K} = \mathbb{Q}$, the second cohomology group of $\mathcal{N}_{\mathbb{A}^1}^\bullet$ vanishes:*

$$\forall\, p \geqslant 1 \qquad \mathrm{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(p)) \simeq \mathrm{CH}^p(\mathbb{A}^1, 2p-2)_{\mathbb{Q}} \simeq \mathrm{CH}^p(\mathbb{Q}, 2p-2)_{\mathbb{Q}} = 0.$$

2.3.3. *Localization sequence.* Let W be a smooth closed of pure codimension d subvariety of a smooth quasi-projective variety X. Let U denote the open complement $U = X \setminus W$. A version adapted to our needs of Theorem 3.4 in [Lev94] gives the localization sequence for higher Chow groups.

Theorem 2.29 ([Lev94])**.** *Let p be a positive integer and l an integer. There is a long exact sequence*
(6)
$$\cdots \longrightarrow \mathrm{CH}^p(U, l+1)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{p-d}(W, l)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{j^*} \mathrm{CH}^p(U, l)_{\mathbb{Q}} \longrightarrow \cdots$$

where $i : W \to X$ denotes the closed immersion and $j : U \to X$ the open one.

Remark 2.30. i_* and j^* are the usual push-forward for proper morphisms and pull-back for flat ones.

In order to study the cycle complex over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we begin by applying the above theorem to the case where $X = \mathbb{A}^1$, $U = \mathbb{P}^1 \setminus 0, 1, \infty$ and $W = \{0, 1\}$.

Corollary 2.31. *We have the following description of* $\mathrm{H}^k(\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}})$*:*

$$\mathrm{H}^k(\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}) \simeq \mathrm{H}^k(\mathcal{N}^\bullet_{\mathbb{Q}}) \oplus \mathrm{H}^{k-1}(\mathcal{N}^\bullet_{\mathbb{Q}}) \otimes \mathbb{Q}L_0 \oplus \mathrm{H}^{k-1}(\mathcal{N}^\bullet_{\mathbb{Q}}) \otimes \mathbb{Q}L_1,$$

where L_0 and L_1 are in cohomological degree 1 *and weight* 1 *(that is of codimension* 1*).*

Proof. The above long exact sequence gives

$$\cdots \longrightarrow \mathrm{CH}^{p-1}(\{0,1\}, l)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(\mathbb{A}^1, l)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\delta}$$
$$\mathrm{CH}^{p-1}(\{0,1\}, l-1)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}^p(\mathbb{A}^1, l-1)_{\mathbb{Q}} \longrightarrow \cdots$$

The map i_* is induced by the inclusions i_0 and i_1 of 0 and 1 in $\mathbb{A}^1$. As $i_0^* : \mathcal{N}^\bullet_{\mathbb{A}^1} \longrightarrow \mathcal{N}^\bullet_{\{0\}}$, and more generally i_x^* for any $\mathbb{K}$ point x of $\mathbb{A}^1$, is a quasi-isomorphism inverse to p^*, the Cartesian diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{Spec}(\mathbb{Q})) \\ \downarrow & & \downarrow{\scriptstyle i_x} \\ \mathrm{Spec}(\mathbb{Q}) & \xrightarrow{i_0} & \mathbb{A}^1 \end{array}$$

shows that $i_{0,*}$ (and respectively $i_{1,*}$) are 0 on cohomology.

In particular, the sequence becomes short exact. Thus, using the homotopy property and the fact that $\mathrm{CH}^p(\{0,1\}, l) \simeq \mathrm{CH}^p(\mathrm{Spec}(Q), l) \oplus \mathrm{CH}^p(\mathrm{Spec}(\mathbb{Q}), l)$ one gets the following short exact sequence

$$0 \longrightarrow \mathrm{CH}^p(\mathrm{Spec}(\mathbb{Q}), l)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{CH}^{p-1}(\mathrm{Spec}(\mathbb{Q}), l-1)^{\oplus 2}_{\mathbb{Q}} \longrightarrow 0$$

Thus, one obtains an isomorphism

$$\mathrm{CH}^p(X, l)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{CH}^p(\mathbb{A}^1, l)_{\mathbb{Q}} \oplus \mathrm{CH}^{p-1}(\mathrm{Spec}(\mathbb{Q}), l-1)^{\oplus 2}_{\mathbb{Q}}.$$

The relation between the cohomology groups of $\mathcal{N}^\bullet_X(p)$ and the higher Chow groups conclude the proof. □

Remark 2.32. The generators L_0 and L_1 can be given in terms of explicit cycles in $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ (see Subsection 2.5).

2.4. **Cycle complex over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and mixed Tate motives.** Levine in [Lev11] makes the link between the category of mixed Tate motive (in the sens of Levine [Lev05] or Voevodsky [Voe00]) over a base X and the cycle complex $\mathcal{N}_X$. . The relation between mixed Tate motives and the cycle complex has been developed before for $X = \mathrm{Spec}(\mathbb{K})$, the spectrum of number field by Bloch and Kriz [BK94].

We now assume that $\mathbb{K}$ is a number field and X still denotes a smooth, quasi-projective variety over $\mathbb{K}$. We will work with $\mathbb{Q}$ coefficients.

Under more general conditions Cisinski and Déglise [CD09] have defined a triangulated category $\mathrm{DM}(X)$ of (effective) motives over a base with the expected property . Levine's work [Lev93, Lev11] shows that if X satisfies the Beilinson-Soulé vanishing conjecture then one obtains a tannakian category $\mathrm{MTM}(X)$ of mixed Tate motives over X as the heart of a t-structure over $\mathrm{DMT}(X)$ the smallest full triangulated subcategory of $\mathrm{DM}(X)$ generated by the Tate motive $\mathbb{Q}_X(n)$. The whole construction is summarized in [Lev11].

Together with defining an avatar of $\mathcal{N}_X$ in $\mathrm{DM}(X)$, Levine [Lev11][Theorem 5.3.2 and beginning of the section 6.6] shows that when the motive of X is in $\mathrm{DMT}(\mathbb{K})$ and satisfies the Beilinson-Soulé conjecture one can identify the tannakian group associated with $\mathrm{MTM}(X)$ with the spectrum of the H^0 of the bar construction (see Section 2.1.3) over the cdga $\mathcal{N}_X$:

$$G_{\mathrm{MTM}(X)} \simeq \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X))).$$

Then, he used a relative bar-construction in order to relate the natural morphisms

$$p^* : \mathrm{DMT}(\mathrm{Spec}(\mathbb{K})) \longrightarrow \mathrm{DMT}(X) \qquad x^* : \mathrm{DMT}(X) \longrightarrow \mathrm{DMT}(\mathrm{Spec}(\mathbb{K})),$$

induced by the structural morphism $p : X \to \mathrm{Spec}(\mathbb{K})$ and a choice of a $\mathbb{K}$-point x, to the motivic fundamental group of X at the based point x defined by Goncharov and Deligne, $\pi^1_{mot}(X,x)$ (see [Del89] and [DG05]).

In particular, applying this to the the case $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ and $\mathbb{K} = \mathbb{Q}$, one has the following result.

Theorem 2.33 ([Lev11][Corollary 6.6.2])**.** *Let x be a $\mathbb{Q}$-point of $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, one has a split exact sequence:*

$$1 \longrightarrow \pi^1_{mot}(X,x) \longrightarrow \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X))) \underset{x^*}{\overset{p^*}{\longrightarrow}} \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_{\mathbb{Q}}))) \longrightarrow 1$$

where p is the structural morphism $p : \mathbb{P}^1 \setminus \{0,1,\infty\} \longrightarrow \mathrm{Spec}(\mathbb{Q})$.

We want to apply results of section 2.1 to the case $A = \mathcal{N}^\bullet_X$ for $X = \mathrm{Spec}(\mathbb{Q})$, $X = \mathbb{A}^1$ and $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$.

Lemma 2.34. *The cdga $\mathcal{N}^\bullet_{\mathrm{Spec}(\mathbb{Q})}$, $\mathcal{N}^\bullet_{\mathbb{A}^1}$ and $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ are cohomologically connected.*

Proof. The case of $\mathcal{N}^\bullet_{\mathbb{A}^1}$ follows by $\mathbb{A}^1$ homotopy invariance from the case of $\mathcal{N}^\bullet_{\mathrm{Spec}(\mathbb{Q})}$ which is deduced from the works of Borel [Bor74] using the relation with the higher Chow groups. One deduces the case of $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ from the $\mathrm{Spec}(\mathbb{Q})$ case using the localization long exact sequence as in Corollary 2.31. □

Let $\mathrm{co}\mathfrak{L}_{\mathbb{Q}}$, $\mathrm{co}\mathfrak{L}_{\mathbb{A}^1}$ and $\mathrm{co}\mathfrak{L}_X$ denote the coLie algebra generating the 1-minimal model of $\mathcal{N}^\bullet_{\mathrm{Spec}(\mathbb{Q})}$, $\mathcal{N}^\bullet_{\mathbb{A}^1}$ and $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ respectively. The relation between the indecomposable of the H^0 of the bar construction and the 1-minimal model (see

[BK94, Lev11] or a short review in section 2.1.3) allows us to reformulate Theorem 2.33 in terms of coLie algebras.

Proposition 2.35. *One has a split exact sequence of coLie algebras:*

$$0 \longrightarrow co\mathfrak{L}_{\mathbb{Q}} \longrightarrow co\mathfrak{L}_X \longrightarrow co\mathfrak{L}_{geom} \longrightarrow 0$$

where $co\mathfrak{L}_{geom}$ is dual to the Lie algebra associated to $\pi^1_{mot}(X,x)$.

In particular $\mathrm{co}\mathfrak{L}_{geom}$ is related to the graded dual of the free Lie algebra on two generators $\mathrm{Lie}(X_0, X_1)$.

2.5. **Algebraic cycles corresponding to polylogarithms.** Now, and until the end of the article, X denotes $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and we assume that $\mathbb{K} = \mathbb{Q}$.

In this section, we present our strategy to build general cycles in $\mathcal{N}^\bullet_X$ corresponding to multiple polylogarithms on the simple case of the polylogarithms, $\mathcal{L}\mathrm{i}_n$. We will pay a special attention to the Totaro cycle which is known to correspond to the function $\mathcal{L}\mathrm{i}_2(z)$ and then explain how the construction is generalized to obtain cycles already present in [BK94] and [GGL09] corresponding to the functions $\mathcal{L}\mathrm{i}_n(z)$.

2.5.1. *Two weight* 1 *examples of cycles generating the* H^1. We want to build cycles in $\mathcal{N}^\bullet_X$ in order to obtain the inductive construction of the 1-minimal model. It will mean to

(1) find in $\mathcal{N}^2_X$ linear combinations of product of already built cycles that are boundaries, that is $d(c)$ for some c in $\mathcal{N}^1_X$ (see Equation (4));
(2) explicitly build the desired c.

But the first step begins with a basis of $\mathrm{H}^1(\mathcal{N}^\bullet_X)$. However, as we want only a description of $\mathrm{co}\mathfrak{L}_X$ relatively to $\mathrm{co}\mathfrak{L}_{\mathbb{Q}}$, we do not want to consider a full basis of $\mathrm{H}^1(\mathcal{N}^\bullet_X)$. We have seen that $\mathrm{H}^1(\mathcal{N}^\bullet_X)$ (Corollary 2.31) is the direct sum of $\mathrm{H}^1(\mathcal{N}^\bullet_{\mathbb{Q}})$ and two copies of $\mathrm{H}^0(\mathcal{N}^\bullet_{\mathbb{Q}})$.

Lemma 2.36. *Let Γ_0 and Γ_1 be respectively the graph of $\rho_0 : X \longrightarrow \mathbb{P}^1 \setminus \{1\} = \square^1$ which sends t to t and the graph of $\rho_1 : X \longrightarrow \mathbb{P}^1 \setminus \{1\} = \square^1$ which sends t to $1-t$. Then, Γ_0 and Γ_1 define admissible algebraic cycles in $X \times \square^1$, applying the projector $\mathcal{A}lt$ on the alternating elements gives two elements L_0 and L_1 in $\mathcal{N}^1_X$ and one has*

$$\mathrm{H}^1(\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}) \simeq \mathrm{H}^1(\mathcal{N}^\bullet_{\mathbb{Q}}) \oplus \mathrm{H}^0(\mathcal{N}^\bullet_{\mathbb{Q}}) \otimes \mathbb{Q}L_0 \oplus \mathrm{H}^0(\mathcal{N}^\bullet_{\mathbb{Q}}) \otimes \mathbb{Q}L_1.$$

Speaking about parametrized cycles, we will usually omit the projector $\mathcal{A}lt$ and write

$$L_0 = [t;t] \qquad \text{and} \qquad L_1 = [t;1-t] \qquad \subset X \times \square^1$$

where the notation $[t; f(t)]$ denotes the set

$$\{(t, f(t) \text{ such that } t \in X)\}.$$

Proof. First of all, one should remark that L_0 and L_1 are codimension 1 cycles in $X \times \square^1 = X \times \square^{2*1-1}$. Moreover as

$$L_0 \cap X \times \{\varepsilon\} = L_0 \cap \mathbb{P}^1 \setminus \{0,1,\infty\} \times \{\varepsilon\} = \emptyset,$$

for $\varepsilon = 0, \infty$, L_0 is admissible (intersect all the faces in the right codimension or not at all) and gives an element of $\mathcal{N}^1_X(1)$. Furthermore, the above intersection tells us that $\partial(L_0) = 0$. Similarly one shows that L_1 gives an element of $\mathcal{N}^1_X(1)$ and that $\partial(L_1) = 0$. Thus L_0 and L_1 gives well defined class in $\mathrm{H}^1(\mathcal{N}^\bullet_X(1))$.

In order to show that they are non trivial, one shows that, in the localization sequence (6), their images by the boundary map

$$\mathrm{H}^1(\mathcal{N}_X^\bullet(1)) \xrightarrow{\delta} \mathrm{H}^0(\mathcal{N}_{\{0\}}^\bullet(0)) \oplus \mathrm{H}^0(\mathcal{N}_{\{1\}}^\bullet(0))$$

are non-zero. It is enough to treat the case of L_0. Let $\overline{L_0}$ be the closure of L_0 in $\mathbb{A}^1 \times \square^1$. Indeed, $\overline{L_0}$ is given by the parametrized cycle

$$\overline{L_0} = [t; t] \subset \mathbb{A}^1 \times \square^1$$

and the intersection with the face $u_1 = 0$ is of codimension 1 in $\mathbb{A}^1 \times \{0\}$ and the intersection with $u_1 = \infty$ is empty. Hence $\overline{L_0}$ is admissible.

Thus, considering the definition of δ, $\delta(L_0)$ is given by the intersection of the differential of $\overline{L_0}$ with $\{0\}$ and $\{1\}$ on respectively, the first and second factor. The above discussion on the admissibility of $\overline{L_0}$ tells us that $\delta(L_0)$ is non zero on the factor $\mathrm{H}^0(\mathcal{N}_{\{0\}}^\bullet(0))$ and 0 on the other factor as the admissibility condition is trivial in $\mathrm{H}^0(\mathcal{N}_{\{0\}}^\bullet(0))$ and the restriction of $\overline{L_0}$ to 1 is empty. The situation is reverse for L_1. □

Later we will consider cycles depending on many parameters and denote by

$$[t; f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \dots, f_n(t, \mathbf{x})] \subset X \times \square^n$$

the (image under the projector $\mathcal{A}lt$ of the) restriction to $X \times \square^n$ of the image of

$$\begin{array}{ccc} X \times (\mathbb{P}^1)^k & \longrightarrow & X \times (\mathbb{P}^1)^n \\ (t, \mathbf{x}) & \longmapsto & (t, f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \dots, f_n(t, \mathbf{x})). \end{array}$$

2.5.2. *A weight* 2 *example: the Totaro cycle.* One considers the linear combination

$$b = L_0 \cdot L_1 \in \mathcal{N}_X^2(2).$$

It is given as a parametrized cycle by

$$b = [t; t, 1 - t] \subset X \times \square^2$$

or in terms of defining equations by

$$T_1V_1 - U_1T_2 = 0 \qquad \text{and} \qquad U_1V_2 + U_2V_1 = V_1V_2$$

where T_1 and T_2 denote the homogeneous coordinates on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and U_i, V_i the homogeneous coordinates on each factor $\square^1 = \mathbb{P}^1 \setminus \{1\}$ of $\square^2$. One sees that the intersection of b with some faces (U_i or $V_i = 0$ for some i's) is empty because T_2 is different from 0 and ∞ in X and because U_i is different from V_i in $\square^1$. This comments insures b is admissible.

Moreover it tells us that $\partial(b) = 0$. So b gives a class

$$[b] \in \mathrm{H}^2(\mathcal{N}_X^\bullet(2)).$$

We will now show that this class is trivial.

Let $\overline{b}$ denote the algebraic closure of b in $\mathbb{A}^1 \times \square^2$. As previously the intersection with $\mathbb{A}^1 \times F$ for any face F of $\square^2$ is empty ; and $\overline{b}$ (after applying the projector $\mathcal{A}lt$) gives

$$\overline{b} \in \mathcal{N}_{\mathbb{A}^1}^2(2).$$

Writing $\partial_{\mathbb{A}^1}$ the differential in $\mathcal{N}_{\mathbb{A}^1}$, one has $\partial_{\mathbb{A}^1}(\overline{b}) = 0$ and a class

$$[b] \in \mathrm{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(2)).$$

As, Corollary 2.28 insures that $\mathrm{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet(2)) = \mathrm{CH}^2(\mathbb{A}^1, 2) = 0$, there exists $\overline{c} \in \mathcal{N}_{\mathbb{A}^1}^1(2)$ such that

$$\partial_{\mathbb{A}^1}(\overline{c}) = \overline{b}.$$

Moreover, one remarks that $\overline{b}|_0 = \overline{b}|_1 = \emptyset$. The multiplication map

$$\mathbb{A}^1 \times \square^1 \times \square^2 \xrightarrow{\mu} A^1 \times \square^2 \quad , \qquad [t; u_1, u_2, u_3] \longmapsto [\tfrac{t}{1-u_1}; u_2, u_3]$$

is flat. Hence, one can consider the pull-back by μ of the cycle $\overline{b}$. This pull-back is given explicitly (after reparametrization) by

$$\mu^*(\overline{b}) = [t; 1 - \frac{t}{x_1}, x_1, 1 - x_1] \subset \mathbb{A}^1 \times \square^3.$$

This is nothing but Totaro's cycle [Tot92], already described in [BK94, Blo91] and gives a well defined element in $\mathcal{N}^1_{A^1}(2)$.

Definition 2.37. Let $L_{01} = \mathrm{Li}^{cy}_2$ denote the cycle

$$L_{01} = [t; 1 - \frac{t}{x_1}, x_1, 1 - x_1] \subset X \times \square^3$$

in $\mathcal{N}^1_X(2)$.

From the parametrized expression above, one sees that:

Lemma 2.38. *The cycle L_{01} satisfies the following properties*

1. $\partial(L_{01}) = b$.
2. *L_{01} extends to $\mathbb{A}^1$ that is it closure in $\mathbb{A}^1 \times \square^3$ gives a well defined element in $\mathcal{N}^1_{\mathbb{A}^1}(2)$.*
3. *$L_{01}|_{t=0} = \emptyset$ and $L_{01}|_{t=1}$ is well defined.*

Remark 2.39. Moreover, L_{01} corresponds to the function $t \mapsto \mathrm{Li}_2(t)$ as shown in [BK94] or in [GGL09].

This conclude the first inductive step of the 1-minimal model construction described at (4).

2.5.3. *Polylogarithms cycles.* By induction, one builds $\mathrm{Li}^{cy}_n = L_{0\cdots 01}$. We define Li^{cy}_1 to be equal to L_1.

Lemma 2.40. *For any integer $n \geqslant 2$ there exists cycles Li^{cy}_n in $\mathcal{N}^1_X(n)$ satisfying*

1. $\partial(\mathrm{Li}^{cy}_n) = L_0 \cdot \mathrm{Li}^{cy}_{n-1}$
2. *Li^{cy}_n extends to $\mathbb{A}^1$ that is it closure in $\mathbb{A}^1 \times \square^{2n-1}$ gives a well defined element in $\mathcal{N}^1_{\mathbb{A}^1}(n)$.*
3. *$\mathrm{Li}^{cy}_n|_{t=0} = \emptyset$ and $\mathrm{Li}^{cy}_n|_{t=1}$ is well defined.*
4. *Li^{cy}_n is explicitly given as a parametrized cycle by*

$$[t; 1 - \frac{t}{x_{n-1}}, x_{n-1}, 1 - \frac{x_{n-1}}{x_{n-2}}, x_{n-2}, \ldots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset X \times \square^{2n-1}$$

Proof. For $n = 2$, we have already defined $\mathrm{Li}^{cy}_2 = L_{01}$ satisfying the expected properties.

Assume that one has built the cycles Li^{cy}_k for $2 \leqslant k \leqslant n$. As previously, let b be the product

$$b = L_0 \cdot \mathrm{Li}^{cy}_{n-1} = [t; t, 1 - \frac{t}{x_{n-2}}, x_{n-2}, 1 - \frac{x_{n-2}}{x_{n-3}}, x_{n-3}, \ldots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1].$$

and $\overline{b}$ its algebraic closure in $\mathbb{A}^1 \times \square^{2n-2}$.

Computing the differential with the Leibniz rule, one gets $\partial(b) = -L_0 \cdot L_0 \cdot \mathrm{Li}^{cy}_{n-2} = 0$ and b gives a class in $\mathrm{H}^2(\mathcal{N}^\bullet_X(n))$.

Using the expression as parametrized cycle one computes the differential of $\overline{b}$ in $\mathcal{N}^\bullet_{\mathbb{A}^1}$

$$\partial_{\mathbb{A}^1}(\overline{b}) = \sum_{i=1}^{2n-2} (\partial^0_{\mathbb{A}^1,i}(\overline{b}) - \partial^\infty_{\mathbb{A}^1,i}(\overline{b})) = 0 \tag{7}$$

as many terms are empty because intersecting with a face $u_i = 0, \infty$ on a factor $\square^1$ leads to a 1 appearing on another $\square^1$ while the other terms cancel after applying the projector $\mathcal{A}lt$.

As in the case of Li^{cy}_2, $\overline{b}$ gives a class in $\mathrm{H}^2(\mathcal{N}^\bullet_{\mathbb{A}^1}(2)) = 0$ by Corollary 2.28 and there exists $\overline{c} \in \mathcal{N}^1_{\mathbb{A}^1}$ such that

$$\partial_{\mathbb{A}^1}(\overline{c}) = \overline{b}.$$

As $\mathrm{Li}^{cy}_{n-1}|_{t=0} = \emptyset$, $\overline{b}|_{t=0} = \emptyset$ and the element $\overline{c}$ is given by the pull-back by the multiplication

$$\mathbb{A}^1 \times \square^1 \times \square^{2n-2} \xrightarrow{\mu} A^1 \times \square^{2n-2} \quad ,$$

given in coordinates by

$$[t; u_1, u_2, \dots, u_{2n-1}] \longmapsto [\tfrac{t}{1-u_1}; u_2, \dots, u_{2n-1}] \quad .$$

Reparametrizing the factor $\mathbb{A}^1$ and the first $\square^1$ factor, one writes $\overline{c} = \mu^*(\overline{b})$ explicitly as a parametrized cycle

$$\overline{c} = [t; 1 - \frac{t}{x_{n-1}}, x_{n-1}, 1 - \frac{x_{n-1}}{x_{n-2}}, x_{n-2}, \dots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset \mathbb{A}^1 \times \square^{2n-1}.$$

Now, let Li^{cy}_n be the restriction of $\overline{c}$ to $\mathcal{N}^1_X(n)$ that is the parametrized cycle

$$\mathrm{Li}^{cy}_{n-1} = [t; 1 - \frac{t}{x_{n-1}}, x_{n-1}, 1 - \frac{x_{n-1}}{x_{n-2}}, x_{n-2}, \dots, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1] \subset X \times \square^{2n-1}.$$

The different properties, $d(\mathrm{Li}^{cy}_n) = L_0 \cdot \mathrm{Li}^{cy}_{n-1}$, extension to $\mathbb{A}^1$, $\mathrm{Li}^{cy}_n|_{t=0} = \emptyset$, can now be derived easily either for the explicit parametric representation or using the properties of $\overline{c}$. □

Remark 2.41. In Equation (7), the fact that $\partial^0_{\mathbb{A}^1,1}(\overline{b}) = 0$ is related to the induction hypothesis $\mathrm{Li}^{cy}_{n-1}|_{t=0} = \emptyset$ as in terms of cycles one has exactly

$$\overline{b} \cap \mathbb{A}^1 \times \{0\} \times \square^{2n-3} \mathrm{Li}^{cy}_{n-1}|_{t=0}.$$

The other terms in the differential are related to the equation satisfied by Li^{cy}_{n-1} in $\mathcal{N}^1_X(n-1)$ and giving

$$\partial(b) = L_0 \cdot L_0 \cdot \mathrm{Li}^{cy}_{n-2} = 0$$

and even if L_0 is not defined in $\mathcal{N}^\bullet_{\mathbb{A}^1}$ the fact that $\mathrm{Li}^{cy}_k|_{t=0} = 0$ insures that the product really correspond to an element in $\mathcal{N}^\bullet_{\mathbb{A}^1}$.

Remark 2.42.

- One finds back the expression given in [BK94] or in [GGL09]..
- Moreover, Li^{cy}_n corresponds to the function $t \mapsto \mathrm{Li}_n(t)$ as shown in [BK94] or in [GGL09].
- The construction is given in full details for more general cycles in Section 4 and is nothing but a direct application of Theorem 4.12 to the word $0 \cdots 01$ (with $n-1$ zero).

  The case of cycles Li^{cy}_k is however simple enough to be treated by itself as the "good" case.
- It is a general fact that pulling back by the multiplication preserve the empty fiber at $t = 0$ property as proved in Proposition 4.9.

2.6. **Admissibility problem at $t=1$ in weight 3.** It seems to the author that the first attempt to define algebraic cycles corresponding not only to polylogarithms but also to multiple polylogarithms was done by Gangl, Goncharov and Levin in [GGL09]. In their work, they have succeeded to build cycles corresponding to the value $\mathrm{Li}_{n_1,\dots,n_k}(x_1,\dots,x_k)$ for fixed parameters x_i in a number field F with the condition $x_i \neq 1$ and $x_i \neq x_j$ for $i \neq j$. However, their cycles are not admissible if one removes the conditions on the x_i. One develops in this section, the first example where such a problem appears, beginning by a review of the general strategy.

2.6.1. *Review of the strategy.* In order to build the 1-minimal model of $\mathcal{N}_X^\bullet$, we have first given generators of $\mathrm{H}^1(\mathcal{N}_X^\bullet)$. Then, assuming, one has already built some cycles c_i in $\mathcal{N}_X^1$, one want to find generators of

$$\ker\left(\mathrm{H}^2\left(\Lambda^2(\mathbb{Q}<c_i>)\right)\longrightarrow \mathrm{H}^2(\mathcal{N}_X^\bullet)\right).$$

In order to do so, we want to find linear combination of products

$$b=\sum \alpha_{i,j} c_i\cdot c_j \in \Lambda^2(\mathbb{Q}<c_i>)$$

such that

- $\partial(b)=0$ (that is $b\in \mathrm{H}^2\left(\Lambda^2(\mathbb{Q}<c_i>)\right)$),
- b is a boundary (that is there exist c in $\mathcal{N}_X^1$ such that $\partial(c)=b$). This tells us that b is in the kernel of $(\mathrm{H}^2\left(\Lambda^2(\mathbb{Q}<c_i>)\right)\to \mathrm{H}^2(\mathcal{N}_X^\bullet)$.

The product $\mathcal{N}_X^1\otimes\mathcal{N}_X^1\longrightarrow\mathcal{N}_X^2$ being anti-commutative, we have above identified the operations in $\Lambda^2(\mathbb{Q}<c_i>)$ and in $\mathcal{N}_X^\bullet$.

The strategy consists in looking for linear combinations

$$b=\sum \alpha_{i,j} c_i\cdot c_j \in \mathcal{N}_X^1$$

satisfying

(1) $\partial(b)=0$,
(2) b extends to $\mathbb{A}^1$ as $\overline{b}\in\mathcal{N}_{\mathbb{A}^1}^2$,
(3) $\partial_{\mathbb{A}^1}(\overline{b})=0$.

Then as $\mathrm{H}^2(\mathcal{N}_{\mathbb{A}^1}^\bullet)=0$ (Corollary 2.28), one gets a $\overline{c}$ given by the pull-back by the multiplication $\mathbb{A}^1\times\square^1\longrightarrow\mathbb{A}^1$ such that

$$\partial_{\mathbb{A}^1}(\overline{c})=\overline{b}.$$

The desired c is the restriction of $\overline{c}$ to $X=\mathbb{P}^1\setminus\{0,1,\infty\}$.

Remark 2.43. The method described above was put in motion for the polylogarithm example with $b=L_0\,\mathrm{Li}_{n-1}^{cy}$. The main result of Section 3 is to give a first general form for the b.

Below is the first example which uses the cycle L_1 in b and where both geometric and combinatorial key points arise.

2.6.2. *The algebraic cycle L_{011}.* The cycle L_{01} was defined previously, so was the cycle $L_{001}=\mathrm{Li}_3^{cy}$ by considering the product

$$b=L_0\cdot L_{01}.$$

Now, one would like to consider also the product

$$b=L_{01}\cdot L_1 \quad \in\mathcal{N}_X^2(3),$$

given as a parametrized cycle by

$$b=[t;1-\frac{t}{x_1},x_1,1-x_1,1-t] \quad \subset X\times\square^4.$$

From this expression, one sees that $\partial(b)=0$ because $t\in X$ can not be equal to 1.

Let $\overline{b}$ be the closure of the defining cycle of b in $\mathbb{A}^1 \times \square^4$, that is

$$\overline{b} = \left\{ (t, 1 - \frac{t}{x_1}, x_1, 1 - x_1, 1 - t) \text{ such that } t \in \mathbb{A}^1,\ x_1 \in \mathbb{P}^1 \right\}.$$

Let F be a face of $\square^4$ and u_i denote the coordinates on each factor $\square^1$. One necessarily has $u_i \neq 1$. If F is contained in an hyperplane defined by $u_2 = \infty$ or $u_3 = \infty$, then, as $u_1 \neq 1$, one gets

$$\overline{b} \cap \mathbb{A}^1 \times F = \emptyset.$$

Similarly, one gets an empty intersection of $\overline{b}$ with a face contained in $u_4 = \infty$ because $t \in \mathbb{A}^1$ is different from ∞. This remark reduces the case of F contained in $u_1 = \infty$ to the case F contained in $u_2 = 0$ which gives an empty intersection as $u_3 \neq 1$. By symmetry, the intersection with F contain in $u_3 = 0$ is also empty.

In order to prove that $\overline{b}$ is admissible and give an element in $\mathcal{N}^2_{\mathbb{A}^1}$ it remains to check the (co)dimension condition on the three remaining faces : $u_1 = 0$, $u_4 = 0$ and $u_1 = u_4 = 0$. The intersection of $\overline{b}$ with the face $u_1 = u_4 = 0$ is empty as $u_2 \neq 1$. The intersection $\overline{b}$ with the face defined by $u_1 = 0$ or $u_4 = 0$ is 1 dimensional and so of codimension 3 in $\mathbb{A}^1 \times F$.

Remark 2.44. Let F^0_4 denote the face of $\square^4$ defined by $u_4 = 0$. The intersection of b with $X \times F^0_4$ is empty as $t \neq 1$ in $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

From the above discussion, one gets a well defined element, written again $\overline{b}$, in $\mathcal{N}^2_{\mathbb{A}^1}(3)$. Computing the differential in $\mathcal{N}^{\mathbb{A}^1}_X$ gives, as the intersection with $u_1 = 0$ is killed by the projector $\mathcal{A}lt$,

$$\partial_{\mathbb{A}^1}(\overline{b}) = -L_{01}|_{t=1} \neq 0$$

and $\overline{b}$ do not gives a class in $\mathrm{H}^2(\mathcal{N}^\bullet_{\mathbb{A}^1})$.

In order to by pass this, one introduces the constant cycle $L_{01}(1)$ in $\mathcal{N}^1_X(2)$ defined by

$$L_{01}(1) = [t; 1 - \frac{1}{x_1}, x_1, 1 - x_1, 1 - t] \subset X \times \square^3.$$

The cycle $L_{01}(1)$ satisfies

$$\forall a \in X \qquad L_{01}|_{t=a} = L_{01}|_{t=1}.$$

and extends to a well defined cycles in $\mathcal{N}^1_{\mathbb{A}^1}(2)$.

Instead of considering the product $L_{01} \cdot L_1$, one looks at the linear combination

$$b = (L_{01} - L_{01}(1)) \cdot L_1 \in \mathcal{N}^2_X(3). \tag{8}$$

As above, one checks that b extends to a well defined element $\overline{b}$ in $\mathcal{N}^2_{\mathbb{A}^1}(3)$. The correction by $-L_{01}(1) \cdot L_1$ insures that

$$\partial(b) = 0, \qquad \partial_{\mathbb{A}^1}(\overline{b}) = 0, \qquad \overline{b}|_{t=0} = \emptyset.$$

Computing the pull-back by the multiplication $\mu : \mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$; one wants to define L_{011} in $\mathcal{N}^\bullet_X(3)$ as the parametrized cycles

$$\begin{aligned} L_{011} = [t; 1 - \frac{t}{x_2}, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1, 1 - x_2] \\ + [t; 1 - \frac{t}{x_2}, 1 - x_2, 1 - \frac{1}{x_1}, x_1, 1 - x_1] \qquad \subset X \times \square^5 \end{aligned} \tag{9}$$

As, $t \neq 1$ in $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, one easily check that L_{011} is an admissible on $X \times \square^5$ and gives a well defined element in $\mathcal{N}^\bullet_X(3)$. An explicit computation gives also that

$$d(L_{011}) = b = (L_{01} - L_{01}(1)) \cdot L_1. \tag{10}$$

Remark 2.45. However, one should emphasis that

- L_{011} is not admissible on $\mathbb{A}^1 \times \square^5$ due to an issue at the point $t = 1$.
- This non-admissibility problem is the same faced by Gangl, Goncharov and Levin in [GGL09].

Section 4 will explain how to obtain general cycles admissible at $t = 1$ and the particular example of a cycle $\mathcal{L}_{011}$ related to L_{011} above will be detailed at Section 5.1.

Remark 2.46. Even if L_{011} is not admissible at $t = 1$, one could go on, looking for "good" linear combination of product. In particular in weight 4, one could consider

$$b = L_0 \cdot L_{011} + L_{001} \cdot L_1 - L_{001}(1) \cdot L_1 + L_{01} \cdot L_{01}(1) \tag{11}$$

and remark that

- The terms $L_0 \cdot L_{011} + L_{001} \cdot L_1$ correspond to a principal part related to the free Lie algebra $\mathrm{Lie}(X_0, X_1)$ as explained in Section 3.2.
- The term $L_{01} \cdot L_{01}(1)$ cancels with the the correction $-L_{01}(1) \cdot L_1$ introduced earlier for L_{011} and insures that $\partial(b) = 0$. It corresponds to "a propagation" of the correction introduced for L_{011}.
- The term $-L_{001}(1) \cdot L_1$ is similar to the correction $-L_{01}(1) \cdot L_1$ introduced earlier for L_{011} and insures that $\partial_{\mathbb{A}^1}(\overline{b}) = 0$.
- The two correction terms are related to the principal part by some special differential on rooted trivalent trees as will be explained in Section 3.4.

3. Combinatorial settings

In this paper a *plane* or *planar* tree is a finite tree whose internal vertices are of valency $\geqslant 3$ and where at each vertex a cyclic ordering of the incident edges are given. We assume that all other vertices are of valency 1 and call them *external vertices*.

A *rooted* tree is a planar tree as above with one distinguished external vertex of valency 1 called its root. In particular a rooted tree has at least one edge. The external vertices which are not the root are called *leaves*.

We will draw trees so that the root vertex is at the top and so that the cyclic order around the vertices is displayed in counterclockwise direction.

3.1. Lyndon words and the free Lie algebra $\mathrm{Lie}(X_0, X_1)$.

The material developed in this section is detailed in full generality in [Reu93, Reu03] and recalls the basic definitions and some properties of the free Lie algebra on two generators and its relations to trivalent trees and Lyndon words.

3.1.1. *Trees and free Lie algebra.* Recall that a *Lie algebra* over $\mathbb{Q}$ is a $\mathbb{Q}$ vector space L, equipped with a bilinear mapping $[\,,\,] : L \otimes L \longrightarrow L$, satisfying the two following properties for any x, y, z in L:

$$[x, x] = 0 \tag{12}$$

$$[[x, y]z] + [[y, z], x] + [[z, x], y] = 0. \tag{Jacobi}$$

Remark 3.1. Note that applying the first relation to $[x + y, x + y]$, one obtains the *antisymmetry* relation

$$[x, y] = -[y, x].$$

Thus, we may rewrite Jacobi identity as

$$[[x, y], z] = [x, [y, z,]] + [[x, z,], y].$$

Definition 3.2. Given a set S, a free Lie algebra on S over $\mathbb{Q}$ is a Lie algebra L over $\mathbb{Q}$ together with a mapping $i : S \to L$ with the following *universal property* :

For each Lie algebra K and each mapping $f : S \to K$, f factors uniquely through L.

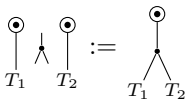
In what follows, we will only consider S to be a set with two elements, either $S = \{0, 1\}$ or $S = \{X_0, X_1\}$.

One is used to see the free Lie algebra on $\{X_0, X_1\}$ as a subspace of $\mathbb{Q} < X_0, X_1 >$ (its enveloping algebra), the space of polynomials in two non commuting variables X_0 and X_1. Let $\mathrm{Lie}(X_0, X_1)$ denote this free Lie algebra.

In order to show the existence of free Lie algebras, one usually uses a tree representation.

Definition 3.3. Let $\mathcal{T}^{tri}_{\mathbb{Q}}$ denote the $\mathbb{Q}$ vector space generated by the set $\mathcal{T}^{tri}$ of rooted, planar, trivalent trees with leaves decorated by 0 and 1.

For two trees T_1, T_2 in $\mathcal{T}^{tri}$ defined $T_1 \curlywedge T_2$ to be the tree obtained by joining the root (marked by a circle around the vertex) of T_1 and T_2 and adding a new root:



The set $\mathcal{T}^{tri}$ is isomorphic to the free magma on $\{0, 1\}$ a branch $\curlywedge$ in a tree corresponding to a bracketing in a well-formed expression.

The composition law $\curlywedge$ extends by bilinearity to $\mathcal{T}^{tri}_{\mathbb{Q}}$. Let I_{Jac} denote the ideal of $\mathcal{T}^{tri}_{\mathbb{Q}}$ generated by the element of the form $T \curlywedge T$ and

$$(T_1 \curlywedge T_2) \curlywedge T_3 + (T_2 \curlywedge T_3) \curlywedge T_1 + (T_3 \curlywedge T_1) \curlywedge T_2.$$

The quotient $\mathcal{T}^{tri}_{\mathbb{Q}}/I_{Jac}$ is a Lie algebra with bracket $[\,,\,]$ given by $\curlywedge$; in fact it is a free Lie algebra on $\{0, 1\}$.

Identifying $\{0, 1\}$ to $\{X_0, X_1\}$ by the obvious morphism and using the correspondence $\curlywedge \leftrightarrow [\,,\,]$, one obtains

Lemma 3.4. *The quotient* $\mathcal{T}^{Lie} = \mathcal{T}^{tri}_{\mathbb{Q}}/I_{Jac}$ *is isomorphic to* $\mathrm{Lie}(X_0, X_1)$.

For T in $\mathcal{T}^{tri}$ let $[T]$ denote its image in $\mathcal{T}^{Lie}$

3.1.2. *Hall set.* Each tree T in $\mathcal{T}^{tri}$ is either a letter $T = \overset{\circ}{|}_a$ for $a \in \{0, 1\}$ or is of the form $T = T_1 \curlywedge T_2$; one writes $S_{\mathcal{T}}$ for the set $\{\overset{\circ}{|}_0, \overset{\circ}{|}_1\}$.

Definition 3.5 ([Reu93])**.** A subset H of $\mathcal{T}^{tri}$ is a *Hall set* if the following conditions hold:

- H as a total order $<$.
- $S_{\mathcal{T}} = \{\overset{\circ}{|}_0, \overset{\circ}{|}_1\} \subset H$.
- for any tree $T = T_1 \curlywedge T_2$ in $H \setminus S_{\mathcal{T}}$, one has $T_2 \in H$ and

$$T < T_2. \tag{13}$$

- For any tree $T = T_1 \curlywedge T_2$ in $\mathcal{T}^{tri} \setminus S_{\mathcal{T}}$, T is in H if and only if

$$T_1,\ T_2 \text{ are in } H \text{ and } T < T_2, \tag{14}$$

$$\text{and either } T_1 \in S_{\mathcal{T}} \text{ or } T_1 = T' \curlywedge T'' \text{ with } T'' \geqslant T_2. \tag{15}$$

Remark 3.6. In [Reu03], Reutenauer begins with a total order on $\mathcal{T}^{tri}$ satisfying

$$T = T_1 \curlywedge T_2 \in \mathcal{T}^{tri} \setminus S_{\mathcal{T}} \Rightarrow T < T_2.$$

Then, he defines the Hall set relative to the order $<$ as the subset of $\mathcal{T}^{tri}$ containing $S_{\mathcal{T}}$ and satisfying the last condition of the above definition (Equations (14) and (15)). This gives the existence of Hall sets.

Zorn's Lemma (even if it may be overwhelming) implies that, beginning with a Hall set as in Definition 3.5, the total order on H induces a total order on $\mathcal{T}^{tri}$ satisfying

$$T = T_1 \curlywedge T_2 \in \mathcal{T}^{tri} \setminus S_{\mathcal{T}} \Rightarrow T < T_2.$$

Theorem 3.7 ([Reu03][Theorem 4])**.** *Let H be a Hall set of $\mathcal{T}^{tri}$. The element $[T]$ for T in H form a basis of $\mathcal{T}^{Lie} \simeq \mathrm{Lie}(X_0, X_1)$.*

We would like to review below the algorithm showing that the elements $[T]$ for $T \in H$ generate $\mathcal{T}^{Lie}$ adding some extra information that will be used latter.

The algorithm described in [Reu03][Section 9] goes essentially as follows. Consider the total order $<$ on $\mathcal{T}^{tri}$ and let $T = T_1 \curlywedge T_2$ with $T_1 < T_2$ be a tree in $\mathcal{T}^{tri}$. Then either T is in H or $T_1 = T' \curlywedge T''$ with $T'' < T_2$ and we can also assume that $T' < T''$. Then one writes in $\mathcal{T}^{Lie}$

$$\begin{aligned}
[T] = [T_1 \curlywedge T_2] = [[T_1], [T_2]] &= \big[[T' \curlywedge T''], [T_2]\big] \\
&= [[[T'], [T'']], [T_2]] \\
&= [[[T'], [T_2]], T''] + [[T'], [[T''], [T_2]]] \\
&= \big[(T' \curlywedge T_2) \curlywedge T''\big] + \big[T' \curlywedge (T'' \curlywedge T_2)\big] \qquad (16)
\end{aligned}$$

and concludes using an induction on both the sum of the degrees of T_1 and T_2 and the maximum of T_1 and T_2. This insures that the algorithm terminates.

In this algorithm, we want for latter use to keep track of

- brackets that are 0, that is there is a subtree of the form $T \curlywedge T$.
- evolution of the position of a distinguished leaf of the original tree.

In [Reu93][proof of Theorem 4.9], Reutenauer gives the same algorithms but going in the other direction : beginning with the smallest subtree T_0 of T which is not in H. For our purpose, we will modify this first algorithm given by Reutenauer.

Let $\mathcal{T}^{tri}_{\mathbb{N},\mathcal{F}in}$ denote the set of finite sequences $\mathfrak{T}$ of triples (T_i, q_i, k_i) in $\mathcal{T}^{tri} \times \mathbb{Z} \times \mathbb{N}^*$ such that for any i, k_i denotes the position (beginning on the left going to the right) of a leaf of T_i.

Let $\mathfrak{T} = (T_i, q_i, k_i)_{1 \leqslant i \leqslant N}$ be an element of $\mathcal{T}^{tri}_{\mathbb{N},\mathcal{F}in}$. Define $Dec(\mathfrak{T})$ in $\mathcal{T}^{tri}_{\mathbb{N},\mathcal{F}in}$ as follows.

(1) for l in $1 \leqslant i \leqslant N$, beginning with $\mathfrak{T}^0 = \mathfrak{T}$, define $\mathfrak{T}^l = (T^l_i, q^l_i, k^l_i)_{1 \leqslant i \leqslant N_l}$ as follows
   Let $j = N + 1 - l$. If T^{l-1}_j is in H then let $\mathfrak{T}^l = \mathfrak{T}^{l-1}$.

(2) Else considers the smallest subtree $T' \curlywedge T''$ of T^{l-1}_j which is not in H. In particular T' and T'' are in H. Then, either $T' < T''$, or $T' = T''$ or $T' > T''$. Let n_j be the number of leaves of T_j and the position of the leaves of T' (resp. T'') in T^l_j to be in $\{b'_j, f'_j\}$ (resp. $\{f'_j + 1, f''_j\}$ for $1 \leqslant b'_j \leqslant f'_j < f''_j$.
   If $\mathbf{T' = T''}$: one defines $\mathfrak{T}^l = \mathfrak{T}^{l-1}$.
   If $\mathbf{T' > T''}$: let T^l_j be the tree obtained from T^{l-1}_j

$$\text{by replacing the subtree} \quad T' \curlywedge T'' \quad \text{by} \quad T'' \curlywedge T' \qquad (17)$$

   and let q^l_j be $-q^{l-1}_j$. Define k^l_j by

$$k^l_j = \begin{cases} k^{l-1}_j & \text{if } k^{l-1}_j < b'_j \text{ or } k^{l-1}_j > f''_j, \\ k^{l-1}_j + f''_j - f'_j & \text{if } b'_j \leqslant k^{l-1}_j \leqslant f'_j, \\ k^{l-1}_j - (f'_j - b'_j + 1) & \text{if } f'_j + 1 \leqslant k^{l-1}_j \leqslant f''_j. \end{cases}$$

Let $N_l = N_{l-1}$ and $\mathfrak{T}^l = (T_i^l, q_i^l, k_i^l)_{1\leqslant i\leqslant N_l}$ be defined by

$$(T_i^l, q_i^l, k_i^l) = (T_i^{l-1}, q_i^{l-1}, k_i^{l-1})$$

for $i \neq j$ and by the triple (T_j^l, q_j^l, k_j^l) for $i = j$.

If $\mathbf{T' < T''}$: as $T' \curlywedge T''$ is not in H, one has $T' = t_1 \curlywedge t_2$ with $t_2 < T''$. Let the position of the leaves of t_1 (resp. t_2) be in $\{b'_j, f'_{j,1}\}$ (resp. $\{f'_{j,1}+1, f'_j\}$). Define $N_l = N_{l-1}+1$, $q_j^l = q_{j+1}^l = q_j^{l-1}$.
Let T_j^l be the tree obtained from T_j^{l-1}

(18) by replacing the subtree $T' \curlywedge T''$ by $(t_1 \curlywedge T'') \curlywedge t_2$,

and let T_{j+1}^l be the tree obtained from T_j^{l-1}

(19) by replacing the subtree $T' \curlywedge T''$ by $t_1 \curlywedge (T'' \curlywedge t_2)$.

This operation is exactly applying Jacobi on a subbracket, that is Equation (16). We also need to see how the position k_j^{l-1} changes and define

$$k_j^l = k_{j+1}^l = \begin{cases} k_j^{l-1} & \text{if } k_j^{l-1} < f'_{j,1}+1 \text{ or } k_j^{l-1} > f''_j, \\ k_j^{l-1} + f''_j - f'_j & \text{if } f'_{j,1}+1 \leqslant k_j^{l-1} \leqslant f'_j, \\ k_j^{l-1} - (f'_j - f'_{j,1}+1) & \text{if } f'_j+1 \leqslant k_j^{l-1} \leqslant f''_j. \end{cases}$$

Now, one defines $\mathfrak{T}^l = (T_i^l, q_i^l, k_i^l)$ as

$$\begin{aligned} (T_i^l, q_i^l, k_i^l) &= (T_i^{l-1}, q_i^{l-1}, k_i^{l-1}) && \text{for } i < j, \\ (T_i^l, q_i^l, k_i^l) &= (T_j^l, q_j^l, k_j^l) && \text{for } i = j, \\ (T_i^l, q_i^l, k_i^l) &= (T_{j+1}^l, q_{j+1}^l, k_{j+1}^l) && \text{for } i = j+1. \\ (T_i^l, q_i^l, k_i^l) &= (T_{i-1}^{l-1}, q_{i-1}^{l-1}, k_{i-1}^{l-1}) && \text{for } i > j+1. \end{aligned}$$

Now, we have $\mathfrak{T}^N$ in $\mathcal{T}^{tri}_{\mathbb{N},\mathcal{F}in}$, a sequence of length N_N and regroup the terms having same T_i and k_i.

(3) For $\mathfrak{T}^l = (T_i^l, q_i^l, k_i^l)_{1\leqslant i\leqslant N_l}$ with $l \leqslant N$, if there exists $1 \leqslant i_0 < j_0 \leqslant N_l$ such that $T_{i_0}^l = T_{j_0}^l$ and $k_{i_0}^l = k_{j_0}^l$, then set $N_{l+1} = N_l - 1$ and $\mathfrak{T}^{l+1} = (T_i^{l+1}, q_i^{l+1}, k_i^{l+1})_{1\leqslant i\leqslant N_{l+1}}$ by

$$\begin{aligned} (T_i^{l+1}, q_i^{l+1}, k_i^{l+1}) &= (T_i^l, q_i^l, k_i^l) && \text{for } i < i_0 \text{ and } i_0 < i < j_0, \\ (T_i^{l+1}, q_i^{l+1}, k_i^{l+1}) &= (T_i^l, q_i^l + q_j^l, k_i^l) && \text{for } i = i_0, \\ (T_i^{l+1}, q_i^{l+1}, k_i^{l+1}) &= (T_{i+1}^l, q_{i+1}^l, k_{i+1}^l) && \text{for } i \geqslant j_0. \end{aligned}$$

This part of the algorithm stops when all the couple (T_i^l, k_i^l) are different for some L large enough ($\leqslant 2N_N + 1$).

The decomposition algorithm described above tells us that beginning with a triple $(T_0, 1, k)$, k being the position of one of the leaves of T_0, the sequence $Dec^{\circ n}(T_0, 1, k)$ is constant for n large enough. Let $\mathfrak{T} = (T_i, q_i, k_i)_{1\leqslant i\leqslant N}$ be its constant values. One gets the following

Lemma 3.8. *With the above notations, the decomposition of $[T_0]$ in $\mathcal{T}^{Lie}$ in terms of $[T]$ for T in H is given by*

$$[T_0] = \sum_{i=1}^{N} q_i [T_i].$$

Remarks that not all the T_i are in the Hall set H. However, those which are not in H contain a subtree of the form $T \curlywedge T$ and thus, the corresponding bracket $[T_i]$ is zero in $\mathcal{T}^{Lie}$.

3.1.3. *Lyndon words.* We are here interested in some particular Hall, the one induced by the Lyndon words. Let S be the set $\{0,1\}$ and S^* denote the set of finite words in the letters $0,1$. Let $<$ be the lexicographic order on S^* such $0<1$.

Definition 3.9 (Lyndon words)**.** A *Lyndon word* W in S^* is a nonempty word which is smaller than all its nontrivial proper right factors; that is $W\neq\emptyset$ and

$$W=UV \text{ with } U,V\neq\emptyset \quad \Rightarrow \quad W<V.$$

Remark that 0 and 1 are Lyndon words.

Example 3.10. The Lyndon words of length $\leqslant 4$ are

$$0,\ 1,\ 01,\ 001,\ 011,\ 0001,\ 0011,\ 0111.$$

They are ordered by the lexicographic order which gives

$$0<0001<001<0011<01<011<0111<1.$$

In order to associate a tree to a Lyndon word, we need the following definition.

Definition 3.11 (Standard factorization)**.** Let W be a word in S^* of length $\geqslant 2$. The standard factorization of W is the decomposition

$$W=UV \text{ with } \begin{cases} U,V\in S^*\setminus\emptyset \\ \text{ and } V \text{ is the smallest nontrivial proper right factor of } W. \end{cases}$$

One has the following property of Lyndon words.

Proposition 3.12 ([Reu93][proof of Theorem 5.1])**.** *Let W be a Lyndon word with standard factorization $W=UV$. Then , U and V are Lyndon words and either U is a letter or has standard factorization $U=U_1U_2$ with $U_2\geqslant V$.*

To any Lyndon word W we associate a tree τ_W in $\mathcal{T}^{tri}$. If $W=0$ or $W=1$, set

$$\tau_0=\overset{\bullet}{\underset{0}{|}} \qquad \tau_1=\overset{\bullet}{\underset{1}{|}}.$$

For a Lyndon word W of length $\geqslant 2$, let $W=UV$ be its standard factorization and set

$$\tau_W=\tau_U \curlywedge \tau_V.$$

Let H_L be the set $\{\tau_W\}$ where W runs through the Lyndon words in the letters $0,1$. Endow H_L with the total order $<$ induced by the ordering of the Lyndon words W given by the lexicographic order on S^*.

For any Lyndon word, let $[W]$ be the image of τ_W in $\mathcal{T}^{Lie}$ and let Lyn be the set of the Lyndon words.

Definition 3.13. For a Lyndon word W, we say that τ_W is a *Lyndon tree* and that $[W]$ is a *Lyndon bracket.*

Theorem 3.14. *The set H_L is a Hall set and the family $([W])_{W\in Lyn}$ forms a basis of $\mathcal{T}^{Lie}$.*

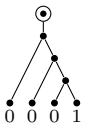
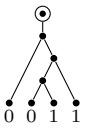
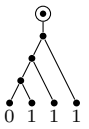
Moreover, a basis of $\mathcal{T}^{Lie}\wedge\mathcal{T}^{Lie}$ is then given by the family $([W_1]\wedge[W_2])$ for W_1,W_2 Lyndon words such that $W_1<W_2$. In these basis, the bracket is then given by

$$[W_1]\wedge[W_2]\overset{[\,,\,]}{\longmapsto}[[W_1],[W_2]]=\sum_{W\in Lyn}\alpha^W_{W_1,W_2}[W]. \tag{20}$$

Example 3.15. In length $\leqslant 3$, one has

$$\tau_0=\overset{\bullet}{\underset{0}{|}},\quad \tau_1=\overset{\bullet}{\underset{1}{|}},\quad \tau_{01}=\ \overset{\bullet}{\underset{0\ 1}{\curlywedge}}\ ,\quad \tau_{001}=\ \overset{\bullet}{\underset{0\ 0\ 1}{\curlywedge}}\ ,\quad \tau_{011}=\ \overset{\bullet}{\underset{0\ 1\ 1}{\curlywedge}}\ ,$$

and in length 4

$\tau_{0001} =$  , $\tau_{0011} =$  , $\tau_{0111} =$  .

3.2. **Trivalent trees and duality.** Let $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ be the quotient of $\mathcal{T}^{tri}_{\mathbb{Q}}$ the ideal (for $\curlywedge$) I_s generated by

$$T_1 \curlywedge T_2 + T_2 \curlywedge T_1.$$

Let T is a tree in $\mathcal{T}^{tri}$ with subtree $T_1 \curlywedge T_2$ and let T' be the tree T in which one has replaced $T_1 \curlywedge T_2$ by $T_2 \curlywedge T_1$. In $\mathcal{T}^{tri,<}_{\mathbb{Q}}$, one has the relation

$$T = -T'.$$

From the total order on H_L, one gets a total order $<$ on $\mathcal{T}^{tri}$. Let $\mathfrak{B}^<$ be the set of trees T in $\mathcal{T}^{tri}$ such that

$$T' = T_1 \curlywedge T_2 \text{ is subtree of } T \quad \Rightarrow \quad T_1 < T_2.$$

Writing $T \in \mathfrak{B}^<$ also for the image of T in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$, one sees that

Lemma 3.16. *The set $\mathfrak{B}^<$ induces a basis of $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ also denoted by $\mathfrak{B}^<$.*

We will now, identify $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ with its dual by the means of the basis $\mathfrak{B}^<$.

Let $I^<_{Jac}$ denote the image of the ideal I_{Jac} in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$. The Lie algebra $\mathcal{T}^{Lie}$ is then isomorphic to the quotient $\mathcal{T}^{tri,<}_{\mathbb{Q}}/I_{Jac}$ and, using the identification between $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ and its dual, one can identify the dual of $\mathcal{T}^{Lie}$ with a subspace $\mathcal{T}^{coL} \subset \mathcal{T}^{tri,<}_{\mathbb{Q}}$.

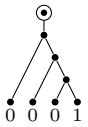
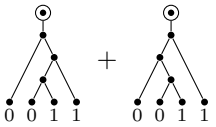
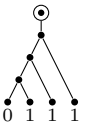
Definition 3.17. Let $(T_{W^*})_{W\in Lyn}$ in $\mathcal{T}^{coL}$ denote the dual basis of the basis $([W])_{W\in Lyn}$ of the free Lie algebra $\mathcal{T}^{Lie}$.

The T_{W^*} are linear combinations of trees in $\mathfrak{B}^<$. One should remark that any Lyndon tree τ_W is in $\mathfrak{B}^<$ and that by definition its coefficient in T_{W^*} is 1.

Example 3.18. Up to length $\leqslant 3$, one has $T_{W^*} = \tau_W$ that is

$T_{0^*} =$  , $T_{1^*} =$, $T_{01^*} =$  , $T_{001^*} =$  , $T_{011^*} =$  .

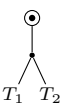
In length 4, appears the first linear combination

$T_{0001^*} =$  , $T_{0011^*} =$  , $T_{0111^*} =$  .

As the Lie bracket on $\mathcal{T}^{Lie}$ is induced by $\curlywedge : \mathcal{T}^{tri} \wedge \mathcal{T}^{tri} \to \mathcal{T}^{tri}$; it is also induced by $\curlywedge$ on $\mathcal{T}^{tri,<}_{\mathbb{Q}}$. By duality, one obtains a differential

$$d_{Lie} : \mathcal{T}^{coL} \longrightarrow \mathcal{T}^{coL} \wedge \mathcal{T}^{coL}$$

dual to the Lie bracket and induced by the map $\mathcal{T}^{tri}_{\mathbb{Q}} \longrightarrow \mathcal{T}^{tri}_{\mathbb{Q}} \wedge \mathcal{T}^{tri}_{\mathbb{Q}}$ also denote by d_{Lie} :

$$d_{Lie} : \text{  } \longmapsto \text{  } . \tag{21}$$

The property that $d_{Lie} \circ d_{Lie} = 0$ on $\mathcal{T}^{coL}$ is dual to the Jacobi identity on $\mathcal{T}^{Lie}$.

Proposition 3.19. *By duality, one has in $\mathcal{T}^{coL}$:*

- $T_{0^*} = \overset{\odot}{\underset{0}{|}}$, $T_{1^*} = \overset{\odot}{\underset{1}{|}}$;
- $d_{Lie}(T_{0^*}) = d_{Lie}(T_{1^*}) = 0$;
- *for all Lyndon words W of length $\geqslant 2$,*

$$d_{Lie}(T_{W^*}) = \sum_{\substack{W_1<W_2\\ W_1,W_2\in Lyn}} \alpha^W_{W_1,W_2} T_{W_1^*} \wedge T_{W_2^*} \tag{22}$$

where the $\alpha^W_{W_1,W_2}$ are defined by equation (20).

Moreover one can build the linear combinations T_{W^*} inductively by

$$T_{W^*} = \sum_{\substack{W_1<W_2\\ W_1,W_2\in Lyn}} \alpha^W_{W_1,W_2} T_{W_1^*} \curlywedge T_{W_2^*} \tag{23}$$

for W of length greater or equal to 2. Here $\curlywedge$ denotes the bilinear map $\mathcal{T}^{coL} \otimes \mathcal{T}^{coL} \longrightarrow \mathcal{T}^{coL}$ induced by $\curlywedge$.

Lemma 3.20. *Let W be a Lyndon word of length greater or equal to 2.*

- *For the lexicographic order one has $0 < W < 1$*
- *A leaf of a tree in the sum T_{W^*} decorated by 1 is always a right leaf.*
- *A leaf of a tree in the sum T_{W^*} decorated by 0 is always a left leaf.*

Proof. Let W be a Lyndon word. As 0 is the smallest non empty word in letters 0 and 1, one has $0 < W$. Writing $W = U\varepsilon$ with ε in $\{0, 1\}$ and U non empty, one has $W < \varepsilon$. Thus, one gets $\varepsilon = 1$ which conclude the first part of the lemma.

Now, the inductive construction of the family T_{W^*} given by Equation (23) (Proposition 3.19)

$$T_{W^*} = \sum_{\substack{W_1<W_2\\ W_1,W_2\in Lyn}} \alpha^W_{W_1,W_2} T_{W_1^*} \curlywedge T_{W_2^*}$$

shows that T_{0^*} is always add as a left factor and T_{1^*} always as a right factor. Induction on the length concludes the lemma. □

Definition 3.21. Let T be a tree in $\mathcal{T}^{tri}$. Its image in $\mathcal{T}^{Lie}$ decomposes on the Lyndon basis as

$$[T] = \sum_{W\in Lyn} c_T^W [W].$$

Duality implies that

Lemma 3.22. *Let W be a Lyndon word. Then T_{W^*} decomposes on the basis $\mathfrak{B}^<$ as*

$$T_{W^*} = \sum_{T\in\mathfrak{B}^<} c_T^W T$$

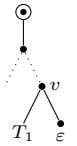
where the c_T^W are the ones defined at Definition 3.21.

In view of Theorem 3.44, we need to express the coefficients c_T^W in terms of the coefficients $c_{T_1}^{U_1}$ and $c_{T_2}^{U_2}$ for some subtrees T_1 and T_2 of T. We give below the necessary definitions and lemmas to give such a decomposition of the c_T^W (Theorem 3.30). The rest of this subsection will be devoted into proving this decomposition.

Definition 3.23. For a tree T, let $Le_{\mathcal{T}}(T) = \{l_1, \ldots, l_n\}$ be the set of its leaves numbered from left to right and let $Le^1_{\mathcal{T}}(T)$ be the set of leaves with decoration equal to 1.

The *position* of a leaf l_i will be its number i and we shall write $i \in Le(T)$ (resp. $i \in Le^1(T)$) to a denote the position of a leaf (resp. of a leaf decorated by 1).

Definition 3.24. Let T be a tree in $\mathcal{T}^{tri}$ and i the position of one of its leaves decorated by $\varepsilon = 0, 1$. Let v denotes the vertex just above this leaf and T_1 the (other) subtree just below v:

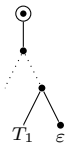


We shall write $\left(T/_{\stackrel{i}{\cdot}}\right)^r$ for the subtree T_1 and ${}^l\left(T/_{\stackrel{i}{\cdot}}\right)$ or $T\setminus^i_{T_1}$ for the tree obtained from T by deleting the subtree T_1 and the i-th leaf and by changing the vertex v into a leaf with decoration ε :

$$\left(T/_{\stackrel{i}{\cdot}}\right)^r := \quad {}^l\left(T/_{\stackrel{i}{\cdot}}\right) = T\setminus^i_{T_1} :=$$

Definition 3.25 (Insertion)**.** Let T_1 and T_2 be two trees in $\mathcal{T}^{tri}$ and i be the position of a leaf in T_2 and ε its decoration. We assume that this leaf is a "right tree". We denote by $T_2 \underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^i T_1$ the tree obtained from T_2 by replacing the i-th leaf by a vertex v with left subtree T_1 and right subtree a leaf decorated by ε :

$$T_2 = \qquad \Rightarrow \qquad T_2 \underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^i T_1 :=$$



In case we need an insertion on a "left leaf", we will use the notation ${}^l\underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^i$.

In the pictures describing definition 3.24 and 3.25 we have drawn the important part in a right subtree, the definitions remains valid in case the considered leaf is in a left subtree.

Definition 3.26. if e denotes a leaf of a tree, we extend the above notation to the leaf e:

$$(T/_{\stackrel{e}{\cdot}})^r, \qquad {}^l(T/_{\stackrel{e}{\cdot}}) = T\setminus^e_{T_1}, \qquad T_2 \underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^e T_1.$$

Let $\mathfrak{B}^=$ denote the set of trees T in $\mathcal{T}^{tri}$ such that there exists one and only one subtree of T of the form $T_1 \overset{\downarrow}{\wedge} T_1$.

Looking closely at the insertion/quotient operation one obtains the following.

Lemma 3.27.

- *Let T be a tree in $\mathfrak{B}^<$ and e a leaf in $Le^1(T)$. Then, there is a unique tuple $(T_1, T_2, f, \varepsilon(T, f))$ with T_1 in $\mathfrak{B}^<$, $T_2 \in \mathfrak{B}^< \cup \mathfrak{B}^=$, f in $Le^1_{T_2}$ and $\varepsilon(T, e)$ in $\{\pm 1\}$ such that*

$$T\setminus^e_{T'} = \varepsilon(T, e)T_2 \qquad \text{and} \qquad (T/_{\stackrel{e}{\cdot}})^r = T_1.$$

- *Let T_1, T_2 be trees in $\mathfrak{B}^< \cup \mathfrak{B}^=$ with T_1 in $\mathfrak{B}^<$ and f in $Le^1_{T_2}$. Then, there exists a unique tree T in $\mathfrak{B}^< \cup \mathfrak{B}^=$ and a unique $\tilde{\varepsilon}(T_1, T_2, f)$ such that*

$$\tilde{\varepsilon}(T_1, T_2, f)T = T_2 \underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^f T_1.$$

and, one write

$$T := \overline{T_2 \underset{\leftarrow}{\overset{\downarrow}{\wedge}}{}^f T_1}.$$

In the case where T is in $\mathfrak{B}^=$, then $\tilde{\varepsilon}(T_1, T_2, f) = 1$.

There is a unique leaf $e \in Le_1(\overline{T_2 \underset{\leftarrow}{\curlywedge} f\ T_1})$ *such that*

$$T \setminus^{e}{}_{T'} = \varepsilon(T,e)T_2 \qquad \text{and} \qquad (T/_{\cdot}^{e})^{r} = T_1$$

and it will be denoted by $\varphi_{T_1,T_2}(f)$ *or simply by* $\varphi(f)$ *if the context is clear enough.*

If T*, as above, is in* $\mathfrak{B}^{<}$ *such that* $T \setminus^{e}{}_{T'}$ *is up to a sign also in* $\mathfrak{B}^{<}$ *then with the above notations*

$$T = \overline{T_2 \underset{\leftarrow}{\curlywedge} f\ T_1} \quad \text{and} \quad \varepsilon(T,e)\varepsilon_{T_1,T_2,f} = 1.$$

Definition 3.28. Let T_0 be a tree in $\mathcal{T}^{tri}$, i the position of one of its leaves and let $\mathfrak{T} = (T_j, q_j, k_j)_{1 \leqslant j \leqslant N}$ be the constant value of $Dec^{\circ n}(T, 1, i)$ for n large enough.

Let T be a tree in $\mathfrak{B}^{<} \cup \mathfrak{B}^{=}$ and k the position of one of its leaves, define the coefficient $c^{T,k}_{T_0,i}$ as

$$c^{T,k}_{T_0,i} = \begin{cases} q_j \text{ if } \exists\, j \text{ such that } T = T_j \text{ and } k = k_j \\ 0 \text{ otherwise.} \end{cases}$$

For any Lyndon word W, the coefficient $c^{\tau_W,k}_{T_0,i}$ will be denoted simply by $c^{W,k}_{T_0,i}$.

Lemma 3.29.

- *Let* T_0 *be a tree in* $\mathcal{T}^{tri}$*,* i *the position of one of its leaves and let* T *be a tree in* $\mathfrak{B}^{<} \cup \mathfrak{B}^{=}$ *then*

$$c^{T}_{T_0} := \sum_{k \in Le(T)} c^{T,k}_{T_0,i}$$

does not depend on i*. Moreover, if* k *denotes the position of one of the leaves of* T*, one has*

$$\sum_{i \in Le(T_0)} c^{T,k}_{T_0,i} = c^{T}_{T_0}.$$

If $T = \tau_W$ *for some Lyndon word* W*, one has* $c^{\tau_W}_{T_0} = c^{W}_{T_0}$ *which makes the above notations consistent.*

- *With the above notations, as the algorithm Dec does not change the leaves, one has for* i *(resp* k*) the position of a leaf decorated by* 1 *of* T_0 *(resp. of* T*)*

$$c^{T}_{T_0} = \sum_{k \in Le^1(T)} c^{T,k}_{T_0,i} \qquad \left(\textit{resp. } c^{T}_{T_0} = \sum_{i \in Le^1(T_0)} c^{T,k}_{T_0,i} \right).$$

- *Let* V *be in* $\mathfrak{B}^{=}$*, and let* $T' \curlywedge T'$ *be its symmetric subtree,* k *be a leaf in* $Le^1(V)$ *in the left factor* T' *and* k' *its symmetric in the right factor* T'*. Let* T_2 *be a tree in* $\mathfrak{B}^{<}$*.*

  Then, one has

$$\sum_{f \in Le^1_{T_2}} c^{V,k}_{T_2,f} = \sum_{f \in Le^1_{T_2}} c^{V,k'}_{T_2,f}.$$

Proof. In order to see that

$$c^{T}_{T_0} = \sum_{k \in Le(T)} c^{T,k}_{T_0,i}$$

does not depend on i, it is enough to remark that the trees arising from Dec do not depend on the marked leaves but only on the original sequence of trees. Now, fix T and k. Let $\mathfrak{T} = (T_j, q_j, k_j)_{1 \leqslant j \leqslant N}$ be the constant value of $Dec^{\circ n}(T_0, 1, i_0)$ for n large enough and some position i_0 of a leaves of T_0 such that

$$T = T_j \text{ for some } j \text{ with } q_j \neq 0.$$

Then, for any position i of a leaf of T_0 there exists integers $j_{i,1}, \dots, j_{i,l_i}$ such that

$$\forall\, l \in \{1, \dots, l_i\}_q \qquad T = T_{i,j_{i,l}} \quad \text{and} \quad q_{i,j_{i,l}} \neq 0$$

where $\mathfrak{T}_i = (T_{i,j}, q_{i,j}, k_{i,j})_{1 \leqslant j \leqslant N}$ stands for the constant value of $Dec^{\circ n}(T_0, 1, i)$ for n large enough (remark that N and the n large enough do not depend on i). As the algorithm Dec does not change the leaves, if T appears then the leaf in position k has to "comes from" some leaves in T_0 and the sum

$$\sum_{\substack{i \in Le(T_0), l \in \{1,\dots,l_i\} \\ \text{s.t} \\ T = T_{i,j_{i,l}} \text{ and } k = k_{i,j_{i,l}}}} q_{i,j_{i,l}}$$

is equal to the total coefficient of T in the decomposition of T_0.

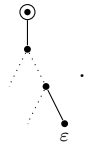
The last part of the lemma is a direct application of the previous point. □

Let W be a Lyndon word and consider T_{W^*} its associated dual tree written on the basis $\mathfrak{B}^<$ as

$$T_{W^*} = \sum_{T \in \mathfrak{B}^<} c_T^W T.$$

Theorem 3.30. *We fix W an Lyndon word. Let T be a tree in $\mathfrak{B}^<$ appearing in T_{W^*}, i the position of one of its leaves decorated by 1.*

As T is in $\mathfrak{B}^<$, the i-th leaves which is decorated by 1 is a right subtree (Lemma 3.20):

$T =$ 

.

Now, let T_1 denote $\left(T/_{\searrow i}\right)^r$, T_2 be ${}^l\left(T/_{\searrow i}\right) = T \setminus^i_{T_1}$ and $j = i - Le^1(T_1)$.

Necessarily, T_1 is in $\mathfrak{B}^<$ and either T_2 is in $\mathfrak{B}^=$ or can be written as $T_2 = \varepsilon(T,i)\overline{T_2}$ with $\overline{T_2}$ in $\mathfrak{B}^<$.

Then, the coefficients c_T^W satisfy

$$c_T^W = \sum_{U_1 \in Lyn} \sum_{U_2 \in Lyn} \sum_{k \in Le^1(U_2)} c_{T_1}^{U_1} \varepsilon(T,i) c_{\overline{T_2},i}^{U_2,k} c^W_{U_2 \underset{\leftarrow}{\curlywedge} k\ U_1} + \sum_{U_1 \in Lyn} \sum_{V \in \mathfrak{B}^=} \sum_{k \in Le^1(V)} c_{T_1}^{U_1} \varepsilon(T,i) c_{\overline{T_2},i}^{V,k} c^W_{V \underset{\leftarrow}{\curlywedge} k\ U_1}. \tag{24}$$

Remark 3.31. The proof also shows the following. Let T in $\mathfrak{B}^=$ and i the position of one of its leaves decorated by 1 such that $T_1 = \left(T/_{\searrow i}\right)^r$ and $T_2 = {}^l\left(T/_{\searrow i}\right) = T \setminus^i_{T_1}$ are both in $\mathfrak{B}^<$ and let $j = i - Le^1(T_1)$. Then, as in the theorem, one has

$$c_T^W = \sum_{U_1 \in Lyn} \sum_{U_2 \in Lyn} \sum_{k \in Le^1(U_2)} c_{T_1}^{U_1} \varepsilon(T,i) c_{\overline{T_2},i}^{U_2,k} c^W_{U_2 \underset{\leftarrow}{\curlywedge} k\ U_1} + \sum_{U_1 \in Lyn} \sum_{V \in \mathfrak{B}^=} \sum_{k \in Le^1(V)} c_{T_1}^{U_1} \varepsilon(T,i) c_{\overline{T_2},i}^{V,k} c^W_{V \underset{\leftarrow}{\curlywedge} k\ U_1}.$$

Proof. Considering the definition of c_T^W, the proof follows from writing down $[T]$ in two different ways in the Lyndon basis. By definition, one has

$$[T] = \sum_{W_0 \in Lyn} c_T^{W_0} [W_0] \qquad \in \mathcal{T}^{Lie}$$

and as T_1 is also in $\mathfrak{B}^<$,

$$[T_1] = \sum_{U_1 \in Lyn} c_{T_1}^{U_1} [U_1] \qquad \in \mathcal{T}^{Lie}.$$

As, $T = T_2 \underset{\leftarrow}{\curlywedge} j \; T_1$, linearity of the Lie brackets gives

$$[T] = \sum_{U_1 \in Lyn} c_{T_1}^{U_1} [T_2 \underset{\leftarrow}{\curlywedge} j \; U_1] \qquad \in \mathcal{T}^{Lie}.$$

Now, $[T_2]$ can be written as

$$[T_2] = \varepsilon(T,i)[\overline{T_2}] = \sum_{U_2 \in Lyn} \sum_{k \in Le^1(U_2)} \varepsilon(T,i) c_{\overline{T_2},j}^{U_2,k} [U_2] + \sum_{V \in \mathfrak{B}^=} \sum_{k \in Le^1(V)} \varepsilon(T,i) c_{\overline{T_2},i}^{V,k} [V].$$

Lemma 3.32 below insures that for any Lyndon word U_1

$$[T_2 \underset{\leftarrow}{\curlywedge} j \; U_1] = \sum_{U_2 \in Lyn} \sum_{k \in Le^1(U_2)} \varepsilon(T,i) c_{\overline{T_2},i}^{U_2,k} [U_2 \underset{\leftarrow}{\curlywedge} k \; U_1] + \sum_{V \in \mathfrak{B}^=} \sum_{k \in Le^1(V)} \varepsilon(T,i) c_{\overline{T_2},i}^{V,k} [V \underset{\leftarrow}{\curlywedge} k \; U_1] \qquad \in \mathcal{T}^{Lie}$$

and decomposing each bracket $[U_2 \underset{\leftarrow}{\curlywedge} k \; U_1]$ and $[V \underset{\leftarrow}{\curlywedge} k \; U_1]$ in the Lyndon basis gives

$$[T] = \sum_{U_1 \in Lyn} \sum_{U_2 \in Lyn} \sum_{k \in Le^1(U_2)} \sum_{W_0 \in Lyn} \varepsilon(T,i) c_{T_1}^{U_1} c_{\overline{T_2},i}^{U_2,k} c_{U_2 \underset{\leftarrow}{\curlywedge} k \; U_1}^{W_0} [W_0] + \sum_{U_1 \in Lyn} \sum_{V \in \mathfrak{B}^=} \sum_{k \in Le^1(V)} \sum_{W_0 \in Lyn} \varepsilon(T,i) c_{T_1}^{U_1} c_{\overline{T_2},i}^{V,k} c_{V \underset{\leftarrow}{\curlywedge} k \; U_1}^{W_0} [W_0] \qquad \in \mathcal{T}^{Lie}$$

which concludes the proof of Theorem 3.30. □

Lemma 3.32. *Let $\mathfrak{T} = (T_j, q_j, k_j)_{1 \leqslant j \leqslant N}$ be a sequence in $\mathcal{T}^{tri}_{\mathbb{N},\mathcal{F}in}$ with k_j in $Le^1(T_j)$ for all j and set*

$$\mathfrak{T}' = Dec(\mathfrak{T}) = (T'_j, q'_j, k'_j)_{1 \leqslant j \leqslant N'}.$$

Let T be a tree in $\mathcal{T}^{tri}$. In the Lie algebra $\mathcal{T}^{Lie}$ one has

$$\sum_{j=1}^{N} q_j [T_j \underset{\leftarrow}{\curlywedge} k_j \; T] = \sum_{j=1}^{N'} q'_j [T'_j \underset{\leftarrow}{\curlywedge} k'_j \; T].$$

Proof. The total number of leaves in the trees involved in equations (17), (18) and (19) stays constant, thus the formulas defining k'_j in terms of the k_j shows that for any j k'_j is the position of a leaf of T'_j. Hence the right hand side of the above equations is well defined.

The second part of the algorithm *Dec* which regroups the different terms of the sequence with same tree and same position commutes with the insertion procedure as it does not change the trees.

So we need only to consider the first part of the algorithm which performs for each tree T_j one operation on the smallest subtree $A \curlywedge B$ which is not a Lyndon tree. Thus, it is enough to prove the above equality in the case where only one of the T_j is changed; say T_N. With the notations from the algorithm, one need to prove that

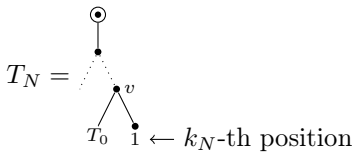
$$\sum_{j=1}^{N} q_j [T_j \underset{\leftarrow}{\curlywedge} k_j \; T] = \sum_{j=1}^{N_1} q_j^1 [T_j^1 \underset{\leftarrow}{\curlywedge} k_j^1 \; T].$$

By definition, $(T_j, q_j, k_j) = (T_j^1, q_j^1, k_j^1)$ for $j \leqslant N-1$ and the $N-1$ first terms of the above sums are equal. We are reduced to show that

$$q_N[T_N \overset{\curvearrowleft k_N}{\leftarrow} T] = \sum_{j=N}^{N_1} q_N^1[T_N^1 \overset{\curvearrowleft k_N^1}{\leftarrow} T] \tag{25}$$

with $N_1 = N$ or $N_1 = N+1$ depending on the smallest subtree $A \curlywedge B$ of T_N which is not a Lyndon tree.

Write T_N as



where v denotes the vertex just above the k_N-th leaf.

If the whole subtree  is moved or not affected at all by operations given at equations (17), (18) and (19), then the identity (25) is satisfied. Similarly if $A \curlywedge B$ is contained in T_0.

Thus, we are interested in the following cases :

- $A = T_0$ and $B = \underset{1}{\odot}$ with $A > B$,
- $A =$  and $B < \underset{1}{\odot}$,
- $A = T_0 = T_0' \curlywedge T_0''$ and $B = \underset{1}{\odot}$ with $T_0'' < B$.

In the first case, identity (25) follows from

$$[[T_0], [[T], \underset{1}{\odot}]] = -[[[T], \underset{1}{\odot}], [T_0]] \qquad \in \mathcal{T}^{Lie}$$

where $[T]$ (resp. $[T_0]$) denotes the image of T (resp. T_0) in $\mathcal{T}^{Lie}$.

The second case, corresponding to operations (18) and (19), follows from Jacobi identity written as

$$[[X,Y],Z] = [X,[Y,Z]] + [[X,Z],Y]$$

applied to $X = [T_0]$, $Y = [[T], \underset{1}{\odot}]$ and $Z = [B]$. Similarly, the third case, also corresponding to operations (18) and (19), follows from the above formula applied to $X = [T_0']$, $Y = [T_0'']$ and $Z = [[T], \underset{1}{\odot}]$. □

3.3. **The differential graded algebra of R-deco forests.** In [GGL09], Gangl, Goncharov and Levin have defined a combinatorial algebra built out of trees, the algebra of R-deco forest, with a differential d_{cy} that imitate the behavior of the differential $\partial_{\mathbb{Q}}$ in $\mathcal{N}_{\mathbb{Q}}$ for cycles related to special linear combinations of trees. Even if we will not use their forest cycling map which maps (particular linear combinations of) trees to (admissible) cycles, an equivalent of the R-deco forest algebra will encode the combinatoric of our problem.

In this subsection, definitions and properties of the forest algebra are recalled. Unless specified otherwise a tree is a planar rooted tree with leaves decorated by 0 and 1. Remark that trees are not assumed to be trivalent has in the previous sections.

Definition 3.33.
- Let T be a planar rooted tree with leaves decorated by 0 and 1 and root decorated by t, 0 or 1. Let $E(T)$ denote the set of edges of T.
- An oriented tree (T, ω) is a tree as above together with a bijective map $\omega : E(T) \longrightarrow \{1, \dots, |(T)|\}$.
- Similarly to Definition 3.23, one defines $Le_{\mathcal{T}}(T)$ and $Le^1_{\mathcal{T}}(T)$ as the sets of leaves of T and the set of leaves decorated by 1 respectively.
- For an oriented tree (T, ω), the orientation of T induces an order on $Le_{\mathcal{T}}(T)$ and $Le^1_{\mathcal{T}}(T)$ respectively and the *position* of a leaf will denote its position with respect to that order and we shall write $i \in Le(T)$ (resp. $i \in Le^1(T)$) for the position of a leaf.
- A *forest* is a disjoint union of planar rooted trees with leaves decorated by 0 and 1.
- The above definitions extend naturally to forests. For a forest F, we shall write $E(F)$, $Le_{\mathcal{T}}(F)$ and $Le^1_{\mathcal{T}}(F)$. Similarly, we will speak of oriented forest (F, ω) and of position of leaves $i \in Le(F)$ and $i \in Le^1(F)$.

Let the weight of a forest F (resp. and oriented (F, ω)) be the number of its leaves $wt(F) = |Le_{\mathcal{T}}(F)|$ and its (cohomological) degree be $e(F) = 2wt(F) - |E(F)|$.

Let $V^k(p)$ be the vector space generated by oriented forests (F, ω) of weight $wt(F) = p$ and such that $e(F) = k$. Adding an extra generator $\mathbb{1}$ in weight 0 and degree 0, $V^0(0) := \mathbb{Q}\mathbb{1}$, we define

$$V^\bullet = \oplus_{p \geqslant 0} \oplus_k V^k(p).$$

Definition 3.34. Disjoint union of forests extends to oriented forests with $\mathbb{1}$ as neutral element as follows. Let $(F_1\omega_1)$ and (F_2, ω_2) be two oriented forests, define

$$(F_1\omega_1) \cdot (F_2, \omega_2) = (F_1 \cup F_2, \omega)$$

where $\omega : E(F_1 \cup F_2) \longrightarrow \{1, \dots, |E(F_1)| + |E(F_2)|\}$ is defined by

$$\omega(e) = \begin{cases} \omega_1(e) & \text{if } e \in F_1 \\ \omega_2(e) + |E(F_1)| & \text{if } e \in F_2 \end{cases}$$

For σ a permutation of $\{1, \dots, n\}$ for some positive integer n, let $\varepsilon(\sigma)$ denote the signature of σ.

Definition 3.35. Let I be the ideal generated by elements of the form
- $(T, \omega) - \varepsilon(\sigma)(T, \sigma \circ \omega)$ for (T, ω) an oriented tree and σ a permutation of $\{1, \dots, \#E(T)\}$
- oriented trees with root decorated by 0, that is $\left(\begin{array}{c} 0 \\ | \\ T \end{array}, \omega \right)$
- the tree $\begin{array}{c} 1 \\ | \\ 0 \end{array}$ with any orientation.

Let $\mathcal{F}^\bullet_{\mathbb{Q}}$ be the (graded) quotient

$$\mathcal{F}^\bullet_{\mathbb{Q}} := V^\bullet / I.$$

The extended disjoint union $\cdot$ makes $\mathcal{F}^\bullet_{\mathbb{Q}}$ into a graded commutative algebra (for the cohomological degree), that is oriented forests "commute" via the rule

$$(F_1, \omega_1) \cdot (F_2, \omega_2) = (-1)^{e(F_1)e(F_2)} (F_2, \omega_2) \cdot (F_1, \omega_1).$$

Definition 3.36. A rooted plane tree has a canonical numbering of its edge, starting from the root edge, which is induced by the cyclic order of the edges at internal vertices. We will speak of its *canonical orientation* or of its *canonical numbering*.

Example 3.37. An example of this canonical ordering is shown at Figure 1; we recall that by convention we draw trees with the root at the top and the cyclic order at internal vertices counterclockwise.

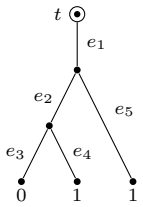


FIGURE 1. A tree with its canonical orientation, that is the canonical numbering of its edge.

Now, we define on $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ a differential of degree 1, that is a linear map

$$d : \mathcal{F}_{\mathbb{Q}}^{\bullet} \longrightarrow \mathcal{F}_{\mathbb{Q}}^{\bullet+1}$$

satisfying $d^2 = 0$ and the Leibniz rule

$$d((F_1, \omega_1) \cdot (F_2, \omega_2)) = d((F_1, \omega_1)) \cdot (F_2, \omega_2) + (-1)^{e(F_1)}(F_1, \omega_1) \cdot d((F_2, \omega_2)).$$

The set of rooted planar trees decorated as above endowed with their canonical orientation forms a set of representative for the permutation relation and it generates $\mathcal{F}_{\mathbb{Q}}^{\bullet}$ as an algebra. Hence, we will define this differential first on trees endowed with their canonical orientation and then extend the definition by Leibniz rule.

The differential of an oriented tree (T, ω) will be a linear combination of oriented forests where the trees appearing arise by contracting an edge of T and fall into two types depending whether the edge in internal or not. We will need the notion of splitting.

Definition 3.38. A *splitting* of a tree T at an internal vertex v is the disjoint union of the trees which arise as $T_i \cup v$ where the T_i are the connected component of $T \setminus v$. Moreover

- the planar structure of T and its decorations of leaves induce a planar structure on each $T_i \cup v$ and decorations of leaves ;
- an ordering of the edges of T induces an orientation of the forest $\sqcup_i (T_i \cup v)$;
- if T as a root r then v becomes the root for all $T_i \cup v$ which do not contain r, and if v has a decoration then it keeps its decoration in all the $T_i \cup v$.

Definition 3.39. Let e be an edge of a tree T. The contraction of T along e denoted T/e is given as follows:

(1) If the tree consists on a single edge, its contraction is the empty tree.
(2) If e is an internal edge, then T/e is the tree obtain from T by contracting e and identifying the incident vertices to a single vertex.
(3) If e is the edge containing the root vertex then T/e is the forest obtained by first contracting e to the internal incident vertex w (which inherit the decoration of the root) and then by splitting at w; w becoming the new root of all trees in the forest T/e.
(4) If e is an external edge not containing the root vertex then T/e is the forest obtained as follow: first one contracts e to the internal incident vertex w (which inherit the decoration of the leaf) and then one performs a splitting at w.

Example 3.40. Two examples are given below. In Figure 2, one contracts the root vertex and in Figure 3, a leaf is contracted.

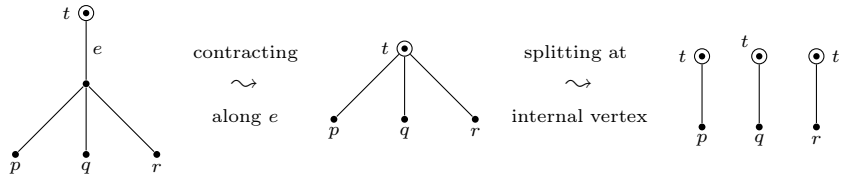


FIGURE 2. Contracting the root

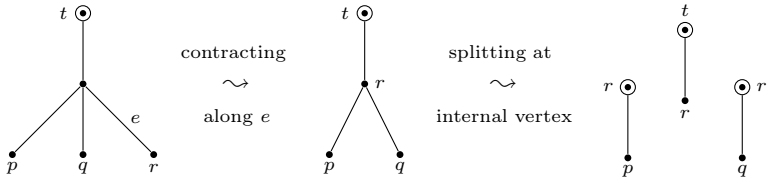


FIGURE 3. Contracting a leaf

Now, Let e be an edge of an oriented tree (T, ω) with ω the canonical orientation of T. As an edge f of T/e is also an edge of T, there is a natural orientation $i_e\omega$ on T/e given as follows :

$$\forall f \in E(T/e) \quad \begin{array}{ll} i_e\omega(f) = \omega(f) & \text{if } \omega(f) < \omega(e) \\ i_e\omega(f) = \omega(f) - 1 & \text{if } \omega(f) > \omega(e). \end{array}$$

Definition 3.41. Let (T, ω) be a tree endowed with it canonical orientation, on defines $d_{cy}(T, w)$ as

$$d_{cy}(T, \omega) = \sum_{e \in E(T)} (-1)^{\omega(e)-1}(T/e, i_e\omega).$$

One extends d_{cy} to all oriented trees by the relation

$$d_{cy}(T, \sigma \circ \omega) = \varepsilon(\sigma) d_{cy}(T, \omega)$$

and to $\mathcal{F}^\bullet_{\mathbb{Q}}$ by linearity and the Leibniz rule.

In particular d_{cy} maps a tree with at most one edge to 0 (which correspond by convention to the empty tree).

As proved in [GGL09], d_{cy}, extended with the Leibniz rule, induces a differential on $\mathcal{F}^\bullet_{\mathbb{Q}}$.

Proposition 3.42. *The map $d_{cy} : \mathcal{F}^\bullet_{\mathbb{Q}} \longrightarrow \mathcal{F}^\bullet_{\mathbb{Q}}$ makes $\mathcal{F}^\bullet_{\mathbb{Q}}$ into a commutative differential graded algebra. In particular $d^2_{cy} = 0$.*

We will give examples of explicit computations of this differential in the next subsection.

3.4. **"Differential equations" for tree sums dual to Lyndon brackets.** The canonical orientation of a tree allows us to define two maps

$$\phi_t : \mathcal{T}^{tri,<}_{\mathbb{Q}} \longrightarrow \mathcal{F}^\bullet_{\mathbb{Q}}, \qquad (\text{resp. } \phi_1 : \mathcal{T}^{tri,<}_{\mathbb{Q}} \longrightarrow \mathcal{F}^\bullet_{\mathbb{Q}})$$

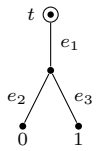
sending a rooted trivalent tree with leaves decorated by 0 and 1 to the same tree with its canonical numbering and the root decorated by t (resp. by 1).

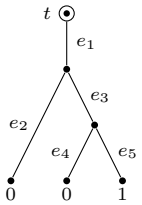
The symmetry relation in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ is compatible with the permutation relation in $\mathcal{F}^\bullet_{\mathbb{Q}}$ as the considered trees in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ are trivalent. We will use the same notation to denote an element τ in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ and its image by ϕ_t and denotes by $\tau(1)$ its image by ϕ_1.

Example 3.43. We give below the image by ϕ_t and ϕ_1 of some (linear combinations of) tress given at example 3.18 together with the numbering of the edges. Up to weight 3, the images by ϕ_t are

$T_{0^*} =$ 

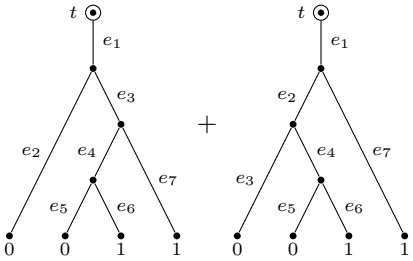
, $T_{1^*} =$ 

, $T_{01^*} =$ 

, $T_{001^*} =$ 

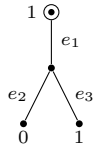
,

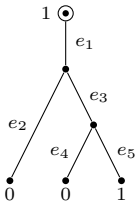
and in weight 4

$T_{0011^*} =$ 

.

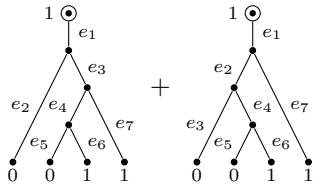
Some images by ϕ_1 are given below:

$T_{01^*}(1) =$ 

, $T_{001^*}(1) =$ 

,

and

$T_{0011^*}(1) =$ 

The linear combination of trivalent trees given by the T_{W^*} have a special behavior under the differential d_{cy} given in the following theorem.

Theorem 3.44. *Let W be a Lyndon word of length $\geqslant 2$. In $\mathcal{F}^\bullet_{\mathbb{Q}}$, the image of T_{W^*} under d_{cy} is decomposable, that is $d_{cy}(T_{W^*})$ is a sum of products. More precisely ,*

$$d_{cy}(T_{W^*}) = \sum_{U<V} \alpha^W_{U,V} T_{U^*} \cdot T_{V^*} + \sum_{U,V} \beta^W_{U,V} T_{U^*} \cdot T_{V^*}(1) \tag{26}$$

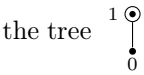
where U and V are Lyndon words, the $\alpha^W_{U,V}$ are the ones defined at Equation (20) *and the $\beta^W_{U,V}$ rational numbers.*

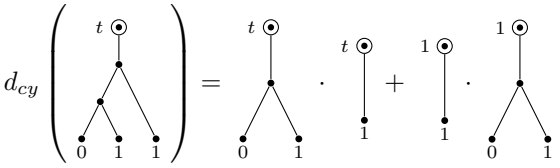
Remark 3.45. From the definition of d_{cy}, one sees Equation (26) involves only W and Lyndon words U, V such that the length of W is equal to the length of U plus the one of V. In particular, $\alpha^W_{U,V} = \beta^W_{U,V} = 0$ as soon as U or V has length greater or equal to W.

The coefficients $\alpha^W_{U,V}$ are defined only for $U < V$. In particular, $\alpha^W_{1,V}$,$\alpha^W_{U,0}$ and $\alpha^W_{U,U}$ are not defined.

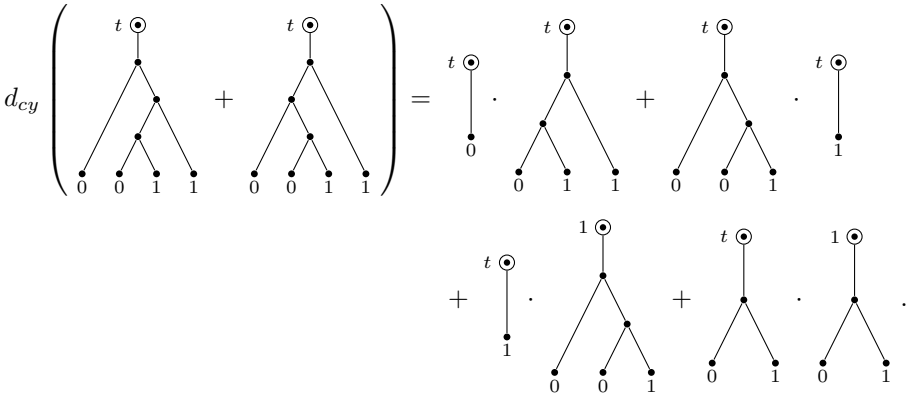
Before proving the above theorem, we give some examples.

Example 3.46. As said before, the trees are endowed with their canonical numbering. We recall that a tree with root decorated by 0 is 0 in $\mathcal{F}^{\bullet}_{\mathbb{Q}}$. As applying an odd permutation to the numbering change the sign of the tree, using the trivalency of the tree T_{W^*} shows that some trees arising from the computation of d_{cy} are 0 in $\mathcal{F}^{\bullet}_{\mathbb{Q}}$ because they contain a symmetric subtree.

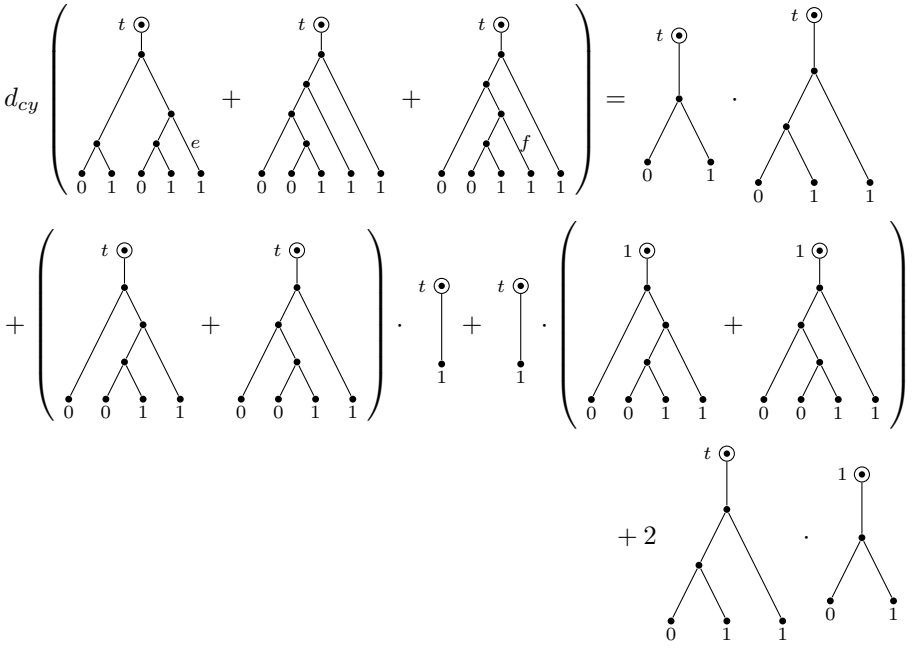
Using the fact that the tree  is 0 in $\mathcal{F}^{\bullet}_{\mathbb{Q}}$, one computes in weight 3, $d_{cy}(T_{011^*})$:



and in weight 4, $d_{cy}(T_{0011^*})$



We give below an example in weight 5, $d_{cy}(T_{01011^*})$:



where the last term arises from the part of the differential associated to edges e and f. In the computations above, we have regrouped terms together and not written the one that were 0. The result can be summarized by

$$\begin{aligned} T_{011^*} =& T_{01^*} \cdot T_{1^*} + T_{1^*} \cdot T_{01^*}(1) \\ T_{0011^*} =& T_{0^*} \cdot T_{011^*} + T_{001^*} \cdot T_{1^*} + T_{1^*} \cdot T_{001^*}(1) + T_{01^*} \cdot T_{01^*}(1) \\ T_{01011^*} =& T_{01^*} \cdot T_{011^*} + T_{0011^*} \cdot T_{1^*} + T_{1^*} \cdot T_{0011^*}(1) + 2T_{011^*} \cdot T_{01^*}(1) \end{aligned}$$

and should be compare with equations (10) and (11).

The proof of Theorem 3.44 will be decomposed in three part:

- proving that terms of the form $\alpha^W_{U,V} T_{U^*} \cdot T_{V^*}$ arise;
- proving that $d_{cy}(T_{W^*})$ is decomposable, in other words that it is a sum of products;
- proving that products that are not terms arising from $\alpha^W_{U,V} T_{U^*} \cdot T_{V^*}$ can be regroup together and give terms of the form $\beta^W_{U,V} T_{U^*} \cdot T_{V^*}(1)$ for some rational number $\beta^W_{U,V}$.

The coefficients $\beta^W_{U,V}$ are in fact integers and the proof gives a precise formula. However the exact expression will not be used latter.

First of all, one decomposes the differential d_{cy} in four parts :

$$d_{cy} = d_{root} + d_{int} + d^0_l + d^1_l$$

where d_{root} is the term corresponding to the root edge, d_{int} the one corresponding to internal edges, d^0_l and d^1_l corresponding respectively to the external edges that are not the root edge with leaves decorated by 0 and 1.

One remarks that d^0_l is zero. Indeed, if e is an external edge with leaf decorated by 0 of a tree T, the corresponding term of $d^0_l(T)$ is given by the forest T/e where one of the tree is of the form

which is 0 in $\mathcal{F}^\bullet_{\mathbb{Q}}$.

Now, let T be a rooted trivalent tree in $\mathcal{T}^{tri}$ as above. It can be endowed with its canonical ordering and root decoration given by t to obtain a tree in $\phi_t(T)$ in $\mathcal{F}^\bullet_{\mathbb{Q}}$. We have defined at Equation (21) a map $d_{Lie} : \mathcal{T}^{tri}_{\mathbb{Q}} \longrightarrow \mathcal{T}^{tri}_{\mathbb{Q}} \wedge \mathcal{T}^{tri}_{\mathbb{Q}}$ that contract and split the root edge of T giving two trees with two new roots. The fact that we have exactly two trees comes form the trivalency hypothesis in $\mathcal{T}^{tri}_{\mathbb{Q}}$. Using again the trivalency of the trees, ϕ_t extends to a map $\mathcal{T}^{tri}_{\mathbb{Q}} \wedge \mathcal{T}^{tri}_{\mathbb{Q}} \longrightarrow \mathcal{F}^\bullet_{\mathbb{Q}}$ where the wedge product is replaced by the product in $\mathcal{F}^\bullet_{\mathbb{Q}}$. As d_{root} consists also in contracting the root edge and splitting, one has

$$d_{root}(\phi_t(T)) = \phi_t(d_{Lie}(T)).$$

In particular, as the T_{W^*} are by definition the dual basis to the basis of $\mathcal{T}^{Lie}$ given by Lyndon brackets and using Proposition 3.19 one finds omitting ϕ_t

$$d_{root}(T_{W^*}) = \sum_{U<V} \alpha^W_{U,V} T_{U^*} \cdot T_{V^*}.$$

Now, let's prove that the non decomposable terms arising from d_{int} cancel each other.

Lemma 3.47. *Let W be a Lyndon in 0 and 1. Then one has*

$$d_{int}(T_{W^*}) = 0$$

Proof. One proves the lemma by induction on the length of W. If the length of W is less or equal to 2, there is nothing to prove as the corresponding tree do not have any internal edges.

Let T be a rooted planar tree (with decoration) and e and edge of T. One has a natural direction on the edges of T, going away from the root. Considering this direction, the edge e goes from a vertex v to a vertex w. The depth of e is the minimal number of edges one has to go trough in order to go from the root vertex to the vertex w. Thus an edge of depth 2 is an edge connected to the edge containing the root vertex.

Using the inductive construction of T_{W^*}

$$T_{W^*} = \sum_{U<V} \alpha U, V^W T_{U^*} \curlywedge T_{V^*}$$

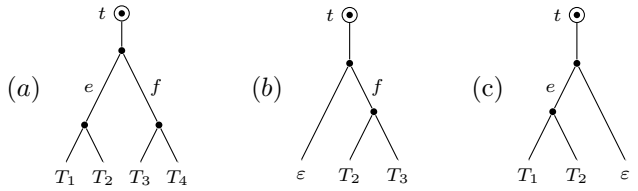
and the fact that T_{W^*} is trivalent, one sees that

$$d_{int}(T_{W^*}) = \begin{array}{l}\text{terms corresponding}\\ \text{to edge of depth 2}\end{array} - \sum_{U<V} \alpha U, V^W d_{int}(T_{U^*}) \curlywedge T_{V^*} + \sum_{U<V} \alpha U, V^W T_{U^*} \curlywedge d_{int}(T_{V^*}).$$

In the above formula, the signs are taking into account the canonical numbering of the respective trees. Using the induction hypothesis, it is enough to check that the terms corresponding to edges of depth 2 cancel each other. Writing T_{W^*} as

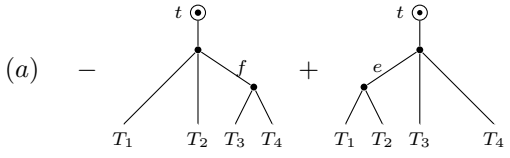
$$T_{W^*} = \sum_{T \in \mathfrak{B}^<} c_T^W T,$$

one considers a tree T in $\mathfrak{B}^<$ such that c_T^W is non zero. As W is of length at least 3, T can be written in one of the following form

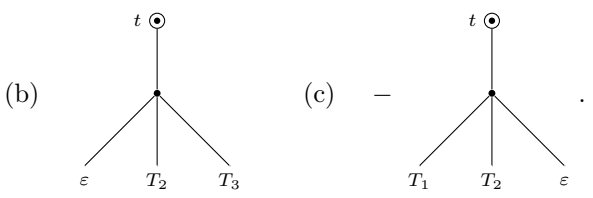


where ε is equal to 0 or 1.

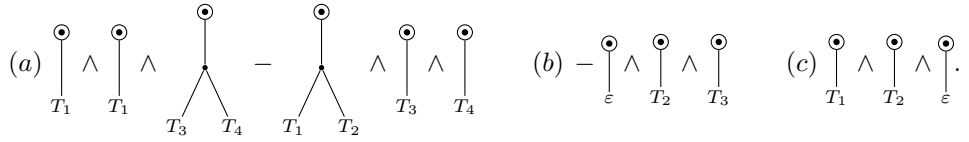
As T is trivalent, edges labeled by e have an even number and those labeled by f an odd number with the natural numbering of T. Computing in each case the terms of d_{int} coming from the depth 2 edges and taking into account the signs arising from the natural numbering, one gets for each cases



and



In the other hand, applying d_{Lie} erases the root and create two new trees. Computing $d^{\circ 2}_{Lie}(T)$ in $\mathcal{T}^{tri}_{\mathbb{Q}}$, one obtains in $\mathcal{T}^{tri}_{\mathbb{Q}} \wedge \mathcal{T}^{tri}_{\mathbb{Q}} \wedge \mathcal{T}^{tri}_{\mathbb{Q}}$ for each cases



Permuting the wedge factors introduces a minus sign when the permutation is a transposition as does permuting the trivalent subtrees in the former formula.

Thus, up to a global minus sign, in $d_{int}(T_{W^*})$ each term arising from the depth 2 edges is given by gluing together the three wedge factor of the corresponding term in $d^{\circ 2}_{Lie}(T_{W^*})$. These different terms cancel each other because $d^{\circ 2}_{Lie}(T_{W^*}) = 0$. As a consequence, in $d_{cy}(T_{W^*})$ terms arising from the depth 2 edges cancel each other and $d_{int}(T_{W^*}) = 0$. □

We will now prove the main part of Theorem 3.44. It is enough to prove that terms in $d_{cy}(T_{W^*})$ coming from the leaves decorated by 1 gives

$$\sum_{U,V} \beta^W_{U,V} T_{U^*} \cdot T_{V^*}(1).$$

Lemma 3.48. *Let W be a word in 0 ans 1. One has*

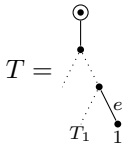
$$d^1_l(T_{W^*}) = \sum_{U,V} \beta^W_{U,V} T_{U^*} \cdot T_{V^*}(1).$$

Proof. The definition of d_{cy} gives $d_{cy}(T_{0^*}) = d_{cy}(T_{1^*}) = 0$ and $d_{cy}(T_{01^*}) = T_{0^*} \cdot T_{1^*}$ and we will assume that W as length greater or equal to 3.

All considered trees T are endowed with their canonical numbering ω_T. The computation of $d^1_l(T_{W^*})$ gives

$$d^1_l(T_{W^*}) = \sum_{T \in \mathfrak{B}^<} c^W_T \, d^1_l(T) = \sum_{T \in \mathfrak{B}^<} c^W_T \sum_{e \in Le^1_T} (T/e, i_e \omega_T).$$

Let T be a tree in $\mathfrak{B}^<$ such that c^W_T is non zero. Lemma 3.20 shows that a leaf e in T decorated by 1 is always a right leaf:



where $T_1 = (T/_{\curvearrowright e})^r$. Thus, if the edge e has number i, one sees that

$$\begin{aligned}(T/e, i_e\omega_T) &= (-1)^{i-1}(-1)^{(2l-1)(2p-1-i)}(T \setminus^e{}_{T_1}, \omega_{T \setminus_{T_1}}) \cdot (T_1(1), \omega_{T_1}) \\ &= (T \setminus^e{}_{T_1}, \omega_{T \setminus_{T_1}}) \cdot (T_1(1), \omega_{T_1})\end{aligned}$$

where $2l-1$ is the number of edges in T_1 and $\cdot$ is the product in $\mathcal{F}^\bullet_{\mathbb{Q}}$; the natural ordering of T_1 begin the same as the one of $T_1(1)$.

Thus, omitting the natural numbering, one can write

$$d^1_l(T_{W^*}) = \sum_{T \in \mathfrak{B}^<} \sum_{e \in Le^1_T} c^W_T \, T \setminus^e{}_{T'} \cdot (T/_{\curvearrowright e})^r (1).$$

Using Lemma 3.27 and the fact that $c^W_{T'} = 0$ for any T' in $\mathfrak{B}^=$, one can rewrite the above sum as

$$d^1_l(T_{W^*}) = \sum_{T_1 \in \mathfrak{B}^<} \sum_{T_2 \in \mathfrak{B}^< \cup \mathfrak{B}^=} \sum_{f \in Le^1_{T_2}} c^{W}_{T_2 \underset{\leftarrow}{\curvearrowright} f \; T_1} \tilde{\varepsilon}(T_1, T_2, f) T_2 \cdot T_1(1)$$

and as the permutation relations kill terms T_2 in $\mathfrak{B}^=$

$$d_l^1(T_{W^*}) = \sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{W}_{T_2\ \overleftarrow{\wedge f}\ T_1}\tilde{\varepsilon}(T_1,T_2,f)T_2\cdot T_1(1).$$

Now, applying Theorem 3.30 and its remark to $\overline{T_2\ \overleftarrow{\wedge f}\ T_1}$ and leaf $e=\varphi_{T_1,T_2}(f)=\varphi(f)$, one decomposes the coefficients $c^{W}_{T_2\ \overleftarrow{\wedge f}\ T_1}$ and, as T_2 is in $\mathfrak{B}^<$, obtains

$$\begin{aligned}
d_l^1(T_{W^*}) = &\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}}\sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)} c^{U_1}_{T_1}\\
&\varepsilon(\overline{T_2\ \overleftarrow{\wedge f}\ T_1},\varphi(f))c^{U_2,k}_{T_2,f}c^{W}_{U_2\ \overleftarrow{\wedge k}\ U_1}\tilde{\varepsilon}(T_1,T_2,f)T_2\cdot T_1(1)\\
&+\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}}\sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)} c^{U_1}_{T_1}\\
&\varepsilon(\overline{T_2\ \overleftarrow{\wedge f}\ T_1},\varphi(f))c^{V,k}_{T_2,f}c^{W}_{V\ \overleftarrow{\wedge k}\ U_1}\tilde{\varepsilon}(T_1,T_2,f)T_2\cdot T_1(1).
\end{aligned}$$

As lemma 3.27 shows that $\varepsilon(\overline{T_2\ \overleftarrow{\wedge f}\ T_1},\varphi(f))\tilde{\varepsilon}(T_1,T_2,f)=1$, one gets

$$\begin{aligned}
d_l^1(T_{W^*}) = &\\
&\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}}\sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)} c^{U_1}_{T_1}c^{U_2,k}_{T_2,f}c^{W}_{U_2\ \overleftarrow{\wedge k}\ U_1}T_2\cdot T_1(1)\\
&+\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}}\sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)} c^{U_1}_{T_1}c^{V,k}_{T_2,f}c^{W}_{V\ \overleftarrow{\wedge k}\ U_1}T_2\cdot T_1(1).
\end{aligned}$$

Now, permuting the summation symbols gives

$$\begin{aligned}
d_l^1(T_{W^*}) = &\\
&\sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)}\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{U_1}_{T_1}c^{U_2,k}_{T_2,f}c^{W}_{U_2\ \overleftarrow{\wedge k}\ U_1}T_2\cdot T_1(1)\\
&+\sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)}\sum_{T_1\in\mathfrak{B}^<}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{U_1}_{T_1}c^{V,k}_{T_2,f}c^{W}_{V\ \overleftarrow{\wedge k}\ U_1}T_2\cdot T_1(1),
\end{aligned}$$

then, collecting terms depending on T_1 leads to

$$\begin{aligned}
d_l^1(T_{W^*}) = &\\
&\sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{U_2,k}_{T_2,f}c^{W}_{U_2\ \overleftarrow{\wedge k}\ U_1}T_2\cdot\left(\sum_{T_1\in\mathfrak{B}^<} c^{U_1}_{T_1}T_1(1)\right)\\
&+\sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{V,k}_{T_2,f}c^{W}_{V\ \overleftarrow{\wedge k}\ U_1}T_2\cdot\left(\sum_{T_1\in\mathfrak{B}^<} c^{U_1}_{T_1}T_1(1)\right),
\end{aligned}$$

that is

$$d_l^1(T_{W^*}) = \sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{U_2,k}_{T_2,f} c^W_{U_2 \underset{\leftarrow}{\curlywedge}k\ U_1} T_2\cdot T_{U_1^*}(1) + \sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)}\sum_{T_2\in\mathfrak{B}^<}\sum_{f\in Le^1_{T_2}} c^{V,k}_{T_2,f} c^W_{V \underset{\leftarrow}{\curlywedge}k\ U_1} T_2\cdot T_{U_1^*}(1).$$

Doing the same for terms in T_2, one has

$$d_l^1(T_{W^*}) = \sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\sum_{k\in Le^1(U_2)} c^W_{U_2 \underset{\leftarrow}{\curlywedge}k\ U_1}\left(\sum_{T_2\in\mathfrak{B}^<}\left(\sum_{f\in Le^1_{T_2}} c^{U_2,k}_{T_2,f}\right)T_2\right)\cdot T_{U_1^*}(1) + \sum_{U_1\in Lyn}\sum_{V\in\mathfrak{B}^=}\sum_{k\in Le^1(V)} c^W_{V \underset{\leftarrow}{\curlywedge}k\ U_1}\left(\sum_{T_2\in\mathfrak{B}^<}\left(\sum_{f\in Le^1_{T_2}} c^{V,k}_{T_2,f}\right)T_2\right)\cdot T_{U_1^*}(1).$$

As, for a fixed k by Lemma 3.29 $\sum_{f\in Le^1_{T_2}} c^{U_2,k}_{T_2,f} = c^{U_2}_{T_2}$, the first term of the above sum is equal to

$$\sum_{U_1\in Lyn}\sum_{U_2\in Lyn}\left(\sum_{k\in Le^1(V)} c^W_{U_2 \underset{\leftarrow}{\curlywedge}k\ U_1}\right)T_{U_2^*}\cdot T_{U_1^*}(1)$$

which is the desired term in $\beta^W_{U_1,U_2}T_{U_2^*}T_{U_1^*}(1)$.

For the second term, V contains a symmetric subtree of the form $T' \curlywedge T'$. If a leaf k in $Le^1(V)$ is not in this symmetric subtree, then the tree $V \underset{\leftarrow}{\curlywedge}k\ U_1$ also contains this symmetric subtree and thus the coefficient $c^W_{V \underset{\leftarrow}{\curlywedge}k\ U_1}$ is 0.

If the leaf k is in the left T' then there exists another leaf k' in $Le^1(V)$ symmetric to k' in the right T' factor. Then, one has

$$c^W_{V \underset{\leftarrow}{\curlywedge}k\ U_1} = -c^W_{V \underset{\leftarrow}{\curlywedge}k'\ U_1}$$

and Lemma 3.29 insures that the remaining terms of the second sum cancel each other. □

3.5. **Relations among the coefficients arising from the differential equations.** From Equation (26), one deduces quadratic relations between coefficients $\alpha^W_{U,V}$ and $\beta^W_{U,V}$ for U, V, W Lyndon words.

We begin by some obvious remarks for the case where U or V is equal to one of the Lyndon word 0 or 1.

From Lemma 3.20, one derives the following facts about coefficients $\alpha^W_{0,V}$, $\alpha^W_{V,1}$, $\beta^W_{V,\varepsilon}$ and $\beta^W_{\varepsilon,V}$ for ε in $\{0,1\}$.

Lemma 3.49. *Let W be a Lyndon word of length greater or equal to 2 and write $d_{cy}(T_{W^*})$ as in Theorem 3.44:*

$$d_{cy}(T_{W^*}) = \sum_{U<V}\alpha^W_{U,V}T_{U^*}\cdot T_{V^*} + \sum_{U,V}\beta^W_{U,V}T_{U^*}\cdot T_{V^*}(1).$$

The following holds :

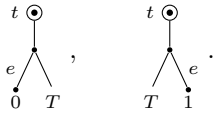
- $\beta^W_{0,V} = \beta^W_{V,0} = 0$,

- $\beta_{U,1} = 0$,
- $\beta^W_{1,U} = \alpha^W_{U,1}$.

In particular, $\beta_{0,0} = \beta_{1,1} = 0$.

Proof. Let W be a Lyndon word of length greater or equal to 2. First, as  is 0 in $\mathcal{F}^\bullet_{\mathbb{Q}}$, one gets $\beta^W_{V,0} = 0$.

In $d_{cy}(T_{W^*})$, trees  and arise only by contracting a root edge or by contracting the edge e in the two following situation



Hence, $\beta^W_{1,V} = \alpha^W_{V,1}$ and as a tree with root decorated by 0 is 0 in $\mathcal{F}^\bullet_{\mathbb{Q}}$, one has $\beta^W_{0,V} = 0$. From the definition of d_{cy}, one sees that a product $TT_{1^*}(1)$ for some tree T can only arise from a subtree of type

As, T_{W^*} is a linear combination of trees in $\mathfrak{B}^<$, none of the appearing trees can contain a symmetric subtree as the one above which insures that $\beta^W_{V,1} = 0$. □

Lemma 3.50. *The family of elements given by*

$$\{T_{W^*} \text{ s.t. } W \text{ Lyndon word}\} \cup \{T_{W^*}(1) \text{ s.t. } W \neq 0, 1 \text{ Lyndon word}\}$$

is linearly independent in $\mathcal{F}^\bullet_{\mathbb{Q}}$.

As a consequence the following families consist of linearly independent elements:

- *For U and V running trough Lyndon words in $0, 1$:*

$$\{T_{U^*}T_{V^*} \,|\, U < V\} \cup \{T_{U^*}T_{V^*}(1) \,|\, 0 < U,\, V \neq 0, 1\}$$

- *for the U_i, running through Lyndon words*

$$\{T_{U_1^*}T_{U_2^*}T_{U_3^*}\} \cup \{T_{U_4^*}T_{U_5^*}T_{U_6^*}(1)\} \cup \{T_{U_7^*}T_{U_8^*}(1)T_{U_9^*}(1)\} \cup \{T_{U_{10}^*}(1)T_{U_{11}^*}(1)T_{U_{12}^*}(1)\},$$

where the U_i are subject to the constraint below
  - $U_1 < U_2 < U_3$, $U_4 < U_5$, $U_8 < U_9$, $U_{10} < U_{11} < U_{12}$
  - $U_6, U_8, U_9, U_{10}, U_{11}, U_{12} \neq 0, 1$

Proof. As a basis of $\mathcal{T}^{coL}$ the elements T_{W^*} where W runs through all Lyndon words are linearly independent in $\mathcal{T}^{tri,<}_{\mathbb{Q}}$ and hence their images by ϕ_t are also linearly independent. Indeed, adding the root decoration t introduce no relation. Moreover, as ϕ_t endowed each tree with its canonical numbering, the permutation relation in $\mathcal{F}^\bullet_{\mathbb{Q}}$ play no role as the canonical numbering produces a set of representative for the tree. It is important to remark here that no products are involved in the element T_{W^*}. A similar argument shows that the elements $T_{W^*}(1)$ ($W \neq 0, 1$ Lyndon word) are linearly independent in $\mathcal{F}^\bullet_{\mathbb{Q}}$. One concludes the first part remarking that in $\mathcal{F}^\bullet_{\mathbb{Q}}$, no relation involves both trees with root decorated by t and 1.

The second part of the lemma which involved products of trees follows from the fact that the permutation relations on a product of tree endowed with their canonical numbering keeps tracts of the order of terms in the product. The trees involved here being all trivalent, their product is anti-commutative. All involved

trees being endowed with their canonical numbering their is no further relation than anticommutativity. □

Now, we described some quadratic relations satisfy by the coefficients $\alpha^W_{U,V}$ and $\beta^W_{U,V}$. Those relations are nothing but writing $d^2_{cy}(T_{W^*}) = 0$ in terms of the above linearly independent families.

Fix W a Lyndon word of length greater or equal to 3 and a, b and c three Lyndon words of smaller length. We define, with the restriction $a < b < c$, a coefficient $r_{a<b<c}$; with the restriction $a < b$, a coefficient $s_{a<b,c}$ and, with the restriction $b < c$ a coefficient $t_{a,b<c}$. In the definitions below, all indices are Lyndon words and we have dropped the superscript W. Latter on, we will generally not write the superscript W when the context is clear enough. We set:

$$\begin{aligned} r^W_{a,b,c} = &\sum_{u<a} \left(\alpha_{u,a}\alpha^u_{b,c} - \alpha_{u,b}\alpha^u_{a,c} + \alpha_{u,c}\alpha^u_{a,b}\right) \\ &+ \sum_{a<u<b} \left(-\alpha_{a,u}\alpha^u_{b,c} - \alpha_{u,b}\alpha^u_{a,c} + \alpha_{u,c}\alpha^u_{a,b}\right) \\ &+ \sum_{b<u<c} \left(-\alpha_{a,u}\alpha^u_{b,c} + \alpha_{b,u}\alpha^u_{a,c} + \alpha_{u,c}\alpha^u_{a,b}\right) \\ &+ \sum_{c<u} \left(-\alpha_{a,u}\alpha^u_{b,c} + \alpha_{b,u}\alpha^u_{a,c} - \alpha_{c,u}\alpha^u_{a,b}\right). \end{aligned} \tag{27}$$

$$\begin{aligned} s^W_{a<b,c} = &\sum_u \beta_{u,c}\alpha^u_{a,b} + \sum_{u<a} \alpha_{u,a}\beta^u_{b,c} + \sum_{u<b} -\alpha_{u,b}\beta^u_{a,c} \\ &+ \sum_{u>a} -\alpha_{a,u}\beta^u_{b,c} + \sum_{u>b} \alpha_{b,u}\beta^u_{a,c} \end{aligned} \tag{28}$$

$$t^W_{a,b<c} = \sum_u \beta_{u,c}\beta^u_{a,b} + \sum_u -\beta_{u,b}\beta^u_{a,c} + \sum_u \beta_{a,u}\left(-\alpha^u_{b,c} - \beta^u_{b,c} + \beta^u_{c,b}\right) \tag{29}$$

Proposition 3.51. *One has the following relations:*

$$r^W_{a<b<c} = s^W_{a<b,c} = t^W_{a,b<c} = 0$$

for a, b and c respecting the constraints from the above definition.

Proof. First, we remark that for any Lyndon word M one has

$$d_{cy}(T_{M^*}(1)) = \sum_{U<V} \alpha^A_{U,V} T_{U^*}(1) \cdot T_{V^*}(1) + \sum_{U,V} \beta^A_{U,V} T_{U^*}(1) \cdot T_{V^*}(1),$$

as one just changes the label of the root which does not change the combinatoric of d_{cy}.

We fix W a Lyndon word of length greater or equal to 3, a, b and c three Lyndon words of smaller length. In the following computations, all indices corresponds to Lyndon words.

Beginning with

$$d_{cy}(T_{W^*}) = \sum_{U<V} \alpha^W_{U,V} T_{U^*} \cdot T_{V^*} + \sum_{U,V} \beta^W_{U,V} T_{U^*} \cdot T_{V^*}(1),$$

one computes $d^2_{cy}(T_{W^*})$ as

$$\begin{aligned} d^2_{cy}(T_{W^*}) = &\sum_{U<V} \alpha^W_{U,V} d_{cy}(T_{U^*}) \cdot T_{V^*} + \sum_{U<V} -\alpha^W_{U,V} T_{U^*} \cdot d_{cy}(T_{V^*}) \\ &+ \sum_{U,V} \beta^W_{U,V} d_{cy}(T_{U^*}) \cdot T_{V^*}(1) + \sum_{U,V} -\beta^W_{U,V} T_{U^*} \cdot d_{cy}(T_{V^*}(1)). \end{aligned}$$

Later on we will omit the $\cdot$ for the product. Developing the expression of d_{cy} on the right hand side, one obtains that $d_{cy}^2(T_{W^*})$ is equal to

$$(A_1^T) \qquad \sum_{U<V} \sum_{V_1<V_2} \alpha^W_{U,V}\alpha^U_{V_1,V_2} T_{V_1^*} T_{V_2^*} T_{V^*}$$

$$(A_2^T) \qquad + \sum_{U<V} \sum_{V_1,V_2} \alpha^W_{U,V}\beta^U_{V_1,V_2} T_{V_1^*} T_{V_2^*}(1) T_{V^*}$$

$$(A_3^T) \qquad + \sum_{U<V} \sum_{U_1<U_2} -\alpha^W_{U,V}\alpha^V_{U_1,U_2} T_{U^*} T_{U_1^*} T_{U_2^*}$$

$$(A_4^T) \qquad + \sum_{U<V} \sum_{U_1,U_2} -\alpha^W_{U,V}\beta^V_{U_1,U_2} T_{U^*} T_{U_1^*} T_{U_2^*}(1)$$

$$(A_5^T) \qquad + \sum_{U,V} \sum_{V_1<V_2} \beta^W_{U,V}\alpha^U_{V_1,V_2} T_{V_1^*} T_{V_2^*} T_{V^*}(1)$$

$$(A_6^T) \qquad + \sum_{U,V} \sum_{V_1,V_2} \beta^W_{U,V}\beta^U_{V_1,V_2} T_{V_1^*} T_{V_2^*}(1) T_{V^*}(1)$$

$$(A_7^T) \qquad + \sum_{U,V} \sum_{U_1<U_2} -\beta^W_{U,V}\alpha^V_{U_1,U_2} T_{U^*} T_{U_1^*}(1) T_{U_2^*}(1)$$

$$(A_8^T) \qquad + \sum_{U,V} \sum_{U_1,U_2} -\beta^W_{U,V}\beta^V_{U_1,U_2} T_{U^*} T_{U_1^*}(1) T_{U_2^*}(1)$$

We expand the above sum in terms of the family

$$\{T_{U_1^*} T_{U_2^*} T_{U_3^*}\} \cup \{T_{U_4^*} T_{U_5^*} T_{U_6^*}(1)\} \cup \{T_{U_7^*} T_{U_8^*}(1) T_{U_9^*}(1)\} \cup \{T_{U_{10}^*}(1) T_{U_{11}^*}(1) T_{U_{12}^*}(1)\}$$

where the U_i satisfy the constraints from Lemma 3.50. In order to do so, one should remark that, as $\overset{1}{\underset{0}{\odot}} = 0$ in $\mathcal{F}^\bullet_{\mathbb{Q}}$, there is no terms in $T_{0^*}(1)$ and that Lemma 3.49 insures that there is no terms in $T_{U_4^*} T_{U_5^*} T_{1^*}(1)$ or in $T_{U_7^*} T_{U_8^*}(1) T_{1^*}(1)$. Obviously $d_{cy}^2(T_{W^*})$ does not give any terms in $T_{U_{10}^*}(1) T_{U_{11}^*}(1) T_{U_{12}^*}(1)$.

We assume now that $a < b < c$. The coefficient of $T_{a^*} T_{b^*} T_{c^*}$ can only come from the sum (A_1^T) and (A_3^T).

From the sum (A_1^T), one gets

$$\begin{cases} \sum_{U<V} \alpha^W_{U,V}\alpha^U_{V_1,V_2} & \text{if } a = V_1 < b = V_2 < c = V \\ \sum_{U<V} -\alpha^W_{U,V}\alpha^U_{V_1,V_2} & \text{if } a = V_1 < b = V < c = V_2 \\ \sum_{U<V} \alpha^W_{U,V}\alpha^U_{V_1,V_2} & \text{if } a = V < b = V_1 < c = V_2 \end{cases}$$

which gives, in terms of words a, b and c and using u as independent variable in the sum signs,

$$\sum_{u<c} \alpha^W_{u,c}\alpha^u_{a,b} + \sum_{u<b} -\alpha^W_{u,b}\alpha^u_{a,c} + \sum_{u<a} \alpha^W_{u,a}\alpha^u_{b,c}.$$

Similarly, the sums (A_3^T) contributes to the coefficient of $T_{a^*} T_{b^*} T_{c^*}$ for

$$\sum_{c<u} -\alpha^W_{c,u}\alpha^u_{a,b} + \sum_{b<u} +\alpha^W_{b,u}\alpha^u_{a,c} + \sum_{a<u} -\alpha^W_{a,u}\alpha^u_{b,c}.$$

Reorganizing the sums, one sees that the coefficient of $T_{a^*} T_{b^*} T_{c^*}$ is exactly $r_{a<b<c}$. Using the fact that d_{cy} is a differential, that is $d_{cy}^2 = 0$, one obtains

$$r_{a<b<c} = 0.$$

We now assume that $a < b$. Lemma 3.49 insures that $s_{a<b,1} = 0$. Thus we can assume that $c \neq 1$. As $T_{0^*}(1)$ is 0 in $\mathcal{F}^\bullet_{\mathbb{Q}}$ we can also discard that case. Thus, we assume that $c \neq 0, 1$.

The coefficient of $T_{a^*}T_{b^*}T_{c^*}(1)$ can only come from the sum (A_2^T), (A_4^T) and (A_5^T).

From the sum (A_2^T), one gets

$$\begin{cases} \sum\limits_{U<V} -\alpha^W_{U,V}\beta^U_{V_1,V_2} & \text{if } a = V_1 < b = V,\ c = V_2 \\ \sum\limits_{U<V} \alpha^W_{U,V}\beta^U_{V_1,V_2} & \text{if } a = V < b = V_1,\ c = V_2 \end{cases}$$

which gives, in terms of words a, b and c and using u as independent variable in the sum signs,

$$\sum_{u<b} -\alpha^W_{u,b}\beta^u_{a,c} + \sum_{u<a} \alpha^W_{u,a}\beta^u_{b,c}.$$

Similarly, the sums (A_4^T) contributes to the coefficient of $T_{a^*}T_{b^*}T_{c^*}$ for

$$\sum_{b<u} \alpha^W_{b,u}\beta^u_{a,c} + \sum_{a<u} -\alpha^W_{a,u}\beta^u_{b,c}.$$

Finally the sum (A_5^T) contributes (with $V_1 = a$, $V_2 = b$, and $V = c$) for

$$\sum_u \beta^W_{u,c}\alpha^u_{a,b}.$$

The coefficient of $T_{a^*}T_{b^*}T_{c^*}(1)$ is then exactly $s_{a<b,c}$. Thus as previously, one obtains

$$s_{a<b,c} = 0.$$

We now compute the coefficient of $T_{a^*}T_{b^*}(1)T_{c^*}(1)$ with the condition $b < c$. Lemma 3.49 insures that $t_{a,b<1} = 0$ and $t_{0,b<c} = 0$. Thus we can assume that $c \neq 1$ and $a \neq 0$. As $T_{0^*}(1)$ is 0 in $\mathcal{F}^\bullet_{\mathbb{Q}}$ we can also discard the case $b = 0$. Thus, we assume that $b, c \neq 0, 1$.

The coefficient of $T_{a^*}T_{b^*}(1)T_{c^*}(1)$ can only come from the sum (A_6^T), (A_7^T) and (A_8^T).

From the sum (A_6^T), one gets

$$\begin{cases} \sum\limits_{U} \beta^W_{U,V}\beta^U_{V_1,V_2} & \text{if } a = V_1,\ b = V_2 < c = V \\ \sum\limits_{U} -\beta^W_{U,V}\beta^U_{V_1,V_2} & \text{if } a = V_1,\ b = V < c = V_2 \end{cases}$$

which gives, in terms of words a, b and c and using u as independent variable in the sum signs,

$$\sum_u \beta^W_{u,c}\beta^u_{a,b} + \sum_u -\beta^W_{u,b}\beta^u_{a,c}.$$

□

From the sum (A_8^T), one gets

$$\begin{cases} \sum\limits_{U} -\beta^W_{U,V}\beta^U_{U_1,U_2} & \text{if } a = U,\ b = U_1 < c = U_2 \\ \sum\limits_{U} \beta^W_{U,V}\beta^U_{U_1,U_2} & \text{if } a = U,\ b = U_2 < c = U_1. \end{cases}$$

So, sum (A_8^T) contributes, in terms of words a, b and c and using u as independent variable in the sum signs, for

$$\sum_u -\beta^W_{a,u}\beta^u_{b,c} + \sum_u \beta^W_{a,u}\beta^u_{c,b}.$$

Finally, from the sum (A_7^T) with $U = a$, $b = U_1$ and $c = U_2$, one gets:

$$\sum_u -\beta^W_{a,u}\alpha^u_{b,c}.$$

Hence, the coefficient of $T_{a^*}T_{b^*}(1)T_{c^*}(1)$, which is 0, is exactly $t_{a,b<c}$; that is putting together the terms coming from (A_7^T) and (A_8^T):

$$\sum_u \beta^W_{u,c}\beta^u_{a,b} + \sum_u -\beta^W_{u,b}\beta^u_{a,c} + \sum_u \beta^W_{a,u}\left(-\beta^u_{b,c} + \beta^u_{c,b} + -\alpha^u_{b,c}\right) = t_{a,b<c} = 0.$$

4. From trees to cycles

In this section we define two "differential systems" for algebraic cycles, one corresponding to cycles with empty at $t = 0$ and another corresponding to cycles with empty fiber at $t = 1$. Then, we show that there exists two families of cycles in $\mathcal{N}_X^{eq,\,1}$ satisfying these systems induced by two families of cycles in $\mathcal{N}_{\mathbb{A}^1}^{eq,\,1}$.

In order to define the systems, we need to twist the coefficients obtained in the tree differential system from Theorem 3.44. In the first subsection, we consider only a combinatorial setting which will be applied later to the cdga $\mathcal{N}^\bullet_{\mathbb{P}^1\setminus\{0,1,\infty\}}$.

4.1. **A combinatorial statement.**

Definition 4.1. Let W be a Lyndon word and U, V two Lyndon words. We set :

$$\begin{aligned} a^W_{U,V} &= \alpha^W_{U,V} + \beta^W_{U,V} - \beta^W_{V,U} && \text{for } U < V \\ b^W_{U,V} &= -\beta^W_{U,V} && \text{for any } U,\, V \end{aligned} \tag{30}$$

and

$$\begin{aligned} a'^W_{U,V} &= -a^W_{U,V} && \text{for } 0 < U < V, \\ b'^W_{U,V} &= a^W_{U,V} + b^W_{U,V} && \text{for } 0 < U < V, \\ b'^W_{V,U} &= -a^W_{U,V} + b^W_{V,U} && \text{for } 0 < U < V, \\ a'^W_{0,V} &= a_{0,V} && \text{for any } V, \\ b'^W_{U,U} &= b^W_{U,U} && \text{for any } U. \end{aligned} \tag{31}$$

It is also convenient to define $b'^W_{0,V} = b'^W_{V,0} = 0$.

As detailed in Remark 4.4, the above definitions correspond to rewriting the differential system (3.44) in terms of two others but related families of independent vectors in $\mathcal{F}^\bullet_{\mathbb{Q}}$.

Consider now the two following differential system in a cdga $(\mathcal{A}, \partial_{\mathcal{A}})$

$$\partial_{\mathcal{A}}(A_W) = \sum_{U<V} a^W_{U,V}A_UA_V + \sum_{U,V} b^W_{U,V}A_UA^1_V, \tag{ED-$\mathfrak{A}$}$$

and

$$\partial_{\mathcal{A}}(A^1_W) = \sum_{0<U<V} a'^W_{U,V}A^1_UA^1_V + \sum_{U,V} b'^W_{U,V}A_UA^1_V + \sum_V a'_{0,V}A_0A_V, \tag{ED-$\mathfrak{A}^1$}$$

Remark 4.2.

- If W is the Lyndon word 0 or 1, then the coefficients $a^W_{U,V}$, $b^W_{U,V}$, $a'^W_{U,V}$ and $b'^W_{U,V}$ are equal to 0.
- Let W be a Lyndon word. By Remark 3.45, if the length of U plus the one of V is not equal to the length of W, then the coefficients $a^W_{U,V}$, $b^W_{U,V}$, $a'^W_{U,V}$ and $b'^W_{U,V}$ are equal to 0. In particular, Equation (ED-$\mathfrak{A}$) and Equation (ED-$\mathfrak{A}^1$) involve only Lyndon words of length smaller than the length of W.

- From Lemma 3.49 and Definition 4.1, one sees that

$$a^W_{U,1} = {a'}^W_{U,1} = b^W_{U,1} = {b'}^W_{U,1} = 0 \quad \text{and} \quad b^W_{0,V} = b^W_{U,0} = {b'}^W_{0,V} = {b'}^W_{V,0} = 0.$$

Proposition 4.3. *Let $(\mathcal{A}, \partial_{\mathcal{A}})$ be a cdga and p be an integer $\geqslant 2$. Assume that there exists element A_U (resp. A^1_V) in $\mathcal{A}$ for any Lyndon word of length k with $2 \leqslant k \leqslant p-1$ satisfying* (ED-$\mathfrak{A}$) *(resp.* (ED-$\mathfrak{A}^1$)*) and elements A_0 and A_1 such that*

$$d(A_0) = d(A_1) = 0.$$

Let W be a Lyndon word of length p. Let $\mathfrak{A}_A$ be defined by

$$\mathfrak{A}_A = \sum_{U<V} a^W_{U,V} A_U A_V + \sum_{U,V} b^W_{U,V} A_U A^1_V,$$

and $\mathfrak{A}_{A^1}$ by

$$\mathfrak{A}_{A^1} = \sum_{U<V} {a'}^W_{U,V} A^1_U A^1_V + \sum_{U,V} {b'}^W_{U,V} A_U A^1_V,$$

Then, one has

$$\partial_{\mathcal{A}}(\mathfrak{A}_A) = \partial_{\mathcal{A}}(\mathfrak{A}_{A^1}) = 0.$$

The definition of $\mathfrak{A}_A$ and $\mathfrak{A}_{A^1}$ only involve only words of length strictly smaller than the one of W.

Proof. From Definition 4.1, we remark that for any W_0 of length $< p$, one has

$$\partial_{\mathcal{A}}(A_{W_0}) - \partial_{\mathcal{A}}(A^1_{W_0}) = \sum_{0<U<V<1} \left(a^{W_0}_{U,V} A_U A_V + a^{W_0}_{U,V} A^1_U A^1_V + a^{W_0}_{U,V} A_V A^1_U - a^{W_0}_{U,V} A_U A^1_V \right). \tag{32}$$

Similarly, one has

$$\mathfrak{A}_A - \mathfrak{A}_{A^1} = \sum_{0<U<V<1} \left(a^W_{U,V} A_U A_V + a^W_{U,V} A^1_U A^1_V + a^W_{U,V} A_V A^1_U - a^W_{U,V} A_U A^1_V \right). \tag{33}$$

First we want to prove that $\partial_{\mathcal{A}}(\mathfrak{A}_A) = 0$. One computes

$$\begin{aligned} \partial_{\mathcal{A}}(\mathfrak{A}_A) = &\sum_{U<V} a^W_{U,V} \partial_{\mathcal{A}}(A_U) A_V + \sum_{U<V} -a^W_{U,V} A_U \partial_{\mathcal{A}}(A_V) \\ &+ \sum_{U,V} b^W_{U,V} \partial_{\mathcal{A}}(A_U) A^1_V + \sum_{U,V} -b^W_{U,V} A_U \partial_{\mathcal{A}}(A^1_V). \end{aligned}$$

Rewriting the above equation, in order to use differences $\partial_{\mathcal{A}}(A_{W_0}) - \partial_{\mathcal{A}}(A_{W_0})$ and thus work only with coefficients $a^{W_0}_{U_0,V_0}$ and $b^{W_0}_{U_0,V_0}$, gives:

$$(B_1^A) \qquad \partial_{\mathcal{A}}(\mathfrak{A}_A) = \sum_{U<V} (a^W_{U,V} - b^W_{V,U})\partial_{\mathcal{A}}(A_U)A_V$$

$$(B_2^A) \qquad + \sum_{U<V} (-a^W_{U,V} - b^W_{U,V})A_U\partial_{\mathcal{A}}(A_V)$$

$$(B_2^{',A}) \qquad + \sum_{U} -b^W_{U,U}A_U\partial_{\mathcal{A}}(A_U)$$

$$(B_3^A) \qquad + \sum_{U<V} b^W_{U,V}\partial_{\mathcal{A}}(A_U)A^1_V$$

$$(B_4^A) \qquad + \sum_{U} b^W_{U,U}\partial_{\mathcal{A}}(A_U)A^1_U$$

$$(B_3^{',A}) \qquad + \sum_{U<V} b^W_{V,U}\partial_{\mathcal{A}}(A_V)A^1_U$$

$$(B_5^A) \qquad + \sum_{U<V} b^W_{U,V}A_U\left(\partial_{\mathcal{A}}(A_V) - \partial_{\mathcal{A}}(A^1_V)\right)$$

$$(B_5^{',A}) \qquad + \sum_{U} b^W_{U,U}A_U\left(\partial_{\mathcal{A}}(A_U) - \partial_{\mathcal{A}}(A^1_U)\right)$$

$$(B_6^A) \qquad + \sum_{U<V} b^W_{V,U}A_V\left(\partial_{\mathcal{A}}(A_U) - \partial_{\mathcal{A}}(A^1_U)\right)$$

The signs are computed using the fact that $\mathcal{N}^\bullet_X$ is graded commutative and the fact that elements A_{W_0} and $A^1_{W_0}$ are of degree 1 while their differentials are of degree 2.

Now, using the induction hypothesis, one can write $\partial_{\mathcal{A}}(\mathfrak{A}_A)$ in terms of the following products (u, v and w are Lyndon words):

$$A_uA_vA_w, \quad A_uA_vA^1_w, \quad A_uA^1_vA^1_w, \text{ and } A^1_uA^1_vA^1_w,$$

which gives

$$\partial_{\mathcal{A}}(\mathfrak{A}_A) = \sum_{u<v<w} r^{cy}_{u<v<c}A_uA_vA_w + \sum_{\substack{u<v,\\ w}} s^{cy}_{u<v,w}A_uA_vA^1_w + \sum_{\substack{u,\\ v<w}} t^{cy}_{u,v<w}A_uA^1_vA^1_w + \sum_{u<v<w} p^{cy}_{u<v<w}A^1_uA^1_vA^1_w.$$

Fix $u<v<w$, terms in $r^{cy}_{u<v<w}$ can not come from the above sums (B_3^A), $(B_3^{',A})$ or (B_4^A). In one hand sum, (B_1^A) gives

$$\sum_{U<V} (a^W_{U,V} - b^W_{V,U}) \sum_{V_1<V_2} a^U_{V_1,V_2}A_{V_1}A_{V_2}A_V$$

plus extra terms which do not contribute to $r^{cy}_{a<b<c}$. In the other hand, sum (B_6^A) gives

$$\sum_{U<V} b^W_{V,U} \sum_{0<V_1<V_2<1} a^U_{V_1,V_2}A_VA_{V_1}A_{V_2}$$

plus extra terms which do not contribute to $r^{cy}_{a<b<c}$. Hence, as $a^{W_0}_{U_0,1} = 0$ for any U_0, W_0, the contribution of (B_1^A) and (B_6^A) is given by the sum

$$\sum_{U<V} \sum_{0<V_1<V_2<1} a^W_{U,V}a^U_{V_1,V_2}A_{V_1}A_{V_2}A_V + \sum_{U<V} \sum_{0<V_2<1} (a^W_{U,V} - b^W_{V,U})a^U_{V_1,V_2}A_0A_{V_2}A_V.$$

Similar remarks hold for sums (B_2^A), $(B_2^{'A})$, (B_5^A) and $(B_5^{'A})$ and the contribution to $r^{cy}_{a<b<c}$ is given by

$$\sum_{U<V} \sum_{0<V_1<V_2<1} a^W_{U,V} a^U_{V_1,V_2} A_{V_1} A_{V_2} A_V \tag{34}$$

$$+ \sum_{U<V} \sum_{0<V_2<1} (a^W_{U,V} - b^W_{V,U}) a^U_{V_1,V_2} A_0 A_{V_2} A_V \tag{35}$$

$$+ \sum_{U<V} \sum_{0<U_1<U_2<1} -a^W_{U,V} a^V_{U_1,U_2} A_U A_{U_1} A_{U_2} \tag{36}$$

$$+ \sum_{U<V} \sum_{0<U_2<1} (-a^W_{U,V} - b^W_{U,V}) a^V_{0,U_2} A_U A_0 A_{U_2} \tag{37}$$

$$+ \sum_{U} \sum_{0<U_2<1} -b^W_{U,U} a^U_{0,U_2} A_U A_0 A_{U_2}. \tag{38}$$

First, we assume that $u > 0$. Then, sums (35), (37), (38) do not contribute to $r^{cy}_{u<v<w}$ and a computation, similar to the computation giving $r_{a<b<c}$ at the previous section, gives (dropping superscripts W)

$$r^{cy}_{u<v<w} = \sum_{m<w} a_{m,w} a^m_{u,v} + \sum_{w<m} -a_{w,m} a^m_{u,v} + \sum_{m<v} -a_{m,v} a^m_{u,w} + \sum_{v<m} a_{v,m} a^m_{u,w} + \sum_{m<u} a_{m,u} a^m_{v,w} + \sum_{u<m} a_{u,m} a^m_{v,w}. \tag{39}$$

Now, we expand each products of the type $a_{m,u} a^m_{v,w}$ as

$$\begin{aligned} a_{m,u} a^m_{v,w} =& (\alpha_{m,u} + \beta_{m,u} - \beta_{u,m})(\alpha^m_{v,w} + \beta^m_{v,w} - \beta^m_{w,v}) \\ =& \alpha_{m,u}\alpha^m_{v,w} + \alpha_{m,u}\beta^m_{v,w} - \alpha_{m,u}\beta^m_{w,v} \\ &+ \beta_{m,u}\alpha^m_{v,w} + \beta_{m,u}\beta^m_{v,w} - \beta_{m,u}\beta^m_{w,v} \\ &- \beta_{u,m}\alpha^m_{v,w} - \beta_{u,m}\beta^m_{v,w} + \beta_{u,m}\beta^m_{w,v}. \end{aligned}$$

Taking care of signs and permutation of the indices, we obtain regrouping some summation signs

$$
\begin{aligned}
r^{cy}_{u<v<w} = &\sum_{m<u} \alpha_{m,u}\alpha^m_{v,w} + \sum_{m<v} -\alpha_{m,v}\alpha^m_{u,w} + \sum_{m<w} \alpha_{m,w}\alpha^m_{u,v} \\
&+ \sum_{u<m} -\alpha_{u,m}\alpha^m_{v,w} + \sum_{v<m} \alpha_{v,m}\alpha^m_{u,w} + \sum_{w<m} -\alpha_{w,m}\alpha^m_{u,v} \\
&+ \sum_{m<u} \alpha_{m,u}\beta^m_{v,w} + \sum_{m<v} -\alpha_{m,v}\beta^m_{u,w} + \sum_{m<w} \alpha_{m,w}\beta^m_{u,v} \\
&+ \sum_{u<m} -\alpha_{u,m}\beta^m_{v,w} + \sum_{v<m} \alpha_{v,m}\beta^m_{u,w} + \sum_{w<m} -\alpha_{w,m}\beta^m_{u,v} \\
&+ \sum_{m<u} -\alpha_{m,u}\beta^m_{w,v} + \sum_{m<v} \alpha_{m,v}\beta^m_{w,u} + \sum_{m<w} -\alpha_{m,w}\beta^m_{v,u} \\
&+ \sum_{u<m} \alpha_{u,m}\beta^m_{w,v} + \sum_{v<m} -\alpha_{v,m}\beta^m_{w,u} + \sum_{w<m} \alpha_{w,m}\beta^m_{v,u} \\
&+ \sum_{m\neq u} \beta_{m,u}\alpha^m_{v,w} + \sum_{m\neq v} -\beta_{m,v}\alpha^m_{u,w} + \sum_{m\neq w} \beta_{m,w}\alpha^m_{u,v} \\
&+ \sum_{m\neq u} -\beta_{u,m}\alpha^m_{v,w} + \sum_{m\neq v} \beta_{v,m}\alpha^m_{u,w} + \sum_{w\neq m} -\beta_{w,m}\alpha^m_{u,v} \\
&+ \sum_{m\neq u} \beta_{m,u}\beta^m_{v,w} + \sum_{m\neq v} -\beta_{m,v}\beta^m_{u,w} + \sum_{m\neq w} \beta_{m,w}\beta^m_{u,v} \\
&+ \sum_{m\neq u} -\beta_{u,m}\beta^m_{v,w} + \sum_{m\neq v} \beta_{v,m}\beta^m_{u,w} + \sum_{m\neq w} -\beta_{w,m}\beta^m_{u,v} \\
&+ \sum_{m\neq u} -\beta_{m,u}\beta^m_{w,v} + \sum_{m\neq v} \beta_{m,v}\beta^m_{w,u} + \sum_{w\neq m} -\beta_{m,w}\beta^m_{v,u} \\
&+ \sum_{m\neq u} \beta_{u,m}\beta^m_{w,v} + \sum_{m\neq v} -\beta_{v,m}\beta^m_{w,u} + \sum_{m\neq w} \beta_{w,m}\beta^m_{v,u}.
\end{aligned}
$$

The two first line of the above sums are equal to $r_{u<v<w}$. The remaining terms are reorganized into the six following sums:

$$
\sum_{m\neq u} \beta_{m,u}\alpha^m_{v,w} + \sum_{m<v} \alpha_{m,v}\beta^m_{w,u} + \sum_{m<w} -\alpha_{m,w}\beta^m_{v,u} + \sum_{v<m} -\alpha_{v,m}\beta^m_{w,u} + \sum_{w<m} \alpha_{w,m}\beta^m_{v,u}
$$

$$
\sum_{m\neq v} -\beta_{m,v}\alpha^m_{u,w} + \sum_{m<u} -\alpha_{m,u}\beta^m_{w,v} + \sum_{m<w} \alpha_{m,w}\beta^m_{u,v} + \sum_{u<m} \alpha_{u,m}\beta^m_{w,v} + \sum_{w<m} -\alpha_{w,m}\beta^m_{u,v},
$$

$$
\sum_{m\neq w} \beta_{m,w}\alpha^m_{u,v} + \sum_{m<u} \alpha_{m,u}\beta^m_{v,w} + \sum_{m<v} -\alpha_{m,v}\beta^m_{u,w} + \sum_{u<m} -\alpha_{u,m}\beta^m_{w,v} + \sum_{v<m} \alpha_{v,m}\beta^m_{w,u},
$$

$$\sum_{m\neq w}\beta_{m,w}\beta^m_{u,v}+\sum_{m\neq v}-\beta_{m,v}\beta^m_{u,w}+\sum_{m\neq u}-\beta_{u,m}\alpha^m_{v,w}+\sum_{m\neq u}-\beta_{u,m}\beta^m_{v,w}+\sum_{m\neq u}\beta_{u,m}\beta^m_{w,v},$$

$$\sum_{m\neq w}-\beta_{m,w}\beta^m_{v,u}+\sum_{m\neq u}\beta_{m,u}\beta^m_{v,w}+\sum_{m\neq v}\beta_{v,m}\alpha^m_{u,w}+\sum_{m\neq v}\beta_{v,m}\beta^m_{u,w}+\sum_{m\neq v}-\beta_{v,m}\beta^m_{w,u},$$

$$\sum_{m\neq v}\beta_{m,v}\beta^m_{w,u}+\sum_{m\neq u}-\beta_{m,u}\beta^m_{w,v}+\sum_{m\neq w}-\beta_{w,m}\alpha^m_{u,v}+\sum_{m\neq w}-\beta_{w,m}\beta^m_{u,v}+\sum_{m\neq w}\beta_{w,m}\beta^m_{v,u}.$$

It is then easy to recognized that

$$r^{cy}_{u<v<w}=r_{u<v<w}+s_{v<w,u}-s_{u<w,v}+s_{u<v,w}+t_{u,v<w}-t_{v,u<w}+t_{w,u<v}$$

as the extra needed equality cases in the sums cancel each other. Finally, using Proposition 3.51, one obtains

$$r^{cy}_{u<v<w}=0.$$

Now, we assume that $u=0$. Sum (34) does not contribute to $r^{cy}_{0<v<w}$ as $0\leqslant U<V$. Sum (36) contribute ($0=U$, $U_1=v$ and $U_2=w$) for

$$\sum_{0<m}a_{0,m}a^m_{v,w}.$$

Similarly, sums (35) and (37) contribute respectively for

$$\sum_{m<v}-(a_{m,v}-b_{m,v})a^m_{0,w}\sum_{m<w}(a_{m,w}-b_{m,w})a^m_{0,v}$$

and

$$\sum_{v<m}-(-a_{v,m}-b_{v,m})a^m_{0,w}\sum_{w<m}(-a_{w,m}-b_{w,m})a^m_{0,v},$$

when sum (38) contributes for

$$b_{v,v}a^v_{0,w}-b_{w,w}a_{0,v}.$$

Thus, with $u=0$ $r^{cy}_{u<v<w}$ can be written as

(40)
$$r^{cy}_{u<v<w}=\sum_{m<w}a_{m,w}a^m_{u,v}+\sum_{w<m}-a_{w,m}a^m_{u,v}+\sum_{m<v}-a_{m,v}a^m_{u,w}+\sum_{v<m}a_{v,m}a^m_{u,w}+\sum_{m<u}a_{m,u}a^m_{v,w}+\sum_{u<m}a_{u,m}a^m_{v,w}+\sum_{m}-b_{w,m}a^m_{u,v}+\sum_{m}-b_{v,m}a^m_{u,w}.$$

The two extra terms arising cancel with

$$+\sum_{m\neq w}\beta_{v,m}(-\alpha^m_{u,w}-\beta^m_{u,w}+\beta^m_{w,u})$$

and

$$+\sum_{m\neq w}\beta_{w,m}(-\alpha^m_{u,v}-\beta^m_{u,v}+\beta^m_{v,u})$$

equality terms adding up to gives the remaining missing equality terms. As $\beta_{0,W_0} = \beta_{W_0,0} = 0$ the other terms appearing in $t_{v,u<w}$ and $t_{w,u<v}$ vanish and one can write for $u = 0$

$$r^{cy}_{u<v<w} = r_{u<v<w} + s_{v<w,u} - s_{u<w,v} + s_{u<v,w} + t_{u,v<w} = 0.$$

Now, we assume only that $u < v$ and we will show that $s^{cy}_{u<v,w} = 0$ by a similar computation. Here all sums (B^A_1), ..., (B^A_6) contribute. Precisely, the contribution to $s^{cy}_{u<v,w}$ is induced by

$$\sum_{U<V} \sum_{V_1,V_2} (a^W_{U,V} - b^W_{V,U}) b^U_{V_1,V_2} A_{V_1} A^1_{V_2} A_V \tag{41}$$

$$+ \sum_{U<V} \sum_{U_1,U_2} (-a^W_{U,V} - b^W_{U,V}) b^V_{U_1,U_2} A_U A_{U_1} A^1_{U_2} \tag{42}$$

$$+ \sum_{U} \sum_{U_1,U_2} -b^W_{U,V} b^U_{U_1,U_2} A_U A_{U_1} A^1_{U_2} \tag{43}$$

$$+ \sum_{U<V} \sum_{V_1<V_2} b^W_{U,V} a^U_{V_1,V_2} A_{V_1} A_{U_2} A^1_V \tag{44}$$

$$+ \sum_{U} \sum_{V_1<V_2} b^W_{U,U} a^U_{V_1,V_2} A_{V_1} A_{V_2} A^1_U \tag{45}$$

$$+ \sum_{U<V} \sum_{U_1<U_2} b^W_{V,U} a^V_{U_1,U_2} A_{V_1} A_{V_2} A^1_U \tag{46}$$

$$+ \sum_{U<V} \sum_{0<U_1<U_2<1} -b^W_{U,V} a^V_{U_1,U_2} A_U A_{U_1} A^1_{U_2} \tag{47}$$

$$+ \sum_{U<V} \sum_{0<U_1<U_2<1} b^W_{U,V} a^V_{U_1,U_2} A_U A_{U_2} A^1_{U_1} \tag{48}$$

$$+ \sum_{U} \sum_{0<U_1<U_2<1} -b^W_{U,U} a^U_{V_1,V_2} A_U A_{U_1} A^1_{U_2} \tag{49}$$

$$+ \sum_{U} \sum_{0<U_1<U_2<1} b^W_{U,U} a^U_{U_1,U_2} A_U A_{U_2} A^1_{U_1} \tag{50}$$

$$+ \sum_{U<V} \sum_{0<V_1<V_2<1} -b^W_{V,U} a^U_{V_1,V_2} A_V A_{V_1} A^1_{V_2} \tag{51}$$

$$+ \sum_{U<V} \sum_{0<V_1<V_2<1} b^W_{V,U} a^U_{V_1,V_2} A_V A_{V_2} A^1_{V_1}. \tag{52}$$

The same types of arguments as before show that sum (41) contributes for

$$\sum_{m<v} -(a_{m,v} - b_{v,m}) b^m_{u,w} + \sum_{m<u} (a_{m,u} - b_{u,m}) b^m_{v,w},$$

sum (42) for

$$-b_{u,u} b^u_{v,w} + b_{v,v} b^v_{u,w}$$

and sum (43) for

$$\sum_{v<m} -(-a_{v,m} - b_{v,m}) b^m_{u,w} + \sum_{u<m} (-a_{u,m} - b_{u,m}) b^m_{v,w}.$$

Sums (44), (45) and (46) contributes to $s^{cy}_{u<v,w}$ for

$$\sum_m b_{m,w} a^m_{u,v}.$$

The above contribution to $s^{cy}_{u<v,w}$ can be written as

$$\sum_m b_{m,w}a^m_{u,v} + \sum_{m<v} -a_{m,v}b^m_{u,w} + \sum_{m<u} a_{m,u}b^m_{v,w} + \sum_{v<m} a_{v,m}b^m_{u,w} + \sum_{u<m} -a_{u,m}b^m_{v,w} + \sum_m b_{v,m}b^m_{u,w} + \sum_m -b_{u,m}b^m_{v,w}.$$

The other sums do not always contribute depending on the relative place of w with respect to $u < v$.

We assume for a time that $u < v < w$. Then, sums (48), (50) and (52) do not contribute. From sums (47), (49) and (51) arises a contribution in

$$\sum_m -b_{u,m}a^m_{v,w} + \sum_m b_{v,m}a^m_{u,w}.$$

Hence, $s^{cy}_{u<v,w}$ is equal to

$$\begin{aligned} s^{cy}_{u<v,w} = &\sum_m b_{m,w}a^m_{u,v} + \sum_m b_{v,m}b^m_{u,w} + \sum_m -b_{u,m}b^m_{v,w} \\ &+ \sum_{m<v} -a_{m,v}b^m_{u,w} + \sum_{m<u} a_{m,u}b^m_{v,w} + \sum_{v<m} a_{v,m}b^m_{u,w} + \sum_{u<m} -a_{u,m}b^m_{v,w} \\ &+ \sum_m -b_{u,m}a^m_{v,w} + \sum_m b_{v,m}a^m_{u,w}. \end{aligned}$$

Expanding $\sum_{m<u} a_{m,u}b^m_{v,w}$ and $\sum_{u<m} -a_{u,m}b^m_{v,w}$ in terms of α's and β's and canceling terms in $\beta_{u,m}$, gives

$$\sum_{m<u} a_{m,u}b^m_{v,w} + \sum_{u<m} -a_{u,m}b^m_{v,w} + \sum_m -b_{u,m}b^m_{v,w} = \sum_{m<u} -\alpha_{m,u}\beta^m_{v,w} + \sum_{u<m} \alpha_{u,m}\beta^m_{v,w} + \sum_m -\beta_{m,u}\beta^m_{v,w}.$$

Similarly, one has

$$\sum_{m<v} -a_{m,v}b^m_{u,w} + \sum_{v<m} a_{v,m}b^m_{u,w} + \sum_m b_{v,m}b^m_{u,w} = \sum_{m<v} \alpha_{m,v}\beta^m_{u,w} + \sum_{v<m} -\alpha_{v,m}\beta^m_{u,w} + \sum_m \beta_{m,v}\beta^m_{u,w}.$$

Using these remarks and expanding $b_{v,m}b^m_{u,w}$ in terms of the α's and β's, $s^{cy}_{u<v,w}$ can be written as

$$\begin{aligned} s^{cy}_{u<v,w} = &\sum_m -\beta_{m,w}\alpha^m_{u,v} + \sum_m -\beta_{m,w}\beta^m_{u,v} + \sum_m \beta_{m,w}\beta^m_{v,u} \\ &+ \sum_{m<u} -\alpha_{m,u}\beta^m_{v,w} + \sum_{u<m} \alpha_{u,m}\beta^m_{v,w} + \sum_m -\beta_{m,u}\beta^m_{v,w} \\ &+ \sum_{m<v} \alpha_{m,v}\beta^m_{u,w} + \sum_{v<m} -\alpha_{v,m}\beta^m_{u,w} + \sum_m \beta_{m,v}\beta^m_{u,w} \\ &+ \sum_m -b_{u,m}a^m_{v,w} + \sum_m b_{v,m}a^m_{u,w}. \end{aligned}$$

We remark that

$$\sum_m -\beta_{m,w}\alpha^m_{u,v} + \sum_{m<u} -\alpha_{m,u}\beta^m_{v,w} + \sum_{u<m} \alpha_{u,m}\beta^m_{v,w} + \sum_{m<v} \alpha_{m,v}\beta^m_{u,w} + \sum_{v<m} -\alpha_{v,m}\beta^m_{u,w} = -s_{u<v,c}.$$

Then, expanding $\sum_m -b_{u,m}a^m_{v,w}$ in terms of $\alpha's$ and β 's, we compute

$$\sum_m -b_{u,m}a^m_{v,w} + \sum_m -\beta_{m,w}\beta^m_{u,v} + \sum_m \beta_{m,v}\beta^m_{u,w} = -t_{u,v<w}$$

and

$$\sum_m -b_{v,m}a^m_{u,w} + \sum_m \beta_{m,w}\beta^m_{v,u} + \sum_m -\beta_{m,u}\beta^m_{v,w} = t_{v,u<w}.$$

Hence, for $u < v < w$

$$s^{cy}_{u<v,w} = -s_{u<v,c} - t_{u,v<w} + t_{v,u<w} = 0.$$

In the case where $u < w < v$, an identical computation shoes that the sum $\sum_m -b_{u,m}a^m_{v,w}$ is replace by $\sum_m b_{u,m}a^m_{w,v}$ and one finds

$$s^{cy}_{u<v,w} = -s_{u<v,c} + t_{u,w<v} + t_{v,u<w} = 0.$$

When $w < u < v$, both sums $\sum_m -b_{u,m}a^m_{v,w}$ and $\sum_m b_{v,m}a^m_{u,w}$ are replaced by $\sum_m b_{u,m}a^m_{w,v}$ and $\sum_m -b_{v,m}a^m_{w,u}$ respectively which gives:

$$s^{cy}_{u<v,w} = -s_{u<v,c} + t_{u,w<v} - t_{v,w<u} = 0.$$

In the case where $w = u$ (resp. $w = v$), there is no contribution in $\sum_m -b_{u,m}a^m_{v,w}$ (resp. in $\sum_m b_{v,m}a^m_{u,w}$) to $s^{cy}_{u<v,w}$. However, in this case a cancellation arises in the other terms; that is for $u = w$

$$\sum_m \beta_{m,w}\beta^m_{v,u} + \sum_m -\beta_{m,u}\beta^m_{v,w} = 0$$

and for $v = w$

$$\sum_m -\beta_{m,w}\beta^m_{u,v} + \sum_m \beta_{m,v}\beta^m_{u,w} = 0.$$

Thus, the above discussion gives in these cases either $s^{cy}_{u<v,w} = -s_{u<v,c}+t_{u,w<v} = 0$ or $s^{cy}_{u<v,w} = -s_{u<v,c} + t_{v,u<w} = 0$ which concludes the case of $s^{cy}_{u<v,w}$.

In order to show that $\partial_{\mathcal{A}}(\mathfrak{A}_A) = 0$, we still need to check that $t^{cy}_{u,v<w} = 0$ for all admissible choices of u, v, and w. We fix Lyndon words u, $v < w$. As by induction, $\partial_{\mathcal{A}}(A_{W_0})$ do not produce any product of the form $A^1_{U_0}A^1_{V_0}$, the sums (B^A_1), (B^A_2) and $(B^{'A}_2)$ do not contribute. Thus the coefficient $t^{cy}_{u,v<w}$ comes from

$$\sum_{U<V}\sum_{V_1,V_2} b^W_{U,V}b^U_{V_1,V_2}A_{V_1}A^1_{V_2}A^1_V \tag{53}$$

$$+\sum_{U<V}\sum_{U_1,U_2} b^W_{V,U}b^V_{U_1,U_2}A_{U_1}A^1_{U_2}A^1_U \tag{54}$$

$$+\sum_{U}\sum_{U_1,U_2} b^W_{U,U}b^U_{U_1,U_2}A_{U_1}A^1_{U_2}A^1_U \tag{55}$$

$$+\sum_{U<V}\sum_{0<U_1<U_2<1} b^W_{U,V}a^V_{U_1,U_2}A_UA^1_{U_1}A^1_{U_2} \tag{56}$$

$$+\sum_{U}\sum_{0<U_1<U_2<1} b^W_{U,V}a^U_{U_1,U_2}A_UA^1_{U_1}A^1_{U_2} \tag{57}$$

$$+\sum_{U<V}\sum_{0<U_1<U_2<1} b^W_{V,U}a^U_{V_1,V_2}A_VA^1_{V_1}A^1_{V_2}. \tag{58}$$

First we should remark that the last three sums do not contribute if either v or w is equal to 0 or 1. In this case, previous comments (cf. Lemma 3.49) insure that the first three sums contribute for 0 as the various products of the form $b^W_{U,V}b^U_{V_1,V_2}$ involved are 0. Thus we can assume that $0 < v < w < 1$.

The last three sums contribute for

$$\sum_{u<m} b_{u,m}a^m_{v,w} + b_{u,u}a^u_{v,w} + \sum_{m<u} b_{u,m}a^m_{v,w} = \sum_m b_{u,m}a^m_{v,w}.$$

Sum (53) contributes for

$$\sum_{m<v} -b_{m,v}b_{u,w} + \sum_{m<w} b_{m,w}b_{u,w},$$

sum (54) contributes for

$$\sum_{m>v} -b_{m,v}b_{u,w} + \sum_{m>w} b_{m,w}b_{u,w},$$

and sum (55) gives the equality case. Finally, one has

$$t^{cy}_{u,v<w} = \sum_m -b_{m,v}b_{u,w} + \sum_m b_{m,w}b_{u,w} + \sum_m b_{u,m}a^m_{v,w} = t_{u,v<w} = 0.$$

As no terms in $A^1_uA^1_vA^1_w$ can arise from $\partial_{\mathcal{A}}(\mathfrak{A}_A)$ we have shown that

$$\partial_{\mathcal{A}}(\mathfrak{A}_A) = 0.$$

We now need to show that $\partial_{\mathcal{A}}(\mathfrak{A}_{A^1}) = 0$. In order to avoid working with the a' and b', we will show that $\partial_{\mathcal{A}}(\mathfrak{A}_A - \mathfrak{A}_{A^1}) = 0$. One has

$$\begin{aligned}\mathfrak{A}_A - \mathfrak{A}_{A^1} &= \sum_{0<U<V<1} a^W_{U,V}\left(A_UA_V + A^1_UA^1_V + A_VA^1_U - A_UA^1_V\right)\\ &= \sum_{0<U<V<1} a^W_{U,V}\left((A_U - A^1_U)(A_V - A^1_V)\right)\end{aligned}$$

and

$$\begin{aligned}\partial_{\mathcal{A}}(\mathfrak{A}_A - \mathfrak{A}_{A^1}) = &\sum_{0<U<V<1} a^W_{U,V}(\partial_{\mathcal{A}}(A_U) - \partial_{\mathcal{A}}(A^1_U))A_V\\ &+ \sum_{0<U<V<1} -a^W_{U,V}(\partial_{\mathcal{A}}(A_U) - \partial_{\mathcal{A}}(A^1_U))A^1_V\\ &+ \sum_{0<U<V<1} -a^W_{U,V}A_U(\partial_{\mathcal{A}}(A_V) - \partial_{\mathcal{A}}(A^1_V))\\ &+ \sum_{0<U<V<1} a^W_{U,V}A^1_U(\partial_{\mathcal{A}}(A_V) - \partial_{\mathcal{A}}(A^1_V)).\end{aligned}$$

Again, using the induction hypothesis, this expression decomposes in terms of products of the form

$$A_uA_vA_w, \quad A_uA_vA^1_w, \quad A_uA^1_vA^1_w, \text{ and } A^1_uA^1_vA^1_w.$$

The computations are closely related to what was done in order to prove that $\partial_{\mathcal{A}}(AL) = 0$ but generally speaking the situation here is much more symmetric. In particular it is easy to see that

$$\partial_{\mathcal{A}}(\mathfrak{A}_A - \mathfrak{A}_{A^1}) = \sum_{u<v<w} (r^{cy}_{u<v<w}A_uA_vA_w - r^{cy}_{u<v<w}A^1_uA^1_vA^1_w) + \text{terms in } A_uA_vA^1_w \text{ and } A_uA^1_vA^1_w.$$

The situations for coefficients of $A_uA_vA^1_w$ and $A_uA^1_vA^1_w$ are very similar, so we will discuss only the case of $A_uA_vA^1_w$.

Computing $\partial_{\mathcal{A}}(\mathfrak{A}_A - \mathfrak{A}_{A^1})$ using the induction hypothesis, the contribution to $A_u A_v A^1_w$ $(u < v)$ comes from

$$\begin{aligned}
&\sum_{0<U<V<1} \sum_{0<V_1<V_2<1} -a^W_{U,V} a^U_{V_1,V_2} A_{V_1} A^1_{V_2} A_V \\
&+ \sum_{0<U<V<1} \sum_{0<V_1<V_2<1} a^W_{U,V} a^U_{V_1,V_2} A_{V_2} A^1_{V_1} A_V \\
&+ \sum_{0<U<V<1} \sum_{0<V_2<V_2<1} -a^W_{U,V} a^U_{V_1,V_2} A_{V_1} A_{V_2} A^1_V \\
&+ \sum_{0<U<V<1} \sum_{0<U_1<U_2<1} a^W_{U,V} a^V_{U_1,U_2} A_U A_{U_1} A^1_{U_2} \\
&+ \sum_{0<U<V<1} \sum_{0<U_1<U_2<1} -a^W_{U,V} a^V_{U_1,U_2} A_U A_{U_2} A^1_{U_1} \\
&+ \sum_{0<U<V<1} \sum_{0<U_1<U_2<1} a^W_{U,V} a^V_{U_1,U_2} A^1_U A_{U_1} A_{U_2}.
\end{aligned}$$

Depending on the relative position of w with respect to u and v not all sums contribute. Assuming that $u < v < w$, the second and fifth sums do not contribute and the coefficient of $A_u A_v A^1_w$ is given by

$$-r^{cy}_{u,v,w} = 0.$$

In the case where $u < w < v$ (resp. $w < u < v$) the coefficient will be given by $r^{cy}_{u,w,v}$ (resp. $-r^{cy}_{w,u,v}$). As previously, in the equality cases ($w = v$ or $w = u$) cancellations arise among the different sums. In the case $w = v$, the second and fifth sums do not contribute and the coefficient of $A_u A_v A^1_w$ is given by

$$\sum_{m<v} a_{m,v} a^m_{u,w} + \sum_{v<m} -a_{v,m} a^m_{u,w} + \sum_{m<w} -a_{m,w} a^m_{u,v} + \sum_{w<m} a_{w,m} a^m_{u,v}$$

which is 0 for $v = w$. □

Remark 4.4. For any Lyndon words, let $T^1_{U^*}$ denote the difference $T_{W^*} - T_{W^*}(1)$. The above computations for $\mathfrak{A}_A$ can be seen as writing the differential $d_{cy}(T_{W^*})$ in terms of the following independent families

$$T_{U^*} \cdot T_{V^*} \text{ for } U < V, \quad T_{U^*} \cdot T^1_{V^*} \text{ for any Lyndon words } U, V$$

and remarking that $d^2_{cy} = 0$. For $\mathfrak{A}_{A^1}$, these computations correspond to the differential $d_{cy}(T^1_{W^*})$ written in terms of

$$T^1_{U^*} \cdot T^1_{V^*} \text{ for } 0 < U < V < 1, \quad T_{U^*} \cdot T^1_{V^*} \text{ and } T_{0^*} \cdot T_{V^*} \text{ for any Lyndon words } U, V$$

together with the fact that $d^2_{cy} = 0$.

In this context showing that $\mathfrak{A}_A$ and $\mathfrak{A}_{A^1}$ have differential 0 is obvious as it is just a change of basis. However, latter on we will not have relations as simple as $T^1_{U^*} = T_{W^*} - T_{W^*}(1)$ and relying on a change of basis argument may still be possible but would certainly demand great attention. Proposition 4.3 will be used to prove Theorem 4.12.

4.2. **Equidimensional cycles.** We recall that the base field is $\mathbb{Q}$ and that all varieties considered below are $\mathbb{Q}$ varieties.

Definition 4.5 (Equidimensionality)**.** Let Y be an irreducible smooth variety

- Let $\mathcal{Z}^p_{eq}(Y, n)$ denote the free abelian group generated by irreducible closed subvarieties $Z \subset Y \times \square^n$ such that for any faces F of $\square^n$, the intersection $Z \cap Y \times F$ is empty or the restriction of $p_1 : Y \times \square^n \longrightarrow Y$ to

$$Z \cap (Y \times F) \longrightarrow Y$$

is equidimensional of relative dimension $\dim(F) - p$.

- We say that elements of $\mathcal{Z}^p_{eq}(Y, n)$ are *equidimensional over Y with respect any faces* or simply *equidimensional*.
- Following the definition of $\mathcal{N}^k_Y(p)$, let $\mathcal{N}_Y^{eq,\,k}(p)$ denote

$$\mathcal{N}_Y^{eq,\,k}(p) = \mathcal{A}lt\left(\mathcal{Z}^p_{eq}(Y, 2p-k) \otimes \mathbb{Q}\right).$$

Definition 4.6. Let C be an element of $\mathcal{N}^\bullet_Y$ decomposed in terms of cycles as

$$C = \sum_{i \in I} q_i Z_i, \qquad q_i \in \mathbb{Q}$$

where I is a finite set and where the Z_i are irreducible closed subvarieties of $Y \times \square^{n_i}$ intersecting all the faces of $\square^{n_i}$ properly (that is in codimension p_i).

- The support of C is define has

$$\mathrm{Supp}(C) = \bigcup_i Z_i.$$

- For C in $\mathcal{N}_Y^{eq,\,k}(p)$, we will say that C has empty fiber at a point y in Y if for any i in I the fiber of $Z_i \longrightarrow Y$ at y is empty.

Proposition 4.7. *Let Y be an irreducible smooth variety.*

(1) *The differential ∂_Y on $\mathcal{N}^\bullet_Y$ induces a differential :*

$$\mathcal{N}_Y^{eq,\,k}(p) \xrightarrow{\partial_Y} \mathcal{N}_Y^{eq,\,k}(p)$$

which makes $\mathcal{N}_Y^{eq,\,\bullet}(p)$ into a sub-complex of $\mathcal{N}^\bullet_Y(p)$.

(2) *$\mathcal{N}_Y^{eq,\,\bullet} = \oplus_{p \geqslant 0} \mathcal{N}_Y^{eq,\,\bullet}(p)$ is a subalgebra (sub-cdga) of $\mathcal{N}^\bullet_Y$.*

(3) *Assume that Z or Z' has an empty fiber at a point y in Y. Then the fiber at y of $\overline{Z} \cdot \overline{Z'}$ is empty.*

Proof. As the generators of $\mathcal{Z}^p_{eq}(Y, 2p-k)$ are equidimensional over Y when intersected with any faces, they stay equidimensional over Y with respect to any faces when intersected with a codimension 1 face because a face intersected with a codimension 1 face is another face or the intersection is empty. This gives the first point.

Let Z (resp. Z') a generator of $\mathcal{Z}^p_{eq}(Y, 2p-k)$ (resp. $\mathcal{Z}^q_{eq}(Y, 2q-l)$) for p, q, k and l integers. By definition, for any face $F \subset \square^{2p-k}$ (resp. $F' \subset \square^{2q-l}$), the projection

$$p_1 : Z \cap (X \times F) \longrightarrow Y \qquad (\text{resp. } p_1 : Z' \cap (X \times F') \longrightarrow Y)$$

is equidimensional of relative dimension $\dim(F) - p$ (resp. $\dim(F') - q$) or the above intersections are empty.

Let F and F' be two faces has above and assume that none of the above intersections is empty. Then,

$$Z \times Z' \cap (Y \times Y \times F \times F') \subset Y \times Y \times \square^{2(p+q)-k-l}$$

is equidimensional over $Y \times Y$ of relative dimension $\dim(F) + \dim(F') - p - q$. In particular for any point x in the image of the diagonal $\Delta : Y \longrightarrow Y \times Y$, one has

$$\begin{aligned}\dim\left(Z \times Z' \cap (Y \times Y \times F \times F')_x\right) = \\ \dim\left(Z \times Z' \cap (\{x\} \times F \times F')\right) = \\ \dim(\{x\}) + \dim(F) + \dim(F') - p - q\end{aligned}$$

and $Z \times Z' \cap (\mathrm{im}(\Delta) \times F \times F')$ is equidimensional over Y of relative dimension $\dim(F) + \dim(F') - p - q$ by any of the two projections $Y \times Y \longrightarrow Y$. If either $Z \cap (Y \times F)$ or $Z' \cap (Y \times F')$ is empty then the intersection

$$Z \times Z' \cap (Y \times Y \times F \times F')$$

is empty and so is $Z \times Z' \cap (\mathrm{im}(\Delta) \times F \times F')$.

From this, we deduce that

$$(\Delta \times \mathrm{id}\,)^{-1}(Z \times Z') \simeq Z \times Z' \cap \left(\mathrm{im}(\Delta) \times \square^{2(p+q)-k-l}\right)$$

is equidimensional over Y with respect to any faces. Hence,

$$Z \cdot Z' = \mathcal{A}lt((\Delta \times \mathrm{id}\,)^{-1}(Z \times Z')) \in \mathcal{N}_Y^\bullet$$

and the product in $\mathcal{N}_Y^\bullet$ induces a cdga structure on $\mathcal{N}_Y^{eq,\bullet}$ which makes it into a sub-cdga.

Moreover, from the above computation, one see that if the fiber of Z is empty at a point y, then, denoting with a subscript y the various fibers at y, one has

$$(\Delta \times \mathrm{id}\,)^{-1}(Z \times Z')_y = Z \times Z' \cap (\{(y,y)\} \times \square^{2(p+q)-k-l}) = Z_y \times Z'_y = \emptyset.$$

The same holds if Z' is empty at y which gives the last point of the proposition. □

In order to compare situation in $\mathcal{N}_X^\bullet$ and in $\mathcal{N}_{\mathbb{A}^1}^\bullet$, we will use the following proposition.

Proposition 4.8. *Let Y_0 be an open dense subset of Y an irreducible smooth variety and let $j : Y_0 \longrightarrow Y$ the inclusion. Then the restriction of cycles from Y to Y_0 induces a morphism of cdga*

$$j^* : \mathcal{N}_Y^{eq,\bullet} \longrightarrow \mathcal{N}_{Y_0}^{eq,\bullet}.$$

Moreover, Let C be in $\mathcal{N}_{Y_0}^\bullet$ and be decomposed in terms of cycles as

$$C = \sum_{i \in I} q_i Z_i, \qquad q_i \in \mathbb{Q}$$

where I is a finite set. Assume that for any i, the Zariski closure $\overline{Z_i}$ of Z_i in $Y \times \square^{n_i}$ intersected with any face F of $\square^{n_i}$ is equidimensional over Y of relative dimension $\dim(F) - p_i$. Define C' as

$$C' = \sum_{i \in I} q_i \overline{Z_i},$$

then,

$$C' \in \mathcal{N}_Y^{eq,\bullet} \qquad \textit{and} \qquad C = j^*(C') \in \mathcal{N}_{Y_0}^{eq,\bullet}.$$

Proof. It is enough to prove the proposition for generators of $\mathcal{N}_Y^{eq,\bullet}$ and $\mathcal{N}_{Y_0}^{eq,\bullet}$.

Let Z (resp. Z') be an irreducible, closed subvariety of codimension p (resp. p') $Y \times \square^{2p-k}$ (resp. of $Y \times \square^{2p'-k'}$) such that for any face F (resp. F') of $\square^{2p-k}$ (resp. of $\square^{2p'-k'}$) the intersection

$$Z \cap (Y \times F) \qquad (\text{resp. } Z' \times (Y \times F'))$$

is equidimensional over Y of relative dimension $\dim(F) - p$ (resp. $\dim(F') - p'$).

Let Z_0 and Z'_0 be the intersections $Z \cap Y_0$ and $Z' \cap Y_0$. As, for any faces F of $\square^{2p-k}$

$$Z_0 \cap (Y_0 \times F) = (Z \cap (Y \times F)) \cap Y_0 \times \square^{2p-k}$$
$$\left(\text{resp. } Z'_0 \cap (Y_0 \times F) = (Z' \cap (Y \times F)) \cap Y_0 \times \square^{2p'-k'}\right),$$

Z_0 and Z'_0 are equidimensional with respect to any face over Y_0 with relative dimension $\dim(F) - p$ (resp. $\dim(F)'_p$). This also shows that j^* commutes with the

differential on $\mathcal{N}_Y^{eq,\bullet}$ and on $\mathcal{N}_{Y_0}^{eq,\bullet}$. In order to show that j^* commute with the product structure, it suffices to remark that

$$Z_0 \times Z_0' = (Z \times Z') \cap Y_0 \times Y_0' \times \square^{2(p+p')-k-k'} \subset Y \times Y \times \square^{2(p+p')-k-k'}.$$

Let C and C' be as in the proposition. The fact that C' is in $\mathcal{N}_Y^{eq,\bullet}$ follows directly from the definition. To prove that

$$C = j^*(C') \in \mathcal{N}_{Y_0}^{eq,\bullet},$$

we can assume that I contains only one element 1 and that $q_1 = 1$. Then it follows from the fact that $Z_1 = \overline{Z_1} \cap Y_0 \subset Y$.

□

Proposition 4.9 (multiplication and equimensionality)**.** *Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the multiplication map sending (x, y) to xy and let $\tau : \square^1 = \mathbb{P}^1 \setminus \{1\} \longrightarrow \mathbb{A}^1$ be the isomorphism sending the affine coordinates u to $\frac{1}{1-u}$. The map τ sends ∞ to 0, 0 to 1 and extends as a map from $\mathbb{P}^1$ to $\mathbb{P}^1$ sending 1 to ∞.*

Maps m and τ are in particular flat and equidimensional of relative dimension 1 and 0 respectively.

Consider the following commutative diagram for a positive integer n

$$\begin{array}{ccc}
\mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{(m\circ(\mathrm{id}_{\mathbb{A}^1} \times \tau))\times \mathrm{id}_{\square^n}} & \mathbb{A}^1 \times \square^n \\
\downarrow{\scriptstyle p_{\mathbb{A}^1 \times \square^1}} & & \downarrow{\scriptstyle p_{\mathbb{A}^1}} \\
\mathbb{A}^1 \times \square^1 & \xrightarrow{m\circ(\mathrm{id}_{\mathbb{A}^1} \times \tau)} & \mathbb{A}^1 \\
\downarrow{\scriptstyle p_{\mathbb{A}^1}} & & \\
\mathbb{A}^1 & &
\end{array}$$

In the following statement, p, k and n will denote positive integers subject to the relation $n = 2p - k$

- *the composition $\widetilde{m} = (m \circ (\mathrm{id}_{\mathbb{A}^1} \tau)) \times \mathrm{id}_{\square^n}$ induces a group morphism*

  $$\mathcal{Z}_{eq}^p(\mathbb{A}^1, n) \xrightarrow{\widetilde{m}^*} \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n)$$

  which extends into a morphism of complexes for any p

  $$\mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}(p) \xrightarrow{\widetilde{m}^*} \mathcal{N}_{\mathbb{A}^1 \times \square^1}^{eq,\bullet}(p).$$

- *Moreover, one has a natural morphism*

  $$h^p_{\mathbb{A}^1,n} : \mathcal{Z}_{eq}^p(\mathbb{A}^1 \times \square^1, n) \longrightarrow \mathcal{Z}_{eq}^p(\mathbb{A}^1, n+1)$$

  given by regrouping the $\square$'s factors.

- *The composition $\mu^* = h^p_{\mathbb{A}^1,n} \circ \widetilde{m}^*$ gives a morphism*

  $$\mu^* : \mathcal{N}_{\mathbb{A}^1}^{eq,\,k}(p) \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq,\,k-1}(p)$$

  sending equidimensional cycles with empty fiber at 0 to equidimensional cycles with empty fiber at 0.

- *Let $\theta : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the involution sending the natural affine coordinate t to $1 - t$. Twisting the multiplication $\widetilde{m}$ by θ via*

  $$\begin{array}{ccc}
  \mathbb{A}^1 \times \square^1 \times \square^n & \longrightarrow & \mathbb{A}^1 \times \square^n \\
  \downarrow{\scriptstyle \theta\times \mathrm{id}_{\square^{n+1}}} & & \uparrow{\scriptstyle \theta\times \mathrm{id}_{\square^n}} \\
  \mathbb{A}^1 \times \square^1 \times \square^n & \xrightarrow{\widetilde{m}} & \mathbb{A}^1 \times \square^n
  \end{array}$$

gives a morphism

$$\nu^* : \mathcal{N}^{eq,\,k}_{\mathbb{A}^1}(p) \longrightarrow \mathcal{N}^{eq,\,k-1}_{\mathbb{A}^1}(p)$$

sending equidimensional cycles with empty fiber at 1 *to equidimensional cycles with empty fiber at* 1.

Proof. It is enough to work with generators of $\mathcal{Z}^p_{eq}(\mathbb{A}^1, n)$. Let Z be an irreducible subvariety of $\mathbb{A}^1 \times \square^n$ such that for any faces F of $\square^n$, the first projection

$$p_{\mathbb{A}^1} : Z \cap (\mathbb{A}^1 \times F) \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension $\dim(F) - p$ or empty. Let F be a face of $\square^n$. First, We want to show that under the projection $\mathbb{A}^1 \times \square^1 \times \square^n \longrightarrow \mathbb{A}^1 \times \square^1$,

$$\widetilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional of relative dimension $\dim(F) - p$ or empty. This follows from the fact that $Z \cap (\mathbb{A}^1 \times F)$ is equidimensional over $\mathbb{A}^1$ and m is flat and equidimensional of relative dimension 1 (hence are $m \times \tau$ and $\widetilde{m}$). The map $\widetilde{m}$ is identity on the $\square^n$ factor, thus for $Z \subset \mathbb{A}^1 \times \square^n$ as above and a codimension 1 face F of $\square^n$, $\widetilde{m}^{-1}(Z)$ satisfies

$$\widetilde{m}^{-1}(Z) \cap (\mathbb{A}^1 \times \square^1 \times F) = \widetilde{m}^{-1}(Z \cap (\mathbb{A}^1 \times F))$$

which makes $\widetilde{m}^*$ into a morphism of complex.

Moreover, assuming that the fiber of Z at 0 is empty, as $\widetilde{m}$ restricted to

$$\{0\} \times \square^1 \times \square^n$$

factors trough the inclusion $\{0\} \times \square^n \longrightarrow \mathbb{A}^1 \times \square^n$, the intersection

$$\widetilde{m}^{-1}(Z) \cap (\{0\} \times \square^1 \times \square^n$$

is empty. Hence the fiber of $\widetilde{m}^{-1}(Z)$ over $\{0\} \times \square^1$ (resp. over $\{0\}$) by $p_{\mathbb{A}^1 \times \square^1}$ (resp. $p_{\mathbb{A}^1} \circ p_{\mathbb{A}^1 \times \square^1}$) is empty.

Now, let Z be an irreducible subvariety of $\mathbb{A}^1 \times \square^1 \times \square^n$ such that for any face F of $\square^n$

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional of relative dimension $\dim(F) - p$. Let F' be a face of

$$\square^{n+1} = \square^1 \times \square^n.$$

The face F' is either of the form $\square^1 \times F$ or of the form $\{\varepsilon\} \times F$ with F a face of $\square^n$ and $\varepsilon \in \{0, \infty\}$. If F' is of the first type, as

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1 \times \square^1$$

is equidimensional and as $\mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$ is equidimensional of relative dimension 1, the projection

$$Z \cap (\mathbb{A}^1 \times \square^1 \times F) \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p + 1 = \dim(F') - p.$$

If F' is of the second type, by symmetry of the role of 0 and ∞, we can assume that $\varepsilon = 0$. Then, the intersection

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F)$$

is nothing but the fiber of $Z \cap (\mathbb{A}^1 \times \square^1 \times F)$ over $\mathbb{A}^1 \times \{0\}$. Hence, it has pure dimension $\dim(F) - p + 1$.

Moreover, denoting with a subscript the fiber, the composition

$$Z \cap (\mathbb{A}^1 \times \{0\} \times F) = \big(Z \cap (\mathbb{A}^1 \times \square^1 \times F)\big)_{\mathbb{A}^1 \times \{0\}} \longrightarrow \mathbb{A}^1 \times \{0\} \longrightarrow \mathbb{A}^1$$

is equidimensional of relative dimension

$$\dim(F) - p = \dim(F') - p.$$

This shows that $h^p_{\mathbb{A}^1,n}$ gives a well define morphism and that it preserves the fiber at a point x in $\mathbb{A}^1$; in particular if Z has an empty fiber at 0, so does $h^p_{\mathbb{A}^1,n}(Z)$.

Finally, the last part of the proposition is deduced from the fact that θ exchanges the role of 0 and 1. □

Remark 4.10. We have remarked that $\widetilde{m}$ sends cycles with empty fiber at 0 to cycles with empty fiber at any point in $\{0\} \times \square^1$. Similarly $\widetilde{m}$ sends cycles with empty fiber at 0 to cycles that also have an empty fiber at any point in $\mathbb{A}^1 \times \{\infty\}$.

From the proof of Levine's Proposition 4.2 in [Lev94], we deduce that μ^* gives a homotopy between $p_0^* \circ i_0^*$ and id where i_0 is the zero section $\{0\} \to \mathbb{A}^1$ and p_0 the projection onto the point $\{0\}$.

Proposition 4.11. *Notations are the ones from Proposition 4.9 above. Let i_0 (resp. i_1) be the inclusion of 0 (resp. 1) in $\mathbb{A}^1$.*

$$i_0 : \{0\} \longrightarrow \mathbb{A}^1 \qquad i_1 : \{1\} \longrightarrow \mathbb{A}^1$$

and let p_0 and p_1 be the corresponding projections $p_\varepsilon : \mathbb{A}^1 \longrightarrow \{\varepsilon\}$ for $\varepsilon = 0, 1$.

Then, μ^* provides a homotopy between

$$p_0^* \circ i_0^* \text{ and } \mathrm{id} : \mathcal{N}^{eq,\bullet}_{\mathbb{A}^1} \longrightarrow \mathcal{N}^{eq,\bullet}_{\mathbb{A}^1}$$

and similarly ν^* provides a homotopy between

$$p_1^* \circ i_1^* \text{ and } \mathrm{id} : \mathcal{N}^{eq,\bullet}_{\mathbb{A}^1} \longrightarrow \mathcal{N}^{eq,\bullet}_{\mathbb{A}^1}.$$

In other words, one has

$$\partial_{\mathbb{A}^1} \circ \mu^* + \mu^* \circ \partial_{\mathbb{A}^1} = \mathrm{id} - p_0^* \circ i_0^* \quad \text{and} \quad \partial_{\mathbb{A}^1} \circ \nu^* + \nu^* \circ \partial_{\mathbb{A}^1} = \mathrm{id} - p_1^* \circ i_1^*$$

The proposition follows from commuting the different compositions involved and the relation between the differential on $\mathcal{N}^{eq,\bullet}_{\mathbb{A}^1 \times \square^1}$ and the one on $\mathcal{N}^{eq,\bullet}_{\mathbb{A}^1}$ via the map $h^p_{\mathbb{A}^1,n}$.

Proof. We denote by $i_{0,\square}$ and $i_{\infty,\square}$ the zero section and the infinity section $\mathbb{A}^1 \longrightarrow \mathbb{A}^1 \times \square^1$. The action of θ only exchanges the role of 0 and 1 in $\mathbb{A}^1$, hence it is enough to prove the statement for μ^*. As previously, in order to obtain the proposition for $\mathcal{N}^{eq,k}_{\mathbb{A}^1}(p)$, it is enough to work on the generators of $\mathcal{Z}^p_{eq}(\mathbb{A}^1, n)$ with $n = 2p - k$.

By the previous proposition 4.9, $\widetilde{m}^*$ commutes with the differential on $\mathcal{Z}^p_{eq}(\mathbb{A}^1, \bullet)$ and on $\mathcal{Z}^p_{eq}(\mathbb{A}^1 \times \square^1, \bullet)$. As the morphism μ^* is defined by $\mu^* = h^p_{\mathbb{A}^1,n} \circ \widetilde{m}^*$, the proof relies on computing $\partial_{\mathbb{A}^1} \circ h^p_{\mathbb{A}^1,n}$. Let Z be a generator of $\mathcal{Z}^p_{eq}(\mathbb{A}^1 \times \square^1, n)$. In particular,

$$Z \subset \mathbb{A}^1 \times \square^1 \times \square^n$$

and $h^p_{\mathbb{A}^1,n}(Z)$ is also given by Z but viewed in

$$\mathbb{A}^1 \times \square^{n+1}.$$

The differentials denoted by $\partial^{n+1}_{\mathbb{A}^1}$ on $\mathcal{Z}^p_{eq}(\mathbb{A}^1, n+1)$ and $\partial^n_{\mathbb{A}^1 \times \square^1}$ on $\mathcal{Z}^p_{eq}(\mathbb{A}^1 \times \square^1, n)$ are both given by intersections with the codimension 1 faces but the first $\square^1$ factor in $\square^{n+1}$ gives two more faces and introduces a change of sign. Namely, using an

extra subscript to indicate in which cycle groups the intersections take place, one has

$$\begin{aligned}
\partial^{n+1}_{\mathbb{A}^1}(h^p_{\mathbb{A}^1,n}(Z)) =& \sum_{i=1}^{n+1}(-1)^{i-1}\left(\partial^0_{i,\mathbb{A}^1}(Z) - \partial^\infty_{i,\mathbb{A}^1}(Z)\right)\\
=&\partial^0_{1,A^1}(Z) - \partial^\infty_{1,\mathbb{A}^1}(Z) - \sum_{i=2}^{n+1}(-1)^{i-2}\left(\partial^0_{i,\mathbb{A}^1}(Z) - \partial^\infty_{i,\mathbb{A}^1}(Z)\right)\\
=&i^*_{0,\square}(Z) - i^*_{\infty,\square}(Z) - \sum_{i=1}^{n}(-1)^{i-1}\left(\partial^0_{i+1,\mathbb{A}^1}(Z) - \partial^\infty_{i+1,\mathbb{A}^1}(Z)\right)\\
=&i^*_{0,\square}(Z) - i^*_{\infty,\square}(Z)\\
&- \sum_{i=1}^{n}(-1)^{i-1}\left(h^p_{\mathbb{A}^1,n-1}\circ\partial^0_{i,\mathbb{A}^1\times\square^1}(Z) - h^p_{\mathbb{A}^1,n-1}\circ\partial^\infty_{i,\mathbb{A}^1\times\square^1}(Z)\right)\\
=&i^*_{0,\square}(Z) - i^*_{\infty,\square}(Z) - h^p_{\mathbb{A}^1,n-1}\circ\partial^n_{\mathbb{A}^1\times\square^1}(Z).
\end{aligned}$$

Thus, one can compute $\partial_{\mathbb{A}^1}\circ\mu^* + \mu^*\circ\partial_{\mathbb{A}^1}$ on $\mathcal{Z}^p_{eq}(\mathbb{A}^1, n)$ as

$$\begin{aligned}
\partial_{\mathbb{A}^1}\circ\mu^* + \mu^*\circ\partial_{\mathbb{A}^1} =&\partial_{\mathbb{A}^1}\circ h_{\mathbb{A}^1,n}\circ\widetilde{m}^* + h_{\mathbb{A}^1,n-1}\circ\widetilde{m}^*\circ\partial_{\mathbb{A}^1}\\
=&i^*_{0,\square}\circ\widetilde{m}^* - i^*_{\infty,\square}\circ\widetilde{m}^* - h_{\mathbb{A}^1,n-1}\circ\partial_{\mathbb{A}^1}\circ\widetilde{m}^*\\
&+ h_{\mathbb{A}^1,n-1}\circ\partial_{\mathbb{A}^1}\circ\widetilde{m}^*\\
=&i^*_{0,\square}\circ\widetilde{m}^* - i^*_{\infty,\square}\circ\widetilde{m}^*.
\end{aligned}$$

The morphism $i^*_{\infty,\square}\circ\widetilde{m}^*$ is induced by

$$\begin{array}{ccccccc}
\mathbb{A}^1 & \xrightarrow{i_{\infty,\square}} & \mathbb{A}^1\times\square^1 & \xrightarrow{\tau} & \mathbb{A}^1\times\mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1\\
x & \longmapsto & (x,\infty) & \longmapsto & (x,0) & \longmapsto & 0
\end{array}$$

which factors through

$$\begin{array}{ccccccc}
\mathbb{A}^1 & \xrightarrow{i_{\infty,\square}} & \mathbb{A}^1\times\square^1 & \xrightarrow{\tau} & \mathbb{A}^1\times\mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1\\
\downarrow{\scriptstyle p_0} & & & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{A}^1}}\\
\mathbb{A}^1 & & & \xrightarrow{\qquad i_0\qquad} & & & \mathbb{A}^1
\end{array}$$

Thus,

$$i^*_{\infty,\square}\circ\widetilde{m}^* = (i_0\circ p_0)^* = p_0^*\circ i_0^*.$$

Similarly $i^*_{0,\square}\circ\widetilde{m}^*$ is induced by

$$\begin{array}{ccccccc}
\mathbb{A}^1 & \xrightarrow{i_{\infty,\square}} & \mathbb{A}^1\times\square^1 & \xrightarrow{\tau} & \mathbb{A}^1\times\mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1\\
x & \longmapsto & (x,0) & \longmapsto & (x,1) & \longmapsto & x
\end{array}$$

which factors through $\mathrm{id}_{\mathbb{A}^1} : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ and one has

$$i^*_{0,\square}\circ\widetilde{m}^* = \mathrm{id}$$

which concludes the proof of the proposition. □

4.3. **Cycles over $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ corresponding to multiple polylogarithms.** Set $\mathcal{L}_0 = L_0$ and $\mathcal{L}_1 = L_1$ where L_0 and L_1 are the cycles in $\mathcal{N}^1_X(1)$ defined in Section 2.5 induced by the graph or $t \mapsto t$ and $t \mapsto 1-t$ from $X \longrightarrow \mathbb{P}^1$.

Consider the two following differential systems

$$\text{(ED-}\mathcal{L}\text{)} \qquad \partial(\mathcal{L}_W) = \sum_{U<V} a^W_{U,V} \mathcal{L}_U \mathcal{L}_V + \sum_{U,V} b^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V$$

and

$$\text{(ED-}\mathcal{L}^1\text{)} \qquad \partial(\mathcal{L}^1_W) = \sum_{0<U<V} a'^W_{U,V} \mathcal{L}^1_U \mathcal{L}^1_V + \sum_{U,V} b'^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V + \sum_V a'_{0,V} \mathcal{L}_0 \mathcal{L}_V$$

where coefficients $a^W_{U,V}$, $b^W_{U,V}$, $a'^W_{U,V}$ and $b'^W_{U,V}$ are the ones defined at Definition 4.1.

These differential equations are exactly the differential system considered in section 4.1.

Theorem 4.12. *Let j be the inclusion $X \hookrightarrow \mathbb{A}^1$. For any Lyndon word of length p greater or equal to 2, there exists two cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ in $\mathcal{N}^1_X(p)$ such that :*

- *$\mathcal{L}_W$, $\mathcal{L}^1_W$ are elements in $\mathcal{N}^{eq,\,1}_X(p)$.*
- *There exists cycles $\overline{\mathcal{L}_W}$, $\overline{\mathcal{L}^1_W}$ in $\mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(p)$ such that*

$$\mathcal{L}_W = j^*(\overline{\mathcal{L}_W}) \qquad \text{and} \qquad \mathcal{L}^1_W = j^*(\overline{\mathcal{L}^1_W}).$$

- *The restriction of $\overline{\mathcal{L}_W}$ (resp. $\overline{\mathcal{L}^1}$) to the fiber $t = 0$ (resp. $t = 1$) is empty.*
- *The cycle $\mathcal{L}_W$ (resp. $\mathcal{L}^1_W$) satisfies the equation* (ED-$\mathcal{L}$) *(resp* (ED-$\mathcal{L}^1$)*) in $\mathcal{N}^\bullet_X$ and the same holds for its extension $\overline{\mathcal{L}_W}$ (resp. $\overline{\mathcal{L}^1_W}$) to $\mathcal{N}^{eq,\,\bullet}_{\mathbb{A}^1}$.*

The rest of the section is essentially devoted into proving the above theorem. Let $A_{\mathcal{L}}$ and $A_{\mathcal{L}^1}$ denote the R.H.S of (ED-$\mathcal{L}$) and (ED-$\mathcal{L}^1$) respectively. The proof works by induction and will be developed as follows

- Reviewing the cycles $\mathcal{L}_{01}$ and $\mathcal{L}^1_{01}$ presented in subsection 2.5 in order to show that they gives the desired cycles for $W = 01$.
- Proving that $A_{\mathcal{L}}$ and $A_{\mathcal{L}^1}$ have differential 0 in $\mathcal{N}^\bullet_X$. This has essentially been proved in Proposition 4.3.
- Extending $A_{\mathcal{L}}$ and $A_{\mathcal{L}^1}$ to $\mathbb{A}^1$ and proving in Lemma 4.14 that the differential stay 0 in $\mathcal{N}^\bullet_{\mathbb{A}^1}$.
- Finally constructing $\mathcal{L}_W$ and $\mathcal{L}^1_W$ by pull-back by the multiplication and pull-back by the twisted multiplication at Lemma 4.15.
- Proving that the pull-back by the (twisted) multiplication preserves the equidimensionality property and has empty fiber at $t = 0$ (resp. $t = 1$) was done at Proposition 4.9.
- Showing that $\mathcal{L}_W$ and $\mathcal{L}^1_W$ satisfy the expected differential equations follows from the homotopy property of the (twisted) multiplication given in Proposition 4.11.

Proof. We initiate the induction with the only Lyndon word of length 2: $W = 01$.

Example 4.13. In Section 2.5, we have already considered the product

$$b = \mathcal{L}_0 \mathcal{L}_1 = [t; t, 1-t]. \subset X \times \square^2.$$

In other word, b is, up to projection on the alternating elements, nothing but the graph of the function $X \longrightarrow (\mathbb{P}^1)^2$ sending t to $(t, 1-t)$. Its closure $\overline{b}$ in $\mathbb{A}^1 \times \square^2$ is induced by the graph of $t \mapsto (t, 1-t)$ viewed as a function from $\mathbb{A}^1$ to $(\mathbb{P}^1)^2$:

$$\overline{b} = [t; t, 1-t] \qquad \subset \mathbb{A}^1 \times \square^2.$$

From this expression, one sees that $\partial_{\mathbb{A}^1}(\overline{b}) = 0$.

Proposition 4.7 already insure that b is equidimensional over X as it is the case for both $\mathcal{L}_0$ and $\mathcal{L}_1$. Then, in order to show that $\overline{b}$ is equidimensional over $\mathbb{A}^1$, it is enough to look the fiber over 0 and 1. In both case, the fiber is empty and $\overline{b}$ is equidimensional over $\mathbb{A}^1$. Now, set

$$\overline{\mathcal{L}_{01}} = \mu^*(\overline{b}) \quad \text{and} \quad \overline{\mathcal{L}_{01}} = \nu^*(\overline{b})$$

where μ^* and ν^* are defined as in Proposition 4.9. The same proposition shows that $\overline{\mathcal{L}_{01}}$ and $\overline{\mathcal{L}^1_{01}}$ are equidimensional over $\mathbb{A}^1$ and more precisely elements of $\mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(2)$.

The fibers at 0 and 1 of $\overline{b}$ being empty and as $\partial_{\mathbb{A}^1}(\overline{b}) = 0$, one conclude from Proposition 4.11 that

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_{01}}) = \partial_{\mathbb{A}^1}(\overline{\mathcal{L}^1_{01}}) = \overline{b}.$$

Finally, we define

$$\mathcal{L}_{01} = j^*(\overline{\mathcal{L}_{01}}) \quad \text{and} \quad \mathcal{L}^1_{01} = j^*(\overline{\mathcal{L}^1_{01}})$$

where j is the inclusion $X \longrightarrow \mathbb{A}^1$ and conclude using Proposition 4.8.

One can explicitly compute the two pull-backs and obtain a parametric representation

$$\mathcal{L}_{01} = [t; 1 - \frac{t}{x}, x, 1 - x], \qquad \mathcal{L}^1_{01} = [t; \frac{x - t}{x - 1}, x, 1 - x].$$

In order to compute the pull-back, one should remark that if $u = 1 - t/x$ then

$$\frac{t}{1 - u} = x.$$

Computing the pull-back by μ^*, is then just rescaling the new $\square^1$ factor which arrives in first position. The case of ν^* is similar but using the fact that for $u = \frac{x-t}{x-1}$ one has

$$\frac{t - u}{1 - u} = x.$$

Let W be a Lyndon word of length p greater or equal to 3. For now on, we assume that Theorem 4.12 holds for any word of length strictly less p. We set

$$A_{\mathcal{L}} = \sum_{U<V} a^W_{U,V} \mathcal{L}_U \mathcal{L}_V + \sum_{U,V} b^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V,$$

and

$$A_{\mathcal{L}^1} = \sum_{U<V} {a'}^W_{U,V} \mathcal{L}^1_U \mathcal{L}^1_V + \sum_{U,V} {b'}^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V,$$

Remark 4.2 shows that $A_{\mathcal{L}}$ and $A_{\mathcal{L}^1}$ only involved Lyndon words U and V such that the sum of the length of U and the length of V equal the one of W; in particular the various coefficients are 0 as soon as U or V has length greater or equal to W.

In order to apply the general strategy detailed in Section 2.6.1, we need first to show $\partial(A_{\mathcal{L}}) = \partial(A_{\mathcal{L}^1}) = 0$.

The induction hypothesis gives the existence of $\mathcal{L}_U$ and $\mathcal{L}^1_V$ for any U and V of smaller length, and by definition $\partial(\mathcal{L}_0) = \partial(Lc_1) = 0$. So the combinatorial Proposition 4.3 shows that

$$\partial(A_{\mathcal{L}}) = \partial(A_{\mathcal{L}^1}) = 0.$$

Lemma 4.14 (extension to $\mathbb{A}^1$)**.** *Let $\overline{A_{\mathcal{L}}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) denotes the algebraic cycles in $\mathcal{Z}(\mathbb{A}^1 \times \square^{2p-2})$ obtained by taking the Zariski closure in $\mathbb{A}^1 \times \square^{2p-2}$ of each term in the formal sum defining $A_{\mathcal{L}}$ (resp. $A_{\mathcal{L}^1}$). Then*

- *$\overline{A_{\mathcal{L}}}$ and $\overline{A_{\mathcal{L}^1}}$ are equidimensional over $\mathbb{A}^1$ with respect to any faces of $\square^{2p-2}$; that is $\overline{A_{\mathcal{L}}}$ and $\overline{A_{\mathcal{L}^1}}$ are in $\mathcal{N}^{eq,\,2}_{\mathbb{A}^1}(p)$.*
- *$A_{\mathcal{L}}$ has empty fiber at 0 and $A_{\mathcal{L}^1}$ has empty fiber at 1.*
- *$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}}) = \partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}}) = 0$*

Proof. Cases of $A_{\mathcal{L}}$ and $A_{\mathcal{L}^1}$ are very similar, thus we will only discuss the case of $A_{\mathcal{L}}$.

Let $\mathcal{L}_U$ and $\mathcal{L}_V$ for U and V Lyndon words different from 0 and 1 of respective length q and q' smaller than p the length of W. Note that in $A_{\mathcal{L}}$ only appears U and V such that $q + q' = p$.

Induction hypothesis tells us that $\mathcal{L}_U$ (resp. $\mathcal{L}^1_U$) and $\mathcal{L}_V$ (resp. $\mathcal{L}^1_V$) extend to equidimensional cycles over $\mathbb{A}^1$ with respect to any faces by taking the Zariski closure in $\mathbb{A}^1 \times \square^\bullet$ of each term of their defining sums; that is

$$\overline{\mathcal{L}_U}, \overline{\mathcal{L}^1_U} \in \mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(q) \quad \text{and} \quad \overline{\mathcal{L}_V}, \overline{\mathcal{L}^1_V} \in \mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(q').$$

Thus, Proposition 4.8 insures that

$$\overline{\mathcal{L}_U \cdot \mathcal{L}_V} = \overline{\mathcal{L}_U} \cdot \overline{\mathcal{L}_V} \qquad \text{and} \qquad \overline{\mathcal{L}_U \cdot \mathcal{L}^1_V} = \overline{\mathcal{L}_U} \cdot \overline{\mathcal{L}^1_V} \in \mathcal{N}^{eq,\,2}_{\mathbb{A}^1}(p).$$

and that the above products have empty fiber at 0 because it is the case for $\overline{\mathcal{L}_U}$.

In order to show that $A_{\mathcal{L}}$ extend in an equidimensional cycle over $\mathbb{A}^1$, it is now enough to study the products $\mathcal{L}_0 \cdot \mathcal{L}_U$ and $\mathcal{L}_1 \cdot \mathcal{L}^1_V$ as thanks to Lemma 3.49 those are the only types of product involving $\mathcal{L}_0$ and $\mathcal{L}_1$ which are not equidimensional over $\mathbb{A}^1$.

The Zariski closure $\overline{\mathcal{L}_0}$ of $\mathcal{L}_0$ in $\mathbb{A}^1 \times \square^1$ is not equidimensional with respect to all the face as in particular

$$\overline{\mathcal{L}_0} \cap (\mathbb{A}^1 \times \{0\}) \longrightarrow \mathbb{A}^1$$

is not dominant. However, $\overline{\mathcal{L}_0}$ is well defined in $\mathcal{N}^1_{\mathbb{A}^1}(1)$ even if it is not equidimensional over $\mathbb{A}^1$ and 0 is the only problematic point. In the other hand $\overline{\mathcal{L}_U}$ is by the induction hypothesis empty at 0. This remark allows us to shows that $\overline{\mathcal{L}_0} \cdot \overline{\mathcal{L}_U}$ is equidimensional over $\mathbb{A}^1$ and have empty fiber at 0.

Now, let $\overline{\mathcal{L}_0}^{\setminus 0}$ and $\overline{\mathcal{L}_U}^{\setminus 0}$ denote respectively the Zariski closure of $\mathcal{L}_0$ and $\mathcal{L}_U$ in $\mathbb{A}^1 \setminus \{0\}$. Let Z be an irreducible component of $\mathrm{Supp}(\mathcal{L}_U)$ and let $\overline{Z}$ (resp. $\overline{Z}^{\setminus 0}$) denotes the Zariski closure of Z in

$$\mathbb{A}^1 \times \square^{2p-1} \qquad (\text{resp. in } \mathbb{A}^1 \setminus \{0\} \times \square^{2p-1}).$$

$\overline{Z}$ (resp. $\overline{Z}^{\setminus 0}$) is then an irreducible component of $\mathrm{Supp}(\overline{\mathcal{L}_U})$ (resp. $\mathrm{Supp}(\overline{\mathcal{L}_U}^{\setminus 0})$) and all irreducible components of $\mathrm{Supp}(\overline{\mathcal{L}_U})$ (resp. $\mathrm{Supp}(\overline{\mathcal{L}_U}^{\setminus 0})$) are of this type.

Let Γ denote the graph of $\mathrm{id} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Then one has

$$\mathcal{L}_0 = \mathcal{A}lt(\Gamma|_{X\times X}), \quad \overline{\mathcal{L}_0}^{\setminus 0} = \mathcal{A}lt(\Gamma|_{\mathbb{A}^1\setminus\{0\}\times\mathbb{A}^1\setminus\{0\}}) \quad \text{and} \quad \overline{\mathcal{L}_0} = \mathcal{A}lt(\Gamma|_{\mathbb{A}^1\times\mathbb{A}^1}).$$

We will write simply Γ_X, $\Gamma_{\mathbb{A}^1\setminus\{0\}}$ and $\Gamma_{\mathbb{A}^1}$ for the restriction of Γ to respectively $X \times X$, $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$ and $\mathbb{A}^1 \times \mathbb{A}^1$.

It is enough to show that $\Gamma_{\mathbb{A}^1} \cdot \overline{Z}$ is equidimensional over $\mathbb{A}^1$ (here $\cdot$ denotes the product in $\mathcal{N}^\bullet_{\mathbb{A}^1}$).

By the induction hypothesis, $\overline{Z}$ is equidimensional with respect to any faces over $\mathbb{A}^1$; in particular Z (resp. $\overline{Z}^{\setminus 0}$) is equidimensional with respect to any faces over X (resp. $\mathbb{A}^1 \setminus \{0\}$). Thus, for any face F of $\square^1$ and any face F' of $\square^{2(p-1)-1}$ one has

$$\Gamma_X \cap (X \times F) \quad \text{is equidimensional over } X$$

$$\big(\text{resp. } \Gamma_{\mathbb{A}^1\setminus\{0\}} \cap \big((\mathbb{A}^1 \setminus \{0\}) \times F\big) \quad \text{is equidimensional over } \mathbb{A}^1 \setminus \{0\}\big)$$

and

$$Z \cap (X \times F') \quad \text{is equidimensional over } X$$

$$\Big(\text{resp. } \overline{Z}^{\setminus 0} \cap \big((\mathbb{A}^1 \setminus \{0\}) \times F'\big) \quad \text{is equidimensional over } \mathbb{A}^1 \setminus \{0\}\Big).$$

Hence, the intersections

$$(\Gamma_X \times Z) \cap (X \times X \times F \times F') \subset X \times X \times \square^{2p-2}$$

and
$$\left(\Gamma_{\mathbb{A}^1\setminus\{0\}} \times \overline{Z}^{\setminus 0}\right) \cap \left(\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\} \times F \times F'\right) \subset \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\} \times \square^{2p-2}$$
are respectively equidimensional over $X \times X$ and over $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$, or empty.

In particular, in the case where the intersection is non-empty,
$$\left(\Gamma_{\mathbb{A}^1\setminus\{0\}} \times \overline{Z}^{\setminus 0}\right) \cap \left(\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\} \times F \times F'\right) \longrightarrow \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$$
is equidimensional of relative dimension $\dim(F) + \dim(F') - p - 2$.

Let x be a point of $\mathrm{im}(\Delta)$. Then, either x is a point of $\mathrm{im}(\Delta_{\mathbb{A}^1\setminus\{0\}})$ and as a point in $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$, the equidimensionality shows that if the intersection below is not empty one has
$$\dim\left(\left(\Gamma_{\mathbb{A}^1} \times \overline{Z}\right) \cap (x \times F \times F'))\right) =$$
$$\dim\left(\left(\Gamma_{\mathbb{A}^1\setminus\{0\}} \times \overline{Z}^{\setminus 0}\right) \cap (x \times F \times F'))\right) = \dim(x) + \dim(F) + \dim(F') - p - 2.$$
Or x is the point $(0,0)$ and, writing Γ_0 (resp. $\overline{Z}_0$) the fiber at 0 under the first projection of $\mathbb{A}^1 \times \square^1$ (resp. $\mathbb{A}^1 \times \square^{2(p-1)-1}$), one has
$$\left(\Gamma_{\mathbb{A}^1} \times \overline{Z}\right) \cap (\{(0,0)\} \times F \times F') = \Gamma_0 \times \overline{Z}_0 \cap (\{(0,0)\} \times F \times F') = \emptyset$$
because by induction hypothesis $\overline{Z}_0 = \emptyset$.

From the above discussion, we obtain that the intersection of
$$(\Delta \times \mathrm{id}\,)^{-1}\left(\Gamma \times \overline{Z}\right) \simeq \Gamma \times \overline{Z} \cap \left(\mathrm{im}(\Delta) \times \square^{2p-2}\right)$$
with any faces is equidimensional over $\mathbb{A}^1$ and that
$$\overline{\mathcal{L}_0} \cdot \overline{Z} = \mathcal{A}lt\left((\Delta \times \mathrm{id}\,)^{-1}(\Gamma \times Z)\right)$$
is equidimensional over $\mathbb{A}^1$ with respect to any faces of $\square^{2p-2}$. Moreover $\overline{\mathcal{L}_0} \cdot \overline{Z}$ has an empty fiber at 0.

Thus, $\overline{\mathcal{L}_0} \cdot \overline{\mathcal{L}_U}$ is equidimensional with respect to any faces and has empty fiber at 0. A similar argument (using 1 instead of 0 and using the fact that $\overline{\mathcal{L}^1_V}$ is empty at 1) shows that $\overline{\mathcal{L}_1} \cdot \overline{\mathcal{L}^1_V}$ is equidimensional over $\mathbb{A}^1$ and has empty fiber at 1.

Now, we need to show that
$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}}) = 0.$$
By induction, terms of the form $\overline{\mathcal{L}_U}$ (resp. $\overline{\mathcal{L}^1_V}$) satisfy the differential system (ED-$\mathcal{L}$) (resp. (ED-$\mathcal{L}^1$)) provide the length of U (resp. V) is greater than 2. Hence, In order to show that $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}}) = 0$, it is enough to show that
$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_0} \cdot \overline{\mathcal{L}_U}) = -\overline{\mathcal{L}_0} \cdot \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_U}) \qquad \text{and} \qquad \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_1} \cdot \overline{\mathcal{L}^1_V}) = -\overline{\mathcal{L}_1} \cdot \partial_{\mathbb{A}^1}(\overline{\mathcal{L}^1_V}).$$
All terms involved in $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}})$ will then satisfy exactly the same differential equations as the ones involved in $\partial(A_{\mathcal{L}}) = 0$. Thus, in order to show that $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}}) = 0$, it will be enough to apply the same computations used to prove that $\partial(A_{\mathcal{L}}) = 0$ (Proposition 4.3).

As previously said, even if $\overline{\mathcal{L}_0}$ is not equidimensional over $\mathbb{A}^1$ it is a well defined element of $\mathcal{N}^1_{\mathbb{A}^1}(1)$ and one has
$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_0}) = [0;0] \qquad \subset \mathbb{A}^1 \times \square^0$$
which is of codimension 1. The differential graded algebra structure on $\mathcal{N}^\bullet_{\mathbb{A}^1}$ shows that
$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_0} \cdot \overline{\mathcal{L}_U}) = [0;0] \cdot \overline{\mathcal{L}_U} - \overline{\mathcal{L}_0} \cdot \partial_{\mathbb{A}^1}(\overline{\mathcal{L}_U}).$$
The product $[0;0]\cdot\overline{\mathcal{L}_U}$ is obtained from the product $[0;0]\cdot\overline{Z}$ where Z is an irreducible component of $\mathrm{Supp}(\mathcal{L}_U)$. The previous computations show that
$$[0;0] \cdot \overline{Z} = \Delta^{-1}(\Gamma_0 \times Z_0) = \emptyset$$

The above equality insures that as a cycle $[0;0]\cdot\overline{\mathcal{L}_U}=0$ and that

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_0}\cdot\overline{\mathcal{L}_U})=-\overline{\mathcal{L}_0}\cdot\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_U});$$

similarly,

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_1}\cdot\overline{\mathcal{L}^1_V})=-\overline{\mathcal{L}_1}\cdot\partial_{\mathbb{A}^1}(\overline{\mathcal{L}^1_V}).$$

Thus, we have obtained that

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}})=0.$$

A similar discussion shows that

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}})=0.$$

□

The equality

$$\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}})=0\qquad\big(\text{resp. }\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}})=0\big)$$

shows that $\overline{A_{\mathcal{L}}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) gives a class in $\mathrm{H}^2(\mathcal{N}^\bullet_{\mathbb{A}^1})$. As Corollary 5 insures that this cohomology group is 0, $\overline{A_{\mathcal{L}}}$ (resp. $\overline{A_{\mathcal{L}^1}}$) is the boundary of some cycle c (resp. c') in $\mathcal{N}^1_{\mathbb{A}^1}$. Lemma 4.15 below gives this c (resp. c') explicitly and, after restriction to X, concludes the proof of Theorem 4.12.

□

Lemma 4.15. *Define $\overline{\mathcal{L}_W}$ and $\overline{\mathcal{L}^1_W}$ in $\mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(p)$ by*

$$\overline{\mathcal{L}_W}=\mu^*(\overline{A_{\mathcal{L}}})\quad\text{and}\quad\overline{\mathcal{L}^1_W}=\nu^*(\overline{A_{\mathcal{L}^1}})$$

where μ^* and ν are the morphisms defined in Proposition 4.9.

Let $j:X\longrightarrow\mathbb{A}^1$ be the natural inclusion of $\mathbb{P}^1\setminus\{0,1,\infty\}$ into $\mathbb{A}^1$ and define $\mathcal{L}_W$ and $\mathcal{L}^1_W$ by

$$\mathcal{L}_W=j^*(\overline{\mathcal{L}_W})\quad\text{and}\quad\mathcal{L}^1_W=j^*(\overline{\mathcal{L}^1_W}).$$

Then $\mathcal{L}_W$ and $\mathcal{L}^1_W$ satisfy conditions of Theorem 4.12.

Proof. As in Proposition 4.11, let i_0 (resp. i_1) be the inclusion of 0 (resp. 1) in $\mathbb{A}^1$:

$$i_0:\{0\}\longrightarrow\mathbb{A}^1\qquad i_1:\{1\}\longrightarrow\mathbb{A}^1,$$

and let p_0 and p_1 be the corresponding projection $p_\varepsilon:\mathbb{A}^1\longrightarrow\{\varepsilon\}$ for $\varepsilon=0,1$.

Proposition 4.9 insures that $\overline{\mathcal{L}_W}$ (resp. $\overline{\mathcal{L}^1_W}$) is equidimensional over $\mathbb{A}^1$ with respect to any faces and has an empty fiber at $t=0$ (resp. $t=1$); in particular $i_0^*(\overline{A_{\mathcal{L}}})=i_1^*(\overline{A_{\mathcal{L}^1}})=0$. Moreover, Proposition 4.11 allows to compute $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W})$ as

$$\begin{aligned}\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W})=&\partial_{\mathbb{A}^1}\circ\mu^*(\overline{A_{\mathcal{L}}})\\=&\,\mathrm{id}\,(\overline{A_{\mathcal{L}}})-p_0^*\circ i_0^*(\overline{A_{\mathcal{L}}})-\mu^*\circ\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}})\\=&\overline{A_{\mathcal{L}}}\end{aligned}$$

because $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}}})=0$ and $i_0^*(\overline{A_{\mathcal{L}}})=0$.

Using again Proposition 4.11, a similar computation gives

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}^1_W})=\overline{A_{\mathcal{L}^1}}$$

because $\partial_{\mathbb{A}^1}(\overline{A_{\mathcal{L}^1}})=0$ and $i_1^*(\overline{A_{\mathcal{L}^1}})=0$.

Now, as

$$\mathcal{L}_W=j^*(\overline{\mathcal{L}_W})\quad\text{and}\quad\mathcal{L}^1_W=j^*(\overline{\mathcal{L}^1_W}),$$

$\mathcal{L}_W$ and $\mathcal{L}^1_W$ are equidimensional with respect to any faces over X by Proposition 4.8 and their closure in $\mathbb{A}^1\times\square^{2p-1}$ are exactly $\overline{\mathcal{L}_W}$ and $\overline{\mathcal{L}^1_W}$. As j^* is a morphism of cdga, $\mathcal{L}_W$ and $\mathcal{L}^1_W$ satisfy the expected differential equations as do $\overline{\mathcal{L}_W}$ and $\overline{\mathcal{L}^1_W}$; that is

$$\partial(\mathcal{L}_W)=A_{\mathcal{L}}\qquad\text{and}\qquad\partial(\mathcal{L}^1_W)=A_{\mathcal{L}^1}.$$

This conclude the proof of the Lemma and of Theorem 4.12 □

5. Concluding remarks

5.1. **Some Examples up to weight 5.** We describe in this section some interesting examples of cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ for W up to length 5.

We have already seen in the previous section, at Example 4.13, the weight 2 examples $\mathcal{L}_{01}$ and $\mathcal{L}^1_{01}$. We recall below their parametric representations

$$\mathcal{L}_{01} = [t; 1 - \frac{t}{x}, x, 1-x], \qquad \mathcal{L}^1_{01} = [t; \frac{x-t}{x-1}, x, 1-x].$$

They satisfy

$$\partial(\mathcal{L}_{01}) = \partial(\mathcal{L}^1_{01}) = \mathcal{L}_0\mathcal{L}_1.$$

In Lemma 2.40, we have defined cycles Li^{cy}_k for any integers $k \geqslant 2$ with $\mathrm{Li}^{cy}_2 = L_{01} = \mathcal{L}_{01}$. Fix k an integer greater or equal to 2 and let W be the Lyndon word

$$W = \underbrace{0\cdots 0}_{k-1 \text{ times}} 1.$$

A simple induction and the construction of the cycle Li^{cy}_k show that

$$\mathrm{Li}^{cy}_k = \mathcal{L}_W.$$

We have previously considered a weight 3 example L_{011} in order to make more apparent where the different problems were. In particular, it satisfies

$$\partial(L_{011}) = (L_{01} - L_{01}(1))L_1.$$

However, the closure of L_{011} over $\mathbb{A}^1$ is not equidimensional as the fiber at 1 is not an admissible cycle. In the next example, we use Theorem 4.12 for the word 011 and give an explicit parametrized description of $\mathcal{L}_{011}$.

Example 5.1 (Weight 3 example)**.** The cycle $\overline{\mathcal{L}_{011}}$ in $\mathcal{N}^{eq,\,1}_{\mathbb{A}^1}(3)$ is defined by

$$\overline{\mathcal{L}_{011}} = \mu^*(-\overline{\mathcal{L}_1}\overline{\mathcal{L}^1_{01}}).$$

The product $\overline{\mathcal{L}_1}\overline{\mathcal{L}^1_{01}}$ is given in terms of parametrized cycle by

$$\overline{\mathcal{L}_1}\overline{\mathcal{L}^1_{01}} = [t; 1-t, \frac{x-t}{x-1}, x, 1-x].$$

Following the comment in Example 4.13, one computes easily the pull-back by μ^* and obtains after restriction to X (and renumbering x as x_1)

$$\mathcal{L}_{011} = -[t; 1 - \frac{t}{x_2}, 1-x_2, \frac{x_1-x_2}{x_1-1}, x_1, 1-x_1].$$

The cycle $\mathcal{L}^1_{011}$ satisfies the same differential equation as $\mathcal{L}_{011}$ but is given by the pull-back ν^*. Thus, a description of $\mathcal{L}^1_{011}$ as parametrized cycle is

$$\mathcal{L}^1_{011} = -[t; \frac{x_2-t}{x_2-1}, 1-x_2, \frac{x_1-x_2}{x_1-1}, x_1, 1-x_1].$$

Computing the differentials of $\mathcal{L}_{011}$ and $\mathcal{L}^1_{011}$ using the above expressions gives back

$$\partial(\mathcal{L}_{011}) = \partial(\mathcal{L}^1_{011}) = -\mathcal{L}_1\mathcal{L}^1_{01}.$$

In weight 4, arises the first linear combination in the differential equation. In weight 5 arises the first case where the differential equation for $\mathcal{L}_W$ and $\mathcal{L}^1_W$ are not the same. There are actually two such examples in weight 5.

Example 5.2. The cycle $\mathcal{L}_{0011}$ satisfies

$$\partial(\mathcal{L}_{0011}) = \mathcal{L}_0\mathcal{L}_{011} - \mathcal{L}_1\mathcal{L}^1_{001} - \mathcal{L}_{01}\mathcal{L}^1_{01}.$$

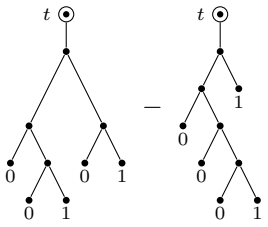
As $\mathcal{L}^1_{001}$ is the restriction of $\nu^*(\overline{\mathcal{L}_0\mathcal{L}_{01}})$, one gets

$$\mathcal{L}^1_{001} = [t; \frac{x_2 - t}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1]$$

and

$$\begin{aligned}(59)\quad \mathcal{L}_{0011} = &-[t; 1 - \frac{t}{x_3}, x_3, 1 - \frac{x_3}{x_2}, 1 - x_2, \frac{x_1 - x_2}{x_1 - 1}, x_1, 1 - x_1]\\ &- [t; 1 - \frac{t}{x_3}, 1 - x_3, \frac{x_2 - x_3}{x_2 - 1}, x_2, 1 - \frac{x_2}{x_1}, x_1, 1 - x_1]\\ &- [t; 1 - \frac{t}{x_3}, 1 - \frac{x_3}{x_2}, x_2, 1 - x_2, \frac{x_1 - x_3}{x_1 - 1}, x_1, 1 - x_1].\end{aligned}$$

Consider the Lyndon word 00101. Its corresponding tree T_{00101^*} is



and computing $d_{cy}(T_{00101^*})$ gives

$$d_{cy}(T_{00101^*}) = T_{001^*}T_{01^*} - T_{0001^*}T_{1^*} - T_{1^*}T_{0001^*}(1).$$

Finally, $\mathcal{L}_{00101}$ and $\mathcal{L}^1_{00101}$ satisfy respectively

$$\partial(\mathcal{L}_{00101}) = \mathcal{L}_{001} \cdot \mathcal{L}_{01} + \mathcal{L}_1\mathcal{L}^1_{0001} \tag{60}$$

and

$$\partial(\mathcal{L}^1_{00101}) = -\mathcal{L}^1_{001} \cdot \mathcal{L}^1_{01} + \mathcal{L}_{001} \cdot \mathcal{L}^1_{01} - \mathcal{L}_0 \cdot \mathcal{L}^1_{001} + \mathcal{L}_1 \cdot \mathcal{L}^1_{0001}. \tag{61}$$

5.2. **A combinatorial representation for the cycles : trees with colored edges.** In this subsection, we give a combinatorial approach to describe cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ as parametrized cycles using trivalent trees with two types of edge.

Definition 5.3. Let $\mathcal{T}^{||}$ be the $\mathbb{Q}$ vector space generated by rooted trivalent trees such that

- the edges can be of two types: $|$ or $\|$;
- the root vertex is decorated by t
- other external vertices are decorated by 0 or 1.

We say that such a tree is a rooted colored tree or simply a colored tree.

We define two bilinear maps $\mathcal{T}^{||} \otimes \mathcal{T}^{||} \longrightarrow \mathcal{T}^{||}$ as follows on the colored trees:

- Let $T_1 \curlywedge T_2$ be the colored tree given by joining the two root of T_1 and T_2 and adding a new root and a new edge of type $|$:

$$T_1 \curlywedge T_2 = \quad \begin{array}{c} \\ \\ \\ T_1 \quad T_2 \end{array}$$

where the dotted edges denote either type of edges.

- Let $T_1 \overset{\|}{\curlywedge} T_2$ be the colored tree given by joining the two root of T_1 and T_2 and adding a new root and a new edge of type $\|$:

$$T_1 \overset{\|}{\curlywedge} T_2 = \quad \begin{array}{c} \bullet \\ \| \\ \bullet \\ T_1 \quad T_2 \end{array}$$

where the dotted edges denote either type of edges.

Definition 5.4. Let $\mathfrak{T}_0$ and $\mathfrak{T}_1$ be the colored tree defined by

$$\mathfrak{T}_0 = \begin{array}{c} t \\ \| \\ 0 \end{array} \quad \text{and} \quad \mathfrak{T}_1 = \begin{array}{c} t \\ | \\ 1 \end{array}.$$

For any Lyndon word W of length greater or equal to 2, let $\mathfrak{T}_W$ (resp. $\mathfrak{T}^1_W$) be the linear combination of colored trees given by

$$\mathfrak{T}_W = \sum_{U<V} a^W_{U,V} \mathfrak{T}_U \overset{|}{\curlywedge} \mathfrak{T}_V + \sum_{U,V} b^W_{U,V} \mathfrak{T}_U \overset{|}{\curlywedge} \mathfrak{T}^1_V,$$

and respectively by

$$\mathfrak{T}^1_W = \sum_{0<U<V} {a'}^W_{U,V} \mathfrak{T}^1_U \overset{\|}{\curlywedge} \mathfrak{T}^1_V + \sum_{U,V} {b'}^W_{U,V} \mathfrak{T}_U \overset{\|}{\curlywedge} \mathfrak{T}^1_V + \sum_{V} {a'}_{0,V} \mathfrak{T}_0 \overset{\|}{\curlywedge} \mathfrak{T}_V.$$

To a colored tree T with p external leaves and a root, one associates a function $f_T : X \times (\mathbb{P}^1)^{p-1} \longrightarrow X \times (\mathbb{P}^1)^{2p-1}$ as follows :

- Endow T with its natural order as trivalent tree.
- This induces a numbering of the edges of T : $(e_1, e_2, \dots, e_{2p-1})$.
- The edges being oriented away from the root, the numbering of the edges induces a numbering of the vertices, $(v_1, v_2, \dots, v_{2p})$ such that the root is v_1.
- Associate variables $x_1, \dots, x_{p-1}$ to each internal vertices such that the numbering of the variable is opposite to the order induced by the numbering of the vertices (first internal vertices has variable x_{p-1}, second internal vertices has variable x_{p-2} and so on).
- For each edge $e_i = \begin{array}{c} a \\ \vdots \\ b \end{array}$ oriented from a to b, define a function

$$f_i(a,b) = \begin{cases} 1 - \dfrac{a}{b} & \text{if } e_i \text{ is of type } |\,, \\ \dfrac{b-a}{b-1} & \text{if } e_i \text{ is of type } \|\,. \end{cases}$$

- Finally $f_T : X \times (\mathbb{P}^1)^{p-1} \longrightarrow X \times (\mathbb{P}^1)^{2p-1}$ is defined by

$$f_T(t, x_1, \dots x_{p-1}) = (t, f_1, \dots, f_{2p-1}).$$

Let $\Gamma(T)$ be the intersection of the the image of f_T with $X \times \square^{2p-1}$. One can formally extend the definition of Γ from $\mathcal{T}^{||}$ into the direct sum $\oplus_{p \geqslant 1} \mathcal{Z}^p(X \times \square^{2p-1})$.

Proposition 5.5. *The map Γ satisfies :*

- *For any Lyndon word of length p, $\Gamma(\mathfrak{T}_W)$ is in $\mathcal{Z}^p_{eq}(X, 2p-1) \otimes \mathbb{Q}$*
- *$\mathcal{A}lt(\Gamma(\mathfrak{T}_0)) = \mathcal{L}_0$ and $\mathcal{A}lt(\Gamma(\mathfrak{T}_1)) = \mathcal{L}_1$*
- *For any Lyndon word of length $p \geqslant 2$,*

$$\mathcal{A}lt(\Gamma(\mathfrak{T}_W)) = \mathcal{L}_W \qquad \text{and} \qquad \mathcal{A}lt(\Gamma(\mathfrak{T}^1_W)) = \mathcal{L}^1_W.$$

Proof. The fact that $\Gamma(\mathfrak{T}_0)$ (resp. $\Gamma(\mathfrak{T}_1)$) is the graph of $t \mapsto t$ (resp. $t \mapsto 1-t$) follows from the definition. Thus one already has $\Gamma(\mathfrak{T}_0)$ (resp. $\Gamma(\mathfrak{T}_1)$) in $\mathcal{Z}^1_{eq}(X,1)$ and

$$\mathcal{A}lt(\Gamma(\mathfrak{T}_0)) = \mathcal{L}_0 \qquad \text{and} \qquad \mathcal{A}lt(\Gamma(\mathfrak{T}_1)) = \mathcal{L}_1.$$

Then, the proposition is deduced by induction because, as already remarked in Example 4.13, in order to compute the pull-back by μ^* one sets the former parameter t to a new variable x_n and parametrizes the new $\square^1$ factor arriving in first position by $1-\frac{t}{x_n}$ (t is again the parameter over X). The case of ν^* is similar but parametrizing the new $\square^1$ factor by $\frac{x_n-t}{x_n-1}$. □

Remark 5.6. Considering that $\mathcal{L}_0$ is empty at 1 and the symmetry of the situation between 0 and 1, one could write $\mathcal{L}^1_0$ instead of $\mathcal{L}_0$ and similarly $\mathfrak{T}^1_0$ instead of $\mathfrak{T}_0$. This cosmetic change of notations will in particular make Definition 4.1 more uniform with respect to the cases where either U or V is equal to 0 or 1.

However, it will add some modifications in the proof of Proposition 4.3 relating relations among a's, a''s, b's and b''s coefficients with relations between α's and β's coefficients.

5.3. **An integral associated to $\mathcal{L}_{011}$.** We present here a sketch of how to associate an integral to the cycle $\mathcal{L}_{011}$. The author will directly follow the algorithm describe in [GGL09][Section 9] and put in detailed practice in [GGL07]. There will be no general review of the direct Hodge realization from Bloch-Kriz motives [BK94][Section 8 and 9]. Gangl, Goncharov and Levine construction seems to consist in setting particular choices of representatives in the intermediate jacobians for their algebraic cycles.

The author will not extend this description and will not generalized here the computations below. Relating Bloch and Kriz approach to the explicit algorithms described by Gangl, Goncharov and Levine and the application to our particular family of cycles $\mathcal{L}_W$ will be the topics of a future paper as it requires, in particular, a family $\mathcal{L}^B_W$ of element in $\mathrm{H}^0(B(\mathcal{N}^\bullet_X))$ not at our disposal yet.

Let's recall the expression of $\mathcal{L}_{011}$ as parametrized cycle:

$$\mathcal{L}_{011} = -[t; 1-\frac{t}{x_2}, 1-x_2, \frac{x_1-x_2}{x_1-1}, x_1, 1-x_1].$$

One wants to bound $\mathcal{L}_{011}$ by an algebraic-topological cycle in a larger bar construction (not described here) introducing topological variables s_i in real simplices

$$\Delta^n_s = \{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant 1\}.$$

Let $d^s : \Delta^n_s \to \Delta^{n-1}_s$ denotes the simplicial differential

$$d^s = \sum_{k=0}^{n} (-1)^k i^*_k$$

where $i_k : \Delta^{n-1}_s \to \Delta^n_s$ is given by the face $s_k = s_{k+1}$ in Δ^n_s with the usual conventions for $k = 0, n$.

Let's define

$$C^{s,1}_{011} = [t; 1-\frac{s_3 t}{x_2}, 1-x_2, \frac{x_1-x_2}{x_1-1}, x_1, 1-x_1]$$

for s_3 going from 0 to 1. Then, $d^s(C^{s,1}_{011}) = \mathcal{L}_{011}$ as $s_3 = 0$ implies that the first cubical coordinate is 1.

Now the algebraic boundary ∂ of $C^{s,1}_{011}$ is given by the intersection with the codimension 1 faces of $\square^5$

$$\partial(C^{s,1}_{011}) = [t; 1-s_3 t, \frac{x_1-s_3 t}{x_1-1}, x_1, 1-x_1].$$

We can again bound this cycle by introducing a new simplicial variable $0 \leqslant s_2 \leqslant s_3$ and the cycle

$$C^{s,2}_{011} = [t; 1 - s_3 t, \frac{x_1 - s_2 t}{x_1 - s_2/s_3}, x_1, 1 - x_1].$$

The intersections with the faces of the simplex $\{0 \leqslant s_2 \leqslant s_3 \leqslant 1\}$ given by $s_2 = 0$ and $s_3 = 1$ lead to empty cycles (as at least one cubical coordinates equals 1). Thus, the simplicial boundary of $C^{s,2}_{011}$ satisfies

$$d^s(C^{s,2}_{011}) = -\partial(C^{s,1}_{011}) = -[t; 1 - s_3 t, \frac{x_1 - s_3 t}{x_1 - 1}, x_1, 1 - x_1].$$

Its algebraic boundary is given by

$$\partial(C^{s,2}_{011}) = -[t; 1 - s_3 t, s_2 t, 1 - s_2 t] + [t; 1 - s_3 t, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}].$$

Finally, we introduce a last simplicial variable $0 \leqslant s_1 \leqslant s_2$ and a purely topological cycle

$$C^{s,3}_{011} = -[t; 1 - s_3 t, s_2 t, 1 - s_1 t] + [t; 1 - s_3 t, \frac{s_2}{s_3}, 1 - \frac{s_1}{s_3}]$$

whose simplicial differential is (up to negligible terms) given by the face $s_1 = s_2$:

$$d^s(C^{s,3}_{011}) = -\partial(C^{2,s}_{011}) = [t; 1 - s_3 t, s_2 t, 1 - s_2 t] - [t; 1 - s_3 t, \frac{s_2}{s_3}, 1 - \frac{s_2}{s_3}]$$

and whose algebraic boundary is 0.

Finally one has

$$(d^s + \partial)(C^{s,1}_{011} + C^{s,3}_{011} + C^{s,3}_{011}) = \mathcal{L}_{011}$$

up to negligible terms.

Now, we fix the situation at the fiber t_0 and following Gangl, Goncharov and Levin, we associate to the algebraic cycle $\mathcal{L}_{011}|_{t=t_0}$ the integral $I_{011}(t_0)$ of the standard volume form

$$\frac{1}{(2i\pi)^3} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}$$

over the simplex given by $C^{s,3}_{011}$. That is :

$$I_{011}(t_0) = -\frac{1}{(2i\pi)^3} \int_{0 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant 1} \frac{t_0\, ds_3}{1 - t_0 s_3} \wedge \frac{ds_2}{s_2} \wedge \frac{t_0\, ds_1}{1 - t_0 s_1} + \frac{1}{(2i\pi)^3} \int_{0 \leqslant s_3 \leqslant 1} \frac{t_0\, ds_3}{1 - t_0 s_3} \int_{0 \leqslant s_1 \leqslant s_2 \leqslant 1} \frac{ds_2}{s_2} \wedge \frac{ds_1}{1 - s_1}.$$

Taking care of the change of sign due to the numbering, the first term in the above sum is (for $t_0 \neq 0$ and up to the factor $(2i\pi)^{-3}$) equal to

$$Li^{\mathbb{C}}_{1,2}(t_0) = \int_{0 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant 1} \frac{ds_1}{t_0^{-1} - s_1} \wedge \frac{ds_2}{s_2} \wedge \frac{ds_3}{t_0^{-1} - s_3}$$

while the second term equals (up to the same multiplicative factor)

$$-Li^{\mathbb{C}}_1(t_0) Li^{\mathbb{C}}_2(1).$$

Globally the integral is well defined for $t_0 = 0$ and, which is the interesting part, also for $t_0 = 1$ as the divergencies as t_0 goes to 1 cancel each other in the above sums. A simple computation and the shuffle relation for $Li^{\mathbb{C}}_1(t_0) Li^{\mathbb{C}}_2(t_0)$ shows that the integral associated to the fiber of $\mathcal{L}_{011}$ at $t_0 = 1$ is

$$(2i\pi)^3 I_{011}(1) = -2Li^{\mathbb{C}}_{2,1}(1) = -2\zeta(2,1).$$

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