# Fragmented deformations of primitive multiple curves

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# FRAGMENTED DEFORMATIONS OF PRIMITIVE MULTIPLE CURVES

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RESUME. A *primitive multiple curve* is a Cohen-Macaulay irreducible projective curve $Y$ that can be locally embedded in a smooth surface, and such that $Y_{red}$ is smooth.

The subject of this paper is the study of deformations of $Y$ in curves with smooth irreducible components, when the number of components is maximal (it is then the multiplicity $n$ of $Y$).

We are particularly interested in deformations in $n$ disjoint smooth irreducible components, which are called *fragmented deformations*. We describe them completely. We give also a characterization of primitive multiple curves having a fragmented deformation.

## SUMMARY



Mathematics Subject Classification : 14M05, 14B20

## 1. INTRODUCTION

A *primitive multiple curve* is an algebraic variety $Y$ over $\mathbb{C}$ wich is Cohen-Macaulay, such that the induced reduced variety $C = Y_{red}$ is a smooth projective irreducible curve, and that every closed point of $Y$ has a neighbourhood that can be embedded in a smooth surface. These curves have been defined and studied by C. Bănică and O. Forster in [1]. The simplest examples are infinitesimal neighbourhoods of projective smooth curves embedded in a smooth surface (but most primitive multiple curves cannot be globally embedded in smooth surfaces, cf. [2], theorem 7.1).

Les $Y$ be a primitive multiple curve with associated reduced curve $C$, and suppose that $Y \neq C$. Let $\mathcal{I}_C$ be the ideal sheaf of $C$ in $Y$. The *multiplicity* of $Y$ is the smallest integer $n$ such that $\mathcal{I}_C^n = 0$. We have then a filtration

$$C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$$

where $C_i$ is the subscheme corresponding to the ideal sheaf $\mathcal{I}_C^i$ and is a primitive multiple curve of multiplicity $i$. The sheaf $L = \mathcal{I}_C/\mathcal{I}_C^2$ is a line bundle on $C$, called the *line bundle on $C$ associated to $Y$*.

**1.1.** *History and motivation* – The deformations of double (i.e. of multiplicity 2) primitive multiple curves (also called *ribbons*) in smooth projective curves have been studied in [14]. In this paper we are interested in deformations of primitive multiple curves $Y$ of any multiplicity $n \geq 2$ in reduced curves having exactly $n$ components which are smooth ($n$ is the maximal number of components of deformations of $Y$). In this case the number of intersection points of two components is exactly $-\deg(L)$. We give some results in the general case (no assumption on $\deg(L)$) and treat more precisely the case $\deg(L) = 0$, i.e. deformations of $Y$ in curves having exactly $n$ disjoint irreducible components.

Let $\pi : \mathcal{C} \to S$ be a flat projective morphism of algebraic varieties, $P$ a closed point of $S$ such that $\pi^{-1}(P) \simeq Y$, $\mathcal{O}_{\mathcal{C}}(1)$ a very ample line bundle on $\mathcal{C}$ and $P$ a polynomial in one variable with rational coefficients. Let

$$\tau : \mathcal{M}_{\mathcal{O}_{\mathcal{C}}(1)}(P) \longrightarrow S$$

be the corresponding relative moduli space of semi-stable sheaves (parametrizing the semi-stables sheaves on the fibers of $\pi$ with Hilbert polynomial $P$ with respect to the restriction of $\mathcal{O}_{\mathcal{C}}(1)$, cf. [20]). In general $\tau$ is not flat (some other examples on non flat relative moduli spaces are given in [17]). For example, if the family $\mathcal{C}$ contains smooth fibers, it is impossible to deform the stable sheaf $\mathcal{O}_C$ on $Y$ in sheaves on the smooth fibers. I conjecture that $\tau$ is flat if all the fibers of $\tau$ are reduced with exactly $n$ components. The reason is that the generic structure of torsion free sheaves on $Y$ (cf. [8]) is more complicated that on smooth curves, and is somehow similar to the generic structure of torsion free sheaves on reducible reduced curves (cf. [23], [24]).

**1.2.** *Maximal reducible deformations* – Let $(S, P)$ be a germ of smooth curve. Let $Y$ be a primitive multiple curve of multiplicity $n \geq 2$ and $k > 0$ an integer. Let $\pi : \mathcal{C} \to S$ be a flat morphism, where $\mathcal{C}$ is a reduced algebraic variety, such that

- For every closed point $s \in S$ such that $s \neq P$, the fiber $\mathcal{C}_s$ has $k$ irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber $\mathcal{C}_P$ is isomorphic to $Y$.

We show that by making a change of variable, i.e. by considering a suitable germ $(S', P')$ and a non constant morphism $\tau : S' \to S$, and replacing $\pi$ with $\pi^*\mathcal{C} \to S'$, we can suppose that $\mathcal{C}$ has exactly $k$ irreducible components, inducing on every fiber $\mathcal{C}_s$, $s \neq P$ the $k$ irreducible components of $\mathcal{C}_s$. In this case $\pi$ is called a *reducible deformation of $Y$ of length $k$*.

We show that $k \leq n$. We say that $\pi$ (or $\mathcal{C}$) is a *maximal reducible deformation of $Y$* if $k = n$.

Suppose that $\pi$ is a maximal reducible deformation of $Y$. We show that if $\mathcal{C}'$ is the union of $i > 0$ irreducible components of $\mathcal{C}$, and $\pi' : \mathcal{C}' \to S$ is the restriction of $\pi$, then $\pi'^{-1}(P) \simeq C_i$, and $\pi'$ is a maximal reducible deformation of $C_i$. Let $s \in S\backslash\{P\}$. We prove that the irreducible components of $\mathcal{C}_s$ have the same genus as $C$. Moreover, if $D_1, D_2$ are distinct irreducible components of $\mathcal{C}_s$, then $D_1 \cap D_2$ consists of $-\deg(L)$ points.

**1.3.** *Fragmented deformations (definition)* – Let $Y$ be a primitive multiple curve of multiplicity $n \geq 2$ and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of $Y$. We call it a *fragmented deformation of* $Y$ if $\deg(L) = 0$, i.e. if for every $s \in S\backslash\{P\}$, $\mathcal{C}_s$ is the disjoint union of $n$ smooth curves. In this case $\mathcal{C}$ has $n$ irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_n$ which are smooth surfaces.

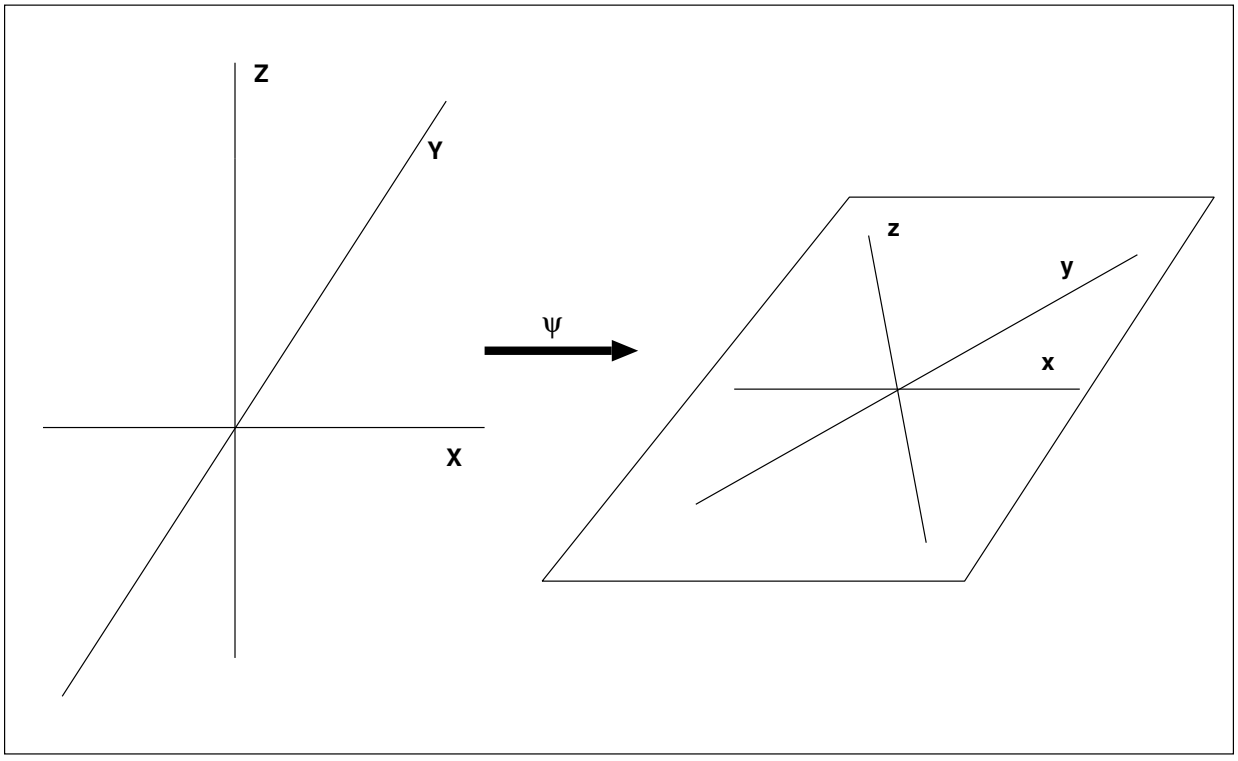
The variety $\mathcal{C}$ appears as a particular case of a *glueing* of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ along $C$ (cf. 4.1.5). We prove (proposition 4.1.5) that such a glueing $\mathcal{D}$ is a fragmented deformation of a primitive multiple curve if and only if every closed point in $C$ has a neighbourhood in $\mathcal{D}$ that can be embedded in a smooth variety of dimension 3. The simplest glueing is the trivial or *initial glueing* $\boldsymbol{\mathcal{A}}$. An open subset $U$ of $\boldsymbol{\mathcal{A}}$ (and $\mathcal{C}$) is given by open subsets $U_1, \ldots, U_n$ of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ respectively, having the same intersection with $C$, and

$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \;=\; \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \cdots \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \cdots = \alpha_{n|C}\},$$

and $\mathcal{O}_{\mathcal{C}}(U)$ appears as a subalgebra of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U)$, hence we have a canonical morphism $\boldsymbol{\mathcal{A}} \to \mathcal{C}$.

We can view elements of $\mathcal{O}_{\mathcal{C}}(U)$ as $n$-tuples $(\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in \mathcal{O}_{\mathcal{C}_i}(U \cap \mathcal{C}_i)$. In particular we can write $\pi = (\pi_1, \ldots, \pi_n)$.

**1.4.** *A simple analogy* – Consider $n$ copies of $\mathbb{C}$ glued at 0. Two extreme examples appear : the trivial glueing $\boldsymbol{\mathcal{A}}_0$ (the set of coordinate lines in $\mathbb{C}^n$), and a set $\mathcal{C}_0$ of $n$ lines in $\mathbb{C}^2$. We can easily construct a bijective morphism $\boldsymbol{\Psi} : \boldsymbol{\mathcal{A}}_0 \to \mathcal{C}_0$ sending each coordinate line to a line in the plane




But the two schemes are of course not isomorphic : the maximal ideal of 0 in $\boldsymbol{\mathcal{A}}_0$ needs $n$ generators, but 2 are enough for the maximal ideal of 0 in $\mathcal{C}_0$.

Let $\pi_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathbb{C}$ be a morphism sending each component linearly onto $\mathbb{C}$, and $\pi_{\boldsymbol{\mathcal{A}}_0} = \pi_{\mathcal{C}_0} \circ \boldsymbol{\Psi} : \boldsymbol{\mathcal{A}}_0 \to \mathbb{C}$. The difference of $\boldsymbol{\mathcal{A}}_\mathbf{0}$ and $\mathcal{C}_0$ can be also seen by using the fibers of 0 : we have

$$\pi_{\mathcal{C}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n)) \quad \text{and} \quad \pi_{\boldsymbol{\mathcal{A}}_0}^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^2) \ .$$

Let $\mathcal{D}$ a general glueing of $n$ copies of $\mathbb{C}$ at 0, such that there exists a morphism $\pi : \mathcal{D} \to \mathbb{C}$ inducing the identity on each copy of $\mathbb{C}$. It is easy to see that we have $\pi^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t]/(t^n))$ if and only if some neighbourhood of 0 in $\mathcal{D}$ can be embedded in a smooth surface.

**1.5.** *Fragmented deformations (main properties)* – Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. Let $I \subset \{1, \ldots, n\}$ be a proper subset, $I^c$ its complement, and $\mathcal{C}_I \subset \mathcal{C}$ the subscheme union of the $\mathcal{C}_i, i \in I$. We prove (theorem 4.3.7) that the ideal sheaf $\mathcal{I}_{\mathcal{C}_I}$ of $\mathcal{C}_I$ is isomorphic to $\mathcal{O}_{\mathcal{C}_{I^c}}$.

In particular, the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of $\mathcal{C}_i$ is generated by a single regular function on $\mathcal{C}$. We show that we can find such a generator such that for $1 \leq j \leq n$, $j \neq i$, its $j$-th coordinate can be written as $\alpha\pi_j^p$, with $p > 0$ and $\alpha \in H^0(\mathcal{O}_S)$ such that $\alpha(P) \neq 0$. We can then suppose that $\alpha = 1$, and the generator can be written as

$$\mathbf{u}_{ij} \ = \ (u_1, \ldots, u_m),$$

with

$$u_i = 0, \quad u_m \ = \ \alpha_{ij}^{(m)} \pi_m^{p_{im}} \ \text{ for } m \neq i, \quad \alpha_{ij}^{(j)} = 1.$$

The constants $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$ have interesting properties (propositions 4.5.2, 4.4.6). Let $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of $\pi$ (or $\mathcal{C}$).

It follows also from the fact that $\mathcal{I}_{\mathcal{C}_i} = (\mathbf{u}_{ij})$ that $Y$ is a *simple* primitive multiple curve, i.e. the ideal sheaf of $C$ in $Y = C_n$ is isomorphic to $\mathcal{O}_{C_{n-1}}$. Conversely, we show in theorem 4.7.1 that if $Y$ is a simple primitive multiple curve, then there exists a fragmented deformation of $Y$.

We give in 4.4 and 4.5 a way to construct fragmented deformations by induction on $n$. This is used later to prove statements on fragmented deformations by induction on $n$.

**1.6.** *$n$-stars and structure of fragmented deformations* – A *$n$-star* of $(S, P)$ is a glueing $\boldsymbol{\mathcal{S}}$ of $n$ copies of $S$ at $P$, together with a morphism $\pi : \boldsymbol{\mathcal{S}} \to S$ which is an identity on each copy of $S$. All the $n$-stars have the same underlying Zariski topological space $S(n)$.

A $n$-star is called *oblate* if some neighbourhood of $P$ can be embedded in a smooth surface. This is the case if and only $\pi^{-1}(0) \simeq \mathrm{spec}(\mathbb{C}[t])/(t^n)$.

Oblate $n$-stars are analogous to fragmented deformations and simpler. We provide a way to build oblate $n$-stars by induction on $n$.

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $Y = C_n$. We associate to it an oblate $n$-star $\boldsymbol{\mathcal{S}}$ of $S$ : for every open subset $U$ of $S(n)$, $\mathcal{O}_{\boldsymbol{\mathcal{S}}}(U)$ is the set of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(U)$ such that $\alpha_i \in \mathcal{O}_S(\pi_i(U \cap \mathcal{C}_i))$ for $1 \leq i \leq n$. We obtain also a canonical morphism

$$\boldsymbol{\Pi} : \mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}.$$

We prove (theorem 5.6.2) that $\boldsymbol{\Pi}$ is flat. Hence it is a flat family of smooth curves, with $\boldsymbol{\Pi}^{-1}(P) = C$. The converse is also true, i.e. starting from an oblate $n$-star of $S$ and a flat

family of smooth curves parametrized by it, we obtain a fragmented deformation of a multiple primitive curve of multiplicity $n$.

**1.7.** *Fragmented deformations of double curves* – Let $Y = C_2$ be a primitive double curve, $C$ its associated smooth curve, $\pi : \mathcal{C} \to S$ a fragmented deformation of $Y$, of spectrum $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$, and $\mathcal{C}_1$, $\mathcal{C}_2$ the irreducible components of $\mathcal{C}$. For $i = 1, 2$, $q > 0$, let $C_i^q$ be the infinitesimal neighbourhood of order $q$ of $C$ in $\mathcal{C}_i$ (defined by the ideal sheaf $(\pi_i^q)$). It is a primitive multiple curve of multiplicity $q$.

It follows from 4.3.5 that $C_1^p$ and $C_2^p$ are isomorphic, and $C_1^{p+1}$, $C_2^{p+1}$ are two extensions of $C_1^p$ in primitive multiple curves of multilicity $p + 1$. According to [6] these extensions are parametrized by an affine space with associated vector space $H^1(C, T_C)$ (where $T_C$ is the tangent bundle of $C$). Let $w \in H^1(C, T_C)$ be the vector from $C_1^{p+1}$ to $C_2^{p+1}$.

Similarly, the primitive double curves with associated smooth curve $C$ such that $\mathcal{I}_C \simeq \mathcal{O}_C$ are parametrized by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ (cf. [2], [6]).

We prove in theorem 6.0.5 that the point of $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ corresponding to $C_2$ is $\mathbb{C}w$.

**1.8.** *Notation:* Let $X$ be an algebraic variety and $Y \subset X$ a closed subvariety. We will denote by $\mathcal{I}_{Y,X}$ (or $\mathcal{I}_Y$ if there is no risk of confusion) the ideal sheaf of $Y$ in $X$.

## 2. Preliminaries

### 2.1. Local embeddings in smooth varieties

**2.1.1. Proposition:** *Let $X$ be an algebraic variety, $x$ a closed point of $X$ and $n$ a positive integer. Then the two following properties are equivalent:*

(i) *There exists a neighbourhood $U$ of $x$ and an embedding $U \subset Z$ in a smooth variety of dimension $n$.*
(ii) *The $\mathcal{O}_{X,x}$-module $m_{X,x}$ (maximal ideal of $x$) can be generated by $n$ elements.*
(iii) *We have* $\dim_{\mathbb{C}}(m_{X,x}/m_{X,x}^2) \leq n$.

*Proof.* It is obvious that (i) implies (ii), and (ii),(iii) are equivalent according to Nakayama's lemma. It remains to prove that (iii) implies (i).

Suppose that (iii) is true. There exists an integer $N$ and an embedding $X \subset \mathbb{P}_N$. Let $\mathcal{I}_X$ be the ideal sheaf of $X$ in $\mathbb{P}_N$. Let $p$ be the biggest integer such that there exists $f_1, \cdots, f_p \in \mathcal{I}_{X,x}$ whose images in the $\mathbb{C}$-vector space $m_{\mathbb{P}_N,x}/m_{\mathbb{P}_N,x}^2$ are linearly independant. Then we have

$$\mathcal{I}_{X,x} \subset (f_1, \cdots, f_p) + m_{\mathbb{P}_N,x}^2.$$

In fact, let $f \in \mathcal{I}_{X,x}$. Since $p$ is maximal, the image of $f$ in $m_{\mathbb{P}_N,x}/m^2_{\mathbb{P}_N,x}$ is a linear combination of those of $f_1, \cdots, f_n$. Hence we can write

$$f = \sum_{i=1}^{p} \lambda_i f_i + g, \quad \text{with} \quad \lambda_i \in \mathbb{C},\ g \in m^2_{\mathbb{P}_N,x}\ ,$$

and our assertion is proved. It follows that we have a surjective morphism

$$\alpha : \mathcal{O}_{X,x}/m^2_{X,x} \longrightarrow \mathcal{O}_{\mathbb{P}_N,x}/\big((f_1, \cdots, f_p) + m^2_{\mathbb{P}_N,x}\big)\ .$$

We have

$$\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m^2_{X,x}) \ \leq\ n+1, \quad \dim_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{P}_N,x}/\big((f_1, \cdots, f_p) + m^2_{\mathbb{P}_N,x}\big)\big) \ = N - p + 1\ .$$

Hence $N - p + 1 \leq n + 1$, i.e. $p \geq N - n$. We can take for $Z$ a neighbourhood of $x$ in the subvariety of $\mathbb{P}_N$ defined by $f_1, \cdots, f_{N-n}$, which is smooth at $x$. □

### 2.2. Flat families of coherent sheaves

Let $(S, P)$ be a smooth germ of curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $\pi : X \to S$ be a flat morphism. If $\mathcal{E}$ is a coherent sheaf on $X$, $\mathcal{E}$ is flat on $S$ at $x \in \pi^{-1}(P)$ if and only if the multiplication by $t : \mathcal{E}_x \to \mathcal{E}_x$ is injective. In particular the multiplication by $t : \mathcal{O}_x \to \mathcal{O}_x$ is injective.

**2.2.1. Lemma:** *Let $\mathcal{E}$ be a coherent sheaf on $X$ flat on $S$. Then, for every open subset $U$ of $X$, the restriction $\mathcal{E}(U) \to \mathcal{E}(U\backslash\pi^{-1}(P))$ is injective.*

*Proof.* Let $s \in \mathcal{E}(U)$ whose restriction to $U\backslash\pi^{-1}(P)$ vanishes. We must show that $s = 0$. By covering $U$ with smaller open subsets we can suppose that $U$ is affine: $U = \mathrm{spec}(A)$. Hence $U\backslash\pi^{-1}(P) = \mathrm{spec}(A_t)$. Let $M = \mathcal{E}(U)$, it is an $A$-module. We have $\mathcal{E}_{|U} = \widetilde{M}$ and $\mathcal{E}(U\backslash\pi^{-1}(P)) = M_t$. Hence if the restriction of $s$ to $U\backslash\pi^{-1}(P)$ vanishes, there exists an integer $n > 0$ such that $t^n s = 0$. Since the multiplication by $t$ is injective (because $\mathcal{E}$ is flat on $S$), we have $s = 0$. □

Let $\mathcal{E}$ be a coherent sheaf on $X$ flat on $S$. Let $\mathcal{F} \subset \mathcal{E}_{|X\backslash\pi^{-1}(P)}$ be a subsheaf. For every open subset $U$ of $X$ we denote by $\overline{\mathcal{F}}(U)$ the subset of $\mathcal{F}(U\backslash\pi^{-1}(P))$ of elements that can be extended to sections of $\mathcal{E}$ on $U$. If $V \subset U$ is an open subset, the restriction $\mathcal{F}(U\backslash\pi^{-1}(P)) \to \mathcal{F}(V\backslash\pi^{-1}(P))$ induces a morphism $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ .

**2.2.2. Proposition:** *$\overline{\mathcal{F}}$ is a subsheaf of $\mathcal{E}$, and $\mathcal{E}/\overline{\mathcal{F}}$ is flat on $S$.*

*Proof.* To prove the first assertion, we must show that if $U$ is an open subset of $X$ and $(U_i)_{i\in I}$ is an open cover of $U$, then

(i) If $s \in \overline{\mathcal{F}}(U)$ is such that for every $i$ we have $s_{|U_i} = 0$, then $s = 0$.
(ii) For every $i \in I$ let $s_i \in \overline{\mathcal{F}}(U_i)$. Then if for all $i$, $j$ we have $s_{i|U_{ij}} = s_{j|U_{ij}}$ , then there exists $s \in \overline{\mathcal{F}}(U)$ such that for every $i \in I$ we have $s_{|U_i} = s_i$.

This follows easily from lemma 2.2.1.

Now we prove that $\mathcal{E}/\overline{\mathcal{F}}$ is flat on $S$. Let $x \in \pi^{-1}(P)$ and $u \in (\mathcal{E}/\overline{\mathcal{F}})_x$ such that $tu = 0$. We must show that $u = 0$. Let $v \in \mathcal{E}_x$ over $u$. Then we have $tv \in \overline{\mathcal{F}}_x$. Let $U$ be a neighbourhood of $x$ such that $tv$ comes from $w \in \overline{\mathcal{F}}(U)$. This means that $w_{|U\backslash\pi^{-1}(P)} \in \mathcal{F}(U\backslash\pi^{-1}(P))$. Since $t$ is invertible on $U\backslash\pi^{-1}(P)$ we can write $w = tw'$, with $w' \in \mathcal{F}(U\backslash\pi^{-1}(P))$. We have then $w' = v$ on $U\backslash\pi^{-1}(P)$. Hence $v \in \overline{\mathcal{F}}_x$ and $u = 0$. □

## 2.3. Primitive multiple curves

(cf. [1], [5], [8]).

Let $C$ be a smooth connected projective curve. A *multiple curve with support* $C$ is a Cohen-Macaulay scheme $Y$ such that $Y_{red} = C$.

Let $n$ be the smallest integer such that $Y = C^{(n-1)}$, $C^{(k-1)}$ being the $k$-th infinitesimal neighbourhood of $C$, i.e. $\mathcal{I}_{C^{(k-1)}} = \mathcal{I}_C^k$ . We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n = Y$ where $C_i$ is the biggest Cohen-Macaulay subscheme contained in $Y \cap C^{(i-1)}$. We call $n$ the *multiplicity* of $Y$.

We say that $Y$ is *primitive* if, for every closed point $x$ of $C$, there exists a smooth surface $S$, containing a neighbourhood of $x$ in $Y$ as a locally closed subvariety. In this case, $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on $C$ and we have $\mathcal{I}_{C_j} = \mathcal{I}_X^j$, $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} = L^j$ for $1 \leq j < n$. We call $L$ the line bundle on $C$ *associated* to $Y$. Let $P \in C$. Then there exists elements $y$, $t$ of $m_{S,P}$ (the maximal ideal of $\mathcal{O}_{S,P}$) whose images in $m_{S,P}/m_{S,P}^2$ form a basis, and such that for $1 \leq i < n$ we have $\mathcal{I}_{C_i,P} = (y^i)$ .

The simplest case is when $Y$ is contained in a smooth surface $S$. Suppose that $Y$ has multiplicity $n$. Let $P \in C$ and $f \in \mathcal{O}_{S,P}$ a local equation of $C$. Then we have $\mathcal{I}_{C_i,P} = (f^i)$ for $1 < j \leq n$, in particular $I_{Y,P} = (f^n)$, and $L = \mathcal{O}_C(-C)$ .

We will note $\mathcal{O}_n = \mathcal{O}_{C_n}$ and we will see $\mathcal{O}_i$ as a coherent sheaf on $C_n$ with schematic support $C_i$ if $1 \leq i < n$.

If $\mathcal{E}$ is a coherent sheaf on $Y$ one defines its *generalized rank* $R(\mathcal{E})$ and *generalized degree* $\mathrm{Deg}(\mathcal{E})$ (cf. [8], 3-). Let $\mathcal{O}_Y(1)$ be a very ample line bundle on $Y$. Then the Hilbert polynomial of $\mathcal{E}$ is

$$P_{\mathcal{E}}(m) \;=\; R(\mathcal{E})\deg(\mathcal{O}_C(1))m + \mathrm{Deg}(\mathcal{E}) + R(\mathcal{E})(1-g)$$

(where $g$ is the genus of $C$).

We deduce from proposition 2.1.1:

**2.3.1. Proposition:** *Let $Y$ be a multiple curve with support $C$. Then $Y$ is a primitive multiple curve if and only if $\mathcal{I}_C/\mathcal{I}_C^2$ is zero, or a line bundle on $C$.*

**2.3.2.** *Parametrization of double curves* - In the case of double curves, D. Bayer and D. Eisenbud have obtained in [2] the following classification: if $Y$ is of multiplicity 2, we have an exact sequence of vector bundles on $C$

$$0 \longrightarrow L \longrightarrow \Omega_{Y|C} \longrightarrow \omega_C \longrightarrow 0$$

which is split if and only if $Y$ is the *trivial curve*, i.e. the second infinitesimal neighbourhood of $C$, embedded by the zero section in the dual bundle $L^*$, seen as a surface. If $Y$ is not trivial, it is completely determined by the line of $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L)$ induced by the preceding exact sequence. The non trivial primitive curves of multiplicity 2 and of associated line bundle $L$ are therefore parametrized by the projective space $\mathbb{P}(\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, L))$.

### 2.4. Simple primitive multiple curves

Let $C$ be a smooth projective irreducible curve, $n \geq 2$ an integer and $C_n$ a primitive multiple curve of multiplicity $n$ and associated reduced curve $C$. Then the ideal sheaf $\mathcal{I}_C$ of $C$ in $C_n$ is a line bundle on $C_{n-1}$.

We say that $C_n$ is *simple* si $\mathcal{I}_C \simeq \mathcal{O}_{n-1}$.

In this case the line bundle on $C$ associated to $C_n$ is $\mathcal{O}_C$. The following result is proved in [10] (théorème 1.2.1):

**2.4.1. Theorem:** *Suppose that $C_n$ is simple. Then there exists a flat family of smooth projective curves $\tau : \mathcal{C} \to \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that $C_n$ is isomorphic to the $n$-th infinitesimal neighbourhood of $C$ in $\mathcal{C}$.*

## 3. Reducible reduced deformations of primitive multiples curves

### 3.1. Connected Components

Let $(S, P)$ be a germ of smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity $n$.

Let $k > 0$ be an integer. Let $\pi : \mathcal{C} \to S$ be a flat morphism, where $\mathcal{C}$ is a reduced algebraic variety, such that

- For every closed point $s \in S$ such that $s \neq P$, the fiber $\mathcal{C}_s$ has $k$ irreducible components, which are smooth and transverse, and any three of these components have no common point.
- The fiber $\mathcal{C}_P$ is isomorphic to $C_n$.

It is easy to see that the irreducible components of $\mathcal{C}$ are reduced surfaces.

Let $Z$ be the open subset of $\mathcal{C}\backslash\mathcal{C}_P$ of points $z$ belonging to only one irreducible component of $\mathcal{C}_{\pi(z)}$. Then the restriction of $\pi : Z \to S\backslash\{P\}$ is a smooth morphism. For every $s \in S\backslash\{P\}$, let $\mathcal{C}'_s = \mathcal{C}_s \cap Z$. It is the open subset of smooth points of $\mathcal{C}_s$.

Let $z \in Z$ and $s = \pi(z)$. There exists a neighbourhood (for the usual topology) $U$ of $s$, isomorphic to $\mathbb{C}$, and a neighbourhood $V$ of $z$ such that $V \simeq \mathbb{C}^2$, $\pi(V) = U$, the restriction of

$\pi : V \to U$ being the projection $\mathbb{C}^2 \to \mathbb{C}$ on the first factor. We deduce easily from that the following facts:

- let $s \in S\backslash\{P\}$ and $C_1$ an irreducible component of $\mathcal{C}_s$. Let $z_1, z_2 \in C_1 \cap Z$. Then there exists neighbourhoods (in $Z$, for the usual topology) $U_1$, $U_2$ of $z_1$, $z_2$ respectively, such that if $y_1 \in U_1$, $y_2 \in U_2$ are such that $\pi(y_1) = \pi(y_2)$, then $y_1$ and $y_2$ belong to the same irreducible component of $\mathcal{C}_{\pi(y_1)}$.
- for every continuous map $\sigma : [0,1] \to S\backslash\{P\}$ and every $z \in Z$ such that $\sigma(0) = \pi(z)$ there exists a lifting of $\sigma$, $\sigma' : [0,1] \to Z$ such that $\sigma'(0) = z$. Moreover, if $\sigma'' : [0,1] \to Z$ is another lifting of $\sigma$ such that $\sigma''(0) = z$, then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$. More generally, if we only impose that $\sigma''(0)$ is in the same irreducible component of $\mathcal{C}_{\sigma(0)}$ as $z$, then $\sigma'(1)$ and $\sigma''(1)$ are in the same irreducible component of $\mathcal{C}_{\sigma(1)}$.

**3.1.1. Lemma:** *Let $\sigma_0, \sigma_1 : [0,1] \to S\backslash\{P\}$ be two continuous maps such that $\sigma_0(0) = \sigma_1(0)$, $s = \sigma_0(1) = \sigma_1(1)$. Suppose that they are homotopic. Let $\sigma'_0$, $\sigma'_1$ be liftings $[0,1] \to Z$ of $\sigma_0$, $\sigma_1$ respectively, such that $\sigma'_0(0) = \sigma'_1(0)$. Then $\sigma'_0(1)$ and $\sigma'_1(1)$ belong to the same irreducible component of $\mathcal{C}'_s$.*

*Proof.* Let

$$\Psi : [0,1] \times [0,1] \longrightarrow S\backslash\{P\}$$

be an homotopy:

$$\Psi(0,t) = \sigma_0(t), \quad \Psi(1,t) = \sigma_1(t), \quad \Psi(t,0) = \sigma_0(0), \quad \Psi(t,1) = \sigma_0(1)$$

for $0 \leq t \leq 1$. For every $u \in [0,1]$ and $\epsilon > 0$ let $I_{u,\epsilon} = [u - \epsilon, u + \epsilon] \cap [0,1]$. By using the local structure of $\pi_{|Z}$ for the usual topology it is easy to see that for every $u \in [0,1]$, there exists an $\epsilon > 0$ such that the restriction of $\Psi$

$$I_{u,\epsilon} \times [0,1] \longrightarrow S\backslash\{P\}$$

can be lifted to a morphism

$$\Psi' : I_{u,\epsilon} \times [0,1] \longrightarrow Z$$

such that $\Psi'(t,0) = \sigma'_0(0)$ for every $t \in I_{u,\epsilon}$. It follows that if $I_{u,\epsilon} = [a_{u,\epsilon}, b_{u,\epsilon}]$, then $\Psi'(a_{u,\epsilon}, 1)$ and $\Psi'(b_{u,\epsilon}, 1)$ are in the same irreducible component of $\mathcal{C}'_{\sigma_0(1)}$. Now we have just to cover $[0,1]$ with a finite number of intervals $I_{u,\epsilon}$ to obtain the result. □

Let $s \in S\backslash\{P\}$, $D_1, \ldots, D_k$ be the irreducible components of $\mathcal{C}'_s$ and $x_i \in D_i$ for $1 \leq i \leq k$. Let $\sigma$ be a loop of $S\backslash\{P\}$ with origin $s$, defining a generator of $\pi_1(S\backslash\{P\})$. Let $i$ be an integer such that $1 \leq i \leq k$. The liftings $\sigma' : [0,1] \to Z$ of $\sigma$ such that $\sigma'(0) = x_i$ end up at a component $D_j$ which does not depend on $x_i$. Hence we can write

$$j \ = \alpha_{\mathcal{C}}(i).$$

**3.1.2. Lemma:** *$\alpha_{\mathcal{C}}$ is a permutation of $\{1, \ldots, k\}$.*

*Proof.* Suppose that $i \neq j$ and $\alpha_{\mathcal{C}}(i) = \alpha_{\mathcal{C}}(j)$. By inverting the paths we find liftings of paths from $D_{\alpha_{\mathcal{C}}(i)}$ to $D_i$ and $D_j$. This contradicts lemma 3.1.1. □

Let $p > 0$ be an integer such that $\alpha_{\mathcal{C}}^p = I_{\{1,\cdots,k\}}$. Let $t$ be a generator of the maximal ideal of $\mathcal{O}_{S,P}$, $K$ the field of rational functions on $S$ and $K' = K(t^{1/p})$. Let $S'$ be the germ of curve corresponding to $K'$, $\theta : S' \to S$ canonical the morphism and $P'$ the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. We have therefore a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \theta} \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where $\rho$ is flat, and for every $s' \in S'$, $\Theta$ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$. We have

$$\alpha_{\mathcal{D}} \ = \ I_{\{1,\ldots,k\}} \ .$$

Let $Z' \subset \mathcal{D}$ be the complement of the union of $\rho^{-1}(P')$ and of the singular points of the curves $\mathcal{D}_{s'}$, $s' \neq P'$ (hence $Z' = \Theta^{-1}(Z)$).

**3.1.3. Proposition:** *The open subset $Z'$ has exactly $k$ irreducible components $Z'_1, \ldots, Z'_k$. Let $\overline{Z'_1}, \ldots, \overline{Z'_k}$ be their closures in $\mathcal{D}$. Then for every $s' \in S'\backslash\{P'\}$, the $Z'_i \cap \mathcal{D}_{s'}$, $1 \leq i \leq k$, are the irreducible components of $\mathcal{D}_{s'}$ minus the intersection points with the other components, and the $\overline{Z'_i} \cap \mathcal{D}_{s'}$ are the irreducible components of $\mathcal{D}_{s'}$.*

**3.1.4. Definition:** *Let $k > 0$ be an integer. We call* reducible deformation of length $k$ of $C_n$ *a flat morphism $\pi : \mathcal{C} \to S$, where $\mathcal{C}$ is a reduced algebraic variety, such that*

- *For every closed point $s \in S$, $s \neq P$, the fiber $\mathcal{C}_s$ has $k$ irreducible components, which are smooth and transverse, and any three of these components have no common point.*
- *The fiber $\mathcal{C}_P$ is isomorphic to $C_n$.*
- *We have $\alpha_{\mathcal{C}} = I_{\{1,\ldots,k\}}$.*

### 3.2. Maximal reducible deformations

Let $(S, P)$ be a germ of smooth curve and $t \in \mathcal{O}_{S,P}$ a generator of the maximal ideal. Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity $n$, with underlying smooth curve $C$. We note $g$ the genus of $C$ and $L$ the line bundle on $C$ associated to $C_n$.

Let $\pi : \mathcal{C} \to S$ be a reducible deformation of length $k$ of $C_n$. Let $Z_1, \ldots, Z_k$ be the closed subvarieties of $\pi^{-1}(S\backslash\{P\})$ such that for every $s \in S\backslash\{P\}$, $Z_{1s}, \ldots, Z_{ks}$ are the irreducible components of $\mathcal{C}_s$ (cf. prop. 3.1.3).

For $1 \leq i \leq k$, we denote by $\mathcal{J}_i$ the ideal sheaf of $Z_1 \cup \cdots \cup Z_i$ in $\pi^{-1}(S\backslash\{P\})$. This sheaf is flat on $S\backslash\{P\}$, and we have

$$0 = \mathcal{J}_k \subset \mathcal{J}_{k-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{O}_{\pi^{-1}(S\backslash\{P\})} \ .$$

The quotients $\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/\mathcal{J}_1$, $\mathcal{J}_i/\mathcal{J}_{i+1}$, $1 \leq i < k$, are also flat on $S\backslash\{P\}$. We obtain the filtration of sheaves on $\mathcal{C}$

$$0 = \overline{\mathcal{J}_k} \subset \overline{\mathcal{J}_{k-1}} \subset \cdots \subset \overline{\mathcal{J}_1} \subset \mathcal{O}_{\mathcal{C}} \ .$$

(cf. 2.2). According to proposition 2.2.2 the quotients $\mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}_1}$ and $\overline{\mathcal{J}_i}/\overline{\mathcal{J}_{i+1}}$, $1 \leq i < n$, are flat on $S$. We have $\mathcal{O}_{\pi^{-1}(S\backslash\{P\})}/\mathcal{J}_1 = \mathcal{O}_{Z_1}$. We denote by $\mathbf{X}_i$ the closed subvariety of $\mathcal{C}$ corresponding to the ideal sheaf $\overline{\mathcal{J}_i}$.

Similarly we consider the ideal sheaf $\mathcal{J}_i'$ of $Z_{i+1} \cup \cdots \cup Z_n$ on $\pi^{-1}(S\backslash\{P\})$, the associated ideal sheaf $\overline{\mathcal{J}_i'}$ on $\mathcal{C}$ and the corresponding subvariety $\mathbf{X}_i'$.

**3.2.1. Proposition:** *We have* $k \leq n$ .

*Proof.* Let $\mathcal{E}_0 = \mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}_1}$ and $\mathcal{E}_i = \overline{\mathcal{J}_i}/\overline{\mathcal{J}_{i+1}}$ for $1 \leq i < n$. The sheaves $\mathcal{E}_{iP}$ are not concentrated on a finite number of points. To see this we use a very ample line bundle $\mathcal{O}(1)$ on $\mathcal{C}$. The Hilbert polynomial of $\mathcal{E}_{iP}$ is the same as that of $\mathcal{E}_{is}$, $s \neq P$, hence it is not constant. So we have $R(\mathcal{E}_i) \geq 1$ (cf. 2.3), and since

$$n \;=\; R(\mathcal{O}_{C_n}) \;= \sum_{i=0}^{k} R(\mathcal{E}_{iP}) \;, \tag{1}$$

we have $k \leq n$. □

**3.2.2. Definition:** *We say that $\pi$ (ou $\mathcal{C}$) is a* maximal reducible deformation *of $C_n$ if $k = n$.*

**3.2.3. Theorem:** *Suppose that $\mathcal{C}$ is a maximal reducible deformation of $C_n$. Then we have, for $1 \leq i < n$*

$$\overline{\mathcal{J}_i}_{,P} \;=\; \mathcal{I}_{C_i,C_n}$$

*and $\mathbf{X}_i$ is a maximal reducible deformation of $C_i$.*

*Proof.* Let $\mathcal{O}_{\mathcal{C}}(1)$ be a very ample line bundle on $\mathcal{C}$.

Let $Q$ be a closed point of $C$. Let $z \in \mathcal{O}_{n,Q}$ be an equation of $C$ and $x \in \mathcal{O}_{n,Q}$ over a generator of the maximal ideal of $Q$ in $\mathcal{O}_{C,Q}$. Let $\mathbf{z}, \mathbf{x} \in \mathcal{O}_{\mathcal{C},Q}$ be over $z, x$ respectively. The maximal ideal of $\mathcal{O}_{nQ}$ is $(x, z)$. The maximal ideal of $\mathcal{O}_{\mathcal{C},Q}$ is generated by $\mathbf{z}, \mathbf{x}, t$. It follows from proposition 2.1.1 that there exists a neighbourhood $U$ of $Q$ in $\mathcal{C}$ and an embedding $j : U \to \mathbb{P}_3$. We can assume that the restriction of $j$ to $\overline{Z_1} \cap U$ is induced by the morphism $\phi : \mathbb{C}[X, Z, T] \to \mathcal{O}_{\overline{Z_1},Q}$ of $\mathbb{C}$-algebras which associates $x$, $z$, $t$ to $X$, $Z$, $T$ respectively.

Since $\mathcal{C}$ is reduced, $U$ is an open subset of a reduced hypersurface of $\mathbb{P}_3$ having $n$ irreducible components, corresponding to $\overline{Z_1}, \ldots, \overline{Z_n}$. It is then clear that $\mathbf{X}_i$, beeing the smallest subscheme of $\mathcal{C}$ containing $Z_1\backslash C, \ldots, Z_i\backslash C$, is the union in $U$ of the first $i$ hypersurface components.

Since $j(\overline{Z_1})$ is an hypersurface, the kernel of $\phi$ is a principal ideal generated by the equation $F$ of the image of $Z_1$.

Recall that $\mathcal{O}_n = \mathcal{O}_{C_n} = (\mathcal{O}_{\mathcal{C}})_P$. We have $R(\mathcal{O}_n/\overline{\mathcal{J}_1}_{,P}) = 1$ according to (1). Hence there exists a nonempty open subset $V$ of $C_n$ such that $\left(\mathcal{O}_n/\overline{\mathcal{J}_1}_{,P}\right)_{|V}$ is a line bundle on $V \cap C$. It follows that the projection $\mathcal{O}_n \to \mathcal{O}_C$ vanishes on $\overline{\mathcal{J}_1}_{,P|V}$. Since $\mathcal{O}_C$ is torsion free this projection vanishes everywhere on $\overline{\mathcal{J}_1}$, i.e. $\overline{\mathcal{J}_1}_P \subset \mathcal{I}_{C,C_n}$, with equality on $V$.

The sheaf $\mathcal{E}_0 = \mathcal{O}_{\mathcal{C}}/\overline{\mathcal{J}_1}$ is the structural sheaf of $\overline{Z_1}$, and the projection $\overline{Z_1} \to S$ is a flat morphism. For every $s \in S\backslash\{P\}$, $(\overline{Z_1})_s$ is a smooth curve. The fiber $(\overline{Z_1})_P$ consists of $C$ and a finite number of embedded points. There exists flat families of curves whose general fiber is smooth

and the special fiber consists of an integral curve and some embedded points (cf. [15], III, Example 9.8.4). We will show that this cannot happen in our case, i.e. we have $\overline{\mathcal{J}_1}_P = \mathcal{I}_{C,C_n}$.

Let $\mathbf{m} = (X, Z, T) \subset \mathbb{C}[X, Z, T]$, and $\mathbf{m}_{Z_1}$ the maximal ideal of $\mathcal{O}_{\overline{Z_1},Q}$. The ideal of $(\overline{Z_1})_P$ in $\mathcal{O}_{n,Q}$ contains $z^q$ and $x^p z$ (for suitable minimal integers $p \geq 0, q > 0$), with $p > 0$ if and only if $Q$ is an embedded point. Hence the ideal of $\overline{Z_1}$ in $\mathcal{O}_{\mathcal{C},Q}$ contains elements of type $\mathbf{x}^p\mathbf{z} - t\alpha$, $\mathbf{z}^q - t\beta$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C},Q}$.

Let $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ be the completion of $\mathcal{O}_{\overline{Z_1},Q}$ with respect to $\mathbf{m}_{Z_1}$ and

$$\widehat{\phi} : \mathbb{C}((X, Z, T)) \longrightarrow \widehat{\mathcal{O}_{\overline{Z_1},Q}}$$

the morphism deduced from $\phi$. We can also see $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ as the completion with respect to $(X, Z, T)$ of $\mathcal{O}_{\overline{Z_1},Q}$ seen as a $\mathbb{C}[X, Z, T]$-module. It follows that $\ker(\widehat{\phi}) = (F)$ (cf. [11], lemma 7.15). Note that $\widehat{\phi}$ is surjective (this is why we use completions). Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}((X, Y, Z))$ be such that $\widehat{\phi}(\boldsymbol{\alpha}) = \alpha$, $\widehat{\phi}(\boldsymbol{\beta}) = \beta$. So we have

$$X^p Z - T\boldsymbol{\alpha},\ Z^q - T\boldsymbol{\beta} \ \in\ \ker(\widehat{\phi})\ .$$

Hence there exists $A, B \in \mathbb{C}((X, Z, T))$ such that $X^p Z - T\boldsymbol{\alpha} = AF$, $Z^q - T\boldsymbol{\beta} = BF$. We can write in an unique way

$$A = A_0 + TA_1,\ B = B_0 + TB_1,\ F = F_0 + TF_1,$$

with $A_0, B_0, F_0 \in \mathbb{C}((X, Z))$ and $A_1, B_1, F_1 \in \mathbb{C}((X, Z, T))$, and we have

$$A_0F_0 = X^pZ,\ B_0F_0 = Z^q\ .$$

Since $F$ is not invertible, it follows that $F_0$ is of the form $F_0 = cZ$, with $c \in \mathbb{C}((X, Z, T))$ invertible. So we have $F = cZ + TF_1$. It follows that $z \in (t)$ in $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$. This implies that this is also true in $\mathcal{O}_{\overline{Z_1},Q}$ : in fact the assertion in $\widehat{\mathcal{O}_{\overline{Z_1},Q}}$ implies that

$$z\ \in\ \bigcap_{n\geq 0}((t) + \mathbf{m}_{Z_1}^n)$$

in $\mathcal{O}_{\overline{Z_1},Q}$, and the latter is equal to $(t)$ according to [18], vol. II, chap. VIII, theorem 9. Hence $z \in (t)$ in $\mathcal{O}_{\overline{Z_1},Q}$, i.e. $p = 0$ and $Q$ is not an embedded point. So there are no embedded points. This implies that $\overline{\mathcal{J}_1}_P = \mathcal{I}_{C,C_n}$. Similarly, if $I_j$ denotes the ideal sheaf of $\overline{Z_j}$ for $1 \leq j \leq n$, we have $I_{j,P} = \mathcal{I}_{C,C_n}$. Since the restriction of $\pi : \overline{Z_j} \to S$ is flat, the curves $\mathcal{E}_{j,s}$, $s \neq P$, have the same genus as $C$, and the same Hilbert polynomial with respect to $\mathcal{O}_{\mathcal{C}}(1)$.

Now we show that $\mathbf{X}'_1$ is a maximal reducible deformation of $C_{n-1}$. We need only to show that $\mathbf{X}'_{1,P} = C_{n-1}$. As we have seen, for $2 \leq j \leq n$, a local equation of $\overline{Z_j}$ at any point $Q \in C$ induces a generator $u_j$ of $\mathcal{I}_{C,C_n,Q}$. Hence $u = \prod_{2\leq j\leq n} u_j$ is a generator of $\mathcal{I}_{C_{n-1},C_n,Q}$. But $u = 0$ on $\mathbf{X}'_1$. It follows that $\mathbf{X}'_{1,P} \subset C_{n-1}$. But the Hilbert polynomial of $\mathcal{O}_{C_{n-1}}$ is the same as that of the structural sheaves of the fibers of the flat morphism $\mathbf{X}'_1 \to S$ over $s \neq P$, hence the same as $\mathcal{O}_{\mathbf{X}'_{1,P}}$. Hence $\mathbf{X}'_{1,P} = C_{n-1}$.

The theorem 3.2.3 is then easily proved by induction on $n$. □

**3.2.4. Corollary :** *Let $s \in S\backslash\{P\}$ and $D_1, D_2$ be two irreducible components of $\mathcal{C}_s$. Then $D_1$ is of genus $g$ and $D_1 \cap D_2$ consists of $-\deg(L)$ points.*

*Proof.* According to theorem 3.2.3, there exists a flat family of smooth curves $\mathbf{C}$ parametrized by $S$ such that $\mathbf{C}_P = C$ and $\mathbf{C}_s = D_1$. So the genus of $D_1$ is equal to that of $C$.

Let us prove the second assertion. Again according to theorem 3.2.3 we can suppose that $n = 2$. We have then $\chi(\mathcal{C}_s) = \chi(C_2) = 2\chi(C) + \deg(L)$. Let $x_1, \ldots, x_N$ be the intersection points of $D_1$ and $D_2$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D_2}(-x_1 - \cdots - x_N) \longrightarrow \mathcal{O}_{\mathcal{C}_s} \longrightarrow \mathcal{O}_{D_1} \longrightarrow 0.$$

Whence $\chi(\mathcal{O}_{\mathcal{C}_s}) = \chi(D_1) + \chi(D_2) - N = 2\chi(\mathcal{O}_C) - N$ (according to the first assertion). Whence $N = -\deg(L)$. □

**3.2.5.** It follows from the previous results that if $\pi : \mathcal{C} \to S$ is a maximal reducible deformation of $C_n$, then we have

(i) $\deg(L) \leq 0$ .
(ii) $\mathcal{C}$ has exactly $n$ irreducible components $\mathcal{C}_1 \ldots, \mathcal{C}_n$.
(iii) For $1 \leq i \leq n$, the restriction of $\pi$, $\pi_i : \mathcal{C}_i \to S$ is a flat morphism , and $\pi_i^{-1}(P) = C$.
(iv) For every nonempty subset $I \subset \{1, \ldots, n\}$, let $\mathcal{C}_I$ be the union of the $\mathcal{C}_i$ such that $i \in I$, and $m$ the number of elements of $I$. Then the restriction of $\pi$, $\pi_I : \mathcal{C}_I \to S$ is a maximal reducible deformation of $C_m$.

The following is immediate, and shows that we need only to consider maximal reducible deformations parametrized by a neighbourhood of 0 in $\mathbb{C}$:

**3.2.6. Proposition:** *Let $t \in \mathcal{O}_S(P)$ be a generator of the maximal ideal, and $\pi : \mathcal{C} \to S$ a maximal reducible deformation of $C_n$. Let $S' \subset S$ an open neighbourhood of $P$ where $t$ is defined and $\mathcal{C}' = \pi^{-1}(U)$, $V = t(U)$. Then $\pi' = t \circ \pi : \mathcal{C}' \to V$ is a maximal reducible deformation of $C_n$.*

## 4. Fragmented deformations of primitive multiple curves

The fragmented deformations of primitive multiple curves are particular cases of reducible deformations.

In this chapter $(S, P)$ denotes a germ of smooth curve. Let $t \in \mathcal{O}_{S,P}$ be a generator of the maximal ideal of $P$. We can suppose that $t$ is defined on the whole of $S$, and that the ideal sheaf of $P$ in $S$ is generated by $t$.

### 4.1. Fragmented deformations and glueing

Let $n > 0$ be an integer and $Y = C_n$ a projective primitive multiple curve of multiplicity $n$.

**4.1.1. Definition:** *Let $k > 0$ be an integer. A* general fragmented deformation of length $k$ *of $C_n$ is a flat morphism $\pi : \mathcal{C} \to S$ such that for every point $s \neq P$ of $S$, the fiber $\mathcal{C}_s$ is a disjoint union of $k$ projective smooth irreducible curves, and such that $\mathcal{C}_P$ is isomorphic to $C_n$.*

We have then $k \leq n$. If $k = n$ we say that $\pi$ (or $\mathcal{C}$) is a *general maximal fragmented deformation* of $C_n$. We suppose in the sequel that it is the case.

The line bundle on $C$ associated to $C_n$ is $\mathcal{O}_C$ (by proposition 3.2.4).

Let $p > 0$ be an integer. Let $K$ be the field of rational functions on $S$ and $K' = K(t^{1/p})$. Let $S'$ be the germ of curve corresponding to $K'$, $\theta : S' \to S$ the canonical morphism and $P'$ the unique point of $\theta^{-1}(P)$. Let $\mathcal{D} = \theta^*(\mathcal{C})$. So we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\rho} & S' \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \theta} \\ \mathcal{C} & \xrightarrow{\pi} & S \end{array}$$

where $\rho$ is flat, and for every $s' \in S'$, $\Theta$ induces an isomorphism $\mathcal{D}_{s'} \simeq \mathcal{C}_{\theta(s')}$.

**4.1.2. Proposition:** *For a suitable choice of $p$, $\mathcal{D}$ has exactly $n$ irreducible components $\mathcal{D}_1, \ldots, \mathcal{D}_n$, and for every point $s \neq P'$ of $S'$, $\mathcal{D}_{1s}, \ldots, \mathcal{D}_{ns}$ are the irreducible components of $\mathcal{D}_s$, for $1 \leq i \leq n$ the restriction of $\rho$: $\mathcal{D}_{is} \to S'$ is flat, and $\mathcal{D}_{P'} = C_n$.*

(See proposition 3.1.3)

**4.1.3. Definition:** *A* fragmented deformation *of $C_n$ is a general maximal fragmented deformation of length $n$ of $C_n$ having $n$ irreducible components.*

We suppose in the sequel that $\mathcal{C}$ is a fragmented deformation of $C_n$, union of $n$ irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

**4.1.4. Proposition:** *Let $I \subset \{1, \ldots, n\}$ a nonempty subset having $m$ elements. Let $\mathcal{C}_I = \cup_{i \in I} \mathcal{C}_i$. Then the restriction of $\pi$, $\mathcal{C}_I \to S$, is flat, and the fiber $\mathcal{C}_{IP}$ is canonically isomorphic to $C_m$.*

(See 3.2.5)

In particular there exists a filtration of ideal sheaves

$$0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_{n-1} \subset \mathcal{O}_{\mathcal{C}}$$

such that for $1 \leq i < n$ and $s \in S \backslash \{P\}$, $\mathcal{I}_{is}$ is the ideal sheaf of $\cup_{j=i}^{n} \mathcal{C}_{js}$, and that $\mathcal{I}_{iP}$ is that of $C_{n-i}$.

**4.1.5. Definition:** *For $1 \leq i \leq n$, let $\pi : \mathcal{C}_i \to S$ be a flat family of smooth projective irreducible curves, with a fixed isomorphism $\pi_i^{-1}(P) \simeq C$. A* glueing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along $C$ *is an algebraic variety $\mathcal{D}$ such that*

- *for $1 \leq i \leq n$, $\mathcal{C}_i$ is isomorphic to a closed subvariety of $\mathcal{D}$, also denoted by $\mathcal{C}_i$, and $\mathcal{D}$ is the union of these subvarieties.*
- *$\coprod_{1 \leq i \leq n} (\mathcal{C}_i \backslash C)$ is an open subset of $\mathcal{D}$.*
- *There exists a morphism $\pi : \mathcal{D} \to S$ inducing $\pi_i$ on $\mathcal{C}_i$, for $1 \leq i \leq n$.*
- *The subvarieties $C = \pi_i^{-1}(P)$ of $\mathcal{C}_i$ coincide in $\mathcal{D}$.*

For example the previous fragmented deformation $\mathcal{C}$ of $C_n$ is a glueing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along $C$.

All the glueings of $\mathcal{C}_1, \cdots, \mathcal{C}_n$ along $C$ have the same underlying Zariski topological space.

Let $\boldsymbol{\mathcal{A}}$ the *initial glueing* of the $\mathcal{C}_i$ along $C$. It is an algebraic variety whose points are the same as those of $\mathcal{C}$, i.e.

$$(\coprod_{i=1}^{n} \mathcal{C}_i)/\sim \quad ,$$

where $\sim$ is the equivalence relation: if $x \in \mathcal{C}_i$ and $y \in \mathcal{C}_j$, $x \sim y$ if and only if $x = y$, or if $x \in \mathcal{C}_{iP} \simeq C$, $y \in \mathcal{C}_{jP} \simeq C$ and $x = y$ in $C$. The structural sheaf is defined by : for every open subset $U$ of $\boldsymbol{\mathcal{A}}$

$$\mathcal{O}_{\boldsymbol{\mathcal{A}}}(U) \;=\; \{(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}_1}(U \cap \mathcal{C}_1) \times \cdots \mathcal{O}_{\mathcal{C}_n}(U \cap \mathcal{C}_n); \alpha_{1|C} = \cdots = \alpha_{n|C}\}.$$

For every glueing $\mathcal{D}$ of $\mathcal{C}_1, \cdots, \mathcal{C}_n$, we have an obvious dominant morphism $\boldsymbol{\mathcal{A}} \to \mathcal{D}$. If follows that the sheaf of rings $\mathcal{O}_{\mathcal{D}}$ can be seen as a subsheaf of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$.

The fiber $D = \boldsymbol{\mathcal{A}}_0$ is not a primitive multiple curve (if $n > 2$): if $\mathcal{I}_{C,D}$ denotes the ideal sheaf of $C$ in $D$ we have $\mathcal{I}_{C,D}^2 = 0$, and $\mathcal{I}_{C,D} \simeq \mathcal{O}_C \otimes \mathbb{C}^{n-1}$ .

**4.1.6. Proposition:** *Let $\mathcal{D}$ be a glueing of $\mathcal{C}_1, \cdots, \mathcal{C}_n$. Then $\pi^{-1}(P)$ is a primitive multiple curve if and only if for every closed point $x$ of $C$, there exists a neighbourhood of $x$ in $\mathcal{D}$ that can be embedded in a smooth variety of dimension 3.*

*Proof.* Suppose that $\pi^{-1}(P)$ is a primitive multiple curve. Then $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ is a principal module at $x$ : suppose that the image of $u \in m_{\mathcal{D},x}$ is a generator. The module $m_{\mathcal{D},x}/\mathcal{I}_C$ is also principal (since it is the maximal ideal of $x$ in $C$) : suppose that the image of $v \in m_{\mathcal{D},x}$ is a generator. Then the images of $u, v, \pi$ generate $m_{\mathcal{D},x}/m_{\mathcal{D},x}^2$, so according to proposition 2.1.1, we can locally embed $\mathcal{D}$ in a smooth variety of dimension 3.

Conversely, suppose that a neighbourhood of $x \in C$ in $\mathcal{D}$ is embedded in a smooth variety $Z$ of dimension 3. The proof of the fact that $\pi^{-1}(P)$ is Cohen-Macaulay is similar to that of theorem 3.2.3. We can suppose that $\pi$ is defined on $Z$. We have $\pi_{|\mathcal{C}_1} = \pi_1 \notin m_{\mathcal{C}_1,x}^2$, so $\pi \notin m_{Z,x}^2$. It follows that the surface of $Z$ defined by $\pi$ is smooth at $x$, and that we can locally embed $\pi^{-1}(P)$ in a smooth surface. Hence $\pi^{-1}(P)$ is a primitive multiple curve. □

## 4.2. Fragmented deformations of length 2

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $C_2$. So $\mathcal{C}$ has two irreducible components $\mathcal{C}_1$, $\mathcal{C}_2$. Let $\mathcal{A}$ be the glueing of $\mathcal{C}_1$ and $\mathcal{C}_2$ along $C$. For every open subset $U$ of $\mathcal{C}$, $U$ is also an open subset of $\mathcal{A}$ and $\mathcal{O}_{\mathcal{C}}(U)$ is a sub-algebra of $\mathcal{O}_{\mathcal{A}}(U)$. For $i = 1, 2$, let $\pi_i : \mathcal{C}_i \to S$ be the restriction of $\pi$. We will also denote $t \circ \pi$ by $\pi$, and $t \circ \pi_i$ by $\pi_i$. So we have $\pi = (\pi_1, \pi_2) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$.

Let $\mathcal{I}_C$ be the ideal sheaf of $C$ in $\mathcal{C}$. Since $C_2 = \pi^{-1}(P)$ we have $\mathcal{I}_C^2 \subset \langle(\pi_1, \pi_2)\rangle$ .

Let $m > 0$ be an integer, $x \in C$, $\alpha_1 \in \mathcal{O}_{\mathcal{C}_1,x}$, $\alpha_2 \in \mathcal{O}_{\mathcal{C}_2,x}$. We denote by $[\alpha_1]_m$ (resp. $[\alpha_2]_m$) the image of $\alpha_1$ (resp. $\alpha_2$) in $\mathcal{O}_{\mathcal{C}_1,x}/(\pi_1^m)$ (resp. $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^m)$).

**4.2.1. Proposition:** **1** − *There exists an unique integer $p > 0$ such that $\mathcal{I}_C/\langle(\pi_1, \pi_2)\rangle$ is generated by the image of $(\pi_1^p, 0)$.*

**2** − *The image of* $(0,\pi_2^p)$ *generates* $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$.

**3** − *For every* $x\in C$, $\alpha\in\mathcal{O}_{\mathcal{C}_1,x}$ *and* $\beta\in\mathcal{O}_{\mathcal{C}_2,x}$, *we have* $(\pi_1^p\alpha,0)\in\mathcal{O}_{\mathcal{C},x}$ *and* $(0,\pi_2^p\beta)\in\mathcal{O}_{\mathcal{C},x}$.

*Proof.* Let $x\in C$ and $u=(\pi_1\alpha,\pi_2\beta)$ whose image is a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ at $x$ ($\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is a locally free sheaf of rank 1 of $\mathcal{O}_C$-modules). Let $\beta_0\in\mathcal{O}_{\mathcal{C}_1,x}$ be such that $(\beta_0,\beta)\in\mathcal{O}_{\mathcal{C},x}$. Then the image of

$$u-(\pi_1,\pi_2)(\beta_0,\beta)=(\pi_1(\alpha-\beta_0),0)$$

is also a generator of $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ at $x$. We can write it $(\pi_1^p\lambda,0)$, where $\lambda$ is not a multiple of $\pi_1$.

Now we show that $p$ is the smallest integer $q$ such that $(\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle)_x$ contains the image of an element of the form $(\pi_1^q\mu,0)$, with $\mu$ not divisible by $\pi_1$. We can write

$$(\pi_1^q\mu,0)\ =\ (u_1,u_2)(\pi_1^p\lambda,0)+(v_1,v_2)(\pi_1,\pi_2)$$

with $(u_1,u_2),(v_1,v_2)\in\mathcal{O}_{\mathcal{C},x}$. So we have $v_2=0$, hence $(v_1,v_2)\in\mathcal{I}_{C,x}$. So we can write $(v_1,v_2)$ as the sum of a multiple of $(\pi_1^p\lambda,0)$ and a multiple of $(\pi_1,\pi_2)$. Finally we obtain $(\pi_1^q\mu,0)$ as

$$(\pi_1^q\mu,0)\ =\ (u_{12},u_{22})(\pi_1^p\lambda,0)+(v_{11},0)(\pi_1,\pi_2)^2.$$

In the same way we see that $(\pi_1^q\mu,0)$ can be written as

$$(\pi_1^q\mu,0)\ =\ (u_{1p},u_{2p})(\pi_1^p\lambda,0)+(v_{1p},0)(\pi_1,\pi_2)^p,$$

which implies immediately that $q\geq p$.

It follows that $p$ does not depend on $x$ and that $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is a subsheaf of $\langle(\pi_1^p,0)\rangle/\langle(\pi_1^{p+1},0)\rangle\simeq\mathcal{O}_C$. Since $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle$ is of degree 0 by (by corollary 3.2.4) it follows that $\mathcal{I}_C/\langle(\pi_1,\pi_2)\rangle\simeq\langle(\pi_1^p,0)\rangle/\langle(\pi_1^{p+1},0)\rangle$, from which we deduce assertion 1- of proposition 4.2.1. The second assertion comes from the fact that $(0,\pi_2^p)=\pi^p-(\pi_1^p,0)$.

To prove the third, we use the fact that there exists $\alpha'\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha,\alpha')\in\mathcal{O}_{\mathcal{C},x}$ (because $\mathcal{C}_1\subset\mathcal{C}$). Hence $(\pi_1^p,0)(\alpha,\alpha')=(\pi_1^p\alpha,0)\in\mathcal{O}_{\mathcal{C},x}$. Similarly, we obtain that $(0,\pi_2^p\beta)\in\mathcal{O}_{\mathcal{C},x}$. □

According to the proof the proposition 4.2.1, for every $x\in C$, $p$ is the smallest integer $q$ such that there exists an element of $\mathcal{O}_{\mathcal{C},x}$ of the form $(\pi_1^q\alpha,0)$ (resp. $(0,\pi_2^q\alpha)$), with $\alpha\in\mathcal{O}_{\mathcal{C}_1,x}$ (resp. $\alpha\in\mathcal{O}_{\mathcal{C}_2,x}$) not vanishing on $C$.

Let $x\in C$ and $\alpha_1\in\mathcal{O}_{\mathcal{C}_1,x}$. Since $\mathcal{C}_1\subset\mathcal{C}$ there exists $\alpha_2\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C},x}$. Let $\alpha_2'\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $(\alpha_1,\alpha_2')\in\mathcal{O}_{\mathcal{C},x}$. We have then $(0,\alpha_2-\alpha_2')\in\mathcal{O}_{\mathcal{C},x}$. So there exists $\alpha\in\mathcal{O}_{\mathcal{C}_2,x}$ such that $\alpha_2-\alpha_2'=\pi_2^p\alpha$. It follows that the image of $\alpha_2$ in $\mathcal{O}_{\mathcal{C}_2,x}/(\pi_2^p)$ is uniquely determined. Hence we have:

**4.2.2. Proposition:** *There exists a canonical isomorphism*

$$\Phi:\mathcal{C}_1^{(p)}\longrightarrow\mathcal{C}_2^{(p)}$$

*between the infinitesimal neighbourhoods of order* $p$ *of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *(i.e.* $\mathcal{O}_{\mathcal{C}_i^{(p)}}=\mathcal{O}_{\mathcal{C}_i}/(\pi_i^p)$*), such that for every* $x\in C$, $\alpha_1\in\mathcal{O}_{\mathcal{C}_1,x}$ *and* $\alpha_2\in\mathcal{O}_{\mathcal{C}_2,x}$, *we have* $(\alpha_1,\alpha_2)\in\mathcal{O}_{\mathcal{C},x}$ *if and only if* $\Phi_x([\alpha_1]_p)=[\alpha_2]_p$. *For every* $\alpha\in\mathcal{O}_{\mathcal{C}_1,x}$ *we have* $\Phi_x(\alpha)_{|C}=\alpha_{|C}$, *and* $\Phi_x(\pi_1)=\pi_2$.

The simplest case is $p=1$. In this case $\Phi : C \to C$ is the identity and $\mathcal{C} = \mathcal{A}$ (the *initial* glueing).

**4.2.3.** *Converse* - Recall that $\boldsymbol{\mathcal{A}}$ denotes the initial glueing of $\mathcal{C}_1, \mathcal{C}_2$ (cf. 4.1.5). Let $\Phi : \mathcal{C}_1^{(p-1)} \to \mathcal{C}_2^{(p-1)}$ be an isomorphism inducing the identity on $C$ and such that $\Phi(\pi_1) = \pi_2$. We define a subsheaf of algebras $\mathcal{U}_\Phi$ of $\mathcal{O}_{\boldsymbol{\mathcal{A}}}$: $\mathcal{U}_\Phi = \mathcal{O}_{\boldsymbol{\mathcal{A}}}$ on $\boldsymbol{\mathcal{A}} \backslash C$, and for every point $x$ of $C$

$$\mathcal{U}_{\Phi,x} \;=\; \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} \;;\; \Phi_x([\alpha_1]_p) = [\alpha_2]_p\} \;.$$

It is easy to see that $\mathcal{U}_\Phi$ is the structural sheaf of an algebraic variety $\mathcal{A}_\Phi$, that the inclusion $\mathcal{U}_\Phi \subset \mathcal{O}_{\boldsymbol{\mathcal{A}}}$ defines a dominant morphism $\boldsymbol{\mathcal{A}} \to \mathcal{A}_\Phi$ inducing an isomorphism between the underlying topological spaces (for the Zariski topology), and that the composed morphisms $\mathcal{C}_i \subset \boldsymbol{\mathcal{A}} \to \mathcal{A}_\Phi$, $i = 1, 2$, are immersions. Moreover, the morphism $\pi : \boldsymbol{\mathcal{A}} \to S$ factorizes through $\mathcal{A}_\Phi$ :

$$\begin{array}{ccccc} \boldsymbol{\mathcal{A}} & \longrightarrow & \mathcal{A}_\Phi & \xrightarrow{\pi_\Phi} & S \\ & \searrow & & \nearrow & \\ & & \pi & & \end{array}$$

and $\pi_\Phi : \mathcal{A}_\Phi \to S$ is flat.

For $2 \leq i \leq p$, let $\Phi^{(i)} : \mathcal{C}_1^{(i)} \to \mathcal{C}_2^{(i)}$ be the isomorphism induced by $\Phi$.

**4.2.4. Proposition:** *$\pi_\Phi^{-1}(P)$ is a primitive double curve.*

*Proof.* Let $x$ be a closed point of $C$. We first show that $\mathcal{I}_{C,x}^2 \subset (\pi)$. Let $u = (\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{C,x}$. Let $\beta' \in \mathcal{O}_{\mathcal{C}_2,x}$ be such that $\Phi_x([\alpha]_p) = [\beta']_p$. We have then $v = (\alpha, \beta') \in \mathcal{O}_{\mathcal{C},x}$. We have $u - \pi v = (0, \pi_2(\beta - \beta')) \in \mathcal{O}_{\mathcal{C},x}$. Therefore $[\pi_2(\beta - \beta')]_p = \Phi_x(0) = 0$. Hence $\pi_2(\beta - \beta') \in (\pi_2^p)$. We can then write

$$u \;=\; \pi v + (0, \pi_2^p \gamma).$$

Let $u' \in \mathcal{I}_{C,x}$, that can be written as $u' = \pi v' + (0, \pi_2^p \gamma')$. We have then

$$uu' \;=\; \pi.\big(\pi v v' + (0, \pi_2\gamma')v + (0, \pi_2\gamma)v' + (0, \pi_2^{2p-1}\gamma\gamma')\big) \;\in\; (\pi).$$

It remains to show that $\mathcal{I}_{C,x}/(\pi) \simeq \mathcal{O}_{C,x}$. We have

$$\begin{array}{rcl} \mathcal{I}_{C,x} & = & \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x}; \Phi_x([\pi_1\alpha]_p) = [\pi_2\beta]_p\} \\ & = & \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x}; \Phi_x^{(p-1)}([\alpha]_{p-1}) = [\beta]_{p-1}\}, \\ (\pi)_x & = & \{(\pi_1\alpha, \pi_2\beta) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x}; \Phi_x([\alpha]_p) = [\beta]_p\}. \end{array}$$

So if $(\pi_1\alpha, \pi_2\beta) \in \mathcal{I}_{C,x}$, we have $w = \Phi_x([\alpha]_p) - [\beta]_p \in (\pi_2^{p-1})_x/(\pi_2^p)_x \simeq \mathcal{O}_{C,x}$. Hence we have a morphism of $\mathcal{O}_{\mathcal{C},x}$-modules

$$\lambda : \mathcal{I}_{C,x} \longrightarrow \mathcal{O}_{C,x}$$

$$(\pi_1\alpha, \pi_2\beta) \longmapsto w$$

whose kernel is $(\pi)_x$. We have now only to show that $\lambda$ is surjective, which follows from the fact that $\lambda(\pi_1^p, 0) = 1$. □

**4.3.** Spectrum of a fragmented deformation and ideals of sub-deformations

Let $\pi : \mathcal{C} \to S$ be a fragmented deformation of $C_n$, $\mathcal{C}_1, \dots, \mathcal{C}_n$ the irreducible components of $\mathcal{C}$. For $1 \leq i \leq n$, let $\pi_i = \pi_{|\mathcal{C}_i}$. As in 4.2, we denote also $t \circ \pi_i$ by $\pi_i$. Let $I = \{i, j\}$ be a subset of $\{1, \dots, n\}$, with $i \neq j$. Then $\pi : \mathcal{C}_I \to S$ is a fragmented deformation of $C_2$. According to 4.2 there exists a unique integer $p > 0$ such that $\mathcal{I}_{C,\mathcal{C}_I}/(\pi)$ is generated by the image of $(\pi_i^p, 0)$ (and also by the image of $(0, \pi_j^p)$). Recall that $p$ is the smallest integer $q$ such that $\mathcal{I}_{C,\mathcal{C}_I}$ contains a non zero element of the form $(\pi_i^q \lambda, 0)$ (or $(0, \pi_j^q \mu)$), with $\lambda_{|C} \neq 0$ (resp. $\mu_{|C} \neq 0$). Let

$$p_{ij} \ = \ p_{ji} \ = \ p,$$

and $p_{ii} = 0$ for $1 \leq i \leq n$. The symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of $\mathcal{C}$.

**4.3.1.** *Generators of* $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$ - Let $i, j \in \{1, \dots, n\}$ be such that $i \neq j$. Let $x \in C$. Since $\mathcal{C}_{\{i,j\}} \subset \mathcal{C}$ there exists an element $\mathbf{u}_{ij} = (u_m)_{1 \leq m \leq n}$ of $\mathcal{O}_{\mathcal{C},x}$ such that $u_i = 0$ and $u_j = \pi_j^{p_{ij}}$. According to proposition 4.1.4, the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$ at $x$.

According to proposition 4.2.1 and the fact that the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_{C,\mathcal{C}_{ij},x}/(\mathcal{I}^2_{C,\mathcal{C}_{ij},x} + (\pi))$, for every integer $m$ such that $m \neq i, j$ and that $1 \leq m \leq n$, $u_m$ is of the form $u_m = \alpha_{ij}^{(m)} \pi_m^{p_{im}}$, with $\alpha_{ij}^{(m)} \in \mathcal{O}_{\mathcal{C}_m,x}$ invertible. Let $\alpha_{ij}^{(i)} = 0$ and $\alpha_{ij}^{(j)} = 1$.

**4.3.2. Proposition:** **1** – $\alpha_{ij|C}^{(m)}$ *is a non zero constant, uniquely determined and independent of* $x$.

**2** – *Let* $\mathbf{a}_{ij}^{(m)} = \alpha_{ij|C}^{(m)} \in \mathbb{C}$. *Then we have, for all integers* $i, j, k, m, q$ *such that* $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$

$$\mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(q)} \ = \ \mathbf{a}_{ik}^{(q)} \mathbf{a}_{ij}^{(m)}.$$

*In particular we have* $\mathbf{a}_{ij}^{(m)} = \mathbf{a}_{ik}^{(m)} \mathbf{a}_{ij}^{(k)}$ *and* $\mathbf{a}_{ij}^{(m)} \mathbf{a}_{im}^{(j)} = 1$.

*Proof.* Let $\mathbf{u}'_{ij}$ having the same properties as $\mathbf{u}_{ij}$. Then $\mathbf{v} = \mathbf{u}'_{ij} - \mathbf{u}_{ij} \in \mathcal{I}^2_{C,x} + (\pi)$. So the image of $\mathbf{v}$ in $\mathcal{O}_{\mathcal{C}_{im},x}$ belongs to $\mathcal{I}^2_{C,\mathcal{C}_{im},x} + (\pi)$. It follows that the $m$-th component of $\mathbf{v}$ is a multiple of $\pi_m^{p_{im}+1}$. Hence $\alpha_{ij|C}^{(m)}$ is uniquely determined. It follows that when $x$ varies the $\alpha_{ij|C}^{(m)}$ can be glued together and define a global section of $\mathcal{O}_C$, which must be a constant. This proves 1-.

Now we prove 2-. There exists $u \in \mathcal{O}_{\mathcal{C},x}$ such that the $k$-th component of $u$ is $\alpha_{ij}^{(k)}$, and $u$ is invertible. Then the image of $(v_m) = \frac{\mathbf{u}_{ij}}{u}$ generates $\mathcal{I}_C/(\mathcal{I}_C^2 + (\pi))$, and $v_k = 1$. Hence according to 1-, we have $v_{m|C} = \mathbf{a}_{ik}^{(m)}$, i.e.

$$\frac{\mathbf{a}_{ij}^{(m)}}{\mathbf{a}_{ij}^{(k)}} \ = \ \mathbf{a}_{ik}^{(m)}.$$

We have the same equality with $m$ instead of $q$, whence 2- is easily deduced. □

Let $p$ an integer such that $1 \leq p < n$, and $(i_1, j_1), \dots, (i_p, j_p)$ $p$ pairs of distinct integers of $\{1, \dots, n\}$. Then the image of $\prod_{m=1}^p \mathbf{u}_{i_m j_m}$ is a generator of $(\mathcal{I}_C^p + (\pi))/(\mathcal{I}_C^{p+1} + (\pi))$.

Let $I \subset \{1, \cdots, n\}$ be a nonempty subset, distinct from $\{1, \cdots, n\}$. Let $i \in \{1, \cdots, n\} \backslash I$. Let

$$\mathbf{u}_{I,i} \;=\; \prod_{j \in I} \mathbf{u}_{ji} \ .$$

Recall that $\mathcal{C}_I = \cup_{j \in I} \mathcal{C}_j \subset \mathcal{C}$.

**4.3.3. Proposition:** *The ideal sheaf of $\mathcal{C}_I$ is generated by $\mathbf{u}_{I,i}$ at $x$.*

*Proof.* According to proposition 4.1.6 there exists an embedding of a neighbourhood of $x$ in a smooth variety of dimension 3. In this variety each $\mathcal{C}_i$ is a smooth surface defined by a single equation. The ideal of the union of the $\mathcal{C}_i$, $i \in I$ is the product of these equations. □

**4.3.4. Proposition:** *Let $i, j, k$ be distinct integers such that $1 \leq i, j, k \leq n$. Then if $p_{ij} < p_{jk}$, we have $p_{ik} = p_{ij}$.*

*Proof.* We can come down to the case $n = 3$ by considering $\mathcal{C}_{\{i,j,k\}}$. We can suppose that $p_{23} \leq p_{12} \leq p_{13}$, and we must show that $p_{23} = p_{12}$. We have

$$\mathbf{u}_{21} = (\pi_1^{p_{12}}, 0, \alpha_{21}^{(3)} \pi_3^{p_{23}}), \quad \mathbf{u}_{31} = (\pi_1^{p_{13}}, \alpha_{31}^{(2)} \pi_2^{p_{23}}, 0).$$

So

$$\mathbf{u}_{31} - \pi^{p_{13}-p_{12}} \mathbf{u}_{21} = \left(0, \alpha_{31}^{(2)} \pi_2^{p_{23}}, -\alpha_{21}^{(3)} \pi_3^{p_{23}+p_{13}-p_{12}}\right) \;\in\; \mathcal{O}_{\mathcal{C},x}.$$

Taking the image of this element in $\mathcal{O}_{\mathcal{C}_{12}x}$, we see that $p_{23} \geq p_{12}$, hence $p_{23} = p_{12}$. □

**4.3.5. Proposition:** **1** − *Let $i$, $j$ be distinct integers such that $1 \leq i, j \leq n$. Then we have $\mathcal{I}_{C,x} = (\mathbf{u}_{ij}) + (\pi)$.*

**2** − *Let $v = (v_m)_{1 \leq m \leq n} \in \mathcal{I}_{C,x}$ such that $v_i$ is a multiple of $\pi_i^p$, with $p > 0$. Then we have $v \in (\mathbf{u}_{ij}) + (\pi^p)$.*

*Proof.* Let $N = 1 + \max_{1 \leq k \leq n}(q_i)$. For every integer $j$ such that $1 \leq j \leq n$ we have $(0, \ldots, 0, \pi_j^{q_j}, 0, \ldots, 0) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$, Hence $\mathcal{I}_C^N \subset (\pi)$. We will show by induction on $k$ that $\mathcal{I}_{C,x} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Taking $k = N$ we obtain 1-.

For $k = 1$ it is obvious. Suppose that it is true for $k - 1 \geq 1$. It is enough to prove that $\mathcal{I}_{C,x}^{k-1} \subset (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. Let $w_1, \ldots, w_{k-1} \in \mathcal{I}_{C,x}$. Since the image of $\mathbf{u}_{ij}$ generates $\mathcal{I}_{C,x}/(\mathcal{I}_{C,x}^2 + (\pi))$, we can write $w_p$ as

$$w_p \;=\; \lambda_p \mathbf{u}_{ij} + \pi \mu_p + \nu_p,$$

with $\lambda_p, \mu_p \in \mathcal{O}_{\mathcal{C},x}$ and $\nu_p \in \mathcal{I}_{C,x}^2$. So we have

$$w_1 \cdots w_{k-1} \;=\; \lambda \mathbf{u}_{ij} + \pi \mu + \nu,$$

with $\lambda, \mu \in \mathcal{O}_{\mathcal{C},x}$ and $\nu_p \in \mathcal{I}_{C,x}^{2k-2}$. Since $2k - 2 \geq k$, we have $w_1 \cdots w_{k-1} \in (\mathbf{u}_{ij}) + (\pi) + \mathcal{I}_{C,x}^k$. This proves 1-.

We prove 2- by induction on $p$. The case $p = 1$ follows 1-. Suppose that it is true for $p - 1 \geq 1$. So we can write $v$ as

$$v \;=\; \lambda \mathbf{u}_{ij} + \pi^{p-1} \mu,$$

with $\lambda, \mu \in \mathcal{O}_{\mathcal{C},x}$. We can write $v_i$ as $v_i = \alpha\pi^p$. So we have $\alpha\pi_i^p = \pi_i^{p-1}\mu_i$, whence $\mu_i = \alpha\pi_i$. Hence $\mu \in \mathcal{I}_{Cx}$. According to 1- we can write $\mu$ as $\mu = \theta\mathbf{u}_{ij} + \pi\tau$, with $\theta, \tau \in \mathcal{O}_{\mathcal{C},x}$. So

$$v \ = \ (\lambda + \pi^{p-1}\theta)\mathbf{u}_{ij} + \pi^p\tau,$$

which proves the result for $p$. □

**4.3.6.** *The ideal sheaves $\mathcal{I}_{\mathcal{C}_I}$* – Recall that $I \subset \{1, \cdots, n\}$ is a nonempty subset, distinct from $\{1, \cdots, n\}$. For every subset $J$ of $\{1, \cdots, n\}$, let $J^c = \{1, \cdots, n\}\backslash J$ and $\mathcal{O}_J = \mathcal{O}_{\mathcal{C}_J}$. It follows from proposition 4.3.3 that $\mathcal{I}_{\mathcal{C}_I}$ is a line bundle on $\mathcal{C}_{I^c}$.

From now on, we suppose that $S \subset \mathbb{C}$ and $P = 0$ (cf. proposition 3.2.6).

**4.3.7. Theorem:** *We have* $\mathcal{I}_{\mathcal{C}_I} \simeq \mathcal{O}_{I^c}$.

*Proof.* By induction on $n$. If $n = 2$ the result follows from proposition 4.2.1 and the fact that $S \subset \mathbb{C}$. Suppose that it is true for $n - 1 \geq 2$. We will prove that it is true for $n$ by induction on the number of elements $q$ of $I^c$. Suppose first that $q = 1$ and let $i$ be the unique element of $I^c$. Then according to proposition 4.3.3, $\mathcal{I}_{\mathcal{C}_I}$ is generated by $(0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)$, so the result is true in this case. Suppose that it is true if $1 \leq q < k < n$, and that $q = k$. Let $K = \{1, \cdots, n-1\}$. We can assume that $I \subset K$.

According to proposition 4.3.3, we have, for every $x \in C$, $\mathcal{I}_{\mathcal{C}_I,x} \simeq \mathcal{O}_{I^c x}$. We have $\mathcal{I}_{\mathcal{C}_K} \subset \mathcal{I}_{\mathcal{C}_I}$, and $\mathcal{I}_{\mathcal{C}_K} \simeq \mathcal{O}_{\{n\}}$. We have

$$\mathcal{I}_{\mathcal{C}_I}/\mathcal{I}_{\mathcal{C}_K} \ = \ \mathcal{I}_{\mathcal{C}_I,\mathcal{C}_K}$$

(the ideal sheaf of $\mathcal{C}_I$ in $\mathcal{C}_K$). From the first induction hypothesis we have

$$\mathcal{I}_{\mathcal{C}_I,\mathcal{C}_K} \ \simeq \ \mathcal{O}_{(I\cup\{n\})^c}.$$

So we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{I}_{\mathcal{C}_I} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0.$$

Now we will compute $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}})$. According to [8], 2.3, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \longrightarrow \mathrm{Hom}(\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}).$$

Since $\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c})$ is concentrated on $\mathcal{C}_{I^c\backslash\{n\}}$, we have

$$\mathrm{Hom}(\mathrm{Tor}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{I^c}), \mathcal{O}_{\{n\}}) \ = \ \{0\}.$$

So we have

$$\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}) \ = \ \mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\backslash\{n\}}, \mathcal{O}_{\{n\}}).$$

Let $\mathcal{J}$ denote the ideal sheaf of $\mathcal{C}_{\{n\}}$ in $\mathcal{C}_{I^c}$. The ideal sheaf of $\mathcal{C}_{I^c\backslash\{n\}}$ is generated by $\mathbf{w} = (0, \ldots, 0, \pi_n^m)$, with $m = \sum_{i\in I^c\backslash\{n\}} p_{in}$. So we have an exact sequence of sheaves on $\mathcal{C}_{I^c}$

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c\backslash\{n\}} \longrightarrow 0 \ ,$$

where $\alpha$ is the multiplication by $\mathbf{w}$. By the induction hypothesis there exists a surjective morphism $\mathcal{O}_{I^c} \to \mathcal{J}$, so we get a locally free resolution of $\mathcal{O}_{I^c\setminus\{n\}}$

$$\mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c} \xrightarrow{\alpha} \mathcal{O}_{I^c} \longrightarrow \mathcal{O}_{I^c\setminus\{n\}} \longrightarrow 0\ ,$$

that can be used to compute $\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\setminus\{n\}}, \mathcal{O}_{\{n\}})$. It follows easily that

$$\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\setminus\{n\}}, \mathcal{O}_{\{n\}}) \ \simeq\ \mathcal{O}_{\{n\}}/(\pi_n^m)\ .$$

We have $\mathcal{H}om(\mathcal{O}_{I^c\setminus\{n\}}, \mathcal{O}_{\{n\}}) = 0$, hence

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\setminus\{n\}}, \mathcal{O}_{\{n\}}) &\simeq H^0(\mathcal{E}xt^1_{\mathcal{O}_{I^c}}(\mathcal{O}_{I^c\setminus\{n\}}, \mathcal{O}_{\{n\}}))\\ &\simeq H^0(\mathcal{O}_{\{n\}}/(\pi_n^m))\\ &\simeq H^0(\mathcal{O}_S/(\pi_n^m))\\ &\simeq \mathbb{C}[\pi_n]/(\pi_n^m).\end{aligned}$$

We will now describle the sheaves $\mathcal{E}$ such that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\{n\}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{I^c\setminus\{n\}} \longrightarrow 0. \tag{2}$$

Let $\nu \in \mathbb{C}[\pi_n]/(\pi_n^m)$ be associated to this exact seqence, and $\overline{\nu} \in H^0(\mathcal{O}_S)$ over $\nu$. Let

$$\tau : \mathcal{O}_{\{n\}} \longrightarrow \mathcal{O}_{\{n\}} \oplus \mathcal{O}_{I^c}$$

$$u \longmapsto (\overline{\nu}u, \mathbf{w}u)$$

Then according to the preceding resolution of $\mathcal{O}_{I^c\setminus\{n\}}$ and the construction of extensions (cf. [7], 4.2), we have $\mathcal{E} \simeq \mathrm{coker}(\tau)$. It is easy to see that if $\nu = -1$ then $\mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c}$. If $\nu$ is invertible, then we have also $\mathcal{E} \simeq \mathcal{O}_{\mathcal{I}^c}$, because the corresponding extension can be obtained from the one corresponding to $\nu = -1$ by multiplying the left morphism of the exact sequence by $\nu$.

A similar construction can be done for extensions of $\mathcal{O}_{I^c,x}$-modules (for every $x \in C$)

$$0 \longrightarrow \mathcal{O}_{\{n\},x} \longrightarrow V \longrightarrow \mathcal{O}_{I^c\setminus\{n\},x} \longrightarrow 0.$$

These extensions are classified by $\mathcal{O}_{\{n\},x}/(\pi_n^m)$, and $\mathcal{O}_{I^c,x}$ corresponds to $-1$.

Conversely we consider extensions

$$0 \longrightarrow \mathcal{O}_{\{n\},x} \xrightarrow{\lambda} \mathcal{O}_{I^c,x} \xrightarrow{\mu} \mathcal{O}_{I^c\setminus\{n\},x} \longrightarrow 0\ .$$

Using the facts that $\mathrm{Hom}(\mathcal{O}_{\{n\},x}, \mathcal{O}_{I^c,x})$ is generated by the multiplication by $\mathbf{w}$ and $\mathrm{Hom}(\mathcal{O}_{I^c,x}, \mathcal{O}_{I^c\setminus\{n\},x})$ by the restriction morphism, it is easy to see that $\lambda$, $\mu$ are unique up to multiplication by an invertible element of $\mathcal{O}_{I^c,x}$. Hence the elements of $\mathrm{Ext}^1_{\mathcal{O}_{I^c,x}}(\mathcal{O}_{I^c\setminus\{n\},x}, \mathcal{O}_{\{n\},x})$ corresponding to the preceding extensions are exactly the invertible elements of $\mathcal{O}_{\{n\},x}/(\pi_n^m)$.

It follows that the extensions (2) where $\mathcal{E}$ is locally free correspond to invertible elements of $\mathbb{C}[\pi_n]/(\pi_n^m)$, and we have seen that in this case we have $\mathcal{E} \simeq \mathcal{O}_{I^c}$. Hence we have $\mathcal{I}_{\mathcal{C}_I} \simeq \mathcal{O}_{I^c}$ and theorem 4.3.7 is proved. □

**4.3.8. Corollary :** *The ideal sheaf of $\mathcal{C}_I$ is globally generated by an element $\mathbf{u}_I$ such that for every integer $i$ such that $1 \le i \le n$ and $i \notin I$, the $i$-th coordinate of $\mathbf{u}_I$ belongs to $H^0(\mathcal{O}_S)$.*

**4.4.** Properties of the fragmented deformations

We use the notations of 4.3.

Let $i$ be an integer such that $1 \leq i \leq n$ and $J_i = \{1, \ldots, n\}\backslash\{i\}$. We denote by $\mathcal{B}$ the image of $\mathcal{O}_{\mathcal{C}}$ in $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is a sheaf of $\mathbb{C}$-algebras on $C$. Let $\mathcal{B}_i$ be the image of $\mathcal{O}_{\mathcal{C}_{J_i}}$ in $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$; it is also a sheaf of $\mathbb{C}$-algebras on $C$. For every point $x$ of $C$ and every $\alpha = (\alpha_m)_{1\leq m\leq n}$ in $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j,x}$, we denote by $b_i(\alpha)$ its image in $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j,x}$ (obtained by forgetting the $i$-th coordinate of $\alpha$).

If $p, k$ are positive integers, with $k \leq n$, $x \in C$ and $\alpha \in \mathcal{O}_{\mathcal{C}_k,x}$, let $[\alpha]_p$ denote the image of $\alpha$ in $\mathcal{O}_{\mathcal{C}_k,x}/\pi_k^p$.

**4.4.1. Proposition:** *There exists a morphism of sheaves of algebras on $C$*

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$$

*such that for every point $x$ of $C$ and all $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\mathcal{C}_{J_i},x}$, $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$, we have $\alpha = (\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\mathcal{C},x}$ if and only if $\Phi_{i,x}(b_i(\alpha)) = [\alpha_i]_{q_i}$.*

*Proof.* Let $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\mathcal{C}_{J_i}x}$. Since $\mathcal{C}_{J_i} \subset \mathcal{C}$, there exists $\alpha_i \in \mathcal{O}_{\mathcal{C}_i,x}$ such that $(\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\mathcal{C},x}$. If $\alpha_i' \in \mathcal{O}_{\mathcal{C}_i}x$ has the same property, we have $(0, \ldots, 0, \alpha_i - \alpha_i', 0, \ldots, 0) \in \mathcal{I}_{J_i x}$. So according to proposition 4.3.3, we have $[\alpha_i]_{q_i} = [\alpha_i']_{q_i}$. Hence we have well defined a morphism of algebras $\theta_x : \mathcal{O}_{\mathcal{C}_{J_i},x} \to \mathcal{O}_{\mathcal{C}_{J_i}}/(\pi_i^{q_i})$ sending $(\alpha_m)_{1\leq m\leq n, m\neq i}$ to $[\alpha_i]_{q_i}$. If $j \in J_i$, we have according to proposition 4.3.3, $\theta_x(0, \ldots, 0, \pi_j^{q_j}, 0, \ldots, 0) = 0$. Hence $\theta_x$ induces a morphism of algebras $\mathcal{B}_{i,x} \to \mathcal{O}_{\mathcal{C}_i,x}/(\pi_i^{q_i})$. □

The morphism $\Phi_i$ has the following properties: for every point $x$ of $C$

(i) For every $\alpha = (\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{B}_{i,x}$, we have $\Phi_{i,x}(\alpha)_{|C} = \alpha_{m|C}$ for $1 \leq m \leq n$, $m \neq i$.
(ii) We have $\Phi_{i,x}((\pi_m)_{1\leq m\leq n, m\neq i}) = \pi_i$.
(iii) Let $j, k \in \{1, \cdots, n\}$ be such that $i, j, k$ are distinct. Let $\mathbf{v}$ be the image of $\mathbf{u}_{jk}$ in $\mathcal{B}_i$. Then there exists $\lambda \in \mathcal{O}^*_{\mathcal{C}_i,x}$ such that $\Phi_{i,x}(\mathbf{v}) = \lambda\pi_i^{p_{ij}}$.
(iv) Let $j$ be an integer such that $1 \leq j \leq n$ and $j \neq i$. Let $\mathbf{v}$ be the image of $\mathbf{u}_{ij}$ in $\mathcal{B}_{i,x}$. Then we have $\ker(\Phi_{i,x}) = (\mathbf{v})$.

**4.4.2.** *Converse* - Let $\mathcal{C}'$ be a glueing of $\mathcal{C}_1, \ldots, \mathcal{C}_{i-1}, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_n$ along $C$, which is a fragmented deformation of a primitive multiple curve of multiplicity $n-1$. Let $(p_{jk})_{1\leq j,k\leq n, j,k\neq i}$ be the spectrum of $\mathcal{C}'$. Let $p_{ij}$, $1 \leq j \leq n, j \neq i$ be positive integers, and $p_{ii} = 0$. For $1 \leq j \leq n$, let $q_j = \sum_{1\leq k\leq n} p_{kj}$.

Let $\mathcal{B}_i$ be the image of $\mathcal{O}_{\mathcal{C}'}$ in $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$ and

$$\Phi_i : \mathcal{B}_i \longrightarrow \mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$$

a morphism of sheaves of algebras on $C$ satisfying properties (i), (ii), (iii) above. Let $\mathcal{A}$ be the subsheaf of algebras of $\boldsymbol{\mathcal{A}}$ defined by : $\mathcal{A} = \boldsymbol{\mathcal{A}}$ on $\boldsymbol{\mathcal{A}}_{top}\backslash C$, and for every point $x$ of $C$, and every $\alpha = (\alpha_m)_{1\leq m\leq n} \in \prod_{m=1}^n \mathcal{O}_{\mathcal{C}_m,x}$, $\alpha \in \mathcal{A}_x$ if and only if $b_i(\alpha) \in \mathcal{B}_{i,x}$ and $\Phi_{i,x}(b_i(\alpha)) = [\alpha_i]_{q_i}$.

It is easy to see that $\mathcal{A}$ is the structural sheaf of a glueing of $C_1, \ldots, C_n$ along $C$, which is a fragmented deformation of a primitive multiple curve of multiplicity $n$, and that $\mathcal{C}' = \mathcal{A}_{\{1,\ldots,i-1,i+1,\ldots,n\}}$.

We give now some applications of the preceding construction.

**4.4.3. Corollary :** *Let $N$ an integer such that $N \geq \max_{1\leq i\leq n}(q_i)$. Let $x \in C$, $\beta \in \mathcal{O}_{\mathcal{C}_1,x} \times \cdots \mathcal{O}_{\mathcal{C}_n,x}$ and $u \in \mathcal{O}_{\mathcal{C},x}$ such that $u_{|C} \neq 0$. Suppose that $[\beta u]_N \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$. Then we have $[\beta]_N \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$.*

*Proof.* By induction on $n$. It is obvious if $n = 1$. Suppose that the lemma is true for $n-1$. Let $I = \{1, \ldots, n-1\}$. So we have $[\beta_{|\mathcal{C}_1\times\cdots\mathcal{C}_{n-1}}]_N \in \mathcal{O}_{\mathcal{C}_I,x}/(\pi_1, \ldots, \pi_{n-1})^N$ by the induction hypothesis. Let $\gamma$ (resp. $v$) be the image of $\beta$ (resp. $u$) in $\mathcal{B}_n$. To show that $[\beta]_N \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$ it is enough to verify that

$$\Phi_n(\gamma) \;=\; [\beta_n]_{q_n}.$$

We have $\Phi_n(\gamma v) \;=\; [\beta_n u_n]_{q_n}$ because $[\beta u]_N \in \mathcal{O}_{\mathcal{C},x}/(\pi^N)$, and $\Phi_n(v) = [u_n]_{q_n}$ because $u \in \mathcal{O}_{\mathcal{C},x}$. So we have

$$\Phi_n(\gamma)[u_n]_{q_n} \;=\; \Phi_n(\gamma)\Phi_n(v) \;=\; \Phi_n(\gamma v) \;=\; [\beta_n u_n]_{q_n} \;=\; [\beta_n]_{q_n}[u_n]_{q_n}.$$

Since $u_{|C} \neq 0$, $[u_n]_{q_n}$ is not a zero divisor in $\mathcal{O}_{\mathcal{C}_n,x}/(\pi_n^{q_n})$, so we have $\Phi_n(\gamma) = [\beta_n]_{q_n}$. □

**4.4.4. Corollary :** *Let $\mathbf{q} = \max_{1\leq i\leq n}(q_i)$ and $p$ the number of integers $i$ such that $1 \leq i \leq n$ and $q_i = \mathbf{q}$. Then we have $p \geq 2$.*

*Proof.* Suppose that $q_i = \mathbf{q}$. Then we have $\pi_i^{q_i-1} \neq 0$ in $\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i})$. Since $\pi_i = \Phi_i((\pi_m)_{1\leq m\leq n, m\neq i})$, we have $(\pi_m^{q_i-1})_{1\leq m\leq n, m\neq i} \neq 0$ in $\mathcal{B}_i$. So we cannot have $q_m < q_i$ for all the $m \neq i$. □

Let $i$ be an integer such that $1 \leq i \leq n$,

$$\mathcal{H} = \prod_{1\leq j\leq n} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^n \qquad (\text{resp.} \quad \mathcal{H}_i = \prod_{1\leq j\leq n, j\neq i} (\pi_j^{q_j-1})/(\pi_j^{q_j}) \simeq \mathcal{O}_C^{n-1}\ ).$$

It is an ideal sheaf of $\prod_{1\leq j\leq n} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$ (resp. $\prod_{1\leq j\leq n, j\neq i} \mathcal{O}_{\mathcal{C}_j}/(\pi_j^{q_j})$ ). Let $\mathcal{J} = \mathcal{H} \cap \mathcal{B}$ (resp. $\mathcal{J}_i = \mathcal{H}_i \cap \mathcal{B}_i$), which is an ideal sheaf of $\mathcal{B}$ (resp. $\mathcal{B}_i$).

**4.4.5. Proposition:** *There exists a unique $\lambda(\mathcal{C}) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$ such that for every $\mathbf{u} = (u_j)_{1\leq j\leq n} \in \mathcal{H}$, we have $\mathbf{u} \in \mathcal{J}$ if and only if $\lambda_1 u_1 + \cdots \lambda_n u_n = 0$. The $\lambda_i$ are all non zero.*

*Proof.* We have $(\pi_m)_{1\leq m\leq n, m\neq i}.\mathcal{J}_i = 0$. Hence $\pi_i\Phi_i(\mathcal{J}_i) = 0$ and $\Phi_i(\mathcal{J}_i) \subset (\pi_i^{q_i-1})/(\pi_j^{q_i})$. The restriction of $\Phi_i$, $\mathcal{J}_i \to (\pi_i^{q_i-1})/(\pi_j^{q_i})$ is a morphism $(n-1)\mathcal{O}_C \to \mathcal{O}_C$ of vector bundles on $C$. The existence of $(\lambda_1, \ldots, \lambda_n)$ follows from that.

If $\lambda_i = 0$, we have $(0, \ldots, 0, \pi_i^{q_i-1}, 0, \ldots, 0) \in \mathcal{O}_{\mathcal{C}}(\mathcal{C})$. This is impossible because according to proposition 4.3.3, $(0, \ldots, 0, \pi_i^{q_i}, 0, \ldots, 0)$ generates the ideal sheaf of $\mathcal{C}_{J_i}$ in $\mathcal{C}$. □

For all distinct integers $i$, $j$ such that $1 \leq i, j \leq n$, let $I_{ij} = \{1, \ldots, n\} \backslash \{i, j\}$. Then according to proposition 4.3.3, $\mathbf{u}_{I_{ij}i}$ generates the ideal sheaf of $\mathcal{C}_{I_{ij}}$. We have $\mathbf{u}_{I_{ij}i} = (b_k)_{1 \leq k \leq n}$, with $b_k = 0$ if $k \neq i, j$, $b_i = \pi_i^{q_i - p_{ij}}$ and

$$b_j \ = \ \big( \prod_{1 \leq m \leq n, m \neq i, j} \alpha_{mi}^{(j)} \big) . \pi_j^{q_j - p_{ij}} .$$

So we have $\pi^{p_{ij}-1} \mathbf{u}_{I_{ij}i} \in \mathcal{J}_i$, which gives the equation

$$\frac{\lambda_i}{\lambda_j} \ = \ - \prod_{1 \leq m \leq n, m \neq i, j} \mathbf{a}_{mi}^{(j)} . \tag{3}$$

**4.4.6. Proposition:** *For all distinct integers $i, j, k$ such that $1 \leq i, j, k \leq n$, we have*

$$\mathbf{a}_{ki}^{(j)} \ = \ -\mathbf{a}_{ik}^{(j)} \mathbf{a}_{ji}^{(k)} \ .$$

*Proof.* We need only to treat the case $n = 3$, and the preceding formula by writing that $\frac{\lambda_1}{\lambda_3} = \frac{\lambda_1}{\lambda_2} . \frac{\lambda_2}{\lambda_3}$, using (3). □

**4.4.7. Proposition:** *Let $(\alpha_1 \pi_1^{m_1}, \ldots, \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$, with $\alpha_1, \ldots, \alpha_n$ invertible. Let $M = m_1 + \cdots + m_n$. then*

$$\big( \frac{1}{\alpha_1} \pi_1^{M - m_1}, \ldots, \frac{1}{\alpha_n} \pi_n^{M - m_n} \big) \ \in \ \mathcal{O}_{\mathcal{C},x} \ .$$

*Proof.* By induction on $n$. It is obvious for $n = 1$. Suppose that it is true for $n - 1 \geq 1$. Let $I = \{1, \ldots, n-1\}$. Then $(\alpha_1 \pi_1^{m_1}, \ldots, \alpha_{n-1} \pi_{n-1}^{m_{n-1}}) \in \mathcal{O}_{\mathcal{C}_{I,x}}$. Hence, by the induction hypothesis, we have

$$\big( \frac{1}{\alpha_1} \pi_1^{M - m_1 - m_n}, \ldots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M - m_{n-1} - m_n} \big) \ \in \ \mathcal{O}_{\mathcal{C}_{I,x}} \ .$$

So there exists $\gamma \in \mathcal{O}_{\mathcal{C}_{n,x}}$ such that

$$u \ = \ \big( \frac{1}{\alpha_1} \pi_1^{M - m_1 - m_n}, \ldots, \frac{1}{\alpha_{n-1}} \pi_{n-1}^{M - m_{n-1} - m_n}, \gamma \big) \ \in \ \mathcal{O}_{\mathcal{C},x} \ .$$

Multiplying by $(\alpha_1 \pi_1^{m_1}, \ldots, \alpha_n \pi_n^{m_n})$ we see that $(\pi_1^{M - m_n}, \ldots, \pi_{n-1}^{M - m_n}, \gamma \alpha_n \pi_n^{m_n}) \in \mathcal{O}_{\mathcal{C},x}$. Substracting $\pi^{M - m_n}$, we find that $(0, \ldots, 0, \gamma \alpha_n \pi_n^{m_n} - \pi_n^{M - m_n}) \in \mathcal{O}_{\mathcal{C},x}$. There exists $\alpha \in \mathcal{O}_{\mathcal{C},x}$ such that the $n$-th coordinate of $\alpha$ is $\alpha_n$, and $\alpha$ is invertible. It follows that $v = (0, \ldots, 0, \gamma \pi_n^{m_n} - \frac{1}{\alpha_n} \pi_n^{M - m_n}) \in \mathcal{O}_{\mathcal{C},x}$. Now we have

$$\pi^{m_n} u - v \ = \ \big( \frac{1}{\alpha_1} \pi_1^{M - m_1}, \ldots, \frac{1}{\alpha_n} \pi_n^{M - m_n} \big) \ \in \ \mathcal{O}_{\mathcal{C},x} \ .$$

□

**4.4.8. Corollary :** *Let $V \subset U$ be open subsets of $\mathcal{C}$, and suppose that $U \cap C \neq \emptyset$. Let $\alpha \in \mathcal{O}_{\mathcal{C}}(V)$ and $\beta \in \mathcal{O}_{\boldsymbol{\mathcal{A}}}(U)$ such that $\beta_{|V} = \alpha$. Then $\beta \in \mathcal{O}_{\mathcal{C}}(U)$.*

(Recall that $\boldsymbol{\mathcal{A}}$ is the *initial glueing* of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ (cf. 4.1.5)).

*Proof.* This can be proved easily by induction on $n$, using proposition 4.4.1. □

### 4.5. Construction of fragmented deformations

Consider a fragmented deformation

$$\pi = \pi^{[n-1]} = (\pi_1, \dots, \pi_{n-1}) : \mathcal{C}^{[n-1]} \longrightarrow S$$

of $C_{n-1}$, with $n-1$ irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_{n-1}$. Let $(p_{ij}^{[n-1]})_{1 \leq i,j < n}$ be its spectrum. For $1 \leq i < n$, let $q_i^{[n-1]} = \sum_{1 \leq j < n} p_{ij}^{[n-1]}$. We denote by $\mathcal{I}_C^{[n-1]}$ the ideal sheaf of $C$ in $\mathcal{C}^{[n-1]}$. Let $\lambda(\mathcal{C}^{[n-1]}) = (\lambda_1, \dots, \lambda_{n-1})$.

Let $p_{1n}, \dots, p_{n-1,n}$ be positive integers, $q_i = q_i^{[n-1]} + p_{in}$ for $1 \leq i < n$, and $q_n = p_{1n} + \cdots + p_{n-1,n}$. Let $\mathbf{u} \in \mathcal{I}_{C,x}^{[n-1]}$ whose image generates $\mathcal{I}_{C,x}^{[n-1]}/((\mathcal{I}_{C,x}^{[n-1]})^2 + (\pi))$, of the form

$$\mathbf{u} \ = \ (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1} \pi_{n-1}^{p_{n-1,n}}),$$

with $\beta_i \in \mathcal{O}_{\mathcal{C}_i,x}$ invertible for $1 \leq i < n$.

Let $\mathcal{B}^{[n-1]}$ be the image of $\mathcal{O}_{\mathcal{C}^{[n-1]}}$ in $\mathcal{O}_{\mathcal{C}_1}/(\pi_1^{q_1}) \times \cdots \times \mathcal{O}_{\mathcal{C}_{n-1}}/(\pi_{n-1}^{q_{n-1}})$ . We will also denote by $\mathbf{u}$ the image of $\mathbf{u}$ in $\mathcal{B}^{[n-1]}$. Let $\mathcal{Q} = \mathcal{B}^{[n-1]}/(\mathbf{u})$ , $\rho : \mathcal{B}^{[n-1]} \to \mathcal{Q}$ the projection and $\pi_n = \rho(\pi)$.

**4.5.1. Proposition:** *We have* $\pi_n^{q_n} = 0$ .

*Proof.* According to proposition 4.4.7 we have

$$v \ = \ \big(\frac{1}{\beta_1}\pi_1^{q_n - p_{1n}}, \dots, \frac{1}{\beta_{n-1}}\pi_n^{q_n - p_{n-1,n}}\big) \ \in \ \mathcal{O}_{\mathcal{C}^{[n-1]}x} \ .$$

Hence $\pi^{q_n} = v\mathbf{u} \in (\mathbf{u})$ in $\mathcal{O}_{\mathcal{C}^{[n-1]}x}$, and $\pi_n^{q_n} = 0$ . □

**4.5.2. Proposition: 1** – *We have* $\pi_n^{q_n - 1} = 0$ *if and only if*

$$\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \ = \ 0 \ .$$

*We suppose now that* $\frac{\lambda_1}{\beta_{1|C}} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1|C}} \neq 0$. *Then*

**2** – *For every* $\epsilon \in \mathcal{B}_x^{[n-1]}$ *such that* $\epsilon_{|C} \neq 0$, *we have* $\pi^{q_n - 1}\epsilon \notin (u)$.

**3** – *For every* $\eta \in \mathcal{B}_x^{[n-1]}/(\mathbf{u})$, *and every integer* $k$ *such that* $1 \leq k < q_n$, *we have* $\pi_n^k \eta = 0$ *if and only if* $\eta$ *is a multiple of* $\pi_n^{q_n - k}$.

**4** – $\mathcal{B}_x^{[n-1]}/(\mathbf{u})$ *is a flat* $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$*-module.*

*Proof.* We have $\pi_n^{q_n-1} = 0$ if and only if $(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) \in (\mathbf{u})$ in $\mathcal{B}_x^{[n-1]}$. We have, in $\mathcal{O}_{\mathcal{C}_{1x}} \times \cdots \times \mathcal{O}_{\mathcal{C}_{n-1,x}}$,

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) \ = \ (\beta_1 \pi_1^{p_{1n}}, \dots, \beta_{n-1,n}\pi_{n-1}^{p_{n-1,n}}).(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1}, \dots, \frac{1}{\beta_p}\pi_1^{q_{n-1}^{[n-1]}-1}),$$

and $\pi_n^{q_n-1} = 0$ if and only if there exists $\eta \in \mathcal{O}_{\mathcal{C}^{[n-1]},x}$, $a_i \in \mathcal{O}_{\mathcal{C}_i,x}$, $1 \leq i < n$, such that

$$(\pi_1^{q_n-1}, \dots, \pi_{n-1}^{q_n-1}) \ = \ \eta\mathbf{u} + (a_1\pi_1^{q_1}, \dots, a_{n-1}\pi_{n-1}^{q_{n-1}}).$$

This equality is equivalent to

$$(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1},\dots,\frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1})-\eta \;=\; (\frac{a_1}{\beta_1}\pi_1^{q_1^{[n-1]}},\dots,\frac{a_{n-1}}{\beta_{n-1}}\pi_{n-1}^{q_{n-1}^{[n-1]}}).$$

Since for $1\leq i<n$, we have $(0,\dots,0,\pi_i^{q_i^{[n-1]}},0,\dots 0)\in\mathcal{O}_{\mathcal{C}^{[n-1]},x}$, we have $\pi_n^{q_n-1}=0$ if and only if

$$(\frac{1}{\beta_1}\pi_1^{q_1^{[n-1]}-1},\dots,\frac{1}{\beta_{n-1}}\pi_1^{q_{n-1}^{[n-1]}-1}) \;\in\; \mathcal{O}_{\mathcal{C}^{[n-1]},x}.$$

So the result of 1- follows from the définition of $\lambda(\mathcal{C}^{[n-1]})$ (cf. prop. 4.4.5), 2- is an easy consequence.

Now we prove 3-, by induction on $k$. Suppose that it is true for $k=1$, and that $\pi_n^k\eta=0$, with $2\leq k<q_n$. We have $\pi_n^{k-1}.\pi_n\eta=0$, so according to the induction hypothesis, $\pi_n\eta$ is a multiple of $\pi_n^{q_n-k+1}$ : $\pi_n\eta=\pi_n^{q_n-k+1}\lambda$. So $\pi_n(\eta-\pi_n^{q_n-k}\lambda)=0$. Since -3 is true for $k=1$, we can write $\eta-\pi_n^{q_n-k}\lambda=\pi_n^{q_n-1}\epsilon$, i.e. $\eta=\pi_n^{q_n-k}(\lambda+\pi_n^{k-1}\epsilon)$, and 3- is true for $k$.

Il remains to prove 3- for $k=1$. Suppose that $\pi_n\eta=0$ (with $\eta\neq 0$). We can write $\eta$ as $\eta=\pi_n^m\theta$, where $\theta$ is not a multiple of $\pi_n$, and $0\leq m<q_n$. Let $\overline{\theta}\in\mathcal{B}_x^{[n-1]}$ be over $\theta$. Since $\mathcal{I}_C=(\mathbf{u})+(\pi)$ according to proposition 4.3.5, the condition "$\theta$ is not a multiple of $\pi_n$" is equivalent to $\overline{\theta}\notin\mathcal{I}_{C,x}$. We have $\pi^{m+1}\overline{\theta}\in(\mathbf{u})$, so according to 2-, we have $m+1\geq q_n$, which proves 3- for $k=1$. The last assertion is an easy consequence of 3-. □

**4.5.3. Example :** Let $N$ be an integer, $s\in\mathcal{O}_{\mathcal{C}^{[n-1]},x}$ invertible, and $k,l$ integers such that $1\leq k,l<n$, $k\neq l$. Suppose that for every integer $i$ such that $1\leq i<n$ and $i\neq k$ we have $N>p_{ik}^{[n-1]}$ and $N\geq q_i^{[n-1]}-q_k^{[n-1]}+p_{ik}^{[n-1]}$. We take $\mathbf{u}=\mathbf{u}_{kl}-s\pi^N$. We have then $\beta_i=\alpha_{kl}^{(i)}$ if $i\neq k$, and $\beta_k=-s$. The condition $\frac{\lambda_1}{\beta_{1|C}}+\dots+\frac{\lambda_{n-1}}{\beta_{n-1|C}}\neq 0$ is fulfilled if and only if

$$\sum_{1\leq i<n,i\neq k}\frac{\lambda_i}{\mathbf{a}_{kl}^{(i)}}-\frac{\lambda_k}{s} \;\neq\; 0.$$

**4.5.4.** *Construction of fragmented deformations* – Suppose that $\frac{\lambda_1}{\beta_{1|C}}+\dots+\frac{\lambda_{n-1}}{\beta_{n-1|C}}\neq 0$. From proposition 4.5.2, 4-, it is easy to prove that

- There exists a flat morphism of algebraic varieties $\tau:Y\to\mathrm{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$ with a canonical isomorphism of sheaves of $\mathbb{C}[\pi_n]/(\pi_n^{q_n})$-algebras $\mathcal{O}_Y\simeq\mathcal{Q}$, such that $\tau^{-1}(*)=C$ (where $*$ is the closed point of $\mathrm{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$).
- There exists a familiy of smooth curves $\mathcal{C}_n$ and a flat morphism $\pi_n:\mathcal{C}_n\to S$ extending $\tau$ (recall that $S$ is a germ). Hence $Y$ is the inverse immage of the subscheme of $\mathcal{C}_n$ corresponding to the ideal sheaf $(\pi_n^{q_n})$. The existence of $\mathcal{C}_n$ can be proved using Hilbert schemes of curves in projective spaces. Of course $\mathcal{C}_n$ need not be unique.

We obtain a glueing $\mathcal{C}$ of $\mathcal{C}_1,\dots,\mathcal{C}_n$ by defining the sheaves of algebras $\mathcal{O}_\mathcal{C}$ (on the Zariski topological space corresponding to the initial glueing $\boldsymbol{\mathcal{A}}$) as in 4.4.2, using for $\Phi_n$ the quotient morphism $\mathcal{B}^{[n-1]}\to\mathcal{Q}$. It is easy to see that $\pi^{-1}(P)$ is a primitive multiple curve $C_n$ of multiplicity $n$ extending $C_{n-1}$, hence $\mathcal{C}$ is a fragmented deformation of $C_n$.

**4.5.5. Remark: 1 −** The multiple curve $C_n$ depends on the choice of the family $\mathcal{C}_n$ extending the family $Y$ parametrized by $\mathrm{spec}(\mathbb{C}[\pi_n]/(\pi_n^{q_n}))$.

**2 −** The multiple curve $C_{n-1}$ is completely defined by $\mathcal{B}^{[n-1]}$, because $(\pi_1^{q_1}) \times \cdots (\pi_{n-1}^{q_{n-1}}) \subset (\pi)$. But it is not enough to know $\mathcal{B}^{[n-1]}$ and $\mathbf{u}$ to define $C_n$. In fact we need $\mathcal{O}_{\mathcal{C}_i}/(\pi_i^{q_i+1})$, $1 \leq i \leq n$.

### 4.6. Basic elements

We use the notations of 4.3 and 4.4.

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a $n$-tuple of positive integers, and

$$\mathbf{\Pi^m} \;=\; (\pi_1^{m_1}) \times \cdots \times (\pi_n^{m_n}).$$

**4.6.1. Definition:** *Let $x \in C$. An element $u$ of $\mathcal{O}_{\mathcal{C},x}$ is called* basic at order $\mathbf{m}$ *if there exists polynomials $P_1, \ldots, P_n \in \mathbb{C}[X]$ such that*

$$u \;\equiv\; (P_1(\pi_1), \ldots, P_n(\pi_n)) \qquad (\text{mod.} \quad \mathbf{\Pi^m}) \ .$$

*If $u = (P_1(\pi_1), \ldots, P_n(\pi_n))$, we say that $u$ is* basic.

Let $\mathbf{q} = (q_1, \ldots, q_n)$. Then according to corollary 4.4.8, if $u$ is basic at order $\mathbf{q}$, then for every $y \in C$, we have $(P_1(\pi_1), \ldots, P_n(\pi_n)) \in \mathcal{O}_{\mathcal{C},y}$. So $(P_1(\pi_1), \ldots, P_n(\pi_n))$ is defined on a neighbourhood of $C$.

**4.6.2. Lemma:** *Let $u, v, w \in \mathcal{O}_{\mathcal{C},x}$ such that $w = uv$ and $w \neq 0$. Suppose that $u$ and $w$ are basic at every order. Then $v$ is basic at every order.*

*Proof.* Let $N$ be a positive integer such that $N \gg 0$ and $\mathbf{N} = (N, \ldots, N)$. Suppose that $w \equiv (Q_1(\pi_1), \ldots, Q_n(\pi_n)) \pmod{(\pi^N)}$, where $Q_1, \ldots, Q_n \in \mathbb{C}[X]$. Let $\mathbf{m} = (m_1, \ldots, m_n)$ be a $n$-tuple of positive integers, and $v = (v_i)_{1 \leq i \leq n}$. Suppose that

$$u \;\equiv\; (P_1(\pi_1), \ldots, P_n(\pi_n)) \qquad (\text{mod.} \quad \mathbf{\Pi^N})$$

Then we have

$$Q_i(\pi_i) \;\equiv\; P_i(\pi_i).v_i \qquad (\text{mod.} \quad (\pi_i^N))$$

for $1 \leq i \leq n$. We can write $P_i(X)$ as $P_i(X) = X^{n_i} R_i(X)$, where $R_i(X) \in \mathbb{C}[X]$ is such that $R_i(0) \neq 0$. Then $Q_i(X)$ is also divisible by $X^{n_i}$: $Q_i(X) = X^{n_i} S_i(X)$, and we have in $\mathcal{O}_{\boldsymbol{\mathcal{A}}x}$ :

$$S_i(\pi_i) \;\equiv\; R_i(\pi_i).v_i \qquad (\text{mod.} \quad (\pi_i^{N'}))$$

for some integer $N' \gg 0$. We can write $R_i(X) = a_i.(1 - X.T_i(X))$, with $a_i \in \mathbb{C}^*$, $T_i \in \mathbb{C}(X)$. We have then

$$v_i \;\equiv\; \frac{S_i(\pi_i)}{a_i} \sum_{p=1}^{m_i - 1} \big(\pi_i T_i(\pi_i)\big)^p \qquad (\text{mod.} \quad \mathbf{\Pi^m}).$$

□

For $1 \leq i \leq n$, let $\mathbf{u}_{(i)} = ((u_{(i)j})_{1 \leq j \leq n}$ be a generator of the ideal sheaf $\mathcal{I}_{\mathcal{C}_i}$ of $\mathcal{C}_i$ in $\mathcal{C}$, such that for $1 \leq j \leq n$, $u_{(i)j} \in \mathbb{C}[\pi_j]$ (cf. corollary 4.3.8).

**4.6.3. Proposition:** *Let $v \in \mathcal{O}_{\mathcal{C},x}$. then $v$ is basic at every order if and only if for every $n$-tuple $\mathbf{m}$ of positive integers, there exists an integer $q > 0$ and $P_1, \dots, P_q \in \mathbb{C}[X]$ such that*

$$v \equiv \sum_{1 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j \qquad (\text{mod.} \quad \mathbf{\Pi}^{\mathbf{m}}).$$

*Proof.* We use the notations of the proof of lemma 4.6.2. Suppose that $v = (v_j)_{1 \leq j \leq n}$ is basic at every order. Let $N$ be a positive integer and $\mathbf{N} = (N, \dots, N)$. We will prove by induction on $q \geq 0$ that we can write $v$ as

$$v \equiv \sum_{0 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j + \gamma_q \mathbf{u}_{(i)}^{q+1} \qquad (\text{mod.} \quad \mathbf{\Pi}^{\mathbf{N}}) \tag{4}$$

with $P_0, \dots, P_q \in \mathbb{C}[X]$, and $\gamma_q \in \mathcal{O}_{\mathcal{C},x}$. This proves proposition 4.6.3 if $q$ and $N$ are big enough. For $q = 0$, we have $v_i \equiv P(\pi_i) (\text{mod } \pi_i^N)$, for some $P \in \mathbb{C}[X]$, and we can take $P_0 = P$. Suppose that the result is true for $q$ and that we have (4). Since $v - \sum_{1 \leq j \leq q} P_j(\pi).\mathbf{u}_{(i)}^j$ is basic at any order, using the same method as in the proof of lemma 4.6.2, we see that $\gamma_q$ is basic at order $\mathbf{N}'$, where $\mathbf{N}' = (N', \dots, N')$, for some integer $N' \gg 0$. As in the case $q = 0$ we have

$$\gamma_q \equiv P_{q+1}(\pi) + \mathbf{u}_{(i)}.\gamma_{q+1} \qquad (\text{mod} \quad \mathbf{\Pi}^{\mathbf{N}'}),$$

with $P_{q+1} \in \mathbb{C}[X]$. Hence

$$v \equiv \sum_{0 \leq j \leq q+1} P_j(\pi).\mathbf{u}_{(i)}^j + \gamma_{q+1} \mathbf{u}_{(i)}^{q+2} \qquad (\text{mod.} \quad \mathbf{\Pi}^{\mathbf{N}})$$

□

**4.6.4. Proposition:** *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ be such that there exists $P_1, \dots, P_{n-1} \in \mathbb{C}[X]$ such that, for $1 \leq i \leq n-1$, we have $\alpha_i \equiv P_i(\pi_i)$ (mod. $(\pi_i^{q_i})$). Then there exists $P_n \in \mathbb{C}[X]$ such that $\alpha_n \equiv P_n(\pi_n)$ (mod. $(\pi^{q_n})$), i.e. $\alpha$ is a basic element of order $\mathbf{q}$.*

*Proof.* By induction on $n$. The case $n = 2$ is an easy consequence of proposition 4.2.2. Suppose that $n \geq 3$ and that the result is true for $n - 1$.

By substracting multiples of $(0, \dots, 0, \pi_i^{q_i}, 0, \dots, 0)$ we may assume that for $1 \leq i \leq n-1$, $\alpha_i \in \mathbb{C}[\pi_i]$. By substracting a regular function on a neighbourhood of $C$ in $\mathcal{C}$, and a multiple of $(\pi_1^{q_1}, 0, \dots, 0)$ we may also assume that $\alpha_1 = 0$. The ideal sheaf of $\mathcal{C}_1$ is generated by $\mathbf{u}_{(1)}$. We can then write $\alpha = \beta \mathbf{u}_{(1)}$, with $\beta = (\beta_i)_{1 \leq i \leq n} \in \mathcal{O}_{\mathcal{C},x}$. We have

$$(\alpha_2, \dots, \alpha_{n-1}) = (\beta_2, \dots, \beta_{n-1}).(u_{(1)2}, \dots, u_{(1)n-1}) \ ,$$

hence by lemma 4.6.2, $(\beta_2, \dots, \beta_{n-1})$ is a basic element at any order. By the induction hypothesis, there exists $Q \in \mathbb{C}[X]$ such that $\beta_n \equiv Q(\pi_n)$ (mod. $(\pi_n^{q_n - p_{1n}})$). Since $u_{(1)n}$ is a multiple of $\pi_n^{p_{1n}}$ (from the definition of $p_{1n}$), it follows that $\alpha_n \equiv u_{(i)n} Q(\pi_n)$ (mod. $(\pi^{q_n})$). □

### 4.7. Simple primitive curves and fragmented deformations

Let $C_n$ be a primitive multiple curve of multiplicity $n$ and associated smooth curve $C$. Let $\mathcal{I}_C$ be the ideal sheaf of $C$ in $C_n$. It is obvious from proposition 4.3.5, 1-, that if there exists a fragmented deformation of $C_n$, then we have $\mathcal{I}_{C,C_n} \simeq \mathcal{O}_{C_{n-1}}$, i.e. $C_n$ is *simple* (cf. 2.4). Conversely we have

**4.7.1. Theorem:** *Let $C_n$ be a simple primitive multiple curve of multiplicity $n$. Then there exists a fragmented deformation of $C_n$.*

*Proof.* According to theorem 2.4.1, there exists a flat family of smooth projective curves $\tau : \mathcal{C} \to \mathbb{C}$ such that $\tau^{-1}(0) \simeq C$ and that $C_n$ is isomorphic to the $n$-th infinitesimal neighbourhood of $C$ in $\mathcal{C}$. Let $\rho_n : \mathbb{C} \to \mathbb{C}$ be the map defined by $\rho_n(z) = z^n$, and $\theta = \rho_n \circ \tau : \mathcal{C} \to \mathbb{C}$. It is a flat morphism, $\theta^{-1}(0) = C_n$, and for every $z \neq 0$ in the image of $\tau$, $\theta^{-1}(z)$ is a disjoint union of $n$ smooth irreducible curves. We can then apply the process of proposition 3.1.3 to obtain the desired fragmented deformation: it is $\mathcal{C} \times_{\mathbb{C}} \mathbb{C}$

$$\begin{array}{ccc} \mathcal{C} \times_{\mathbb{C}} \mathbb{C} & \xrightarrow{\pi} & \mathbb{C} \\ \downarrow & & \downarrow{\scriptstyle \rho_n} \\ \mathcal{C} & \xrightarrow{\theta} & \mathbb{C} \end{array}$$

□

**4.7.2. Remark:** let $(p_{ij})$ be the spectrum of the fragmented deformation constructed in the proof of theorem 4.7.1. Then it is easy to see that $p_{ij} = 1$ for $1 \leq i, j \leq n$, $i \neq j$. If $x \in C$, then $(\mathcal{C} \times_{\mathbb{C}} \mathbb{C})_x = \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathbb{C},x}} \mathcal{O}_{\mathbb{C},x}$, and if $t = I_{\mathbb{C}} \in \mathcal{O}_{\mathbb{C},x}$, we have for $1 \leq k \leq n$

$$(\pi_1, \ldots, \pi_{k-1}, 0, \pi_{k+1}, \ldots, \pi_n) \;=\; \frac{1}{n-1}(1 \otimes t - e^{\frac{2ki\pi}{n}}(t \otimes 1)) \; .$$

## 5. Stars of a curve

### 5.1. Definitions

Let $S$ be a smooth irreducible curve, and $P \in S$ (we can also take for $(S, P)$ a germ of smooth curve). Let $n$ be a positive integer.

**5.1.1. Definition:** *A $n$-*star *(or more simply, a* star*) of $(S, P)$ is an algebraic variety $\boldsymbol{\mathcal{S}}$ such that*

(i) *$\boldsymbol{\mathcal{S}}$ is the union of $n$ irreducible components $S_1, \ldots, S_n$, with fixed isomorphisms $S_i \simeq S$, $1 \leq i \leq n$.*
(ii) *For $1 \leq i < j \leq n$, $S_i \cap S_j$ has only one closed point, namely $P$.*
(iii) *There exists a morphism $\pi : \boldsymbol{\mathcal{S}} \to S$, which is the identity on each component $S_i$.*

All the $n$-stars of $(S,P)$ have the same underlying Zariski topological space $S(n)$ and set of closed points. The latter is $(\bigcup_{1\leq i\leq n}\widehat{S}_i)/\sim$, where $\widehat{S}_i$ is the set of closed points of $S_i$, and the equivalence relation $\sim$ is defined by: for $x\in\widehat{S}_i$ and $y\in\widehat{S}_j$, $x\sim y$ if and only if $i=j$ and $x=y$, or $x=P\in\widehat{S}_i$ and $y=P\in\widehat{S}_j$. An open subset of $\boldsymbol{\mathcal{S}}$ is defined by open subsets $U_1$ of $S_1$,..., $U_n$ of $S_n$, such that for $1\leq i<j\leq n$, we have $P\in U_i$ if and only if $P\in U_j$.

The *initial* star $\boldsymbol{\mathcal{S}}_0$ of $(S,P)$ is defined as follows: for every open subset $U$ of $S(n)$, $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0}(U)$ is the set of $(\alpha_1,\dots,\alpha_n)\in\mathcal{O}_{S_1}(U\cap S_1)\times\cdots\mathcal{O}_{S_n}(U\cap S_n)$ such that if $P\in U$ then $\alpha_1(P)=\cdots=\alpha_n(P)$ .

For every $n$-star $\boldsymbol{\mathcal{S}}$ of $(S,P)$, there is a unique dominant morphism $\boldsymbol{\mathcal{S}}_0\to\boldsymbol{\mathcal{S}}$ inducing the identity on each component. So $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ is a subring of $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0,P}$.

Note that (iii) is equivalent to

(iii)' For every $\alpha\in\mathcal{O}_{S,P}$, we have $(\alpha,\dots,\alpha)\in\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$.

**5.1.2. Definition:** *An* oblate $n$-star *(or more simply, an* oblate star*) of* $(S,P)$ *is a* $n$-star $\boldsymbol{\mathcal{S}}$ *such that some neighbourhood of* $P$ *in* $\boldsymbol{\mathcal{S}}$ *can be embedded in a smooth surface.*

**5.1.3. Proposition:** *A* $n$*-star* $\boldsymbol{\mathcal{S}}$ *is oblate if and only if* $\pi^{-1}(P)\simeq\operatorname{spec}(\mathbb{C}[X]/(X^n))$.

(cf. prop. 4.1.6).

Let $I\subset\{1,\dots,n\}$ be a nonempty subset. Let $\boldsymbol{\mathcal{S}}^{(I)}=\bigcup_{i\in I}S_i\subset\boldsymbol{\mathcal{S}}$. If $\boldsymbol{\mathcal{S}}$ is oblate then $\boldsymbol{\mathcal{S}}^{(I)}$ is oblate too.

## 5.2. Properties of oblate stars

Let $\boldsymbol{\mathcal{S}}$ be an oblate $n$-star of $S$. Recall that $t$ denotes a generator of the maximal ideal of $P$ in $S$. We will denote this generator on $S_i\subset\boldsymbol{\mathcal{S}}$ by $t_i$. We will also denote by $\pi$ the element $t\circ\pi$ of the maximal ideal of $P$ in $\boldsymbol{\mathcal{S}}$. Let $\mathcal{I}_P$ be the ideal sheaf of $P$ in $\boldsymbol{\mathcal{S}}$.

We begin with 2-stars:

**5.2.1. Proposition:** *Suppose that* $n=2$. *Then*

**1** – *There exists a unique integer* $p>0$ *such that* $\mathcal{I}_{P,P}/(\pi)$ *is generated by the image of* $(t_1^p,0)$.

**2** – *The image of* $(0,t_2^p)$ *is also a generator of* $\mathcal{I}_{P,P}/(\pi)$.

**3** – $(0,t_2^p)$ *(resp.* $(t_1^p,0)$*) is a generator of the ideal sheaf of* $S_1$ *(resp.* $S_2$*) at* $P$.

**4** – $\mathcal{O}_{S^{(2)},P}$ *consists of pairs* $(\alpha,\beta)\in\mathcal{O}_{S,P}\times\mathcal{O}_{S,P}$ *such that* $\alpha-\beta\in(t^p)$.

Now suppose that $n\geq 2$. Let $I=\{i,j\}\subset\{1,\dots,n\}$, with $i\neq j$. Then $S_i\cup S_j\subset\boldsymbol{\mathcal{S}}$ is a 2-star of $S$. Hence by proposition 5.2.1 there exists a unique integer $p_{ij}>0$ such that $\mathcal{I}_{P,P}/(\pi)$ (on

$S_i \cup S_j$) is generated by the image of $(t_i^{p_{ij}}, 0)$ (and also by the image of $(0, t_j^{p_{ij}})$). Let $p_{ii} = 0$. Then the symmetric matrix $(p_{ij})_{1 \leq i,j \leq n}$ is called the *spectrum* of $\boldsymbol{\mathcal{S}}$.

There exists an element $v_{ij} = (\nu_m)_{1 \leq m \leq n}$ such that $\nu_i = 0$ and $\nu_j = t_j^{p_{ij}}$. For every integer $m$ such that $1 \leq m \leq n$, $m \neq i, j$, there exists an invertible element $\beta_{ij}^{(m)} \in \mathcal{O}_{S,P}$ such that $\nu_m = \beta_{ij}^{(m)} t_m^{p_{im}}$. Let $\beta_{ij}^{(i)} = 0$, $\beta_{ij}^{(j)} = 1$.

**5.2.2. Proposition:** *Let* $\mathbf{b}_{ij}^{(m)} = \beta_{ij}^{(m)}(P) \in \mathbb{C}$. *Then we have, for all integers* $i, j, k, m, q$ *such that* $1 \leq i, j, k, m, q \leq n$, $i \neq j$, $i \neq k$

$$\mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(q)} \;=\; \mathbf{b}_{ik}^{(q)} \mathbf{b}_{ij}^{(m)}.$$

*In particular we have* $\mathbf{b}_{ij}^{(m)} = \mathbf{b}_{ik}^{(m)} \mathbf{b}_{ij}^{(k)}$ *and* $\mathbf{b}_{ij}^{(m)} \mathbf{b}_{im}^{(j)} = 1$.

*For all distinct integers* $i, j, k$ *such that* $1 \leq i, j, k \leq n$, *we have*

$$\mathbf{b}_{ki}^{(j)} \;=\; -\mathbf{b}_{ik}^{(j)} \mathbf{b}_{ji}^{(k)} \;.$$

(cf. prop. 4.3.2 and 4.4.6).

Let $p$ an integer such that $1 \leq p < n$, and $(i_1, j_1), \ldots, (i_p, j_p)$ $p$ pairs of distinct integers of $\{1, \ldots, n\}$. Then the image of $\prod_{m=1}^{p} \mathbf{v}_{i_m j_m}$ is a generator of $(\mathcal{I}_{P,P}^{p} + (\pi))/(\mathcal{I}_{P,P}^{p+1} + (\pi))$.

Let $I \subset \{1, \cdots, n\}$ be a nonempty subset, distinct from $\{1, \cdots, n\}$. Let $i \in \{1, \cdots, n\} \backslash I$. Let

$$\mathbf{v}_{I,i} \;=\; \prod_{j \in I} \mathbf{v}_{ji} \;.$$

**5.2.3. Proposition:** *The ideal sheaf of* $\boldsymbol{\mathcal{S}}^{(I)}$ *in* $\boldsymbol{\mathcal{S}}$ *is generated by* $\mathbf{v}_{I,i}$ *at* $P$.

(cf. prop. 4.3.3).

Note that if $I = \{1, \cdots, n\} \backslash \{i\}$ then $\mathbf{v}_{I,i|S_j} = 0$ if $j \neq i$, and $\mathbf{v}_{I,i|S_i} = t_i^{q_i}$, with $q_i = \sum_{1 \leq j \leq n} p_{ij}$.

Let $i$ be an integer such that $1 \leq i \leq n$ and $J_i = \{1, \ldots, n\} \backslash \{i\}$. Let $\mathcal{K}_i$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}}$ in $\prod_{1 \leq j \leq n, j \neq i} \mathcal{O}_{S_j}/(t_j^{q_j})$. We can view $\mathcal{K}_i$ as a $\mathbb{C}$-algebra. For every $\alpha = (\alpha_m) \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$, let $k_i(\alpha)$ be the image of $\alpha$ in $\mathcal{K}_i$.

**5.2.4. Proposition:** *There exists a morphism of* $\mathbb{C}$*-algebras*

$$\Psi_i : \mathcal{K}_i \longrightarrow \mathcal{O}_{S_{i,P}}/(t_i^{q_i})$$

*such that for every* $(\alpha_m)_{1 \leq m \leq n, m \neq i} \in \mathcal{O}_{\boldsymbol{\mathcal{S}}^{(J_i)},P}$, $\alpha_i \in \mathcal{O}_{S_{j,P}}$, *we have* $\alpha = (\alpha_m)_{1 \leq m \leq n} \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ *if and only if* $\Psi_i(k_i(\alpha)) = [\alpha_i]_{q_i}$.

(cf. prop. 4.4.1).

The morphism $\Psi_i$ has the following properties:

(i) For every $(\alpha_m)_{1\leq m\leq n, m\neq i} \in \mathcal{O}_{\boldsymbol{\mathcal{S}}^{(J_i)},P}$, we have $\Psi_i(\alpha)(P) = \alpha_m(P)$ for $1 \leq m \leq n, m \neq i$.
(ii) We have $\Psi_i((t_m)_{1\leq m\leq n, m\neq i}) = t_i$.
(iii) Let $j, k \in \{1, \cdots, n\}$ be such that $i, j, k$ are distinct. Let $\mathbf{w}$ be the image of $\mathbf{v}_{jk}$ in $\mathcal{B}_i$. Then there exists $\lambda \in \mathcal{O}^*_{S_i,P}$ such that $\Psi_i(\mathbf{w}) = \lambda t_i^{p_{ij}}$.
(iv) Let $j$ be an integer such that $1 \leq j \leq n$ and $j \neq i$. Let $\mathbf{w}$ be the image of $\mathbf{v}_{ij}$ in $\mathcal{K}_i x$. Then we have $\ker(\Psi_i) = (\mathbf{w})$.

**5.2.5.** *Converse* – Let $\boldsymbol{\mathcal{S}}^{[n-1]}$ be a $(n-1)$-star of $S$, with components $S_1, \ldots, S_{n-1}$, of spectrum $(p_{jk})_{1\leq j,k\leq n-1}$. Let $p_{nj} = p_{jn}$, $1 \leq j < n$ be positive integers, and $p_{nn} = 0$. For $1 \leq j \leq n$, let $q_j = \sum_{1\leq k\leq n} p_{kj}$.

Let $S_n$ be another copy of $S$. Let $\mathcal{K}_n$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]}}$ in $\prod_{1\leq j\leq n-1} \mathcal{O}_{S_j}/(t_j^{q_j})$ and

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n}/(t_n^{q_n})$$

a morphism of $\mathbb{C}$-algebras satisfying properties (i), (ii), (iii) above. Let $\mathcal{K}$ be the subsheaf of algebras of $\mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$ defined by: $\mathcal{K} = \mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$ on $\boldsymbol{\mathcal{S}}_0\backslash\{P\}$, and for every $\alpha = (\alpha_m)_{1\leq m\leq n} \in \mathcal{O}_{\boldsymbol{\mathcal{S}}_0,P}$, $\alpha \in \mathcal{K}_P$ if and only if $\Psi_n(\alpha') = [\alpha_n]_{q_n}$ (where $\alpha'$ is the image of $(\alpha_m)_{1\leq m\leq n-1}$ in $\mathcal{K}_n$).

It is easy to see that $\mathcal{K}$ is the structural sheaf of an oblate $n$–star of $S$.

Let $\mathcal{H} = \prod_{1\leq j\leq n}(t_j^{q_j-1})/(t_j^{q_j}) \simeq \mathbb{C}^n$ and $\mathcal{K}$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}}$ in $\prod_{1\leq j\leq n} \mathcal{O}_{S_j}/(t_j^{q_j})$. We can view $\mathcal{K}$ as a $\mathbb{C}$-algebra. Let $\mathcal{J} = \mathcal{H} \cap \mathcal{K}$.

**5.2.6. Proposition:** *There exists a unique* $\lambda(\boldsymbol{\mathcal{S}}) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_n(\mathbb{C})$ *such that for every* $\mathbf{u} = (u_j)_{1\leq j\leq n} \in \mathcal{H}$, *we have* $\mathbf{u} \in \mathcal{J}$ *if and only if* $\lambda_1 u_1 + \cdots \lambda_n u_n = 0$. *The* $\lambda_i$ *are all non zero.*

(cf. prop. 4.4.5).

For all distinct integers $i$, $j$ such that $1 \leq i, j \leq n$, we have

$$\frac{\lambda_i}{\lambda_j} \;=\; -\prod_{1\leq m\leq n, m\neq i,j} \mathbf{b}^{(j)}_{mi}.$$

## 5.3. Construction of oblate stars of a curve

Consider an oblate $(n-1)$-star of $S$, $\boldsymbol{\mathcal{S}}^{[n-1]}$, with $n-1$ irreducible components $S_1, \ldots, S_{n-1}$, copies of $S$. Let $(p_{ij}^{[n-1]})_{1\leq i,j<n}$ be its spectrum. For $1 \leq i < n$, let $q_i^{[n-1]} = \sum_{1\leq j<n} p_{ij}^{[n-1]}$. We denote by $\mathcal{I}_P^{[n-1]}$ the ideal of $P$ in $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]},P}$. Let $\lambda(\boldsymbol{\mathcal{S}}^{[n-1]}) = (\lambda_1, \ldots, \lambda_{n-1})$.

Let $p_{1n}, \ldots, p_{n-1,n}$ be positive integers, $q_i = q_i^{[n-1]} + p_{in}$ for $1 \leq i < n$, and $q_n = p_{i1} + \cdots + p_{n-1,n}$. Let $\mathbf{u} \in \mathcal{I}_{P,P}^{[n-1]}$ whose image generates $\mathcal{I}_P^{[n-1]}/((\mathcal{I}_P^{[n-1]})^2 + (\pi))$, of the form

$$\mathbf{u} \;=\; (\beta_1 t_1^{p_{1n}}, \ldots, \beta_{n-1} t_{n-1}^{p_{n-1,n}}),$$

with $\beta_i \in \mathcal{O}_{S_i,P}$ invertible for $1 \leq i < n$.

Let $\mathcal{K}^{[n-1]}$ be the image of $\mathcal{O}_{\boldsymbol{\mathcal{S}}^{[n-1]}}$ in $\mathcal{O}_{S_1}/(t_1^{q_1}) \times \cdots \times \mathcal{O}_{S_{n-1}}/(t_{n-1}^{q_{n-1}})$ . We will also denote by $\mathbf{u}$ the image of $\mathbf{u}$ in $\mathcal{K}^{[n-1]}$. Let $\mathcal{Q} = \mathcal{K}^{[n-1]}/(\mathbf{u})$ , $\rho : \mathcal{K}^{[n-1]} \to \mathcal{Q}$ the projection and $t_n = \rho(\pi)$.

**5.3.1. Proposition:** **1** – *We have* $t_n^{q_n} = 0$ .

**2** – *We have* $t_n^{q_n-1} = 0$ *if and only if*

$$\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \;=\; 0 \;.$$

*We suppose now that* $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. *Then*

**3** – *For every* $\epsilon \in \mathcal{K}^{[n-1]}$ *such that* $\epsilon(P) \neq 0$, *we have* $t^{q_n-1}\epsilon \notin (u)$.

**4** – *For every* $\eta \in \mathcal{K}^{[n-1]}/(\mathbf{u})$, *and every integer* $k$ *such that* $1 \leq k < q_n$, *we have* $t_n^k \eta = 0$ *if and only if* $\eta$ *is a multiple of* $t_n^{q_n-k}$.

**5** – $\mathcal{K}^{[n-1]}/(\mathbf{u})$ *is a flat* $\mathbb{C}[t_n]/(t_n^{q_n})$*-module.*

**5.3.2.** *Construction of stars of a curve* – Suppose that $\frac{\lambda_1}{\beta_1(P)} + \cdots + \frac{\lambda_{n-1}}{\beta_{n-1}(P)} \neq 0$. From proposition 5.3.1, 5-, it is easy to prove, using 5.2.5, that there is a unique oblate $n$-star $\boldsymbol{\mathcal{S}}$ such that $\boldsymbol{\mathcal{S}}^{[n-1]}$ is the union $\bigcup_{1\leq i\leq n-1} S_i$ in $\boldsymbol{\mathcal{S}}$ and $\Psi_n$ is the quotient map $\mathcal{K}_n = \mathcal{K}^{[n-1]} \to \mathcal{Q}$.

## 5.4. Morphisms of stars

Recall that if $\boldsymbol{\mathcal{S}}$ is an oblate $n$-star of $S$, then we have a canonical inclusion of sheaves of algebras (on the underlying topological space $S(n)$ of $\boldsymbol{\mathcal{S}}$) $\mathcal{O}_{\boldsymbol{\mathcal{S}}} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}_0}$.

Let $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{S}}'$ be oblate $n$-stars of $S$, with irreducible components $S_1, \ldots, S_n$, and $f : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{S}}'$ a morphism inducing the identity on all the components. Such a morphism exists if and only if $\boldsymbol{\mathcal{S}}' \subset \boldsymbol{\mathcal{S}}$, and in this case $f$ is unique and is induced by the previous inclusion. Let $(p_{ij})$ (resp. $(p'_{ij})$) be the spectrum of $\boldsymbol{\mathcal{S}}$ (resp. $\boldsymbol{\mathcal{S}}'$).

**5.4.1. Proposition:** *We have* $p_{ij} \leq p'_{ij}$ *for* $1 \leq i, j \leq n$. *If* $f$ *is not the identity morphism then there exists* $i$, $j$ *such that* $p_{ij} < p'_{ij}$.

*Proof.* Let $I = \{i, j\}$. Then $f$ induces a morphism $\boldsymbol{\mathcal{S}}^{(I)} \to \boldsymbol{\mathcal{S}}'^{(I)}$. So we have $\mathcal{O}_{\boldsymbol{\mathcal{S}}'^{(I)},P} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}^{(I)},P}$. From proposition 5.2.1, 4-, it follows that $p_{ij} \leq p'_{ij}$.

Suppose now that $p'_{ij} = p_{ij}$ for $1 \leq i, j \leq n$. We must prove that $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}'$, i.e. that $\mathcal{O}_{\boldsymbol{\mathcal{S}}'^{(I)},P} = \mathcal{O}_{\boldsymbol{\mathcal{S}}^{(I)},P}$. This is done by induction on $n$. For $n = 2$ it is obvious. Suppose that it is true for $n-1$. Let $I = \{1, \ldots, n-1\}$. Then $f$ induces a morphism $f_{n-1} : \boldsymbol{\mathcal{S}}^{(I)} \to \boldsymbol{\mathcal{S}}'^{(I)}$. It follows from the induction hypothesis that $\boldsymbol{\mathcal{S}}^{(I)} = \boldsymbol{\mathcal{S}}'^{(I)}$. Since the integers $q_i$ are the same for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}'$, the algebras $\mathcal{K}_n$ for $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}'$ (cf. proposition 5.2.4) are also the same. Now let $\alpha \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$, and let $\beta \in \mathcal{K}_n$ be the image of $\alpha$. Let $\alpha' \in \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ be such that its image in $\mathcal{K}_n$ is also $\beta$. Then $\alpha - \alpha'$ belongs to the ideal generated by the $(0, \ldots, 0, t_i^{q_i}, 0 \ldots, 0)$, $1 \leq i \leq n$, which is included in $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. Hence $\alpha \in \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. □

**5.4.2. Lemma:** *Suppose that $f$ is not the identity morphism. Then there exists an ideal $\mathcal{I} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and $u \in \mathcal{I}$, $v \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ such that*

$$u \otimes v \neq 0 \qquad in \qquad \mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$$

*and $uv = 0$.*

*Proof.* Let $q_1 = \sum_{i=1}^{n} p_{1i}$, $q_1' = \sum_{i=1}^{n} p_{1i}'$. According to proposition 5.4.1 we can assume that $q_1 < q_1'$. Let $u$ be a generator of the ideal of $S_1$ in $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and $\mathcal{I} = (u)$. Let $v = (t_1^{q_1}, 0, \ldots, 0)$. We have $uv = 0$. We have to prove that $u \otimes v \neq 0$. We need only to find an $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-module $M$ and a $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-bilinear map

$$\phi : \mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{\boldsymbol{\mathcal{S}},P} \longrightarrow M$$

such that $\phi(u \otimes v) \neq 0$. We take $M = \mathcal{O}_{S_1,P}/(t_1^{q_1'})$, which is a quotient of $\mathcal{O}_{\boldsymbol{\mathcal{S}}'}$. It is easy to verify that

$$\phi : ((\lambda_i)_{1\leq i\leq n} u, (w_i)_{1\leq i\leq n}) \longmapsto \lambda_1 w_1 \pmod{t_1^{q_1'}}$$

is well defined, bilinear, and that $\phi(u \otimes v) \neq 0$. □

**5.4.3. Corollary:** *Suppose that $f$ is not the identity morphism. Let $Y$ be an algebraic variety and $g : Y \to S$ a morphism such that $g^* : \mathcal{O}_{\boldsymbol{\mathcal{S}},P} \to \mathcal{O}_{Y,P}$ is injective. Then $f \circ g : Y \to S'$ is not flat.*

*Proof.* We use the notations of the proof of lemma 5.4.2. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{\boldsymbol{\mathcal{S}},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P} \\
\downarrow{\scriptstyle \lambda_S} & & \downarrow{\scriptstyle \lambda_Y} \\
\mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{\boldsymbol{\mathcal{S}},P} & \xrightarrow{I_{\mathcal{I}} \otimes g^*} & \mathcal{I} \otimes_{\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}} \mathcal{O}_{Y,P} \\
\downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \mu_Y} \\
\mathcal{O}_{\boldsymbol{\mathcal{S}},P} & \xrightarrow{g^*} & \mathcal{O}_{Y,P}
\end{array}$$

where $\lambda_S(\alpha) = u \otimes \alpha$, $\mu_S(u \otimes \alpha) = u\alpha$, and $\lambda_Y, \mu_Y$ are defined similarly. It follows that $\mu_Y(u \otimes g^*v) = 0$. We will show that $u \otimes g^*v \neq 0$, and this will imply that $f \circ g$ is not flat. Let $w = (t_1^{q_1'}, 0, \ldots, 0)$. Then we have $\mathcal{I} \simeq \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}/(w)$, and from the exact sequence of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$-modules $0 \to (w) \to \mathcal{O}_{\boldsymbol{\mathcal{S}}',P} \to \mathcal{I} \to 0$ we deduce that $\ker(\lambda_Y) = (w).\mathcal{O}_{Y,P}$. Suppose that $u \otimes g^*v = 0$. Then $g^*v$ is a multiple of $w$: $g^*v = w.a$, for some $a \in \mathcal{O}_{Y,P}$. But we have $w = g^*\pi^{q_1'-q_1}v$. Hence $g^*v.(1 - g^*\pi^{q_1'-q_1}) = 0$. Since $1 - g^*\pi^{q_1'-q_1}$ is invertible, we have $g^*v = 0$, which is false since $g^*$ is injective. Hence $u \otimes g^*v \neq 0$. □

### 5.5. Structure of ideals

Let $\boldsymbol{\mathcal{S}}$ be an oblate $n$-star of $S$.

**5.5.1. Proposition:** *Let $\mathcal{I} \subset \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ be a proper ideal. Then*

**1 -** *There exists a positive integer $k$ such that $k \leq n$ and a filtration by ideals*

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}$$

*such that, for $1 \leq i \leq k$ there exists a positive integer $j$ such that $j \leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$-modules.*

**2 -** *If $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$, then $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$ and $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$.*

*Proof.* We prove **1-** by induction on $n$. The case $n = 1$ is trivial. Suppose that $n > 1$ and that the result is true for $n-1$. Let $\mathcal{J}_1$ be the ideal sheaf of $S_1 \subset \boldsymbol{\mathcal{S}}$, and $\boldsymbol{\mathcal{S}}' = S_2 \cup \cdots \cup S_{n-1} \subset \boldsymbol{\mathcal{S}}$. We can view $\mathcal{J}_1$ as an ideal of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$. We can suppose that $\mathcal{I} \not\subset \mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$, i.e that some element of $\mathcal{I}$ has a nonzero first coordinate. Let $m$ be the smallest positive integer such that $\mathcal{I}$ contains an element $u$ of the form

$$u \;=\; (t^m, \alpha_2, \ldots, \alpha_n)\ .$$

Then every element $v$ of $\mathcal{I}$ can be written as

$$v \;=\; \lambda u + v'\ ,$$

with $\lambda \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$ and $v' \in \mathcal{J}_1 \cap \mathcal{I}$, and the first coordinate of $\lambda$ is uniquely determined. It follows that $\mathcal{I}/(\mathcal{J}_1 \cap \mathcal{I}) \simeq \mathcal{O}_{S_1,P}$. We can apply the recurrence hypothesis to the ideal $\mathcal{J}_1 \cap \mathcal{I}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}}',P}$ and get a filtration of it, from which we deduce the filtration of $\mathcal{I}$. This proves **1-** for $n$.

Now we prove **2-**. Let $\alpha \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P} \backslash \mathcal{I}_{S_j}$. Let $u \in \mathcal{I}_i$ be over a generator of $\mathcal{I}_i/\mathcal{I}_{i+1}$. Then the image of $\alpha u$ in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, i.e. $\alpha u \notin \mathcal{I}_{i+1}$. Hence $\alpha \notin \mathcal{I}_{i+1}$, and $\mathcal{I}_{i+1} \subset \mathcal{I}_{S_j}$. Let $v_i = (0, \ldots, 0, t_i^{q_i}, 0, \ldots, 0) \in \mathcal{O}_{\boldsymbol{\mathcal{S}},P}$. Then the image of $v_i u$ in $\mathcal{I}_i/\mathcal{I}_{i+1}$ is not zero, hence $u \notin \mathcal{I}_{S_j}$ and $\mathcal{I}_i \not\subset \mathcal{I}_{S_j}$. □

### 5.6. Star associated to a fragmented deformation

We keep the notations of chapter 4. Let $n \geq 2$ be an integer, $\pi : \mathcal{C} \to S$ a fragmented deformation of $C_n$, and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ the irreducible components of $\mathcal{C}$.

Recall that $S(n)$ is the underlying (Zariski) topological space of any $n$-star of $S$. Let $\mathcal{C}^{top}$ be the underlying topological space of $\mathcal{C}$. We have an obvious continuous map $\boldsymbol{\pi} : \mathcal{C}^{top} \to S(n)$. Let $\mathcal{A}_n$ be the sheaf of algebras on $S(n)$ defined by: for every open subset $U$ of $S(n)$, $\mathcal{A}_n(U)$ is the algebra of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C}}(\boldsymbol{\pi}^{-1}(U))$ such that $\alpha_i \in \mathcal{O}_{S_i}(U \cap S_i)$ for $1 \leq i \leq n$.

According to corollary 4.4.8, for every $x \in C$, $\mathcal{A}_{n,P}$ is the algebra of $(\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{\mathcal{C},x}$ such that $\alpha_i \in \mathcal{O}_{S,P}$ for $1 \leq i \leq n$.

**5.6.1. Proposition:** *The sheaf $\mathcal{A}_n$ is the structural sheaf of an oblate $n$-star of $S$.*

*Proof.* By induction on $n$. The case $n = 1$ is obvious. Suppose that $n > 1$ and that the result is true for $n-1$. Let $\mathcal{C}' = \mathcal{C}_1 \cup \cdots \mathcal{C}_{n-1} \subset \mathcal{C}$, and $\mathcal{A}_{n-1}$ the corresponding oblate $(n-1)$-star of $S$. Let

$$\Phi_n : \mathcal{B}_n \longrightarrow \mathcal{O}_{\mathcal{C}_n}/(\pi_n^{q_n})$$

be the morphism of proposition 4.4.1. According to proposition 4.6.4, $\Phi_n$ induces a morphism

$$\Psi_n : \mathcal{K}_n \longrightarrow \mathcal{O}_{S_n,P}/(t_n^{q_n}) \ .$$

By the definitions of $\mathcal{A}_n$ and $\Phi_n$, if $u = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}_{S_1,P} \times \cdots \times \mathcal{O}_{S_1,P}$, then $u \in \mathcal{A}_{n,P}$ if and only if $\Psi_n(u') = v$, where $u'$ (resp. $v$) is the image of $u$ in $\mathcal{K}_n$ (resp. $\mathcal{O}_{S_n,P}/(t_n^{q_n})$). The result follows then from 5.2.5. □

We denote by $\mathcal{S}(\mathcal{C})$ (or more simply $\mathcal{S}$) the oblate $n$-star corresponding to $\mathcal{A}_n$, so $\mathcal{O}_{\mathcal{S}(\mathcal{C})} = \mathcal{A}_n$. From the definition of $\mathcal{A}_n$ we get a canonical morphism

$$\mathbf{\Pi} : \mathcal{C} \longrightarrow \mathcal{S}$$

such that $\mathbf{\Pi}_{|\mathcal{C}_i} = \pi_i : \mathcal{C}_i \to S_i$ for $1 \leq i \leq n$.

**5.6.2. Theorem:** *The morphism* $\mathbf{\Pi}$ *is flat.*

*Proof.* We need only to prove that $\mathbf{\Pi}$ is flat at any point $x$ of $C$. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{S},P}$ be a proper ideal. We have to show that the canonical morphism of $\mathcal{O}_{\mathcal{S},P}$-modules

$$\tau = \tau_{\mathcal{I}} : \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I} \longrightarrow \mathcal{O}_{\mathcal{C},x}$$

is injective. According to proposition 5.5.1 there is a filtration by ideals

$$\{0\} = \mathcal{I}_{k+1} \subset \mathcal{I}_k \subset \cdots \subset \mathcal{I}_1 = \mathcal{I}$$

such that, for $1 \leq i \leq k$ there exists a positive integer $j$ such that $j \leq n$ and an isomorphism $\mathcal{I}_i/\mathcal{I}_{i+1} \simeq \mathcal{O}_{S_j,P}$ of $\mathcal{O}_{\boldsymbol{\mathcal{S}},P}$-modules. We will prove the injectivity of $\tau$ by induction on $k$.

Recall that for $1 \leq j \leq n$, $\mathcal{I}_{S_j,P} = \mathcal{I}_{S_j,\mathcal{S},P}$ is a principal ideal, generated by an element $u_j$ which is also a generator of $\mathcal{I}_{\mathcal{C}_j,x} = \mathcal{I}_{\mathcal{C}_j,\mathcal{C},x}$ (cf. corollary 4.3.8 and proposition 5.2.3), and that the only zero coordinate of $u_j$ is the $j$-th.

Suppose that $k = 1$, so $\mathcal{I}$ is isomorphic to $\mathcal{O}_{S_j,P}$ for some $j$. Let $u$ be a generator of $\mathcal{I}$ and $w \in \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}$, that can be written as $w = v \otimes u$, $v \in \mathcal{O}_{\mathcal{C},x}$. Suppose that $\tau(v \otimes u) = vu = 0$. Since $\mathcal{I}$ is annihilated by $\mathcal{I}_{S_j,P}$, we have $\mathcal{I} \subset \big((0, \ldots, 0, t_j^{q_j}, 0, \ldots, 0)\big)$. Since $vu = 0$, the $j$-th component of $v$ is zero, i.e. $v \in \mathcal{I}_{\mathcal{C}_j,x}$. Hence $v$ is a multiple of $u_j$ : $v = \alpha u_j$. We have then

$$\begin{aligned} w &= \alpha u_j \otimes u \\ &= \alpha \otimes u_j u \ \ (\text{because} \ \ u_j \in \mathcal{O}_{\mathcal{S},P}) \\ &= 0 \quad (\text{because} \ \ u_j u = 0) \ . \end{aligned}$$

Hence $\tau$ is injective.

Suppose that the result is true for $k - 1 \geq 1$ and that the filtration of $\mathcal{I}$ is of length $k$. According to proposition 5.5.1, **1-**, we have $\mathcal{I}/\mathcal{I}_2 \simeq \mathcal{O}_{S_j,P}$ for some $j$. Let $u \in \mathcal{I}$ be such that its image in $\mathcal{I}/\mathcal{I}_2$ is a generator, and $w \in \mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}$ such that $\tau(w) = 0$. We can write $w$ as $w = \alpha \otimes v + \beta \otimes u$, with $\alpha, \beta \in \mathcal{O}_{\mathcal{C},x}$ and $v \in \mathcal{I}_2$. Since $\alpha v + \beta u = 0$, we have $\beta u \in \mathcal{O}_{\mathcal{C},x}\mathcal{I}_2$,

and $\mathcal{O}_{\mathcal{C},x}\mathcal{I}_2 \subset \mathcal{I}_{\mathcal{C}_j}$ by proposition 5.5.1, **2-**, i.e. the $j$-th coordinate of $\beta u$ is zero. By proposition 5.5.1, **2-**, the $j$-th coordinate of $u$ does not vanish, hence the $j$-th coordinate of $\beta$ is zero, i.e. $\beta \in \mathcal{I}_{\mathcal{C}_j}$. Hence $\beta$ is a multiple of $u_j$ : $\beta = \gamma u_j$. We have then

$$\beta \otimes u = \gamma u_j \otimes u = \gamma \otimes u_j u,$$

and $u_j u \in \mathcal{I}_2$ (because its image in $\mathcal{I}/\mathcal{I}_2$ vanishes). It follows that $w$ is the image of an element $w'$ of $\mathcal{O}_{\mathcal{C},x} \otimes_{\mathcal{O}_{\mathcal{S},P}} \mathcal{I}_2$. We have $\tau_{\mathcal{I}_2}(w') = 0$, hence by the induction hypothesis $w' = 0$. It follows that we have also $w = 0$. □

**5.6.3. Remark:** If $\mathcal{S}'$ is an oblate $n$-star of $S$, and if $\Pi' : \mathcal{C} \to \mathcal{S}'$ is a flat morphism compatible with the projections to $S$, then we have $\mathcal{S}' = \mathcal{S}(\mathcal{C})$ and $\Pi' = \Pi$. This is an easy consequence of corollary 5.4.3.

**5.6.4.** *Converse* - Let $\pi : \mathcal{S} \to S$ be an oblate $n$-star of $S$. Let $\mathbf{\Pi} : \mathcal{C} \to \mathcal{S}$ be a flat morphism such that for every closed point $s \in \mathcal{S}$, $\mathbf{\Pi}^{-1}(s)$ is a smooth irreducible projective curve. Let $C = \mathbf{\Pi}^{-1}(P)$ and $\tau = \pi \circ \mathbf{\Pi} : \mathcal{C} \to S$. Then $C_n = \tau^{-1}(P)$ is a primitive multiple curve of multiplicity $n$ ans associated smooth curve $C$, and $\mathcal{C}$ is a frangmented deformation of $C_n$. This is an easy consequence of proposition 4.1.6.

## 6. Classification of fragmented deformations of length 2

Let $\pi : \mathcal{C} \to \mathbb{C}$ be a fragmented deformation of length 2. The corresponding double curve $C_2$ is $\pi^{-1}(0)$. Suppose that the spectrum of $\mathcal{C}$ is $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$. This means that the infinitesimal neighbourhoods of order $p$ of $C$ in $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic, i.e. we have an isomorphic of sheaves of algebras on $C$

$$\Phi : \mathcal{O}_{\mathcal{C}_1}/(\pi_1^p) \longrightarrow \mathcal{O}_{\mathcal{C}_2}/(\pi_2^p) \ ,$$

and for every point $x$ of $C$, we have

$$\mathcal{O}_{\mathcal{C},x} \ = \ \{(\alpha_1, \alpha_2) \in \mathcal{O}_{\mathcal{C}_1,x} \times \mathcal{O}_{\mathcal{C}_2,x} \ ; \ \alpha_2 \ (\mathrm{mod}\ \pi_2^p) = \Phi(\alpha_1 \ (\mathrm{mod}\ \pi_1^p))\} \ .$$

Let $C_i^k$ denote the infinitesimal neighbourhood of order $k$ of $C$ in $\mathcal{C}_i$, $i = 1, 2$, $k > 0$. It is a primitive multiple curve of multiplicity $k$ and associated smooth curve $C$, and we have $C_1^p = C_2^p$. Hence $C_1^{p+1}$ and $C_2^{p+1}$ appear as extensions of $C_1^p$ in primitive multiple curves of multiplicity $p+1$. According to [6] and [10] these extensions are classified by $H^1(C, T_C)$ ($T_C$ beeing the tangent sheaf on $C$). More precisely, we say that two such extensions $D$, $D'$ are *isomorphic* if there exists an isomorphism $D \simeq D'$ leaving $C_1^p$ invariant. Then if $\mathcal{H}$ is the set of isomorphism classes of such extensions, a bijection $\lambda : H^1(C, T_C) \to \mathcal{H}$ is defined in [6], such that $\lambda(0) = C_1^{p+1}$.

On the other hand, it follows from [2], [6] that the primitive double curves with associated smooth curve $C$ and associated line bundle $\mathcal{O}_C$ are classified by $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ .

**6.0.5. Theorem:** *The point of* $\mathbb{P}(H^1(C, T_C)) \cup \{0\}$ *corresponding to* $C_2$ *is* $\mathbb{C}.\lambda^{-1}(C_2^{p+1})$.

*Proof.* According to [6], there exists an open covering $(U_i)_{i\in I}$ of $C$ such that for $k=1,2$, the open subset of $C_k^{p+1}$ corresponding to $U_i$ is isomorphic to $U_i \times \mathrm{spec}(C[t]/(t^{p+1}))$. Here $t$ is $\pi_1$ on $\mathcal{C}_1$ and $\pi_2$ on $\mathcal{C}_2$. We obtain then cocycles $(\theta_{ij}^{(k)})_{i,j\in I}$, where $\theta_{ij}^{(k)}$ is an automorphism of $U_{ij} \times \mathrm{spec}(C[t]/(t^{p+1}))$. We can also suppose that $\omega_{C|U_i}$ is trivial, for every $i \in I$. Let $dx_{ij} = dx$ be a generator of $\omega_C(U_{ij})$. Since the ideal sheaf of $C$ in $C_k^{p+1}$ is the trivial sheaf on $C_k^p$, we can write, using the notations of [6], $\theta_{ij}^{(k)} = \phi_{\mu_{ij}^{(k)},1}$ , with $\mu_{ij}^{(k)} \in \mathcal{O}_C(U_{ij})[t]/(t^p)$ , i.e. for every $\alpha \in \mathcal{O}_C(U_i)$, we have, at the level of regular functions

$$\theta_{ij}^{(k)}(\alpha) \;=\; \sum_{m=0}^{p} \frac{1}{m!}(\mu_{ij}^{(k)}t)^m \frac{d^m\alpha}{dx^m} \;,$$

and $\theta_{ij}^{(k)}(t) = t$ . Since $C_1^p = C_2^p$ we can suppose that $\mu_{ij}^{(1)} \equiv \mu_{ij}^{(2)} \pmod{t^{p-1}}$ . Hence $\tau_{ij} = \mu_{ij}^{(2)} - \mu_{ij}^{(1)} \in (t^{p-1})/(t^p) \simeq \mathcal{O}_C(U_i)$. The family $(\tau_{ij})$ is (in some sense) a cocycle representing $\lambda^{-1}(C_2^{p+1})$ (cf. [6], [10]).

We have $(\pi_1^{p+1}) + (\pi_2^{p+1}) \subset (\pi)$ in $\mathcal{O}_{\mathcal{C}}$. Hence $C_2 = \pi^{-1}(0)$ is contained in the subscheme $Z$ of $\mathcal{C}$ corresponding to the ideal sheaf $(\pi_1^{p+1}) + (\pi_2^{p+1})$. We have

$$\begin{aligned}\mathcal{O}_Z(U_{ij}) &= \{(\alpha_1,\alpha_2) \in \mathcal{O}_{\mathcal{C}_1}(U_{ij})/(t^{p+1}) \times \mathcal{O}_{\mathcal{C}_2}(U_{ij})/(t^{p+1}) \;;\; \Phi(\alpha_1 \bmod t^p) = \alpha_2 \bmod t^p\} \\ &= \{(\alpha_1,\alpha_2) \in \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \times \mathcal{O}_C(U_{ij})[t]/(t^{p+1}) \;;\; \alpha_1 \equiv \alpha_2 \bmod t^p\}.\end{aligned}$$

To obtain $\mathcal{O}_{C_2}(U_{ij})$, we have just to quotient by $\pi = (t,t)$, and we obtain

$$\mathcal{O}_{C_2}(U_{ij}) \;=\; \mathcal{O}_Z(U_{ij})/(t,t) \;\simeq\; \mathcal{O}_C(U_{ij})[z]/(z^2) \;,$$

the last isomorphism beeing

$$(a_0 + a_1t + \cdots + a_{p-1}t^{p-1} + \alpha t^p, a_0 + a_1t + \cdots + a_{p-1}t^{p-1} + \beta t^p) \;\mapsto\; \alpha_0 + (\beta-\alpha)z.$$

Now we can explicit the automorphism of $\mathcal{O}_C(U_{ij})[z]/(z^2)$ induced by $\theta_{ij}$ (these isomorphisms will define the cocycle corresponding to $C_2$). It is easy to see that this isomorphism is $\phi_{\tau_{ij},1}$, which proves theorem 6.0.5. □

## References


[1] Bănică, C., Forster, O. *Multiple structures on space curves.* In Contemporary Mathematics 58, Proc. of Lefschetz Centennial Conf. (1986), AMS, 47-64.

[2] Bayer, D., Eisenbud, D. *Ribbons and their canonical embeddings.* Trans. of the Amer. Math. Soc. 347, 3 (1995), 719-756.

[3] Bhosle Usha N. *Generalized parabolic bundles and applications to torsion free sheaves on nodal curves.* Arkiv for Matematik 30 (1992), 187-215.

[4] Bhosle Usha N. *Picard groups of the moduli spaces of vector bundles.* Math. Ann. 314 (1999) 245-263.

[5] Drézet, J.-M. *Faisceaux cohérents sur les courbes multiples* . Collect. Math. 57, 2 (2006), 121-171.

[6] Drézet, J.-M. *Paramétrisation des courbes multiples primitives* Adv. in Geom. 7 (2007), 559-612.

[7] Drézet, J.-M. *Déformations des extensions larges de faisceaux* . Pacific Journ. of Math. 220, 2 (2005), 201-297

[8] Drézet, J.-M. *Faisceaux sans torsion et faisceaux quasi localement libres sur les courbes multiples primitives.* Mathematische Nachrichten 282, No.7 (2009), 919-952.

[9] Drézet, J.-M. *Sur les conditions d'existence des faisceaux semi-stables sur les courbes multiples primitives.* Pacific Journ. of Math. 249, No. 2 (2011), 291-319.

[10] Drézet, J.-M. *Courbes multiples primitives et déformations de courbes lisses.* To appear in Annales de la Faculté des Sciences de Toulouse.
[11] Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry.* GTM 150, Springer-Verlag (1995)
[12] Eisenbud, D., Green, M. *Clifford indices of ribbons.* Trans. of the Amer. Math. Soc. 347, 3 (1995), 757-765.
[13] Godement, R. *Théorie des faisceaux.* Actualités scientifiques et industrielles 1252, Hermann, Paris (1964).
[14] González, M. *Smoothing of ribbons over curves.* Journ. für die reine und angew. Math. 591 (2006), 201-235.
[15] Hartshorne, R. *Algebraic geometry.* (Grad. Texts in Math., Vol. 52). Berlin-Heidelberg-New York: Springer 1977.
[16] Inaba, M.-A. *On the moduli of stable sheaves on some nonreduced projective schemes.* Journ. of Alg. Geom. 13 (2004), 1-27.
[17] Inaba, M.-A. *On the moduli of stable sheaves on a reducible projective scheme and examples on a reducible quadric surface.* Nagoya Math. J. (2002), 135-181.
[18] Samuel, P., Zariski, O. *Commutative Algebra (vol. I, II).* GTM 28,29 Springer-Verlag (1975)
[19] Seshadri, C.S. *Fibrés vectoriels sur les courbes algebriques.* Astérisque 96 (1982).
[20] Simpson, C.T. *Moduli of representations of the fundamental group of a smooth projective variety I.* Publ. Math. IHES 79 (1994), 47-129.
[21] Sun, X. *Degeneration of moduli spaces and generalized theta functions.* Journ. of Alg. Geom. 9 (2000), 459-527.
[22] Sun X., *Degeneration of SL(n)-bundles we have reducible curve.* Proc. of Algebraic Geometry in East Asia (Japan, 2001) and math.AG/0112072.
[23] Teixidor i Bigas, M. *Moduli spaces of (semi-)stable vector bundles on tree-like curves.* Math. Ann. 290 (1191), 341-348.
[24] Teixidor i Bigas, M. *Moduli spaces of vector bundles on reducible curves.* Amer. J. of Math. 117 (1995), 125-139.
[25] Teixidor i Bigas, M. *Compactifications of moduli spaces of (semi)stable bundles on singular curves: two points of view. Dedicated to the memory of Fernando Serrano.* Collect. Math. 49 (1998), 527-548.

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