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EXCEPTIONAL BUNDLES AND MODULI SPACES OF STABLE SHEAVES ON $\mathbb{P}_n$

JEAN–MARC DRÉZET

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1. Introduction

In this paper I try to show how the exceptional bundles can be useful to study vector bundles on the projective spaces. The exceptional bundles appeared in [5], and they were used to describe the ranks and Chern classes of semi-stable sheaves. In [1] the generalized Beilinson spectral sequence, built with exceptional bundles, was defined, and it was used in [2] and [3] to describe some moduli spaces of semi-stable sheaves on $\mathbb{P}_2$. The general notion of exceptional bundle and helix, on $\mathbb{P}_n$ and many other varieties, is due mainly to A.L. Gorodentsev and A.N. Rudakov (cf. [7] , [14]). A.N. Rudakov described completely in [12] the exceptional bundles on $\mathbb{P}_1 \times \mathbb{P}_1$, and used them in [13] to describe the ranks and Chern classes of semi-stable sheaves on this variety. The exceptional vector bundles on $\mathbb{P}_3$ have been studied (cf [4], [10], [11]) but they have not yet been used to describe semi-stable sheaves on $\mathbb{P}_3$. On higher $\mathbb{P}_n$ almost nothing is known.

In the second part of this paper, new invariants of coherent sheaves of non-zero rank are defined. In some cases they are more convenient than the Chern classes.

In the third part the exceptional bundles and helices are defined, and their basic properties are given.

In the fourth part, I define some useful hypersurfaces in the space of invariants of coherent sheaves on $\mathbb{P}_n$. On $\mathbb{P}_2$, this space is $\mathbb{R}^2$, with coordinates (μ, Δ), where μ is the *slope* and Δ the *discriminant* of coherent sheaves, as defined in [5]. On $\mathbb{P}_n$, the space of invariants is $\mathbb{R}^n$, and the coordinates are the invariants defined in the second part.

In the fifth part, the description of ranks and Chern classes of semi-stable sheaves on $\mathbb{P}_2$ is recalled. The ranks and Chern classes of semi-stable sheaves on $\mathbb{P}_3$ are not known, and in this case I can only try to formulate the problem correctly, using the notions of the fourth part.

In the sixth part, some partial results are given on the description of the simplest moduli spaces of semi-stable sheaves on $\mathbb{P}_n$. A moduli space is simple when the corresponding point in the space of invariants belongs to many hypersurfaces defined in part 4 (in this case a suitable generalized Beilinson spectral sequence applied to the sheaves of this moduli space is supposed to degenerate). In the case of $\mathbb{P}_3$, $n \geq 3$, many questions remain open.

2. Logarithmic invariants

Let X be a projective smooth algebraic variety of dimension n, E a vector bundle (or coherent sheaf) on X, of rank $r > 0$. The *logarithmic invariants* $\Delta_i(E) \in A^i(E) \otimes \mathbb{Q}$ of E are defined formally by the following formula :

$$log(ch(E)) = log(r) + \sum_{i=1}^{n} (-1)^{i+1} \Delta_i(E),$$

where $ch(E)$ is the Chern character of E. For example, we have

$$\Delta_1(E) = \frac{c_1}{r} \ , \ \Delta_2(E) = \frac{1}{r}(c_2 - \frac{r-1}{2r} c_1^2),$$

$$\Delta_3(E) = \frac{1}{r}(\frac{c_3}{2} + c_1 c_2(\frac{1}{r} - \frac{1}{2}) + c_1^3(\frac{1}{3r^2} - \frac{1}{2r} + \frac{1}{6})),$$

$$\begin{aligned} \Delta_4(E) \ &= \ \frac{1}{r}(\frac{c_4}{6} + c2^2(\frac{1}{2r} - \frac{1}{12}) + c_1 c_3(\frac{1}{2r} - \frac{1}{6}) \\ &\quad + c_1^2 c_2(\frac{1}{r^2} - \frac{1}{r} + \frac{1}{6}) + c_1^4(\frac{1}{4r^3} - \frac{1}{2r^2} + \frac{7}{24r} - \frac{1}{24}), \end{aligned}$$

where for $1 \leq i \leq n$, c_i is the i-th Chern class of E. The first invariant is the *slope* and the second the *discriminant* of E.

Proposition 2.1. *Let L be a line bundle, E,F vector bundles on X. Then*

(1) $\Delta_1(L) = c_1(L)$ *and* $\Delta_i(L) = 0$ *if* $i > 1$.
(2) $\Delta_i(E \otimes F) = \Delta_i(E) + \Delta_i(F)$ *if* $1 \leq i \leq n$. *Thus* $\Delta_i(E \otimes L) = \Delta_i(E)$ *if* $2 \leq i \leq n$.
(3) $\Delta_i(E^*) = (-1)^i \Delta_i(E)$ *if* $1 \leq i \leq n$.

Démonstration. This is clear from the definition of the Δ_i's. □

Since $ch(E)/r$ is a polynomial in $\Delta_1(E), \ldots, \Delta_n(E)$, the Riemann-Roch theorem on X can be written in the following way :

$$\frac{\chi(E)}{r} = P(\Delta_1(E), ..., \Delta_n(E)),$$

where P is a polynomial with rational coefficients that depends only on X. If X is a surface with fundamental class K, we have

$$P(\Delta_1, \Delta_2) = \frac{\Delta_1(\Delta_1 - K)}{2} + \chi(\mathcal{O}_X) - \Delta_2.$$

If X is a volume with fundamental class K, and if c_2 is the second Chern class of the tangent bundle of X, we have

$$P(\Delta_1, \Delta_2, \Delta_3) = \Delta_3 - \Delta_1\Delta_2 + \frac{1}{2}K\Delta_2 + \frac{1}{6}\Delta_1^3 - \frac{1}{4}K\Delta_1^2 + \frac{1}{12}(K^2 + c_2)\Delta_1 + \chi(\mathcal{O}_X).$$

In particular, for $X = \mathbb{P}_3$,

$$P(\Delta_1, \Delta_2, \Delta_3) = \Delta_3 - \Delta_1\Delta_2 - 2\Delta_2 + \binom{\Delta_1 + 3}{3}.$$

For $X = \mathbb{P}_4$,

$$\begin{aligned} P(\Delta_1, \Delta_2, \Delta_3, \Delta_4) \quad &= \quad -\Delta_4 + \Delta_1\Delta_3 + \frac{1}{2}\Delta_2(\Delta_2 - \Delta_1^2) + \frac{5}{2}(\Delta_3 - \Delta_1\Delta_2) \\ &\quad + \binom{\Delta_1 + 4}{4}. \end{aligned}$$

In the case of $\mathbb{P}_n$, let Γ be the hyperplane of elements of rank 0 in the Grothendieck group $K(\mathbb{P}_n)$. Then we have a surjective map

$$(\Delta_1, \ldots, \Delta_n) \ : \ K(\mathbb{P}_n)\backslash\Gamma \longrightarrow \mathbb{Q}^n.$$

Two elements of $K(\mathbb{P}_n)\backslash\Gamma$ are in the same fiber of this map if and only if they are colinear.

3. Exceptional bundles

3.1. Definition of exceptional bundles

Let E be an algebraic vector bundle on a smooth projective irreducible algebraic variety X. Then E is called *exceptional* if $H^i(X, Ad(E)) = 0$ for every i. If X is one of the varieties considered here (a projective space or a smooth quadric surface) then E is exceptional if and only if E is *simple* (i.e. the only endomorphisms of E are the homotheties) and $\mathrm{Ext}^i(E, E) = 0$ for every $i \geq 1$.

For example, on $\mathbb{P}_n$ the line bundles are exceptional. So is the tangent bundle. In general, if E is an exceptional bundle and L a line on X, then $E \otimes L$ is also exceptional.

3.2. Helices

Suppose that $X = \mathbb{P}_n$, with $n \geq 2$. An infinite sequence $(E_i)_{i\in\mathbb{Z}}$ of exceptional bundles is called *exceptional* if the following three conditions are satisfied :

(1) The sequence is *periodical*, i.e for all $i \in \mathbb{Z}$ we have
$$E_{i+n+1} \simeq E_i(n+1).$$
(2) There exists an integer i_0 such that for $i_0 \leq i < j \leq i_0 + n$ we have
$$\mathrm{Ext}^k(E_i, E_j) = 0 \text{ if } k > 0,$$
$$\mathrm{Ext}^k(E_j, E_i) = 0 \text{ for all } k.$$
(3) For every integer j, the canonical morphism
$$ev_j : E_j \otimes \mathrm{Hom}(E_{j-1}, E_j) \longrightarrow E_j$$
$$(\text{resp. } ev_j^* : E_{j-1} \longrightarrow E_j \otimes \mathrm{Hom}(E_{j-1}, E_j)^* \ )$$
is surjective (resp. injective).

If $\sigma = (E_i)_{i\in\mathbb{Z}}$ is a sequence of exceptional bundles, let $\tau(\sigma)$ denote the sequence $(E'_i)_{i\in\mathbb{Z}}$, where $E'_i = E_{i-1}$ for each i. Suppose that σ satisfies condition 1. Then any subsequence $(E_i, \ldots, E_{i+n})$ is called a *foundation* or a *basis* of σ. Suppose that σ is exceptional. Then it is not difficult to see that condition 2 above is verified for every integer i_0, and that $\ker(ev_j)$ and $\mathrm{coker}(ev_j^*)$ are exceptional bundles. We can thus define two new sequences of exceptional bundles, associated to σ and j *mod* $(n+1)$. The first sequence
$$L_j(\sigma) = (E'_i)_{i\in\mathbb{Z}}$$
is defined by :
$$\begin{aligned} E'_i &= E_i \text{ if } i \neq j-1 \ (mod\ n+1) \ \ and \ \ i \neq j \ (mod\ n+1), \\ E'_{j-1+k(n+1)} &= \ker(ev_j)(k(n+1)), \\ E'_{j+k(n+1)} &= E_{j-1+k(n+1)}, \end{aligned}$$
for all k. The second sequence $R_{j-1}(\sigma)$ is defined in the same way, by replacing in σ each pair $(E_{j-1+k(n+1)}, E_{j+k(n+1)})$ by
$$(E_{j+k(n+1)}, \mathrm{coker}(ev_j^*)(k(n+1))).$$
The sequence $L_j(\sigma)$ is called the *left mutation* of σ at E_j and $R_{j-1}(\sigma)$ the *right mutation* of σ at E_{j-1}. For these two sequences, conditions 1 and 2 above are satisfied.

Suppose that condition 3 is also satisfied for $L_j(\sigma)$, i.e. that it is an exceptional sequence. Then it has a foundation of type
$$(E_{j-1}, E_{j+1}, \ldots, E_{j+n-1}, F_1),$$
where F_1 is an exceptional bundle. It is possible to define
$$L_j^2(\sigma) \ = \ L_{j-1} \circ \ L_j(\sigma).$$
Suppose that this is again an exceptional sequence. Then it has a foundation of type
$$(E_{j-1}, E_j, \ldots, E_{j+n-2}, F_2, E_{j+n-1}),$$
where F_2 is an exceptional bundle. It is then possible to define
$$L_j^3(\sigma) \ = \ L_{j-2} \circ \ L_j^2(\sigma).$$
If this process can be continued, we can define the exceptional sequence $L_j^k(\sigma)$, for $1 \leq k \leq n$, which has, if $k \leq n-1$, a foundation of type
$$(E_{j-1}, E_{j+1}, \ldots, E_{j+n-k}, F_k, E_{j+n-k+1}, \ldots, E_{j+n-1}),$$

where F_k is an exceptional bundle. In particular, $L^{n-1}(\sigma)$ has a foundation of type

$$(E_{j-1}, E_{j+1}, F_{n-1}, E_{j+2}, \ldots, E_{j+n-1}),$$

so $L_j^n(\sigma)$ has a foundation of type

$$(E_{j-1}, F_n, E_{j+1}, \ldots, E_{j+n-1}).$$

The exceptional bundle $F_k(-n)$, for $1 \leq k \leq n$, is denoted by $L^{(k)}E_j$, and $L^{(0)}E_j = E_j$.

A *helix* is an exceptional sequence σ such that for every integer j, the sequences $L_j^k(\sigma)$ are defined, for $1 \leq k \leq n$, and such that

$$L_j^n(\sigma) \;=\; \tau(\sigma).$$

The last condition means that $F_n \simeq E_j$.

The helices have an interesting property : any left or right mutation of a helix is a helix. So it is possible to define infinitely many helices and exceptional bundles simply by making successive mutations of one helix. The simplest helix is the sequence $(\mathcal{O}(i))_{i\in\mathbb{Z}}$. The helices that can be obtained by successive mutations of this helix are called *constructive helices* and the corresponding exceptional bundles are the *constructive exceptional bundles*. All helices and vector bundles on $\mathbb{P}_2$ are constructive (see [1, 5, 7]), and so are all helices on $\mathbb{P}_3$ (see [11]).

All the mutation transformations defined above can be expressed in terms of τ and L_0 only. For example, we have $L_j = \tau^i \circ L_0 \circ \tau^{-j}$. There are some relations among τ and L_0 :

$$L_0^n = \tau,$$
$$L_0 \circ \tau \circ L_0 \circ \tau^{-1} \circ L_0 \circ \tau = \tau \circ L_0 \circ \tau^{-1} \circ L_0 \circ \tau \circ L_0,$$
$$L_0 \circ \tau^n = \tau^n \circ L_0,$$
$$L_0 \circ \tau^i \circ L_0 \circ \tau^{-i} = \tau^i \circ L_0 \circ \tau^{-i} \circ L_0 \;\text{ if }\; 2 \leq i \leq n-1.$$

3.3. Generalized Beilinson spectral sequence

Let $(E_0, \ldots, E_n)$ be a foundation of a constructive helix. Then the sequence

$$(L^{(n)}E_n, L^{(n-1)}E_{n-1}, \ldots, L^{(1)}E_1, E_0)$$

is a foundation of the helix $L_n \circ L_{n-1} \circ \ldots \circ L_1(\sigma)$. There exists a canonical resolution of the diagonal Δ of $\mathbb{P}_n \times \mathbb{P}_n$:

$$0 \longrightarrow p_1^* L^{(n)} E_n \otimes p_2^* E_n^* \longrightarrow p_1^* L^{(n-1)} E_{n-1} \otimes p_2^* E_{n-1}^* \longrightarrow \ldots$$
$$\longrightarrow \ldots p_1^* L^{(1)} E_1 \otimes p_2^* E_1^* \longrightarrow p_1^* E_0 \otimes p_2^* E_0^* \xrightarrow{\phi} \mathcal{O}_\Delta \longrightarrow 0,$$

where p_1, p_2 denote the projections $\mathbb{P}_n \times \mathbb{P}_n \longrightarrow \mathbb{P}_n$ and ϕ is the trace morphism. It follows easily that for every coherent sheaf $\mathcal{E}$ on $\mathbb{P}_n$ there exists a spectral sequence $E_r^{p,q}$ of coherent sheaves on $\mathbb{P}_n$, converging to $\mathcal{E}$ in degree 0 and to zero in other degrees, such that the only possibly non-zero $E_1^{p,q}$ terms are

$$E_1^{p,q} \;=\; E_{-p}^* \otimes H^q(\mathcal{E} \otimes L^{(-p)} E_{-p}),$$

for $-n \leq p \leq 0,\ 0 \leq q \leq n$. The morphisms $d_1^{p,q}$ come from the morphisms in the preceding resolution of $\mathcal{O}_\Delta$. This spectral sequence is called the *generalized Beilinson spectral sequence* associated to $\mathcal{E}$ and the foundation $(E_0, \ldots, E_n)$. If this foundation is $(\mathcal{O}(i))_{0 \leq i \leq n}$, the generalized Beilinson spectral sequence if of course the ordinary Beilinson spectral sequence.

From the generalized Beilinson spectral sequence one can deduce the *generalized Beilinson complex*

$$0 \longrightarrow X_{-n} \longrightarrow X_{-n+1} \longrightarrow \ldots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n \longrightarrow 0$$

where for $-n \leq k \leq n$

$$X_k \ = \bigoplus_{p+q=k} E_1^{p,q}.$$

This complex is exact in non-zero degrees and its cohomology in degree 0 is isomorphic to $\mathcal{E}$.

4. The geometry associated to exceptional bundles

4.1. The space of invariants and its canonical hypersurface

Consider the space $\mathbb{R}^n$, with coordinates $(\Delta_1, \ldots, \Delta_n)$. Then to each coherent sheaf $\mathcal{E}$ on $\mathbb{P}_n$ with non-zero rank one associates the point

$$(\Delta_1(\mathcal{E}), \ldots, \Delta_n(\mathcal{E}))$$

of $\mathbb{R}^n$, which will be also denoted by $\mathcal{E}$. Recall that there exists a polynomial P in n variables with rational coefficients, such that for every coherent sheaf $\mathcal{E}$ on $\mathbb{P}_n$ with non-zero rank we have

$$\chi(\mathcal{E}) \ = \ rk(\mathcal{E}).P(\Delta_1(\mathcal{E}), \ldots, \Delta_n(\mathcal{E})).$$

The hypersurface H of $\mathbb{R}^n$ defined by the equation

$$P(0, 2\Delta_2, 0, 2\Delta_4, \ldots) = 0$$

is called the *canonical hypersurface*. If $\mathcal{E}$ is a stable sheaf on $\mathbb{P}_n$, then $\mathcal{E}$ belongs to the halfspace

$$P(0, 2\Delta_2, 0, 2\Delta_4, ...) < 0$$

if and only if the expected dimension of the moduli space of semi-stable sheaves that contains $\mathcal{E}$ is strictly positive.

For example, on $\mathbb{P}_2$, the equation of H is

$$\Delta_2 = \frac{1}{2}.$$

On $\mathbb{P}_3$ it is

$$\Delta_2 = \frac{1}{4},$$

and on $\mathbb{P}_4$

$$\Delta_4 = \Delta_2^2 - \frac{35}{6}\Delta_2.$$

Question 1. *Is the expected dimension of a moduli space of semi-stable sheaves on $\mathbb{P}_n$ always nonnegative ?*

4.2. Hypersurfaces associated to exceptional bundles and limit hypersurfaces

Let E be an exceptional bundle on $\mathbb{P}_n$. Then to E one associates the hypersurface $S(E)$ of $\mathbb{R}^n$ defined by the equation

$$P(\Delta_1(E) - \Delta_1, \ldots, \Delta_n(E) - \Delta_n) \;=\; 0.$$

It contains the points corresponding to sheaves $\mathcal{E}$ such that

$$\chi(\mathcal{E}, E) \;=\; \sum_{i=0}^{n} (-1)^i \dim(\mathrm{Ext}^i(\mathcal{E}, E)) \;=\; 0.$$

To define the *limit hypersurfaces* we need to consider an *exceptional pair*, i.e. a pair (E_0, E_1) of exceptional bundles that can be inserted as a pair of consecutive elements in some helix. Then it follows from the definition of a helix that there exists a sequence $(F_i)_{i\in\mathbb{Z}}$ of exceptional bundles such that $F_0 = E_0$, $F_1 = E_1$, and for every integer i we have an exact sequence

$$0 \longrightarrow F_{i-1} \longrightarrow F_i \otimes \mathrm{Hom}(F_i, F_{i+1}) \longrightarrow F_{i+1} \longrightarrow 0.$$

Then the hypersurfaces $S(F_i)$ have a limit when i tends to $+\infty$ or $-\infty$. It is possible to define more complicated limit hypersurfaces (limits of limits, and so on). Let $C(E)$ denote the intersection of $S(E)$ and the canonical hypersurface H.

In the case of $\mathbb{P}_2$, the curve $S(E)$ is a parabola in $\mathbb{R}^2$, of equation

$$\Delta_2 \;=\; \frac{1}{2}(\Delta_1(E) - \Delta_1)^2 + \frac{3}{2}(\Delta_1(E) - \Delta_1) + \frac{1}{2} + \frac{1}{2rk(E)^2},$$

and $C(E)$ consists of two points on the line H. The limit points on H coincide with the non-limit points.

In the case of $\mathbb{P}_3$, the equation of the surface $S(E)$ is

$$\Delta_3 \;=\; -\frac{1}{6}z^3 + (\Delta_2 + \frac{5}{12} - \frac{1}{4rk(E)^2})z + \Delta_3(E),$$

with $z = \Delta_1 - \Delta_1(E) - 2$. The curve $C(E)$ is obtained by taking $\Delta_2 = 1/4$ in the preceding equation. In this case, the limit curves are distinct from the non-limit ones. Some other surfaces and curves may be interesting in the case of $\mathbb{P}_3$: the images of the preceding ones by the translations

$$(\Delta_1, \Delta_2, \Delta_3) \longrightarrow (\Delta_1, \Delta_2, \Delta_3 + k),$$

where k is an integer. In [11], Nogin proved that the semi-orthogonal bases of $K(\mathbb{P}_3)$ are the sequences

$$([E_0] \otimes \alpha^k, [E_1] \otimes \alpha^k, [E_2] \otimes \alpha^k, [E_3] \otimes \alpha^k),$$

(E_0, E_1, E_2, E_3) beeing a foundation of a helix, α the class of the ideal sheaf of a point and k an integer. The multiplication by α^k corresponds to the preceding translation in $\mathbb{R}^3$.

4.3. The case of $\mathbb{P}_1 \times \mathbb{P}_1$

The space of invariants is here $\mathbb{R}^3$ with coordinates (a, b, Δ_2), a, b beeing the two coordinates of Δ_1. The equation of H is

$$\Delta_2 \ = \ \frac{1}{2}.$$

The surfaces $S(E)$ are quadrics, and the corresponding conics $C(E)$ have been used in [12]. It is also possible here to define the notion of limit surface (or curve). This case is similar to the case of $\mathbb{P}_3$ (cf. [4]).

5. Existence theorems

5.1. The existence theorem on $\mathbb{P}_2$

Let $\mathcal{E}$ be a stable coherent sheaf on $\mathbb{P}_2$, not exceptional, and E an exceptional bundle such that $rk(E) < rk(\mathcal{E})$ and $\mid \Delta_1(E) - \Delta_1(\mathcal{E}) \mid \leq 1$. Then we have

$$\begin{array}{ll} \chi(E, \mathcal{E}) \leq 0 & \text{if} \quad \Delta_1(\mathcal{E}) \leq \Delta_1(E), \\ \chi(\mathcal{E}, E) \leq 0 & \text{if} \quad \Delta_1(\mathcal{E}) > \Delta_1(E). \end{array}$$

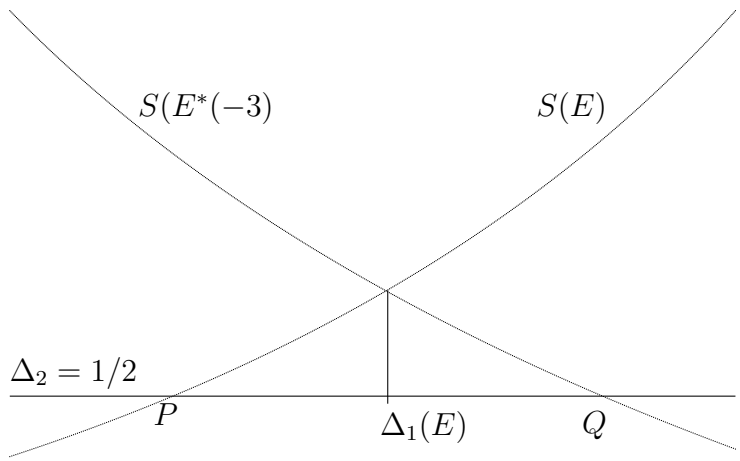
The first condition means that the point $\mathcal{E}$ in $\mathbb{R}^2$ is over the curve $S(E^*(-3))$[1] and the second that it is over the curve $S(E)$. Conversely the following is proved in [5] :

Theorem 5.1. *Let $q = (\Delta_1, \Delta_2)$ be a point in $\mathbb{Q}^2$. Suppose that for every exceptional bundle E such that*

$$\mid \Delta_1(E) - \Delta_1 \mid \leq 1$$

the point q is over $S(E^*(-3))$ if $\Delta_1(\mathcal{E}) \leq \Delta_1(E)$, and over $S(E)$ if $\Delta_1(\mathcal{E}) > \Delta_1(E)$. Then for every triple (r, c_1, c_2) of integers, with $r > 0$ such that $\Delta_i(r, c_1, c_2) = \Delta_i$ for i=1,2, there exists a stable vector bundle of rank r and Chern classes c_1, c_2.

[1]this means that $\Delta_2(E)$ is greater than, or equal to the Δ_2 coordinate of the point of $S(E^*(-3))$ whose first coordinate is $\Delta_1(E)$.



The coordinates of the point P (resp. Q) above are $(\Delta_1(E)-x_E, 1/2)$ (resp. $(\Delta_1(E)+x_E, 1/2)$), where x_E is the smallest root of the equation

$$x^2 + 3x + \frac{1}{3rg(E)^2} \;=\; 0.$$

Let $I_E =]\Delta_1(E) - x_E, \Delta_1(E) + x_E[$, and $\mathcal{E}xc$ be the set of isomorphism classes of exceptional bundles on $\mathbb{P}_2$. Let $M(r, c_1, c_2)$ denote the moduli space of semi-stable coherent sheaves on $\mathbb{P}_2$, of rank r and Chern classes c_1, c_2. The preceding theorem can be improved and one obtains easily the final form of the existence theorem on $\mathbb{P}_2$:

Theorem 5.2. (1) *The family of intervals $(I_E)_{E\in\mathcal{E}xc}$ is a partition of $\mathbb{Q}$.*

(2) *There exists a unique mapping*

$$\delta : \mathbb{Q} \longrightarrow \mathbb{Q}$$

such that for all integers r, c_1, c_2 with $r \geq 1$ one has

$$\dim(M(r, c_1, c_2)) \geq 0 \Longleftrightarrow \Delta_2 \geq \delta(\Delta_1),$$

(with $\Delta_1 = \frac{c_1}{r}$ and $\Delta_2 = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)$.)

(3) *If E in an exceptional bundle on $\mathbb{P}_2$, δ is given on $]\Delta_1(E) - x_E, \Delta_1(E)]$ by $S(E^*(-3))$ and on $[\Delta_1(E), \Delta_1(E) + x_E[$ by $S(E)$.*

5.2. The existence theorem on $\mathbb{P}_1 \times \mathbb{P}_1$

A.N. Rudakov has proved in [13] a result analogous to theorem 5.1, using the exceptional bundles on $\mathbb{P}_1 \times \mathbb{P}_1$, but it seems more difficult than in the case of $\mathbb{P}_2$ to obtain the analogous result to theorem 5.2.

Question 2. *It is easy to deduce from Rudakov's result that there exists a surface S in $\mathbb{R}^3$, defined by an equation*

$$\Delta_2 \ = f(\Delta_1),$$

such that for all integers r, a, b, c_2 with $r \geq 1$, there exists a stable non-exceptional coherent sheaf on $\mathbb{P}_1 \times \mathbb{P}_1$ of rank r and Chern classes $(a, b), c_2$ if and only if the associated point in $\mathbb{R}^3$ (whose coordinates are the corres- ponding Δ_1, Δ_2) is over[2] S . It would be interesting to give a description of S. It should be made of pieces of the surfaces $S(E)$ or perhaps of the limit surfaces defined in section 4.3 .

5.3. The existence theorem on $\mathbb{P}_n$, $n \geq 3$

In this case, almost nothing is known, except for rank-2 stable reflexive sheaves on $\mathbb{P}_3$ (cf. [8]).

Question 3. *Is there a surface S in $\mathbb{R}^3$, of equation*

$$\Delta_2 \ = \ f(\Delta_1, \Delta_3),$$

such that for all integers r, c_1, c_2, c_3 with $r \geq 1$, the moduli space $M(r, c_1, c_2, c_3)$ of semi-stable sheaves on $\mathbb{P}_3$ of rank r and Chern classes c_1, c_2, c_3 has a positive dimension if and only if the associated point of $\mathbb{R}^3$ (whose coordinates are the corresponding $\Delta_1, \Delta_2, \Delta_3$) is over[3] S ?

In particular, do the gaps *in c_3 found in* [8] *can be filled if one allows non-reflexive rank-2 stable sheaves ?*

If it exists, is S made of pieces of the $S(E)$ and the limit surfaces ?

6. Descriptions of moduli spaces of semi-stable sheaves using exceptional bundles

6.1. The case of $\mathbb{P}_2$

Let E be an exceptional bundle on $\mathbb{P}_2$, and Δ_1 a rational number such that $\Delta_1(E) - x_E < \Delta_1 \leq \Delta_1(E)$. There exists exceptional bundles F,G such that (E, F, G) is a foundation of a helix. Then to study moduli spaces of semi-stable sheaves $M(r, c_1, c_2)$ such that $c_1/r = \Delta_1$ it is convenient to use the Beilinson spectral sequence associated to $(G^*(3), F^*(3), E^*(3))$. We obtain a good description of $M(r, c_1, c_2)$ if $\Delta_2 = \delta(\Delta_1)$, i.e. if the point of $\mathbb{R}^2$ corresponding to $M(r, c_1, c_2)$ lies on the curve $S(E^*(-3))$ (or more generally in some cases where $M(r, c_1, c_2)$ is *extremal*, i.e if $\dim(M(r, c_1, c_2)) > 0$ and $\dim(M(r, c_1, c_2 - 1)) \leq 0$).

[2]this means that $\Delta_2 \geq f(\Delta_1)$.
[3]this means that $\Delta_2 \geq f(\Delta_1, \Delta_3)$.

Suppose that $\Delta_2 = \delta(\Delta_1)$. Let H be the exceptional bundle cokernel of the canonical map

$$F \longrightarrow G \otimes \mathrm{Hom}(F, G)^*,$$

and

$$m = -\chi(\mathcal{E} \otimes H^*),\ k = -\chi(\mathcal{E} \otimes G^*),$$

where $\mathcal{E}$ is a coherent sheaf of rank r and Chern classes c_1, c_2. Then $m > 0$ and $k > 0$. If $\mathcal{E}$ is semi-stable then the only two non zero $E_1^{p,q}$ terms in the Beilinson spectral sequence associated to (G^*, F^*, E^*) and $\mathcal{E}$ are $F(-3) \otimes H^0(\mathcal{E} \otimes H^*(3))$ and $G(-3) \otimes H^0(\mathcal{E} \otimes G^*(3))$. So the spectral sequence degenerates and we have an exact sequence

$$0 \longrightarrow F(-3) \otimes \mathbb{C}^m \longrightarrow G(-3) \otimes \mathbb{C}^k \longrightarrow \mathcal{E} \longrightarrow 0.$$

Consider now the vector space

$$W \ = \ \mathrm{Hom}(F \otimes \mathbb{C}^m, G \otimes \mathbb{C}^k) \ = \ L(\mathrm{Hom}(F, G)^* \otimes \mathbb{C}^m, \mathbb{C}^k),$$

with the obvious action of the reductive group

$$G_0 = (GL(m) \times GL(k))/\mathbb{C}^*.$$

This action can be linearized in an obvious way, so we have the notion of semi-stable (or stable) point of $\mathbb{P}_W$. A non-zero element of W will be called *semi-stable* (resp. *stable*) if its image in $\mathbb{P}_W$ is. Let $q = \dim(W)$ and

$$N(q, m, k) \ = \ \mathbb{P}(W)^{ss}/G_0.$$

which is a projective variety. The following result is proved in [2] :

Theorem 6.1. (1) *Let α be a non-zero element of W, f the corresponding morphism of vector bundles. Then f is injective as a morphism of sheaves, and* $\mathrm{coker}(f)$ *is semi-stable (resp. stable) if and only if α is semi-stable (resp. stable).*

(2) *The map $f \longmapsto \mathrm{coker}(f)$ defines an isomorphism*

$$N(q, m, k) \ \simeq \ M(r, c_1, c_2).$$

There is a similar result for some other extremal moduli spaces of semi-stable sheaves (cf. [3]). In this case we have to consider morphisms of the following type

$$(E(-3) \otimes \mathbb{C}^h) \oplus (F(-3) \otimes \mathbb{C}^m) \longrightarrow G(-3) \otimes \mathbb{C}^k,$$

and the group acting on the space of such morphisms is non reductive.

There is a canonical isomorphism

$$N(q, m, k) \ \simeq \ N(q, k, qk - m).$$

Hence to $N(q, k, qk - m)$ is associated another moduli space $M(r', c_1', c_2')$ which is canonically isomorphic to $M(r, c_1, c_2)$, with

$$\Delta_1(E) - x_E < \frac{c_1'}{r'} < \frac{c_1}{r} \leq \Delta_1(E).$$

Finally, we obtain an infinite sequence of moduli spaces of semi-stable sheaves all isomorphic to $M(r, c_1, c_2)$. This phenomenon does not occur for moduli spaces such that $\Delta_2 > \delta(\Delta_1)$. Here are some examples of descriptions of moduli spaces obtained with the preceding theorem :

$$M(1,0,1) \simeq M(3,-1,2) \simeq M(8,-3,8) \simeq \mathbb{P}_2,$$

$$M(4,-2,4) \simeq M(24,-10,60) \simeq M(140,-58,1740) \simeq \mathbb{P}_5,$$

$$M(4,-1,3) \simeq M(11,-4,13) \simeq M(29,-11,73) \simeq N(3,2,3).$$

Question 4. *It is also possible to study more complicated moduli spaces (non extremal ones) using some foundation of a helix. We can for example obtain descriptions of moduli spaces by monads. What is the best choice for the generalized Beilinson spectral sequence ? Or are there foundations of helices that would lead to more interesting monads than those obtained from the classical Beilinson spectral sequence ?*

6.2. The case of $\mathbb{P}_n$, $n \geq 3$

If we want to use exceptional bundles to study a moduli space of semi-stable sheaves on $\mathbb{P}_n$, we have to choose a foundation of helix such that the Beilinson spectral sequence associated to it and to sheaves in this moduli space is as simple as possible. This is the case of course if many $E_1^{p,q}$ are zero. Let $(E_0,\dots,E_n)$ be a foundation of helix, $r, c_1,\dots,c_n$ integers with $r \geq 1$, $\mathcal{E}$ a coherent sheaf on $\mathbb{P}_n$ of rank r and Chern classes $c_1,\dots,c_n$, and $E_r^{p,q}$ the Beilinson spectral sequence associated to $(E_0,\dots,E_n)$ and $\mathcal{E}$. Then, if $0 \leq i \leq n$, we can hope that the terms $E_1^{p,-i}$ will be zero for all p only if $\chi(\mathcal{E} \otimes L^{(i)}E_i) = 0$. This means that the point corresponding to $\mathcal{E}$ in the space of invariants belongs to the hypersurface $S(L^{(i)}E_i)$. Of course this is not sufficient.

Suppose for example that all the $E_1^{p,-i}$, for $0 \leq i \leq n-2$, vanish, and that $E_1^{n,-n} = 0$. Then we have an exact sequence

$$0 \longrightarrow E_n^* \otimes H^{n-1}(\mathcal{E} \otimes L^{(n)}E_n) \longrightarrow E_{n-1}^* \otimes H^{n-1}(\mathcal{E} \otimes L^{(n-1)}E_{n-1}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

In general the vanishing of the cohomology groups necessary to obtain the above exact sequence are very hard to verify.

Let E,F be exceptional bundles on $\mathbb{P}_n$ which are consecutive terms in some helix, and m,k two positive integers. We want to study morphisms

$$E \otimes \mathbb{C}^m \longrightarrow F \otimes \mathbb{C}^k.$$

Let

$$W \;=\; \mathrm{Hom}(E \otimes \mathbb{C}^m, F \otimes \mathbb{C}^k) \;=\; L(\mathbb{C}^m \otimes \mathrm{Hom}(E,F)^*, \mathbb{C}^k).$$

on which acts the reductive group G_0. Recall the characterization of semi-stable and stable points of W:

Proposition 6.2. *Let α be a non-zero element of W. Then α is semi-stable (resp. stable) if and only if for every non-zero subspace $H \subset \mathbb{C}^m$, if*

$$K = \alpha(\mathrm{Hom}(E,F)^* \otimes H),$$

we have $\dfrac{\dim(K)}{\dim(H)} \ \geq \ \dfrac{k}{m}$ *(resp. $>$).*

Let $q = \dim(\mathrm{Hom}(E,F))$. Then we have $\dim(N(q,m,k)) > 0$ if and only if

$$x_q \ < \ \frac{m}{k} \ < \ \frac{1}{x_q},$$

where x_q is the smallest root of the equation

$$X^2 - qX + 1 \ = \ 0.$$

Suppose that there exist an injective morphism of sheaves

$$E \otimes \mathbb{C}^m \longrightarrow F \otimes \mathbb{C}^k.$$

Let $\mathcal{E}$ be its cokernel. Then it is easy to see that the preceding inequalities are verified if and only if the expected dimension of the moduli space of semi-stable sheaves with the same invariants as $\mathcal{E}$ is positive.

The first problem is the injectivity of stable maps.

Proposition 6.3. *Let x be a point of $\mathbb{P}_n$. Suppose that the canonical map*

$$ev_x : E_x \otimes \mathrm{Hom}(E,F) \longrightarrow F_x$$

is stable (for the action of $(GL(E_x) \times GL(F_x))/\mathbb{C}^*$). Suppose that

$$\frac{k}{m} \ \geq \ \chi(E^* \otimes F) - \frac{rk(F)}{rk(E)}.$$

Then the morphism of vector bundles associated to a semi-stable element of W is injective on the complement of a finite set, and injective if the preceding inequality is strict.

The proof uses the same arguments as on $\mathbb{P}_2$. Of course, the stability of ev_x is independent of x.

Question 5. *Is ev_x always stable ?*

The answer is yes on $\mathbb{P}_2$.

If we allow $\frac{n}{m}$ to be smaller than the bound in the preceding proposition, it may happen that a semi-stable morphism is non injective on some subvariety of $\mathbb{P}_n$. For example let (E_0, E_1, E_2, E_3) de a foundation of some helix on $\mathbb{P}_3$. It follows easily from the generalized Beilinson spectral sequence that if C is a smooth curve in $\mathbb{P}_3$, and $\mathcal{F}$ a vector bundle on C such that

$$H^0(L^{(3)}E_3 \otimes i_*\mathcal{F}) = H^1(L^{(3)}E_3 \otimes i_*\mathcal{F}) = 0,$$

(where i is the inclusion of C in $\mathbb{P}_3$) then there exists an exact sequence

$$0 \longrightarrow H^0(L^{(2)}E_2 \otimes i_*\mathcal{F}) \otimes E_2^* \longrightarrow H^0(L^{(1)}E_1 \otimes i_*\mathcal{F}) \otimes E_1^* \longrightarrow \dots$$

$$\ldots \longrightarrow H^0(E_0 \otimes i_*\mathcal{F}) \otimes E_0^* \longrightarrow i_*\mathcal{F} \longrightarrow 0.$$

In this case, we get a morphism

$$H^0(L^{(2)}E_2 \otimes i_*\mathcal{F}) \otimes E_2^* \longrightarrow H^0(L^{(1)}E_1 \otimes i_*\mathcal{F}) \otimes E_1^*$$

which is non injective along C. For example, this happens if C is a degree 19 curve not on a cubic, $\mathcal{F}$ a general line bundle on C of degree $g+2$ on C (where g is the genus of C), and

$$(E_0, E_1, E_2, E_3) = (\mathcal{O}, Q(3), Q^*(4), Q_3^*(4)),$$

where Q (resp. Q_3) is the cokernel of the canonical morphism $\mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes H^0(\mathcal{O}(1))^*$ (resp. $\mathcal{O}(-3) \longrightarrow \mathcal{O} \otimes H^0(\mathcal{O}(3))^*$). In this case we have a morphism

$$Q(-4) \otimes \mathbb{C}^{10} \longrightarrow Q^*(-3) \otimes \mathbb{C}^{11}$$

which is non injective along C.

Question 6. *What is the smallest number z such that if $\frac{k}{m} > z$, then every semi-stable morphism is injective on a nonempty open subset of $\mathbb{P}_n$?*

The next problem is the relation between the (semi-)stability of morphisms and the (semi-)stability of cokernels. On $\mathbb{P}_n$, $n \geq 3$, no general result is known. The only non trivial case where the problem is completely solved is on $\mathbb{P}_3$, with $E = \mathcal{O}(-2)$, $F = \mathcal{O}(-1)$, $m = 2$, $k = 4$. In this case, R.M. Miro-Roig and G. Trautmann have proved in [9] that the (semi-)stability of the map is equivalent to the (semi-)stability of the cokernel, and it follows that the moduli space $M(2, 0, 2, 4)$ is isomorphic to $N(4, 2, 4)$.

Question 7. *In which cases is there an equivalence between the (semi-)stability of morphisms and the (semi-)stability of cokernels ?*

Suppose that every (semi-)stable morphism is injective on a nonempty open subset of $\mathbb{P}_n$, and that m and k are relatively prime (so $N(q, m, k)$ is smooth). Then there exists a *universal cokernel* on $N(q, m, k) \times \mathbb{P}_n$, i.e a coherent sheaf $\mathcal{F}$ on $N(q, m, k) \times \mathbb{P}_n$, flat on $N(q, m, k)$, such that for every stable morphism α, if $\pi(\alpha)$ denotes its image in $N(q, m, k)$, then $\mathcal{F}_{\pi(\alpha)}$ is isomorphic to coker(α) (for every closed point y in $N(q, m, k)$, $\mathcal{F}_y$ denotes the restriction of $\mathcal{F}$ to $\{y\} \times \mathbb{P}_n$). This family of sheaves on $\mathbb{P}_n$ is a *universal deformation* at each point of $N(q, m, k)$. It is also *injective*, i.e. if y, y' are distinct points of $N(q, m, k)$ then the sheaves $\mathcal{F}_y$, $\mathcal{F}_{y'}$ on $\mathbb{P}_n$ are not isomorphic.

Question 8. *Let S be a smooth projective variety, $\mathcal{F}$ a coherent sheaf on $S \times \mathbb{P}_n$, flat on S, such that for every closed point s, $\mathcal{F}_s$ has no torsion, $\mathcal{F}$ is a universal deformation of $\mathcal{F}_s$, and such that if s, s' are distinct points of S, the sheaves $\mathcal{F}_s$ and $\mathcal{F}_{s'}$ are not isomorphic. Does it follow that S is a component of a moduli space of stable sheaves on $\mathbb{P}_n$, and $\mathcal{F}$ the universal sheaf ?*

Références

[1] Drézet, J.-M. *Fibrés exceptionnels et suite spectrale de Beilinson généralisée sur* $\mathbb{P}_2(\mathbb{C})$. Math. Ann. 275 (1986), 25-48.

[2] Drézet, J.-M. *Fibrés exceptionnels et variétés de modules de faisceaux semi-stables sur* $\mathbb{P}_2(\mathbb{C})$. Journ. Reine Angew. Math. 380 (1987), 14-58.

[3] Drézet J.-M. *Variétés de modules extrémales de faisceaux semi-stables sur* $\mathbb{P}_2(\mathbb{C})$. Math. Ann. 290 (1991), 727-770.

[4] Drézet, J.-M. *Sur les équations vérifiées par les invariants des fibrés exceptionnels.* Forum Math. 8 (1996), 237-265.

[5] Drézet, J.-M., Le Potier, J. *Fibrés stables et fibrés exceptionnels sur* $\mathbb{P}_2$. Ann. Ec. Norm. Sup. 18 (1985), 193-244.

[6] Gorodentsev, A.L. *Exceptional bundles on surfaces with a moving anticanonical class.* Math. USSR Izvestiya. AMS transl. 33 (1989), 67-83.

[7] Gorodentsev, A.L., Rudakov, A.N. *Exceptional vector bundles on projective spaces.* Duke Math. Journ. 54 (1987), 115-130.

[8] Miro-Roig, R.M. *Gaps in Chern Classes of Rank 2 Stable Reflexive Sheaves.* Math. Ann. 270 (1985), 317-323.

[9] Miro-Roig, R.M., Trautmann, G. *The Moduli Scheme* $M(0,2,4)$ *over* $\mathbb{P}_3$. Math. Z. 216 (1994), 283-315.

[10] Nogin, D.Y. *Helices of period 4 and Markov type equations.* To appear in Math. USSR Izvestiya.

[11] Nogin D.Y. *Helices on some Fano threefolds : constructivity of semi-orthogonal bases of* K_0. Ann. Scient. Ec. Norm. Sup. 27 (1994), 1-44.

[12] Rudakov A.N. *Exceptional vector bundles on a quadric.* Math. USSR Izvestiya. AMS transl. 33 (1989), 115-138.

[13] Rudakov A.N. *A description of Chern classes of semi-stable sheaves on a quadric surface.* J. reine angew. Math. 453 (1994), 113-135.

[14] Rudakov A.N. et al. *Helices and Vector Bundles : seminaire Rudakov.* London Math. Soc. Lect. Note Series 148. Cambridge Univ. Press (1990).

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