Propagation of a Tsunami wave

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Abstract : The importance of the study to the propagation of a *Tsunami* wave became from the complex phenomenon that represent and its natural disasters which represent a major risk for populations. To model this phenomena, we will consider the BBM-BBM [1] Boussinesq system [2] with a variable bottom in space and apply this system, first, using a mesh generated through a photo of the Mediterranean sea, then using a mesh generated through an imported xyz bathymetry for the East part of the Mediterranean sea.

We choose here to use FreeFem++ [10] software which simplify the construction of the domain in particular, one of the advantage of FreeFem++ is that we can build a mesh through a photo and we can easily export bathymetric data in order to consider more realistic simulations.

1 Introduction

We consider here the numerical simulation of the BBM-BBM Boussinesq system in 2D over a variable bottom

$$\begin{aligned} \eta_t + \nabla\cdot((D+\eta)V) - b\nabla\cdot\left(D^2\nabla\eta_t\right) &= 0, \\ \mathbf{V}_t + \nabla\eta + \tfrac{1}{2}\nabla|\mathbf{V}|^2 - dD^2\Delta\mathbf{V}_t &= 0, \end{aligned} \tag{1}$$

where b and d are positive parameters (in the sequel, they are equal to $1/6$). This system is a type of Boussinesq systems derived in [1, 5] as approximations to the three-dimensional Euler equations describing irrotational free surface flow of an ideal fluid $\Omega \subset \mathbb{R}^3$ which is limited from below by the bottom $-D(x, y)$ and from above by the free surface elevation $\eta(x, y, t)$ (cf. Figure 1).

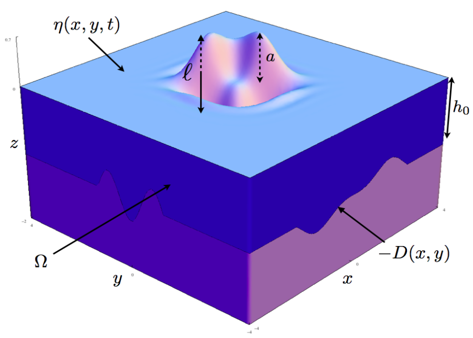


Figure 1: The domain Ω.

1. Benjamin, Bona and Mahony (BBM)

The variables in (1) are non-dimensional and unscaled : $\mathbf{X} = (x, y) \in \Omega$ and $t > 0$ are proportional to position along the channel and time, respectively, $\eta = \eta(\mathbf{X}, t)$ is proportional to the deviation of the free surface from its rest position, $\mathbf{V} = \mathbf{V}(\mathbf{X}, t) = \begin{pmatrix} u(\mathbf{X}, t) \\ v(\mathbf{X}, t) \end{pmatrix} = (u, v)^T = (u; v)$ is proportional to the horizontal velocity of the fluid at some height, $\nabla \bullet = \begin{pmatrix} \partial_x \bullet \\ \partial_y \bullet \end{pmatrix}$ is the gradient, $\nabla \cdot (\star\, ; \bullet\,) = \partial_x \star + \partial_y \bullet$ is the divergence and $\Delta \bullet = \partial_{xx} \bullet + \partial_{yy} \bullet$ is the laplacian.

Remark. In our study, we suppose that $\eta = \mathcal{O}(a)$, where the amplitude a is the difference between the surface of water and the zero level, and the wave length $\lambda = \mathcal{O}(\ell)$, in addition we limit ourselves to the case where $\eta + D > 0$ (there is no dry zone) since we are in a big deep water wave regime.

The paper is organized as follows: in Section 2, we present the discretization of (1) in space an in time. Then, in Section 3, we present a method to build a mesh from a photo using FreeFem++ and in Section 4, another method to build a mesh through an imported bathymetry with FreeFem++ was detailed. Finally, in Section 5 we simulate the propagation of a wave, that looks like a *Tsunami* generated by an earthquake, in the Mediterranean sea over the BBM-BBM systems (1) with a flat bottom and with a variable one in space using the mesh generated from a photo of the Mediterranean and from the xyz bathymetry of a part of the Mediterranean sea.

2 Discretization of the Boussinesq system

We let Ω be a convex, plane domain, let $\mathcal{T}_h$ denote a regular, quasiuniform triangulation of Ω with triangles of maximum size $h < 1$, let $V_h = \{v_h \in C^0(\bar{\Omega}); v_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}$ denote a finite-dimensional subspace of $H^1(\Omega) = \{u \in L^2(\Omega) \text{ s.t. } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)\}$ where $\mathbb{P}_1$ is the set of polynomials of $\mathbb{R}$ of degrees ≤ 1 and let $\langle \cdot; \cdot \rangle$ denote the L^2 inner product on Ω.
Consider the weak formulation of the system (1), find $\eta_h, u_h, v_h \in V_h$ such that $\forall \phi_h \in V_h$ we have :

$$\begin{aligned}
\langle \eta_{ht}; \phi_h \rangle + b\Big\langle D^2 \nabla \eta_{ht}; \nabla \phi_h \Big\rangle + \Big\langle \nabla \cdot (u_h; v_h) + \eta_{hx} u_h + \eta_h u_{hx} + \eta_{hy} u_h + \eta_h u_{hy}; \phi_h \Big\rangle &= 0; \\
\Big\langle \left(Id + d\nabla\left(D^2\right) \cdot \nabla\right) u_{ht}; \phi_h \Big\rangle + d\Big\langle D^2 \nabla u_{ht}; \nabla \phi_h \Big\rangle + \langle \eta_{hx} + u_h u_{hx} + v_h v_{hx}; \phi_h \rangle &= 0; \\
\Big\langle \left(Id + d\nabla\left(D^2\right) \cdot \nabla\right) v_{ht}; \phi_h \Big\rangle + d\Big\langle D^2 \nabla v_{ht}; \nabla \phi_h \Big\rangle + \langle \eta_{hy} + u_h u_{hy} + v_h v_{hy}; \phi_h \rangle &= 0.
\end{aligned} \tag{2}$$

We note also that when using a mesh adaptation method integrated in FreeFem++, a $\mathbb{P}_2$-finite element space is used for better resolution. This method gives comparable results with $\mathbb{P}_1$-finite element space without mesh adaptation technique however we have to more refine in the domain.

Remark. The time discretization of the problem is realized using an explicit second order Runge-Kutta where the study of it convergence in the flat bottom case is detailed in [6].

3 Mesh generated through an imported a photo with FreeFem++

We present here a method to build a mesh from a photo using Photoshop® and a FreeFem++ script made by F. Hecht [9].

Through Google Earth®, we can download pictures of the earth for areas of interest. In this section, we choose the picture of the Mediterranean sea. The Mediterranean is being very large, in order to have better resolution, we take pictures of many parts (cf. Figure 2) that subsequently assembled using Photoshop® so as to obtain a complete picture of the Mediterranean (cf. Figure 3).

Using also Photoshop®, we can eliminate the dry areas that circumvent the Mediterranean (cf. Figure 4).

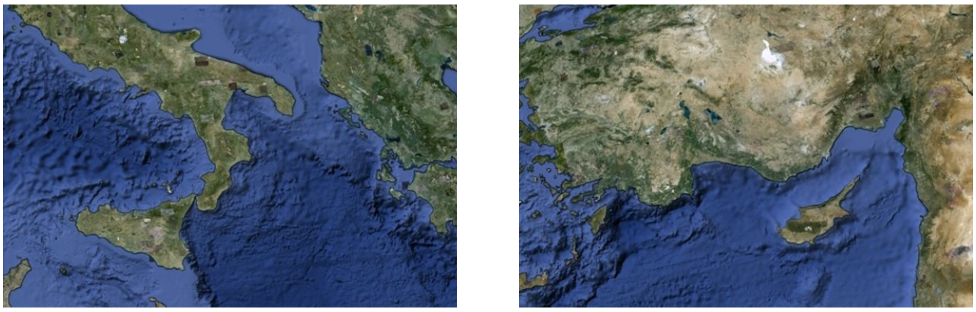


Figure 2: The pictures of two parts of the Mediterranean sea.

Figure 3: The Mediterranean sea after assembly with Photoshop®.

Then we convert the jpg photo to a pgm photo which can be read by FreeFem++ using in a terminal window :

```
convert Medit_sea.jpg Medit_sea.pgm
```

In order to generate the mesh of the Mediterranean sea domain (cf. Figure 5), we use a FreeFem++ script made by F. Hecht which can be downloaded from the following link
http://www.freefem.org/ff2a3/Stic-FF2A3-2010/edp/Chesapeake/Chesapeake-mesh.edp.

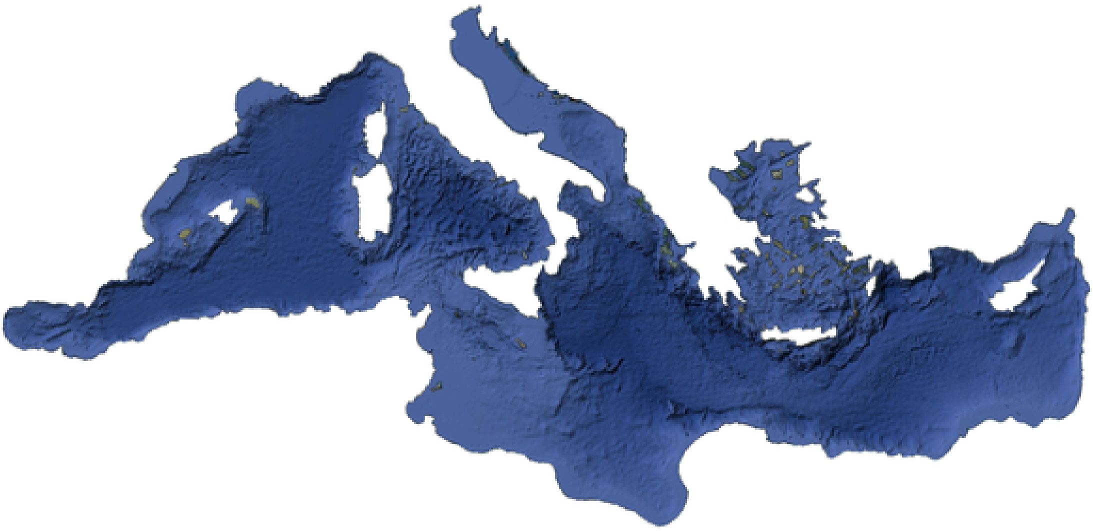


Figure 4: The Mediterranean sea.

Figure 5: The mesh generated by FreeFem++ for the Mediterranean sea.

Because of problem of non-smoothing boundary, we met some difficulties in the generation of the mesh in some regions as in the Figure 6 when using mesh adaptation technique. On the other side, without using the mesh adaptation technique, we no longer have this problem with the mesh generation since the mesh is generated at the beginning.

Recently, and in order to smooth the boundary, F. Hecht has write another script and the result are better as shown in Figure 7. This script could be downloaded from the following link: http://lamfa.u-picardie.fr/sadaka/FreeFem++/generate_a_mesh_using_a_photo.zip, and for more details see also [8].

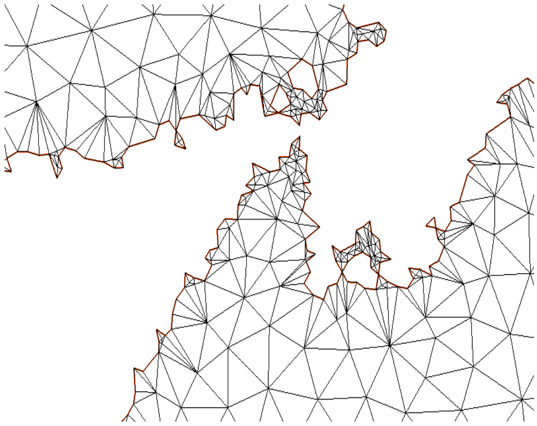


Figure 6: Mesh problem using the script “Chesapeake-mesh.edp”.

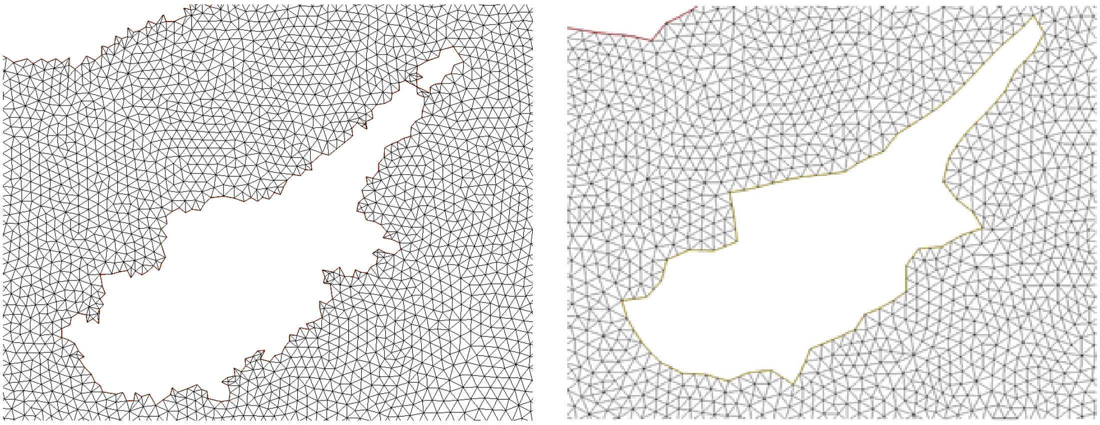


Figure 7: The mesh around Cyprus using the script of F. Hecht (at left : the old one and at right : the new one).

4 Mesh generated through an imported xyz bathymetry with FreeFem++

In order to consider more realistic case *i.e.* the Mediterranean sea with a bathymetry, O. Pantz writes a script [11] with FreeFem++ that allows, through the xyz bathymetry of a chosen area of the earth (downloaded from the NOAA [2] website, cf. Figure 8), to generate the mesh of the area where the amplitude is zero while using a level-set method to smooth the boundary (cf. Figure 9). In addition, we can through this script, on the one hand use the mesh adaptation method without having problem of generation of the mesh and in the other hand, we can have different label for each part of the boundary, which facilitates the use of different types of boundary condition.

Using the `triangulate` function in FreeFem++, we start by reading the xyz downloaded data , then we separate the positive part from the negative one of the bathymetry by a curve which will be the boundary of our domain, we use the level-set method in order to smooth the boundary (cf. Figure 9). The details of this script could be downloaded from the following link: http://lamfa.u-picardie.fr/sadaka/FreeFem++/generate_a_mesh_using_xyz_bathymetry.zip

2. http://www.ngdc.noaa.gov/mgg/gdas/gd_designagrid.html

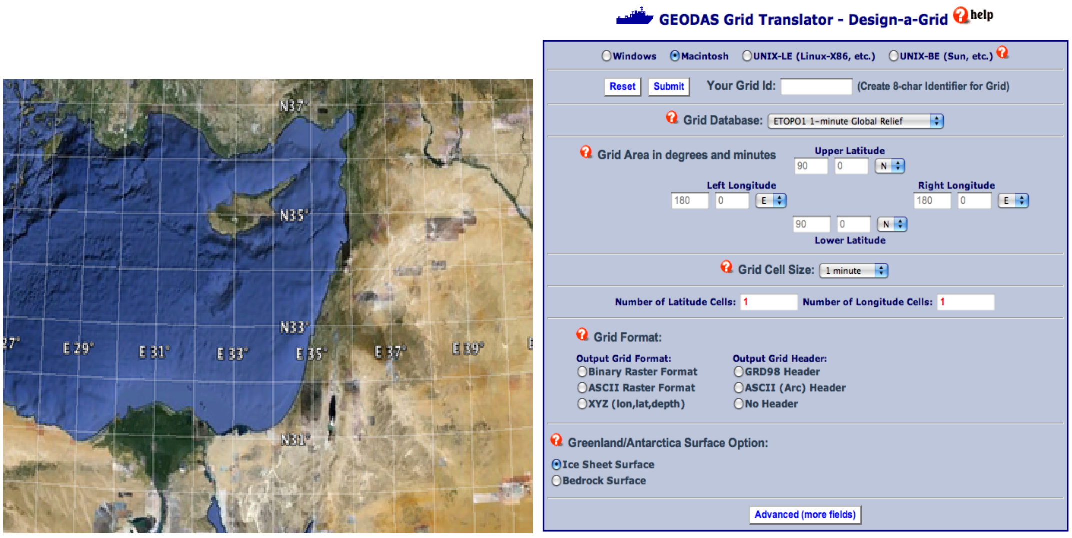


Figure 8: Importation of bathymetry datas through the NOAA website using Google Earth®.

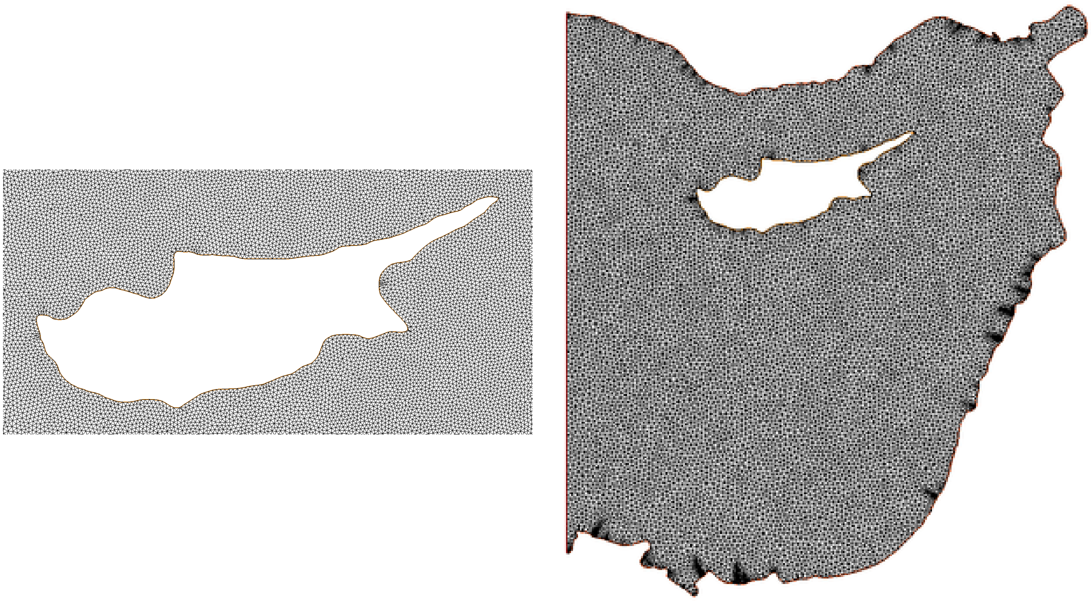


Figure 9: The mesh around Cyprus (left) and the mesh generated (right) using the script of O. Pantz.

5 Numerical simulations

In this section, we will use FreeFem++ in order to simulate the propagation of a wave, that looks like a *Tsunami* generated by an earthquake, in the Mediterranean sea over the BBM-BBM systems (1) with a flat bottom $-D(x,y) = -1$ and with a variable one in space using the mesh generated from a photo of the Mediterranean sea (5) and from the xyz bathymetry of a part of the Mediterranean sea (7).

5.1 Propagation of a *Tsunami* in the Mediterranean sea with a flat bottom.

Inspiring from [4], we simulate here, the propagation of a wave, that looks like a *Tsunami* generated by an earthquake, in the Mediterranean sea over the BBM-BBM system with a flat bottom $-D(x,y) = -1$, where Dirichlet boundary conditions are taken on the hole boundary of our domain for η, u and v and as initial datas :

$$\eta_{h0} = 0.1 \cdot \left\{ 1 - 1/(1 + 10^3 \cdot \exp\left[- \left(\frac{-70 + (y - PY) - 0.2 \cdot (x - PX)}{10} \right)^2 \right.\right.$$
$$\left.\left. - \left(\frac{-13 \cdot 10^4 + 0.2 \cdot (y - PY) + (x - PX) \cdot 10^3}{10^{(7/2)}} \right)^2 \right] \right\},$$
$$u_{h0}(x,y) = v_{h0}(x,y) = 0.$$

Here $PX = 2270, PY = 500$, the step time is $\Delta t = 0.1$.

In the Figure 10, we represent the maximum of the propagation of the solution at $t = 1000 sec$, for the same datas above using $\mathbb{P}_1$-finite element space and without mesh adaptation technique.

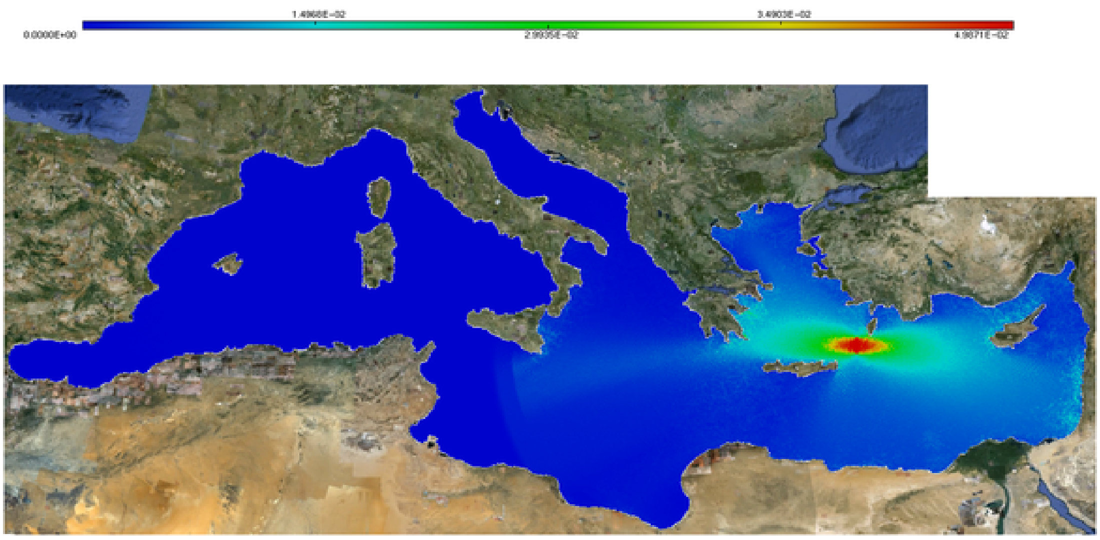


Figure 10: The maximum of the propagation of the solution of a *Tsunami* in the Mediterranean sea for $t = 1000$ sec. Video available on :
http://lamfa.u-picardie.fr/sadaka/films/tsunami_mediterranee_MAX_P1.avi.

On the other side, we present in the Figures 11 and 12 the propagation of the same wave, using a mesh adaptation method and a $\mathbb{P}_2$-finite element space for better resolution. This method gives comparable results with $\mathbb{P}_1$-finite element space without mesh adaptation technique, where in the last case, we no longer have a problem with the mesh generation, however we have to more refine, this makes the computation very heavy.

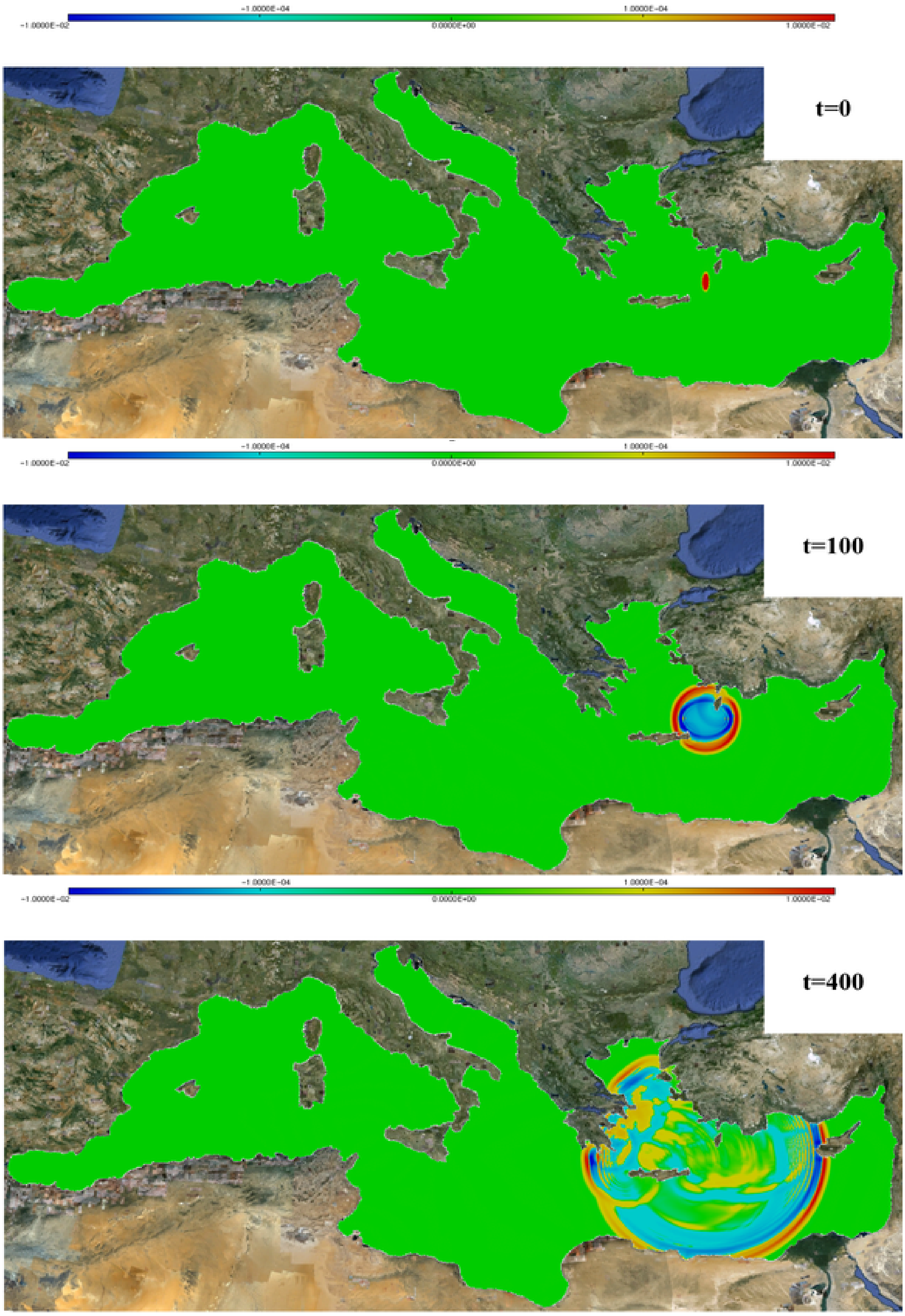


Figure 11: Propagation of the solution of a *Tsunami* in the Mediterranean sea for $t = \{0, 100, 400\} sec$. The *Tsunami* is generated near Crete.
Video available on : http://lamfa.u-picardie.fr/sadaka/films/tsunami_mediterranee.avi.

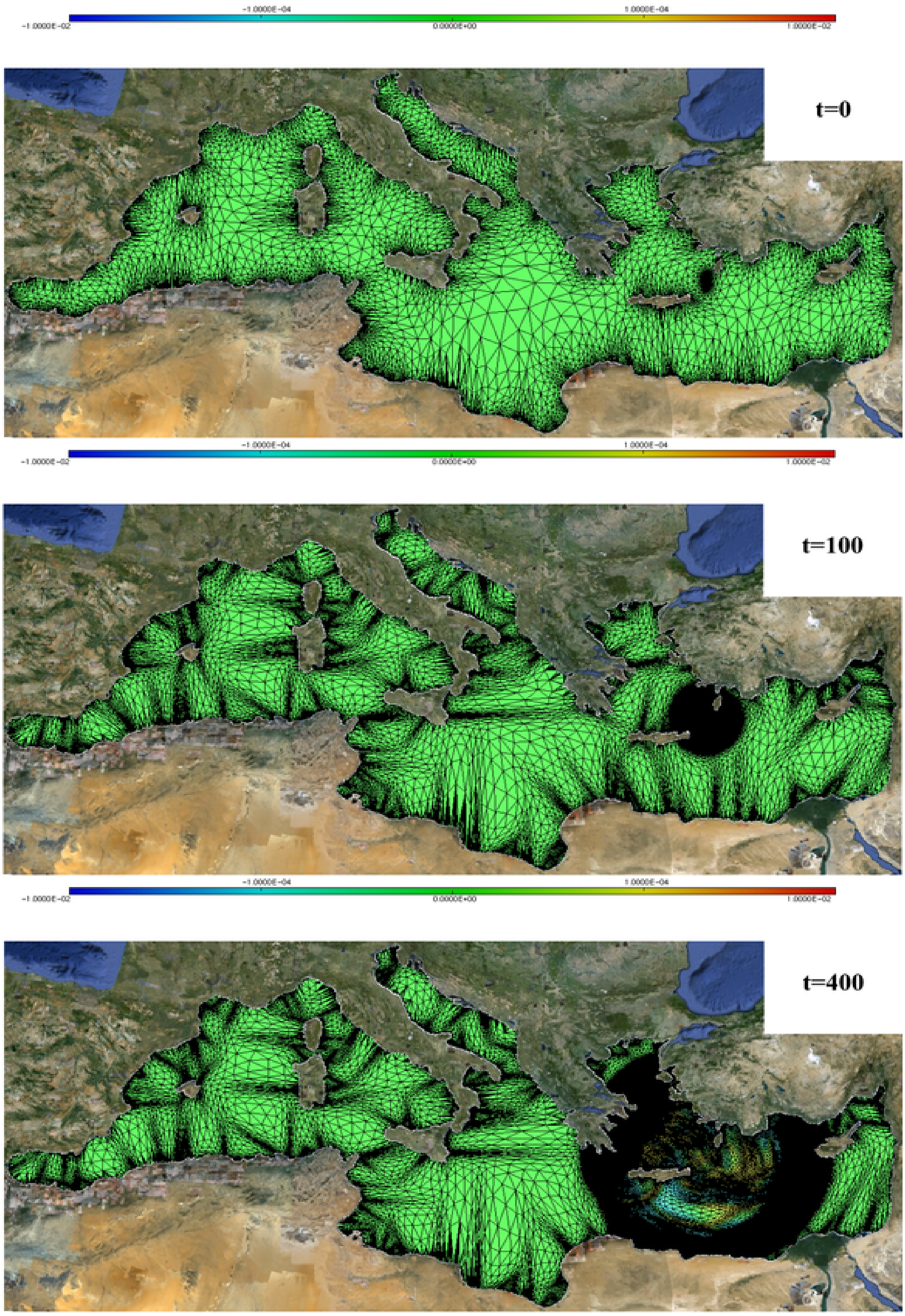


Figure 12: Propagation of the solution and the mesh of a *Tsunami* in the Mediterranean sea for $t = \{0, 100, 400\} sec$.
Video available on : http://lamfa.u-picardie.fr/sadaka/films/tsunami_mediterranee_mesh.avi.

5.2 Propagation of a *Tsunami* in the Mediterranean sea with realistic bathymetry datas.

In order to consider more realistic case *i.e.* the Mediterranean sea with a bathymetry, the bottom $-D(x, y)$ of our study domain will be the same of the one downloaded , but since we are in a big deep water wave regime *i.e.* $\eta(x, y, t) + D(x, y) > 0$, all the points of the bathymetry that are greater than or equal to $-10m$ will be equal to $-10m$ (cf. Figure 13).

We remark also that to download the bathymetry data of the NOAA website, we must know the degree of Latitude and of Longitude of our domain. The spherical shape of the earth was taken into account, even if it does not play significant role because of the small spatial scale of the experiments. So, since the radius of the earth near the equator is $R = 6378, 137km$, we take into account that :

$$1 \text{ degree of Latitude} = \pi \cdot R/180 = 111, 263km,$$

$$\text{and } 1 \text{ degree of Longitude} = \cos(1 \text{ degree of Latitude} \cdot \pi/180) \cdot \pi \cdot R/180.$$

We limit our study for this example by making a translation of our domain considering that 1 degree of Latitude is equal to 1 degree of Longitude is equal to $100km$. We use an homogeneous Dirichlet boundary conditions for η_h, u_h and v_h for the shoreline boundary, and an homogeneous Neumann boundary conditions η_h, u_h and v_h at the open sea. We use also $\mathbb{P}_1$-finite element space and we start from the following initial datas :

$$\eta_{h0} = 0.01 \cdot \exp\left(-\left(\frac{x - 33.5 \cdot 100}{3}\right)^2 \left(\frac{y - 33.8 \cdot 100}{10}\right)^2\right),$$

$$u_{h0}(x, y) = v_{h0}(x, y) = 0.$$

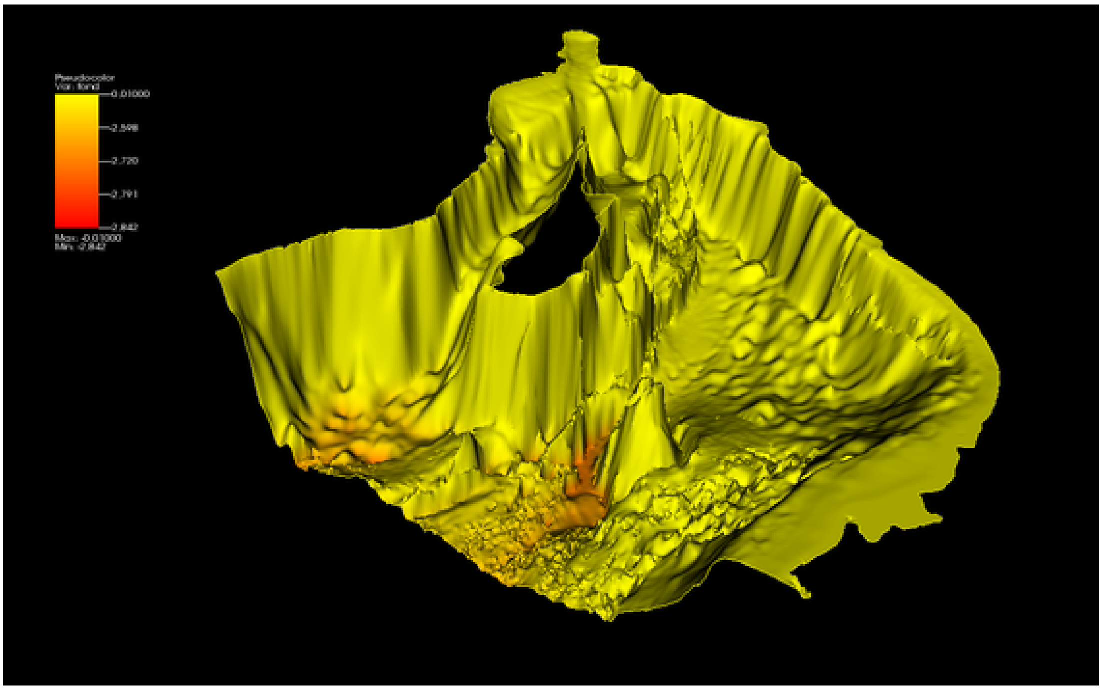
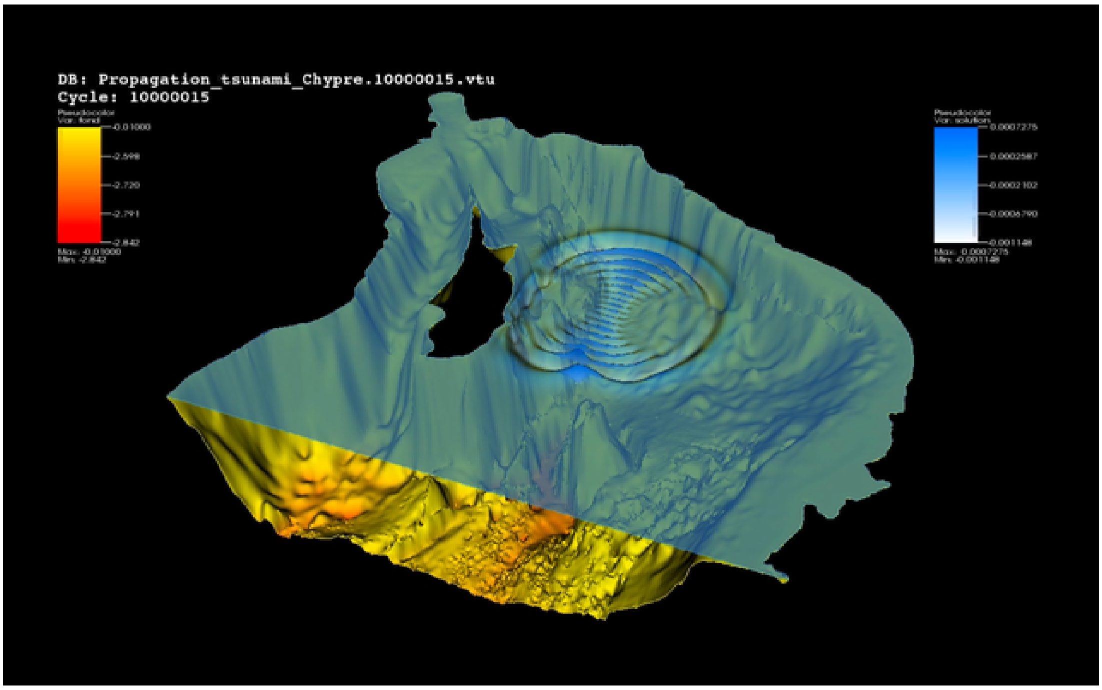


Figure 13: The bottom of the bathymetry downloaded from the NOAA website, (min $= -2.842km$ and max $= -0.01km$).

We present in Figure 14 the propagation of a wave generated near Cyprus in the Mediterranean sea thats looks like the one generated through an earthquake. We present in the Figure 15 the maximum of the propagation of the solution for $t = 230sec$. As we can see also in the same figure that a mesh adaptation technique is used too without having any difficulties in mesh generation.



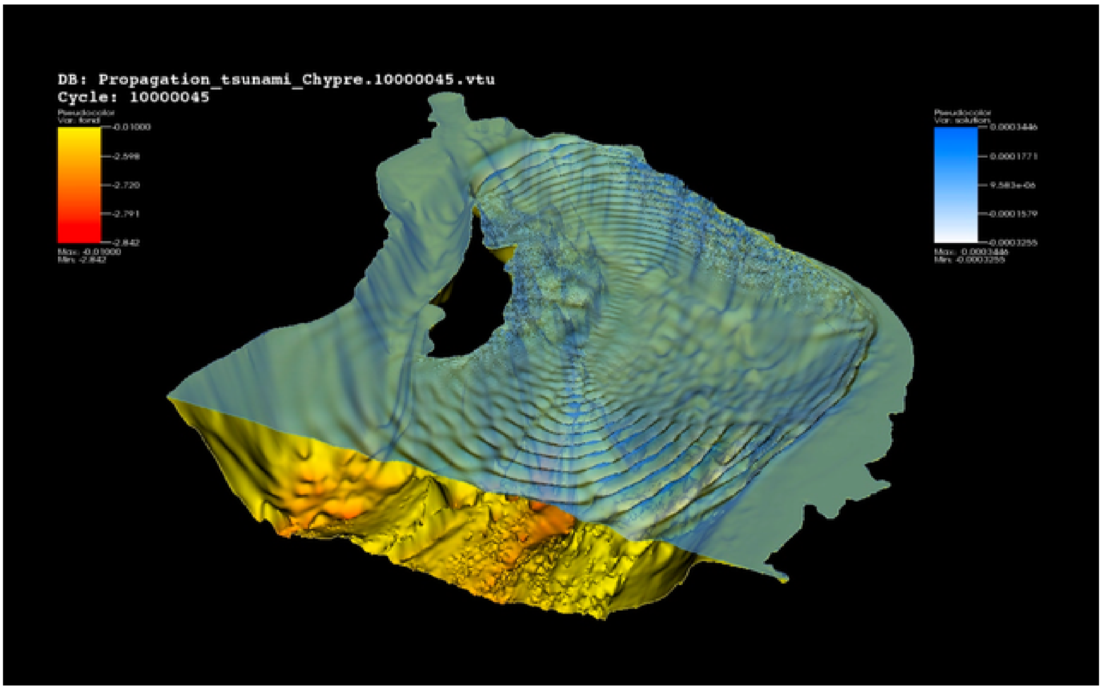


Figure 14: Propagation of a *Tsunami* in the Mediterranean sea, generated near Cyprus for $t = 75$ and $225 sec$.
Video available on : http://lamfa.u-picardie.fr/sadaka/films/propagation_tsunami_Chypre_P1.mpeg.

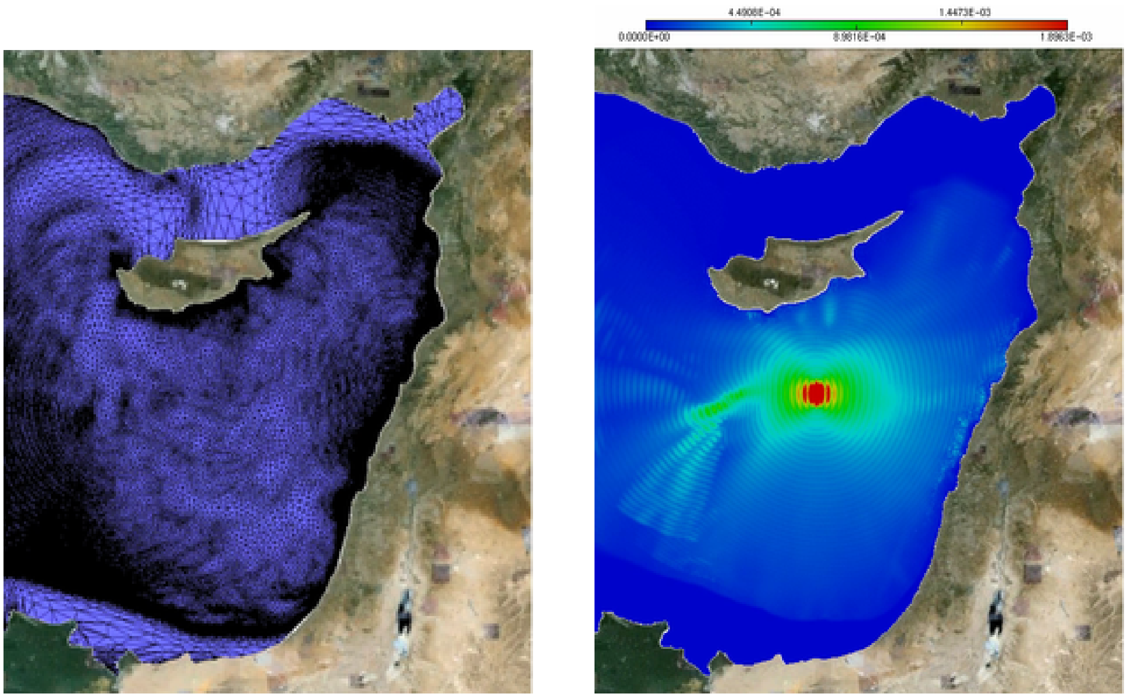


Figure 15: Left : mesh adaptation around Cyprus after the script of O. Pantz.
Video available on : http://lamfa.u-picardie.fr/sadaka/films/chypre_adaptmesh.avi
Right : the maximum of the propagation of the solution of a *Tsunami* in the Mediterranean sea for $t = 230sec$.
Video available on : http://lamfa.u-picardie.fr/sadaka/films/max_solution_tsunami_Chypre_P1.avi.

6 Conclusion and Outlook

We show in this paper, the utilities of FreeFem++ for the Boussinesq systemby building the domain, on the one hand through a photo taken from Google Earth® and on the other hand through an xyz bathymetry downloaded from the NOAA website. Concerning the simulation of a tsunami in the Mediterranean sea, the digital computing environment that we developed allows the integration of realistic data (bathymetry and geography) in a relatively simple framework.

As we see, the mesh generation through a picture presents some difficulties, other then smoothing the boundary, we are limited to takes only homogeneous Dirichlet boundary condition on the hole boundary of our domain for η, u and v. In the other hand, for the simulation of a *Tsunami* near Cyprus, we remark that an artificial reflection come from the open sea after $t = 230sec$ which is due to the Neumann boundary conditions. In this case we must take some absorbent boundary condition which is an open problem. We see also, near the shoreline some oscillation because here, we are not in the big deep water wave regime and we must solve another equations such as Shallow Water equations [7]. Another approach is to use the sponge layer conditions as in [3].

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