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# Study of a numerical scheme for miscible two-phase flow in porous media

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## Abstract

We study the convergence of a finite volume scheme for a model of miscible two-phase flow in porous media. In this model, one phase can dissolve into the other one. The convergence of the scheme is proved thanks to an estimate on the two pressures, which allows to prove some estimates on the discrete time derivative of some nonlinear functions of the unknowns. Monotony arguments allow to show some properties on the limits of these functions. A key point in the scheme is to use particular averaging formula for the dissolution function arising in the space term.

KEYWORDS. Two-phase flow in porous medium, dissolution, finite volume methods, convergence study.

AMS CLASSIFICATION. 65N08, 65N12.

## 1 Introduction

We consider the following problem: approximate the given model

$$q = p_w - p_g, \tag{1.1a}$$

$$\Phi(x)\partial_t (S(q)\rho_w) - \operatorname{div} (\rho_w k_w(S(q))\nabla p_w) = f_w, \tag{1.1b}$$

$$\begin{aligned} \Phi(x)\partial_t (S(q)\rho_w X(p_g) + (1 - S(q))\rho_g) \\ - \operatorname{div} (\rho_w k_w(S(q))X(p_g)\nabla p_w + \rho_g k_g(S(q))\nabla p_g + D\rho_w S(q)\nabla X(p_g)) = f_g, \end{aligned} \tag{1.1c}$$

together with boundary and initial conditions. This problem arises in different engineering frameworks. In the framework of hydrology, it represents the two-phase flow of the water phase and the air gaseous phase in the underground. In the framework of nuclear waste management, it represents the propagation of gaseous hydrogen, produced by acid attack of metallic containers containing the nuclear waste, into porous soils. This model also holds for CO<sub>2</sub> storage, gas production, and other situations. In Problem (1.1),  $p_\beta$  is the pressure of the phase  $\beta$  ( $\beta = w, g$ ),  $q$  is the difference between the two pressures, called the capillary pressure,  $\Phi$  is the porosity (volume occupied by the two fluid phases per unit volume),  $S(q)$  is the water saturation (volume occupied by the water phase per unit fluid volume),  $\rho_\beta$  is

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the mass density of the phase  $\beta$  ( $\beta = w, g$ ),  $k_\beta$  is the mobility of the phase  $\beta$  ( $\beta = w, g$ ) (including the absolute permeability and the viscosity of the phase),  $X$  is the mass fraction of the gaseous component dissolved in the water phase,  $D$  is the diffusivity coefficient of the dissolved gas phase in the liquid phase and  $f_\beta$  is the source term of the phase  $\beta$  ( $\beta = w, g$ ).

Problem (1.1) has been the subject of many theoretical and numerical studies in the case where  $X = 0$ . Let us recall the reference book [11] or the convergence study of the industrial phase-by-phase upstream weighting scheme [10] for example, and let us mention the study done in [5], considering a case where the porous medium is nonrigid. In [9], the authors study the convergence of a large class of discretization methods, namely the gradient schemes.

In [4], the authors study the two-phase flow model, assuming compressibility and mass exchange between the phases (with a mass exchange rate, out of thermodynamical equilibrium between the phases), and they show the existence of a weak solution to this model. In [3], the authors consider Problem (1.1), with an additional diffusion term, assuming the thermodynamical equilibrium, and they show the existence of a weak solution. Our aim is to study the convergence of a numerical scheme to the solution of Problem (1.1).

Let us emphasize that this study is the source of a series of difficulties. The problem posed by the first estimation is that it leads to consider nonlinear functions of the unknowns as test functions. Although this does not provide particular difficulties in the continuous case (thanks to Stampacchia's results), it prevents from using general schemes for the discretization of the space terms (like it is done in [9]). On the contrary, we are led to use two-point flux approximation for the space terms, in the same spirit of [10] (and further works like [1] in the case of one compressible phase). Note that such a difficulty also arises in [7] or [2], for the study of numerical schemes for two-phase flow with discontinuous capillary forces, or more generally in some elliptic problems with irregular data [6] where nonlinear test functions of the unknown must be used. Let us now give the precise hypotheses, all together denoted in the following by **(H)**, that are done in this paper:

**( $\Omega T$ )**  $\Omega \subset \mathbb{R}^d$  with  $d = 1, 2, 3$ ,  $T \in (0, +\infty)$ .

**(Ini)**  $p_w^0 \in \mathbf{L}^2(\Omega)$  and  $p_g^0 \in \mathbf{L}^2(\Omega)$ .

**(R)**  $\rho_w = \text{const}$  and  $\rho_g = \text{const}$ .

**(S)**  $S : \mathbb{R} \rightarrow [S_{min}, S_{max}]$  is a non-decreasing Lipschitz continuous function with  $0 < S_{min} < S_{max} < 1$ . We denote by  $L_S$  the corresponding Lipschitz constant.

**(X)**  $X : \mathbb{R} \rightarrow [0, \bar{X}]$  is a non-decreasing Lipschitz continuous function with  $X(0) = 0$ .

**(D)**  $D \geq 0$  is a possibly null positive constant.

**(K)**  $k_w \in \mathbf{C}^{0,1}([0, 1])$  is a positive, non-decreasing function with  $k_w^{min} := k_w(S_{min}) > 0$  and  $k_g \in \mathbf{C}^{0,1}([0, 1])$  is a positive, non-increasing function with  $k_g^{min} := k_g(S_{max}) > 0$ .

**( $\Phi$ )**  $\Phi : \Omega \rightarrow \mathbb{R}$  is a measurable function with  $0 < \Phi_{min} \leq \Phi(x) \leq \Phi_{max}$ .

**(f)**  $f_i \in \mathbf{L}^2(\Omega \times (0, T))$ , for  $i = w, g$  are given source terms.

**Remark 1.1** (Nonhomogeneous Dirichlet boundary conditions and gravity terms). *We could as well consider the case of nonhomogeneous Dirichlet boundary conditions and the presence of gravity terms, which would only lead to additional terms in the estimates and in the convergence results.*

Let us mention that Hypothesis **(S)** is a regularization of the physical situation, where  $S_{min} = 0$  and  $S_{max} = 1$ . This regularization is done for enabling some estimates and convergence results. This hypothesis can probably be relaxed considering the phase-by-phase upstream weighting scheme, but our paper is focused on the difficulties due to the numerical approximation of the function  $X$  in the space terms. Therefore we have preferred to use simpler arguments, deduced from Hypothesis **(S)**. Let us also emphasize that we do not use  $D > 0$  for proving the convergence of the scheme, which means that the present analysis apply in case where Problem (1.1) is a regularization of the immiscible two-phase flow problem by the addition of a dissolution term.

Let us now provide the weak sense that is considered for a solution to Problem (1.1).

**Definition 1.2** (Weak sense of a continuous solution). *Under Hypotheses **(H)**, we say that  $(p_w, p_g)$  is a weak solution to (1.1) if:*

$$p_g \in \mathbf{L}^2((0, T); \mathbf{H}_0^1), p_w \in \mathbf{L}^2((0, T); \mathbf{H}_0^1) \text{ and, for all } \varphi \in \mathbf{C}_c^\infty(\Omega \times [0, T]) \quad (1.2a)$$

$$\begin{aligned} \int_0^T \int_\Omega -\Phi(x) \rho_w S(q) \partial_t \varphi \, dx dt + \int_0^T \int_\Omega k_w(S(q)) \rho_w \nabla p_w \nabla \varphi \, dx dt \\ = \int_\Omega \Phi(x) \rho_w S(q^0) \varphi(x, 0) \, dx + \int_0^T \int_\Omega f_w \varphi \, dx dt, \end{aligned} \quad (1.2b)$$

$$\begin{aligned} \int_0^T \int_\Omega -\Phi(x) \left( \rho_w S(q) X(p_g) + (1 - S(q)) \rho_g \right) \partial_t \varphi \, dx dt \\ + \int_0^T \int_\Omega \left( k_w(S(q)) \rho_w X(p_g) \nabla p_w + k_g(S(q)) \rho_g \nabla p_g + D \rho_w S(q) \nabla X(p_g) \right) \nabla \varphi \, dx dt \\ = \int_\Omega \Phi(x) \left( \rho_w S(q^0) X(p_g^0) - (1 - S(q^0)) \rho - g \right) \varphi(x, 0) \, dx + \int_0^T \int_\Omega f_g \varphi \, dx dt, \end{aligned} \quad (1.2c)$$

where  $\mathbf{C}_c^\infty(\Omega \times [0, T])$  denotes the set of the restrictions of all functions of  $\mathbf{C}_c^\infty(\Omega \times (-\infty, T))$  to  $(\Omega \times [0, T])$ .

This paper is organized as follows. In Section 2, we show briefly the continuous method used in order to prove the estimates, assuming sufficient regularity of the solution. This section details in particular the use of nonlinear test functions. Then, after presenting the discrete scheme in Section 3, we mimic the techniques used in section 2 to prove the estimates which hold in the discrete setting. From these estimates, we deduce the existence of at least one solution to the scheme and conclude the necessary compactness results, leading to the convergence proof of the scheme. In Section 4, simple numerical test cases show how some physical features are reproduced by this simple model. A short conclusion is finally given.

## 2 Estimates in the continuous setting, assuming that the solution is regular

In this section, we provide the computations which enable to take into account the dissolution function  $X$ . For the sake of simplicity, we write the following estimates in the 1D case

assuming  $D = 0$ , which does not change the principles of the computations. The ideas and techniques applied in this section will then be carried over to the discrete setting in 3.

**Lemma 2.1.** *Let us assume that Hypotheses **(H)** hold under the additional hypotheses that  $d = 1$ ,  $f_w = f_g = 0$  and that there exists a solution  $(p_w, p_g)$  of Problem (1.2) which is sufficiently regular. Then, there exists a constant  $C$  depending on the initial conditions  $p_g^0$  and  $p_w^0$  and on  $k_g^{min}$ ,  $k_w^{min}$  and  $\Phi_{min}$ , such that for any solution of (1.2), we have*

$$\|\partial_x p_g\|_{\mathbf{L}^2((0,T),\Omega)} \leq C \quad \text{and} \quad \|\partial_x p_w\|_{\mathbf{L}^2((0,T),\Omega)} \leq C \quad (2.1)$$

*Proof.* Multiplying equation (1.1b) by  $\frac{p_w}{\rho_w}$  and equation (1.1c) by  $\frac{p_g}{\rho_g}$  and add the obtained equations, we have

$$\begin{aligned} & \Phi(x) \partial_t \left( S \frac{\rho_w}{\rho_g} X + (1 - S) \right) p_g + \Phi(x) \partial_t (S) p_w \\ &= \partial_x \left( \frac{\rho_w}{\rho_g} k_w(S) X \partial_x p_w + k_g(S) \partial_x p_g \right) p_g + \partial_x \left( k_w(s) \partial_x p_w \right) p_w \end{aligned} \quad (2.2)$$

Define

$$\tilde{S}(p) := \int_0^p S'(\tau) \tau \, d\tau, \quad \tilde{X}(p) := \int_0^p X'(\tau) \tau \, d\tau \quad \text{and} \quad \alpha = \frac{\rho_w}{\rho_g}. \quad (2.3)$$

Then, the left hand side of (2.2) becomes

$$\begin{aligned} & \Phi(x) \partial_t \left( S(q) \alpha X(p_g) + (1 - S(q)) \right) p_g + \Phi(x) \partial_t (S(q)) p_w \\ &= \Phi(x) \partial_t (S(q)) q + \Phi(x) \alpha \partial_t \left( S(q) X(p_g) + 1 \right) p_g \\ &= \Phi(x) \left[ \partial_t \left( \tilde{S}(q) \right) + \alpha S \partial_t (X(p_g)) p_g + \alpha X(p_g) p_g \partial_t (S(q)) \right] \\ &= \Phi(x) \left[ \partial_t \left( \tilde{S}(q) \right) + \partial_t \left( \alpha S(q) \tilde{X}(p_g) \right) \right] - \left[ \alpha \left( \tilde{X}(p_g) - X(p_g) p_g \right) \partial_x \left( k_w(S(q)) \partial_x p_w \right) \right] \end{aligned}$$

and we can write (2.2) as

$$\begin{aligned} & \Phi(x) \left[ \partial_t \left( \tilde{S}(q) \right) + \partial_t \left( \alpha S(q) \tilde{X}(p_g) \right) \right] \\ &= \partial_x \left( \alpha k_w(S) X \partial_x p_w + k_g(S) \partial_x p_g \right) p_g + \partial_x \left( k_w(s) \partial_x p_w \right) p_w \\ & \quad + \alpha \left( \tilde{X}(p_g) - X(p_g) p_g \right) \partial_x \left( k_w(S(q)) \partial_x p_w \right) \end{aligned}$$

Recall that  $\tilde{X}(0) = 0$ , and  $\frac{d}{dp} (X(p)p - \tilde{X}(p)) = X(p)$ , such that integration yields

$$\begin{aligned} & \int_0^T \int_{\Omega} k_w(s) \left( \partial_x p_w \right)^2 \, dx dt + \int_0^T \int_{\Omega} k_g(s) \left( \partial_x p_g \right)^2 \, dx dt \\ &= \int_{\Omega} \Phi(x) \left[ \left( \tilde{S}(q^0) - \tilde{S}(q(T)) \right) + \left( \alpha S(q^0) \tilde{X}(p_g^0) - S(q(T)) \tilde{X}(p_g(T)) \right) \right] \, dx \end{aligned} \quad (2.4)$$

Recall that  $S$  and  $X$  are Lipschitz continuous and bounded, such that

$$0 \leq \tilde{S}(q) \leq \frac{L_S}{2} q^2 \quad \text{and} \quad 0 \leq \alpha S(q) \tilde{X}(p_g) \leq \alpha \frac{L_X}{2} p_g^2. \quad (2.5)$$

We have thus,

$$\begin{aligned} k_w^{min} \|\partial_x p_w\|_{\mathbf{L}^2((0,T)\times\Omega)}^2 + k_g^{min} \|\partial_x p_g\|_{\mathbf{L}^2((0,T)\times\Omega)}^2 \\ \leq \Phi_{max} \frac{L_S}{2} \|q^0\|_{\mathbf{L}^2(\Omega)}^2 + \Phi_{max} \alpha \frac{L_X}{2} \|p_g^0\|_{L^2(\Omega)}^2 \end{aligned} \quad (2.6)$$

□

**Lemma 2.2.** *Let us assume that Hypotheses **(H)** hold under the additional hypotheses that  $d = 1$ ,  $f_w = f_g = 0$  and that there exists a solution  $(p_w, p_g)$  of Problem (1.2) which is sufficiently regular. Then, there exists a constant  $C$  depending on the initial conditions  $p_g^0$  and  $p_w^0$  and on  $k_w^{min}$ ,  $k_w^{max}$  and  $\Phi_{min}$  such that for any solution of (1.2), we have*

$$\|\Phi \partial_t S(q)\|_{\mathbf{L}^2((0,T);\mathbf{H}^{-1}(\Omega))} \leq C \quad \text{and} \quad \left\| \Phi \partial_t (S(q)X(p_g)) \right\|_{\mathbf{L}^2((0,T);\mathbf{H}^{-1}(\Omega))}^2 \leq C \quad (2.7)$$

*Proof.* Let  $\omega \in \mathbf{H}_0^1(\Omega)$ . Then,

$$\begin{aligned} \left( \int_{\Omega} \left( \Phi(x) \partial_t S(q) \right) \omega \, dx \right)^2 &= \left( \int_{\Omega} k_w(S(q)) \partial_x p_w \partial_x \omega \, dx \right)^2 \\ &\leq (k_w^{max})^2 \int_{\Omega} (\partial_x p_w)^2 \, dx \|\partial_x \omega\|_{\mathbf{L}^2(\Omega)}^2 \end{aligned}$$

Taking the supremum and integrating in time yields

$$\begin{aligned} \int_0^T \|\Phi \partial_t S(q)\|_{\mathbf{H}^{-1}(\Omega)}^2 \, dt &= \int_0^T \left( \sup_{\substack{\omega \in \mathbf{H}_0^1(\Omega) \\ \|\partial_x \omega\|_{\mathbf{L}^2(\Omega)}}} \left| \int_{\Omega} \left( \Phi(x) \partial_t S(q) \right) \omega \, dx \right| \right)^2 \, dt \\ &\leq \left( \frac{k_w^{max}}{\Phi_{min}} \right)^2 \|\partial_x p_w\|_{\mathbf{L}^2((0,T)\times\Omega)}^2 \end{aligned}$$

Using the estimate of lemma 2.1 this proves the first estimate.

Adding equation (1.1b) and (1.1c), we get

$$\Phi \partial_t (\alpha S X) - \partial_x \left( (\alpha X + 1) k_w(S) \partial_x p_w + k_g(S) \partial_x p_g \right) = 0.$$

Using Hölder and Young inequalities, we have therefore

$$\begin{aligned} \alpha^2 \left( \left| \int_{\Omega} \partial_t (\Phi S X) \omega \, dx \right| \right)^2 &= \left| \int_{\Omega} (\alpha X + 1) k_w(S) \partial_x p_w \partial_x \omega \, dx + \int_{\Omega} k_g(S) \partial_x p_g \partial_x \omega \, dx \right|^2 \\ &\leq \|\omega\|_{\mathbf{L}^2(\Omega)}^2 \left( (\alpha \bar{X} + 1)^2 (k_w^{max})^2 \int_{\Omega} (\partial_x p_w)^2 \, dx + (k_g^{max})^2 \int_{\Omega} (\partial_x p_g)^2 \, dx \right) \\ &\quad + (\alpha \bar{X} + 1) k_w^{max} k_g^{max} \left( \left( \int_{\Omega} |\partial_x p_w \partial_x \omega| \, dx \right)^2 + \left( \int_{\Omega} |\partial_x p_g \partial_x \omega| \, dx \right)^2 \right) \end{aligned}$$

$$\leq ((\alpha\bar{X} + 1)k_w^{max} + k_g^{max})\|\omega\|_{\mathbf{L}^2(\Omega)}^2 \left( (\alpha\bar{X} + 1)(k_w^{max}) \int_{\Omega} (\partial_x p_w)^2 dx + (k_g^{max}) \int_{\Omega} (\partial_x p_g)^2 dx \right)$$

Taking the supremum and integrating in time, we get

$$\|\partial_t(S(q)X(p_g))\|_{\mathbf{L}^2((0,T);\mathbf{H}^{-1}(\Omega))}^2 \leq C_1\|\partial_x p_w\|_{\mathbf{L}^2((0,T)\times\Omega)} + C_2\|\partial_x p_w\|_{\mathbf{L}^2((0,T)\times\Omega)},$$

where  $C_1$  and  $C_2$  only depend on  $\alpha$ ,  $\bar{X}$ ,  $\Phi$ ,  $k_w$  and  $k_g$ .  $\square$

### 3 Finite volume scheme with two-point flux approximation

We assume Hypotheses **(H)**, and we consider a given mesh  $\mathcal{T}$  with the following properties.

**(T)** Denote by  $\mathcal{E}$  the set of edges (in 2D) or sides (in 3D) respectively. Then,

- (i) For every  $K \in \mathcal{T}$  there exists  $\mathcal{E}_K \subset \mathcal{E}$  such that  $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$  and  $\mathcal{E} = \bigcup_{K \in \mathcal{T}} \mathcal{E}_K$ .
- (ii) For  $\sigma \in \mathcal{E}$  such that  $\bar{K} \cap \bar{L} = \bar{\sigma} \neq \emptyset$  for some  $K \in \mathcal{T}$  and  $L \in \mathcal{T}$ , we write also  $K|L$ .
- (iii) The family of cell centers  $\mathcal{P} := (x_K)_{K \in \mathcal{T}}$  is such that  $x_K \in \bar{K}$  for all  $K \in \mathcal{T}$  and if  $\sigma = K|L$ , we have  $x_K \neq x_L$  and the straight line  $D_{K,L}$  connecting  $x_K$  and  $x_L$  is orthogonal to  $K|L$ .
- (iv) For any  $\sigma \in \mathcal{E}$  with  $\sigma \subset \partial\Omega$  and  $\sigma \in \mathcal{E}_K$ , we have  $x_K \notin \sigma$  and the straight line through  $x_K$  orthogonal to  $\sigma$  is denoted by  $D_{K|\sigma}$ . We define  $y_\sigma := D_{K|\sigma} \cap \sigma$

For simplicity, we introduce the following notation.

- The  $(d-1)$ -dimensional measure of  $\sigma$  is denoted by  $|\sigma|$ .
- The set of interior edges is denoted by  $\mathcal{E}_{int}$ , the set of boundary edges by  $\mathcal{E}_{ext}$ .
- The euclidean distance of  $x_K$  and  $x_L$  for  $L \in \mathcal{N}_K$  is denoted by  $d_\sigma$  (or  $d_{K|L}$ ). For  $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$ , the euclidean distance between  $x_K$  and  $y_\sigma$  is denoted by  $d_{K|\sigma}$ .
- For  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}$ , we define  $y_\sigma := d_{K|L} \cap \sigma$  and denote by  $d_{K|\sigma}$  the straight line connecting  $x_K$  and  $y_\sigma$ .
- For  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}$  we denote by  $\mathcal{D}_{K|L}$  the diamond spanned by the diagonals  $\sigma$  and  $d_{K|L}$ . For  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}$  we denote by  $\mathcal{D}_{K|\sigma}$  the tetrahedron spanned by  $\sigma$  and  $D_{K|\sigma}$ .
- We define  $h_m := \max \left\{ \max_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} d_{K|L}, \max_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} d_{K|\sigma}, \max_{\sigma \in \mathcal{E}} |\sigma| \right\}$ .
- We denote by  $\mathcal{X}_{\mathcal{T}}$  the set of all functions  $u : \Omega \rightarrow \mathbb{R}$  such that  $u(x) = \sum_{K \in \mathcal{T}} u_K \chi_K(x)$  with  $u_K = \text{const}$ .

For any sequence  $t^{(0)} = 0 < t^{(1)} \dots < t^{(N)} = T$ , we denote the discrete time step by  $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$  for  $n = 0, \dots, N-1$ . Furthermore,  $h$  denotes the maximum value of the diameter of all  $K \in \mathcal{T}$  and  $\delta t^{(n+\frac{1}{2})}$ ,  $n = 0, \dots, N-1$ .

**Definition 3.1** (discrete bilinear forms and norm). For  $u, v \in \mathcal{X}(\mathcal{T})$  and for a family  $(\alpha_\sigma)_{\sigma \in \mathcal{E}}$  of real values, we define the scalar product

$$[u, v]_{\alpha, \mathcal{T}} = \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \alpha_{K|L} \frac{|\sigma|}{d_{K|L}} (u_K - u_L)(v_K - v_L) + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} \alpha_{K|\sigma} \frac{|\sigma|}{d_{K|\sigma}} u_K v_K,$$

we define the following discrete operator

$$\operatorname{div}_K(\alpha, u) = [u, 1_K]_{\alpha, \mathcal{T}} = \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int} \\ \sigma = K|L}} \alpha_{K|L} \frac{|\sigma|}{d_{K|L}} (u_K - u_L) + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{-ext} \\ \sigma = K|\sigma}} \alpha_{K|\sigma} \frac{|\sigma|}{d_{K|\sigma}} u_K, \quad \forall K \in \mathcal{T},$$

and the following norm

$$\|u\|_{\mathcal{T}} = ([u, u]_{1, \mathcal{T}})^{1/2} = \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{d_{K|L}} (u_K - u_L)^2 + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} \alpha_{K|\sigma} \frac{|\sigma|}{d_{K|\sigma}} u_K^2 \right)^{1/2}.$$

Let  $p_g^{(0)} \in \mathcal{X}_{\mathcal{T}}$  and  $p_w^{(0)} \in \mathcal{X}_{\mathcal{T}}$  be given piecewise constant initial conditions. Then, we define the implicit scheme for a discretization of problem (1.2) by

$$\begin{aligned} \Phi_K &= \int_K \Phi(x) dx, & f_{\alpha, K}^{(n+1)} &= \frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_K f_w(x, t) dx dt, \quad \alpha = w, g \\ p_w^{(n+1)} &\in X_{\mathcal{T}, 0}, p_g^{(n+1)} \in X_{\mathcal{T}, 0}, & q^{(0)} &= p_w^{(0)} - p_g^{(0)}, q^{(n+1)} = p_w^{(n+1)} - p_g^{(n+1)}, \\ S_K^{(0)} &= S(q_K^{(0)}), S_K^{(n+1)} = S(q_K^{(n+1)}), & X_K^{(0)} &= X(p_{g, K}^{(0)}), X_K^{(n+1)} = X(p_{g, K}^{(n+1)}), \\ \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} &= \frac{S_K^{(n+1)} - S_K^{(n)}}{\delta t^{(n+\frac{1}{2})}}, & \delta_{\mathcal{T}}^{(n+\frac{1}{2})} (S_K X_K)^{(n+1)} &= \frac{(S_K^{(n+1)} X_K^{(n+1)}) - (S_K^{(n)} X_K^{(n)})}{\delta t^{(n+\frac{1}{2})}}, \\ \Psi_{K|\sigma}^{n+1} p_{g, K}^{n+1} &= \int_0^{p_{g, K}^{(n+1)}} X(p) dp, & \Psi_{K|L}^{n+1} (p_{g, L}^{(n+1)} - p_{g, K}^{(n+1)}) &= \int_{p_{g, K}^{(n+1)}}^{p_{g, L}^{(n+1)}} X(p) dp \\ k_{g, K|L}^{(0)} &= k_g(S_{K|L}^{(0)}), k_{g, K|L}^{(n+1)} = k_g(S_{K|L}^{(n+1)}), & k_{w, K|L}^{(0)} &= k_w(S_{K|L}^{(0)}), k_{w, K|L}^{(n+1)} = k_w(S_{K|L}^{(n+1)}) \\ k_{g, K|\sigma}^{(0)} &= k_g(S_{K|\sigma}^{(0)}), k_{g, K|\sigma}^{(n+1)} = k_g(S_{K|\sigma}^{(n+1)}), & k_{w, K|\sigma}^{(0)} &= k_w(S_{K|\sigma}^{(0)}), k_{w, K|\sigma}^{(n+1)} = k_w(S_{K|\sigma}^{(n+1)}) \\ S_{K|\sigma} &\in [\min\{S_K, S(0)\}, \max\{S_K, S(0)\}], & S_{K|L} &= S_{L|K} \in [\min\{S_K, S_L\}, \max\{S_K, S_L\}] \\ \Phi_K \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} - \operatorname{div}_K(k_w^{(n+1)}, p_w^{(n+1)}) &= f_{w, K}^{(n+1)} \end{aligned} \tag{3.1a}$$

$$\begin{aligned} \Phi_K \alpha \delta_{\mathcal{T}}^{(n+\frac{1}{2})} (S_K X_K)^{(n+1)} - \Phi_K \delta t^{(n+\frac{1}{2})} S_K - \alpha \operatorname{div}_K(k_w^{(n+1)} \Psi^{(n+1)}, p_w^{(n+1)}) \\ - \operatorname{div}_K(k_g^{(n+1)}, p_g^{(n+1)}) - \alpha D \operatorname{div}_K(S_{\mathcal{E}}^{(n+1)}, X(p_g^{(n+1)})) = f_{g, K}^{(n+1)}, \end{aligned} \tag{3.1b}$$

$\forall K \in \mathcal{T}, \forall n \in \{0, \dots, N-1\}$ .

Note that  $\Psi$  fulfills

$$\Psi_{K|L} = \frac{\int_{p_{g, K}^{(n+1)}}^{p_{g, L}^{(n+1)}} X(p) dp}{p_{g, L} - p_{g, K}} \in [\min\{X_K^{(n+1)}, X_L^{(n+1)}\}, \max\{X_K^{(n+1)}, X_L^{(n+1)}\}],$$



$$\Psi_{K|\sigma} = \frac{\int_0^{p_{g,K}^{(n+1)}} X(p) dp}{p_{g,K} - 0} \in [0, X_K^{(n+1)}]$$

and is therefore a mean value at the cell boundary, especially chosen to fit the requirements of the algorithm.

To prove the convergence of the above defined algorithm, we proceed as follows. In Section 3.1, we prove estimates analogous to the ones in the continuous setting of section 2. These estimates will be used to prove some estimates on time and space translates of  $S$  and  $SX$  necessary in order to deduce the relative compactness of the sequences  $S_m$  and  $(SX)_m$  by the Kolmogorov-Riesz theorem. It remains then to show in section 3.2 that the obtained limit is a solution to (1.2).

### 3.1 Existence of a discrete solution and discrete estimates

**Lemma 3.2.** *Let Hypotheses (H) hold. Then there exists at least one solution to scheme (3.1) and a constant  $C > 0$  depending on the initial conditions  $p_g^{(0)} \in \mathcal{X}_{\mathcal{T},0}$  and  $p_w^{(0)} \in \mathcal{X}_{\mathcal{T},0}$  and on  $\Omega$ ,  $k_g^{min}$ ,  $k_w^{min}$ ,  $f_w$ ,  $f_g$ ,  $\alpha$ ,  $\bar{X}$  and  $\Phi$  such that for any solution of (3.1) we have*

$$\int_0^T \|p_g\|_{\mathcal{T}}^2 dt \leq C \qquad \int_0^T \|p_w\|_{\mathcal{T}}^2 dt \leq C \qquad (3.2)$$

*Proof.* For any non-decreasing function  $F$  satisfying  $F(0) = 0$ , we define  $\tilde{F}(r) := \int_0^r F'(r) r dr$ . Note that  $\tilde{F}(r) \geq 0$  therefore,

$$\tilde{F}(b) - \tilde{F}(a) = \int_a^b F'(r) r dr = b(F(b) - F(a)) - \int_a^b (F(r) - F(a)) dr \leq b(F(b) - F(a)). \quad (3.3)$$

Now, let  $F_{\mathcal{T}}^{(n+1)}(x) = \sum_{K \in \mathcal{T}} F(r_K^{(n+1)}) \chi_K(x)$ . Using (3.3) we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_K \in \mathcal{T} \left( F_{\mathcal{T}}^{(n+1)}(x) - F_{\mathcal{T}}^{(n)}(x) \right) r_{\mathcal{T}}^{(n+1)} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \chi_K(x) \left( F(r_K^{(n+1)}) - F(r_K^{(n)}) \right) r_K^{(n+1)} \\ &\geq \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \chi_K(x) \left( \tilde{F}(r_K^{(n+1)}) - \tilde{F}(r_K^{(n)}) \right) \\ &\geq - \sum_{K \in \mathcal{T}} \chi_K(x) \left( \tilde{F}(r_K^0) \right) \\ &= -F_{\mathcal{T}}^{(0)}(x). \end{aligned} \qquad (3.4)$$

For any quantity with time index  $(n+1)$  (say  $Q^{(n+1)}$ ) in (3.1), we denote with the notation  $Q$  (without the time index) the function of the time, equal to  $Q^{(n+1)}$  in the interval  $(t^{(n)}, t^{(n+1)})$ . Now, we multiply equation (3.1a) by  $\delta t^{(n+\frac{1}{2})} p_w^{(n+1)}$  and equation (3.1b) by  $\omega = \delta t^{(n+\frac{1}{2})} p_g^{(n+1)}$ . Adding the two equations so obtained and summing on  $K \in \mathcal{T}$  and on  $n = 0, \dots, N-1$ , we have

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \left( S_K^{(n+1)} - S_K^{(n)} \right) q_K^{(n+1)} \qquad (3.5a)$$

$$+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \alpha \Phi_K \left( S_K^{(n+1)} X_K^{(n+1)} - S_K^{(n)} X_K^{(n)} \right) p_{g,K}^{(n+1)} \quad (3.5b)$$

$$+ \int_0^T [p_w, p_w]_{k_w, \mathcal{T}} dt \quad (3.5c)$$

$$+ \int_0^T [p_g, p_g]_{k_g, \mathcal{T}} dt \quad (3.5d)$$

$$+ \int_0^T \alpha [p_w, p_g]_{k_w \Psi, \mathcal{T}} dt \quad (3.5e)$$

$$+ \int_0^T \alpha D[X(p_g), p_g]_{S_\varepsilon, \mathcal{T}} dt \quad (3.5f)$$

$$= \int_0^T \sum_{K \in \mathcal{T}} (f_{w,K} p_{w,K} + f_{g,K} p_{g,K}) dt. \quad (3.5g)$$

To obtain the desired estimates, we observe that (3.5f) is non-negative and that the terms in (3.5c) and (3.5d) can be bounded from above by

$$\int_0^T \left( [p_w, p_w]_{k_w, \mathcal{T}} + [p_g, p_g]_{k_g, \mathcal{T}} \right) dt \geq k_w^{\min} \int_0^T \|p_w\|_{\mathcal{T}}^2 dt + k_g^{\min} \int_0^T \|p_g\|_{\mathcal{T}}^2 dt.$$

Thanks to (3.4) and (2.5), the term (3.5a) becomes

$$\begin{aligned} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K (S_K^{(n+1)} - S_K^{(n)}) q_K^{(n+1)} &\geq -\frac{LS}{2} \sum_{K \in \mathcal{T}} \Phi_K \left( q_K^{(0)} \right)^2 \\ &\geq -\frac{\Phi_{\max} LS}{2} \|q_{\mathcal{T}}^0\|_{L^2(\Omega)}^2 \end{aligned}$$

Using again (3.4) and (2.5), the term (3.5b) can be written as

$$\begin{aligned} &\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \alpha (S_K^{(n+1)} X_K^{(n+1)} - S_K^{(n)} X_K^{(n)}) p_{g,K}^{(n+1)} \\ &\geq \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \alpha \left( S_{\mathcal{D}}^{(n)} \left( \tilde{X}_K^{(n+1)} - \tilde{X}_K^{(n)} \right) + X_K^{(n+1)} p_{g,K}^{(n+1)} \left( S_K^{(n+1)} - S_K^{(n)} \right) \right) dx \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \alpha \left( \left( S_K^{(n+1)} \tilde{X}_K^{(n+1)} - S_K^{(n)} \tilde{X}_K^{(n)} \right) + \left( X_K^{(n+1)} p_{g,K}^{(n+1)} - \tilde{X}_K^{(n+1)} \right) \left( S_K^{(n+1)} - S_K^{(n)} \right) \right) \\ &\geq - \sum_{K \in \mathcal{T}} \Phi_K \alpha S_K^{(0)} \tilde{X}_K^{(0)} + \int_0^T \sum_{K \in \mathcal{T}} \alpha \hat{X}_K f_{w,K} dt \\ &\quad - \alpha \int_0^T [p_w, \hat{X}]_{k_w, \mathcal{T}} dt, \end{aligned} \quad (3.6)$$

where  $\hat{X}_K = X_K p_{g,K} - \tilde{X}_K = \int_0^{p_{g,K}} X(p) dp$ . Note that  $|\hat{X}_K| \leq \bar{X} |p_{g,K}|$ . We then remark that, by definition of  $\Psi_{K|L}$ ,

$$[p_w, \hat{X}]_{k_w, \mathcal{T}} = [p_w, p_g]_{k_w \Psi, \mathcal{T}},$$

term which vanishes when combining with (3.5e). We finally turn to a bound of the term arising in (3.5g), combined with the last but one term issued from (3.6). We easily get, thanks to the Young inequality and to the discrete Poincaré inequality [8, Lemma 9.1 p.765], that

$$\begin{aligned} & \int_0^T \sum_{K \in \mathcal{T}} (f_{w,K}(p_{w,K} - \alpha \hat{X}_K) + f_{g,K} p_{g,K}) dt \\ & \leq \frac{\text{diam}(\Omega)}{2} \left( \frac{\|f_w\|_{\mathbf{L}^2(\Omega \times (0,T))}^2}{k_w^{\min}} + \frac{\| |f_g| + \alpha \bar{X} |f_w| \|_{\mathbf{L}^2(\Omega \times (0,T))}^2}{k_g^{\min}} \right) \\ & \quad + \frac{1}{2} \left( k_w^{\min} \int_0^T \|p_w\|_{\mathcal{T}}^2 dt + k_g^{\min} \int_0^T \|p_g\|_{\mathcal{T}}^2 dt \right). \end{aligned}$$

Combining all results and the fact that the term in (3.5f) is nonnegative, we have

$$\begin{aligned} & \frac{1}{2} \left( k_g^{\min} \int_0^T \|p_g\|_{\mathcal{T}}^2 dt + k_w^{\min} \int_0^T \|p_w\|_{\mathcal{T}}^2 dt \right) \\ & \leq \Phi_{\max} \left( \alpha \frac{L_X}{2} \|p_{g,\mathcal{T}}^{(0)}\|_{L^2(\Omega)} + \frac{L_S}{2} \|q_T^0\|_{L^2(\Omega)}^2 \right) + C_1 \left( \|f_w\|_{\mathbf{L}^2(\Omega \times (0,T))}^2 + \|f_g\|_{\mathbf{L}^2(\Omega \times (0,T))}^2 \right), \end{aligned} \quad (3.7)$$

where  $C_1$  only depends on the data listed in the statement of the lemma, which concludes the proof of the estimates.

To deduce the existence of at least one solution, we substitute the functions  $S$  and  $X$  in (3.1) by  $S_\lambda = \lambda S + (1 - \lambda)$  and  $X_\lambda = \lambda X + (1 - \lambda)$ . Note that the estimates derived above hold also for solutions of the modified scheme. Furthermore, the modified scheme has a solution for  $\lambda = 0$  (that is  $p_w = 0$  and  $p_g = 0$ ), such that we can deduce by a classical topological degree argument the existence of at least one solution for  $\lambda = 1$  and thus for (3.1).  $\square$

To show the estimates on  $\delta_{\mathcal{T}} S_{\mathcal{T}}$  and  $\delta_{\mathcal{T}}(S_{\mathcal{T}} X_{\mathcal{T}})$ , we define the following dual norm.

**Definition 3.3.** *Let  $\mathcal{T}$  be a mesh on  $\Omega$  that satisfies **(T)**. Then, we define the discrete dual semi-norm on  $\mathbf{L}^2(\Omega)$  by*

$$|w|_{*,\mathcal{T}} := \sup \left\{ \sum_{K \in \mathcal{T}} \Phi_K w_K v_K \mid v \in \mathcal{X}_{\mathcal{T},0} \text{ with } \|v\|_{\mathcal{T}} = 1 \right\}, \quad \forall w \in \mathbf{L}^2(\Omega).$$

**Lemma 3.4.** *Let Hypotheses **(H)** hold. Then, there exists a constant  $C$  depending on the initial conditions  $p_{g,\mathcal{T}}^0$  and  $p_{w,\mathcal{T}}^0$ , on  $f_w$ ,  $f_g$ ,  $k_w$ ,  $k_g$  and on  $T$  such that for any solution of (3.1), we have*

$$\int_0^T |\delta_{\mathcal{T}} S_{\mathcal{T}}(\cdot, t)|_{*,\mathcal{T}}^2 dt \leq C \quad \text{and} \quad \int_0^T |\delta_{\mathcal{T}}(S_{\mathcal{T}} X_{\mathcal{T}})(\cdot, t)|_{*,\mathcal{T}}^2 dt \leq C \quad (3.8)$$

*Proof.* Let  $\omega \in \mathcal{X}_{\mathcal{T},0}$ . From (3.1a), we get

$$\left| \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_{\mathcal{T}}^{(n+1)} \right|_{*,\mathcal{T}}^2 = \left( \sup_{\substack{\omega \in \mathcal{X}_{\mathcal{T},0} \\ \|\omega\|_{\mathcal{T}}=1}} \sum_{K \in \mathcal{T}} \Phi_K \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} \omega_K \right)^2$$

$$\begin{aligned}
&= \left( \sup_{\substack{\omega \in \mathcal{X}_{\mathcal{T},0} \\ \|\omega\|_{\mathcal{T}}=1}} \left( [p_w^{(n+1)}, \omega]_{k_w^{(n+1)}, \mathcal{T}} + \sum_{K \in \mathcal{T}} f_{w,K}^{(n+1)} \omega_K \right) \right)^2 \\
&\leq (k_w^{max})^2 \|p_w^{(n+1)}\|_{\mathcal{T}}^2 + \text{diam}(\Omega) \|f_w^{(n+1)}\|_{\mathbf{L}^2(\Omega)}^2,
\end{aligned}$$

by the Cauchy-Schwarz inequality, the boundedness of  $k_w$  from above and the discrete Poincaré inequality. Multiplying by  $\delta t^{(n+\frac{1}{2})}$ , sum

$$\int_0^T |\delta_{\mathcal{T}} S_{\mathcal{T}}(\cdot, t)|_{*, \mathcal{T}}^2 dt \leq C$$

To show the second estimate, we add equations (3.1a) and (3.1b) and follow the same steps as above to obtain

$$\begin{aligned}
&\left| \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_{\mathcal{T}}^{(n+1)} X_{\mathcal{T}}^{(n+1)} \right|_{*, \mathcal{T}}^2 \\
&\leq \left( (k_w^{max})^2 (\alpha \bar{X} + 1)^2 \|p_w\|_{\mathcal{T}}^2 + \left( (k_g^{max})^2 + (\alpha D \bar{X})^2 \right) \|p_g\|_{\mathcal{T}}^2 + \text{diam}(\Omega) \|f_g^{(n+1)}\|_{\mathbf{L}^2(\Omega)}^2 \right).
\end{aligned}$$

As for the first estimate, multiplication by  $\delta t^{(n+\frac{1}{2})}$ , summation on  $n$  and application of (3.7) yields the second estimate.  $\square$

### 3.2 Convergence of the scheme

**Lemma 3.5.** *Let Hypotheses **(H)** hold. Furthermore, let  $\mathcal{T}_m$  be a sequence of space-time discretizations satisfying **(T)** with  $h_m \rightarrow 0$  as  $m \rightarrow \infty$ . We assume that  $p_g^{(0)} \in \mathcal{X}_{\mathcal{T}}$  and  $p_w^{(0)} \in \mathcal{X}_{\mathcal{T}}$  are given such that  $p_g^{(0)} \in \mathbf{L}^2(\Omega)$  and  $p_w^{(0)} \in \mathbf{L}^2(\Omega)$ .*

*If  $p_{w, \mathcal{T}_m}$ ,  $p_{g, \mathcal{T}_m}$ ,  $S_{\mathcal{T}_m}$  and  $X_{\mathcal{T}_m}$  satisfy the scheme (3.1) for all  $m \in \mathbb{N}$ , the families  $(S_{\mathcal{T}_m})_{m \in \mathbb{N}}$  and  $(S_{\mathcal{T}_m} X_{\mathcal{T}_m})_{m \in \mathbb{N}}$  are relatively compact in  $\mathbf{L}^2((0, T) \times \Omega)$ .*

*Proof.* The relative compactness of the family  $(S_{\mathcal{T}_m})_{m \in \mathbb{N}}$  can be shown exactly as in [9, Lemma 2.6]. To show the relative compactness of  $(S_{\mathcal{T}_m} X_{\mathcal{T}_m})_{m \in \mathbb{N}}$ , we extend the functions  $S_{\mathcal{T}_m}$  and  $X_{\mathcal{T}_m}$  by 0 outside of  $\Omega \times (0, T)$ . Recall that  $S$  and  $X$  are Lipschitz continuous and bounded from above. Let  $\tau \in (0, T)$ . For  $t \in (0, T - \tau)$  we have then

$$\begin{aligned}
&\int_{\Omega} \Phi(x) \left( (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t) \right)^2 dx \\
&\leq \bar{X} L_S \sum_{K \in \mathcal{T}_m} \Phi_K \left( (S_K X_K)(t + \tau) - (S_K X_K)(t) \right) (q_K(t + \tau) - q_K(t)) \\
&\quad + L_X \sum_{K \in \mathcal{T}_m} \Phi_K \left( (S_K X_K)(t + \tau) - (S_K X_K)(t) \right) (p_{g,K}(t + \tau) - p_{g,K}(t))
\end{aligned}$$

Using Definition 3.3, taking the square root and integrating in time, we obtain

$$\sqrt{\Phi_{min}} \int_0^{T-\tau} \left\| (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t) \right\|_{\mathbf{L}^2(\Omega)} dt$$

$$\begin{aligned}
&\leq \sqrt{L_{SX}} \int_0^{T-\tau} |(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)|_{*, \mathcal{T}_m}^{\frac{1}{2}} \\
&\quad \cdot \left( \|q_m(\cdot, t + \tau) - q_m(\cdot, t)\|_{\mathcal{T}_m}^{\frac{1}{2}} + \|p_{g,m}(\cdot, t + \tau) - p_{g,m}(\cdot, t)\|_{\mathcal{T}_m}^{\frac{1}{2}} \right) dt \\
&\stackrel{Young}{\leq} \frac{\sqrt{L_{SX}}}{\sqrt{\tau}} \int_0^{T-\tau} |(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)|_{*, \mathcal{T}_m} dt \\
&\quad + \frac{\sqrt{L_{SX}\tau}}{2} \int_0^{T-\tau} \|q_m(\cdot, t + \tau) - q_m(\cdot, t)\|_{\mathcal{T}_m} dt \\
&\quad + \frac{\sqrt{L_{SX}\tau}}{2} \int_0^{T-\tau} \|p_{g,m}(\cdot, t + \tau) - p_{g,m}(\cdot, t)\|_{\mathcal{T}_m} dt
\end{aligned}$$

with  $L_{SX} = \max\{\bar{X}L_S, L_X\}$ . Now, we get by the definition of  $q$

$$\int_0^{T-\tau} \|p_{g,m}(\cdot, t + \tau) - p_{g,m}(\cdot, t)\|_{\mathcal{T}_m} dt \leq 2 \int_0^T \|p_{g,m}\|_{\mathcal{T}_m} dt \leq 2\sqrt{T} \left( \int_0^T \|p_{g,m}\|_{\mathcal{T}_m}^2 dt \right)^{\frac{1}{2}} \quad (3.9)$$

and

$$\begin{aligned}
\int_0^{T-\tau} \|q_m(\cdot, t + \tau) - q_m(\cdot, t)\|_{\mathcal{T}_m} dt &\leq 2 \int_0^T (\|p_{w,m}\|_{\mathcal{T}} + \|p_{g,m}\|_{\mathcal{T}}) dt \\
&\leq 2\sqrt{T} \left( \left( \int_0^T \|p_{w,m}\|_{\mathcal{T}_m}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T \|p_{g,m}\|_{\mathcal{T}_m} dt \right)^{\frac{1}{2}} \right). \quad (3.10)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\int_0^{T-\tau} |(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)|_{*, \mathcal{T}_m} dt &\leq \tau \int_0^T |\delta_{\mathcal{T}}(S_{\mathcal{T}} X_{\mathcal{T}})(\cdot, t)|_{*, \mathcal{T}} dt \\
&\leq \tau\sqrt{T} \left( \int_0^T |\delta_{\mathcal{T}}(S_{\mathcal{T}} X_{\mathcal{T}})(\cdot, t)|_{*, \mathcal{T}}^2 dt \right)^{\frac{1}{2}} \quad (3.11)
\end{aligned}$$

Gathering the results (3.9)–(3.11) and applying lemmas 3.2 and 3.4, we can conclude the existence of a constant  $C$  such that for all  $\tau \in \mathbb{R}$  the following estimate holds:

$$\int_{\mathbb{R}} \|(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t + \tau) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)\|_{\mathbf{L}^2(\Omega)} dt \leq TC\sqrt{|\tau|}.$$

It remains to show the convergence of the space translates. For every  $\xi \in \mathbb{R}^d$  we define the domain  $\Omega^\xi := \left\{ x \in \Omega \mid x + \xi \in \Omega \right\}$ . Then, the Lipschitz continuity of  $S$  and  $X$  yield

$$\begin{aligned}
&\|(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x + \xi, t) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)\|_{\mathbf{L}^2(\mathbb{R})}^2 \\
&\leq \tilde{C}|\xi| + \|(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(\cdot + \xi, t) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2 \\
&\leq \tilde{C}|\xi| + L_S^2 \bar{X}^2 \|q_{\mathcal{T}_m}(\cdot + \xi, t) - q_{\mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2 + L_X^2 \|p_{g, \mathcal{T}_m}(\cdot + \xi, t) - p_{g, \mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2
\end{aligned}$$

$$\leq \tilde{C}|\xi| + \bar{X}^2 L_S^2 \|p_{g, \mathcal{T}_m}(\cdot + \xi, t) - p_{g, \mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2 + (\bar{X}^2 L_S^2 + L_X^2) \|p_{g, \mathcal{T}_m}(\cdot + \xi, t) - p_{g, \mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2$$

A classical result (see e.g. [8, Lemma 9.3]) yields

$$\begin{aligned} \|p_{g, \mathcal{T}_m}(\cdot + \xi, t) - p_{g, \mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2 &\leq \|p_g\|_{\mathcal{T}_m}^2 |\xi| (|\xi| + C \text{size}(\mathcal{T})) \\ \|p_{w, \mathcal{T}_m}(\cdot + \xi, t) - p_{w, \mathcal{T}_m}(\cdot, t)\|_{\mathbf{L}^2(\Omega^\xi)}^2 &\leq \|p_w\|_{\mathcal{T}_m}^2 |\xi| (|\xi| + C \text{size}(\mathcal{T})). \end{aligned}$$

Integration in time and application of lemma 3.2 yields

$$\int_0^T \|(S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x + \xi, t) - (S_{\mathcal{T}_m} X_{\mathcal{T}_m})(x, t)\|_{\mathbf{L}^2(\mathbb{R})}^2 dt \leq TC|\xi|$$

We can therefore conclude the relative compactness of the families  $(S_{\mathcal{T}_m})_{m \in \mathbb{N}}$  and  $(S_{\mathcal{T}_m} X_{\mathcal{T}_m})_{m \in \mathbb{N}}$  in  $\mathbf{L}^2((0, T) \times \Omega)$  by the Kolmogorov-Riesz Theorem.  $\square$

**Lemma 3.6.** *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^d$ ,  $d \geq 1$  and let  $S : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded continuous nondecreasing function. If there is a sequence  $(q_n)_{n \in \mathbb{N}} \subset \mathbf{L}^2(\Omega)$  such that*

1. *there exists  $q \in \mathbf{L}^2(\Omega)$  such that  $(q_n)_{n \in \mathbb{N}}$  converges weakly to  $q$  in  $\mathbf{L}^2(\Omega)$*
2. *there exists a function  $\mathbf{S} \in \mathbf{L}^2(\Omega)$  such that  $(S_n)_{n \in \mathbb{N}}$  converges to  $\mathbf{S}$  in  $\mathbf{L}^2(\Omega)$ , where  $S_n := S(q_n)$ .*

Then  $\mathbf{S}(x) = S(q(x))$ .

*Proof.* The proof of this theorem is analogous to the proof of [9, lemma 2.7].  $\square$

**Corollary 3.7.** *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^d$ ,  $d \geq 1$  and let  $S : \mathbb{R} \rightarrow \mathbb{R}$  and  $X : \mathbb{R} \rightarrow \mathbb{R}$  be two bounded continuous non-decreasing functions with  $0 < S_{\min}$ . If there are sequences  $(q_n)_{n \in \mathbb{N}} \subset \mathbf{L}^2(\Omega)$  and  $(p_n)_{n \in \mathbb{N}}$  such that*

1. *there exists  $q \in \mathbf{L}^2(\Omega)$  such that  $(q_n)_{n \in \mathbb{N}}$  converges weakly to  $q$  in  $\mathbf{L}^2(\Omega)$*
2. *there exists  $p \in \mathbf{L}^2(\Omega)$  such that  $(p_n)_{n \in \mathbb{N}}$  converges weakly to  $p$  in  $\mathbf{L}^2(\Omega)$*
3. *there exists a function  $\mathbf{S} \in \mathbf{L}^2(\Omega)$  such that  $(S_n)_{n \in \mathbb{N}}$  converges (strongly) to  $\mathbf{S}$  in  $\mathbf{L}^2(\Omega)$ , where  $S_n := S(q_n)$ .*
4. *there exists a function  $\mathbf{X}_\mathbf{S} \in \mathbf{L}^2(\Omega)$  such that  $((SX)_n)_{n \in \mathbb{N}}$  converges (strongly) to  $\mathbf{X}_\mathbf{S}$  in  $\mathbf{L}^2(\Omega)$ , where  $(SX)_n := S(q_n)X(p_n)$ .*

Then  $X(p_n)$  converges strongly to  $\mathbf{X} = X(p)$ .

*Proof.* Since  $0 < S_{\min}$ , we can write

$$A_n = \int_{\Omega} (X(p_n) - X(a))(p_n - a) dx = \int_{\Omega} \left( \frac{(SX)_n}{S_n} - X(a) \right) (p_n - a) dx \geq 0.$$

Defining  $\mathbf{X} := \frac{\mathbf{X}_\mathbf{S}}{\mathbf{S}}$ , we get by strong / weak convergence

$$\lim_{n \rightarrow \infty} A_n = \int_{\Omega} (\mathbf{X} - X(a))(p - a) dx \geq 0.$$

The above inequality holds especially for  $a = p - t\varphi$  with  $t > 0$  and  $\varphi \in \mathbf{C}_c^\infty(\Omega)$ . Dividing by  $t > 0$  we obtain

$$\int_{\Omega} (\mathbf{X} - X(p - t\varphi))\varphi \, dx \geq 0$$

and for  $t \rightarrow 0$

$$\int_{\Omega} (\mathbf{X} - X(p))\varphi \, dx \geq 0.$$

As the above inequality also holds for  $-\varphi$ , we obtain

$$\int_{\Omega} (\mathbf{X} - X(p))\varphi \, dx = 0. \quad (3.12)$$

□

**Lemma 3.8.** *Let Hypotheses **(K)** hold and let  $\mathcal{T}_m$  be a sequence of space-time discretizations satisfying **(T)** with  $h_m \rightarrow 0$  as  $m \rightarrow \infty$ . Furthermore define*

$$\begin{aligned} K_{w,\mathcal{T}_m} &:= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} k_w(S_{K|L})\chi_{\mathcal{D}_{K|L}} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} k_w(S_{K|\sigma})\chi_{\mathcal{D}_{K|\sigma}} \\ K_{g,\mathcal{T}_m} &:= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} k_g(S_{K|L})\chi_{\mathcal{D}_{K|L}} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} k_g(S_{K|\sigma})\chi_{\mathcal{D}_{K|\sigma}}. \end{aligned}$$

Then,  $(K_{w,\mathcal{T}_m})_{m \in \mathbb{N}}$  converges to  $k_w(S)$  in  $\mathbf{L}^2((0, T) \times \Omega)$  and  $(K_{g,\mathcal{T}_m})_{m \in \mathbb{N}}$  converges to  $k_g(S)$  in  $\mathbf{L}^2((0, T) \times \Omega)$ .

*Proof.* Denote by  $L_{K_w}$  the Lipschitz constant of  $k_w$  and recall that

$$h_m := \max \left\{ \max_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} d_{K|L}, \max_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} d_{K|\sigma}, \max_{\sigma \in \mathcal{E}} |\sigma| \right\}.$$

Recall that  $S_{K|L} \in [\min\{S_K, S_L\}, \max\{S_K, S_L\}]$  and  $S_{K|\sigma} \in [0, S_K]$ . We deduce therefore that  $(S_{K|L} - S_K)^2 \leq (S_L - S_K)^2$  and  $(S_{K|\sigma} - S_K)^2 \leq (S_K)^2$  and obtain

$$\begin{aligned} & \int_{\Omega} (K_{w,\mathcal{T}_m} - k_w(S_{\mathcal{T}_m}))^2 \, dx \\ &= \sum_{K \in \mathcal{T}_m} \left( \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int} \\ \sigma = K|L}} (k_w(S_{K|L}) - k_w(S_K))^2 \frac{1}{d} |\sigma| d_{K,\sigma} + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{-ext} \\ \sigma = K|\sigma}} (k_w(S_{K|\sigma}) - k_w(S_K))^2 \frac{1}{d} |\sigma| d_{K,\sigma} \right) \\ &\leq L_{K_w} \frac{h_m}{d} \sum_{K \in \mathcal{T}_m} \left( \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{d_{K|L}} (S_{K|L} - S_K)^2 d_{K|L} + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{-ext} \\ \sigma = K|\sigma}} \frac{|\sigma|}{d_{K|\sigma}} (S_{K|\sigma} - S_K)^2 d_{K,\sigma} \right) \\ &\leq L_{K_w} \frac{h_m^2}{d} \sum_{K \in \mathcal{T}_m} \left( \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{d_{K|L}} (S_L - S_K)^2 + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{-ext} \\ \sigma = K|\sigma}} \frac{|\sigma|}{d_{K|\sigma}} (S_K)^2 \right) \end{aligned}$$

$$\begin{aligned}
&\leq L_{K_w} L_S \frac{h_m^2}{d} \sum_{K \in \mathcal{T}_m} \left( \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{d_{K|L}} (q_L - q_K)^2 + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{-ext} \\ \sigma = K|\sigma}} \frac{|\sigma|}{d_{K|\sigma}} (q_K)^2 \right) \\
&\leq L_{K_w} L_S \frac{h_m^2}{d} \left( \|p_g\|_{\mathcal{T}_m} + \|p_w\|_{\mathcal{T}_m} \right)^2
\end{aligned}$$

Integration in time and application of lemma 3.2 yields

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} (K_{w, \mathcal{T}_m} - k_w(S_{\mathcal{T}_m}))^2 dx dt = \lim_{m \rightarrow \infty} C h_m^2 = 0,$$

since  $h_m \rightarrow 0$  for  $m \rightarrow \infty$ . Due to the regularity of  $k_w$  and the strong convergence of  $S_{\mathcal{T}_m}$  the assertion follows. Analogously, we can prove the convergence of  $K_g$  to  $k_g(S)$  in  $\mathbf{L}^2((0, T) \times \Omega)$ .  $\square$

**Lemma 3.9.** *Let Hypotheses **(K)** hold and let  $\mathcal{T}_m$  be a sequence of space-time discretizations satisfying **(T)** with  $h_m \rightarrow 0$  as  $m \rightarrow \infty$ . Furthermore define*

$$\Psi_{\mathcal{T}_m} := \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \Psi_{K|L} \chi_{\mathcal{D}_{K|L}} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} \Psi_{K|\sigma} \chi_{\mathcal{D}_{K|\sigma}}.$$

Then,  $(\Psi_{\mathcal{T}_m})_{m \in \mathbb{N}}$  converges to  $X(p_g)$  in  $\mathbf{L}^2((0, T) \times \Omega)$ .

*Proof.* First, recall that  $X$  is monotone and non-decreasing such that we get by definition of  $\Psi_{K|L}$ :

$$\begin{aligned}
(\Psi_{K|L} - X(p_{g,K}))^2 &= \left( \frac{\int_{p_{g,K}}^{p_{g,L}} X(p) - X(p_{g,K}) dp}{p_{g,L} - p_{g,K}} \right)^2 \\
&\leq \left( \frac{(X(p_{g,L}) - X(p_{g,K}))(p_{g,L} - p_{g,K})}{p_{g,L} - p_{g,K}} \right)^2 \\
&= (X(p_{g,L}) - X(p_{g,K}))^2
\end{aligned}$$

and analogously

$$(\Psi_{K|\sigma} - X(p_{g,K}))^2 = \left( \frac{\int_0^{p_{g,K}} X(p) - X(p_{g,K}) dp}{p_{g,K}} \right)^2 \leq \left( \frac{X(p_{g,K}) p_{g,K}}{p_{g,K}} \right)^2 = X(p_{g,K})^2$$

Analogously to the proof of lemma 3.8 we get

$$\int_{\Omega} (\Psi_{\mathcal{T}_m} - X(p_{g, \mathcal{T}_m}))^2 dx \leq \frac{h_m^2}{d} L_X \|p_g\|_{\mathcal{T}_m}.$$

Again, integration in time and application of lemma 3.2 yields

$$\lim_{m \rightarrow \infty} \|\Psi_{\mathcal{T}_m} - X(p_{g, \mathcal{T}_m})\|_{\mathbf{L}^2((0, T) \times \Omega)} \leq \lim_{m \rightarrow \infty} C h_m^2 = 0.$$

$\square$



**Lemma 3.10.** Denote by  $\nabla_m p := d \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{p_L - p_K}{d_{K|L}} \chi_{\mathcal{D}_{K|L}} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} \frac{p_K}{d_{K|\sigma}} \chi_{\mathcal{D}_{K|\sigma}} \right)$  the discrete gradient associated to the mesh  $\mathcal{T}_m$ . Then,  $\nabla_m p_g$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $\nabla p_g$  and  $\nabla_m p_w$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $\nabla p_w$ .

*Proof.* Let  $\varphi \in (\mathbf{C}_c^\infty)^d$  by an arbitrary test function. We denote by  $n_{K|L}$  the normal vector of  $\sigma$  directed into  $L$  and by  $\varphi_{K|L} \cdot n_{K|L} = \frac{1}{|\mathcal{D}_{K|L}|} \int_{\mathcal{D}_{K|L}} \varphi \cdot n_{K|L} dx$  the integral mean of  $\varphi \cdot n_{K|L}$  on the diamond  $\mathcal{D}_{K|L}$ . The quantities  $\varphi_{K|\sigma}$  and  $n_{K|\sigma}$  are defined analogously. Then,

$$\begin{aligned}
& \int_{\Omega} \varphi (\nabla_m p_g) - p_{g, \mathcal{T}_m} \operatorname{div} \varphi dx \\
&= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} d \frac{|\sigma| d_{K|L} (p_{g,L} - p_{g,K})}{d d_{K|L}} \varphi_{K|L} \cdot n_{K|L} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} d \frac{|\sigma| d_{K|\sigma} p_{g,K}}{d d_{K|\sigma}} \varphi_{K|\sigma} \cdot n_{K|\sigma} \\
&\quad - \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} (p_{g,L} - p_{g,K}) \int_{|\sigma|} \varphi \cdot n_{K|L} ds + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} p_{g,K} \int_{|\sigma|} \varphi \cdot n_{K|\sigma} ds \right) \\
&= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} |\sigma| (p_{g,L} - p_{g,K}) \left( \frac{1}{|\mathcal{D}_{K|L}|} \int_{\mathcal{D}_{K|L}} \varphi_{K|L} \cdot n_{K|L} dx - \frac{1}{|\sigma|} \int_{|\sigma|} \varphi \cdot n_{K|L} ds \right) \\
&\quad + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} |\sigma| p_{g,K} \left( \frac{1}{|\mathcal{D}_{K|\sigma}|} \int_{\mathcal{D}_{K|\sigma}} \varphi_{K|\sigma} \cdot n_{K|\sigma} dx - \frac{1}{|\sigma|} \int_{|\sigma|} \varphi \cdot n_{K|\sigma} ds \right)
\end{aligned}$$

Since  $\varphi \in (\mathbf{C}_c^\infty)^d$ , we can bound

$$\frac{1}{|\mathcal{D}_{K|L}|} \int_{\mathcal{D}_{K|L}} \varphi_{K|L} \cdot n_{K|L} dx - \frac{1}{|\sigma|} \int_{|\sigma|} \varphi \cdot n_{K|L} ds \leq Ch_m$$

and

$$\frac{1}{|\mathcal{D}_{K|\sigma}|} \int_{\mathcal{D}_{K|\sigma}} \varphi_{K|\sigma} \cdot n_{K|\sigma} dx - \frac{1}{|\sigma|} \int_{|\sigma|} \varphi \cdot n_{K|\sigma} ds \leq Ch_m.$$

Thus,

$$\begin{aligned}
\int_{\Omega} \varphi (\nabla_m p_g) - p_{g, \mathcal{T}_m} \operatorname{div} \varphi dx &\leq Ch_m \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} |\sigma| (p_{g,L} - p_{g,K}) + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} |\sigma| p_{g,K} \right) \\
&\leq Ch_m \|p_g\|_{\mathcal{T}_m} \left( \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} |\sigma| d_{K|L} + \sum_{\substack{\sigma \in \mathcal{E}_{ext} \\ \sigma \in \mathcal{E}_K}} |\sigma| d_{K|\sigma} \right)
\end{aligned}$$

$$\leq Cd|\Omega| \|p_g\|_{\mathcal{T}_m} h_m.$$

Integration in time and application of lemma 3.2 yields

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} \varphi (\nabla_m p_g) - p_{g, \mathcal{T}_m} \operatorname{div} \varphi \, dx dt = 0.$$

Since  $p_{g, \mathcal{T}_m}$  converges weakly to  $p_g$ , this proves the weak convergence of  $\nabla_m p_g$  to  $\nabla p_g$ .  $\square$

**Theorem 3.11** (Convergence of the scheme). *Let Hypotheses **(H)** hold and let  $\mathcal{T}_m$  be a sequence of space-time discretizations satisfying **(T)** with  $h_m \rightarrow 0$  as  $m \rightarrow \infty$ . Let  $(p_{g, \mathcal{T}_m}^{(0)})_{m \in \mathbb{N}}$  and  $(p_{w, \mathcal{T}_m}^{(0)})_{m \in \mathbb{N}}$  be a family of given initial conditions such that  $\lim_{m \rightarrow \infty} \|p_{g, \mathcal{T}_m}^{(0)} - p_g^{(0)}\|_{\mathbf{L}^2(\Omega)} = 0$  and  $\lim_{m \rightarrow \infty} \|p_{w, \mathcal{T}_m}^{(0)} - p_w^{(0)}\|_{\mathbf{L}^2(\Omega)} = 0$ .*

*If  $p_{g, \mathcal{T}_m}$  and  $p_{w, \mathcal{T}_m}$  fulfill the scheme (3.1) for all  $m \in \mathbb{N}$ , then there exist  $p_g \in \mathbf{L}^2((0, T) \times \Omega)$ ,  $p_w \in \mathbf{L}^2((0, T) \times \Omega)$  and  $q = p_w - p_g$  such that (up to a subsequence)*

1.  $(p_{g, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $p_g$  as  $m \rightarrow \infty$ .
2.  $(p_{w, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $p_w$  as  $m \rightarrow \infty$ .
3.  $(\nabla_m p_{g, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $p_g$  as  $m \rightarrow \infty$ .
4.  $(\nabla_m p_{w, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges weakly in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $p_w$  as  $m \rightarrow \infty$ .
5.  $(S(q_{\mathcal{T}_m}))_{m \in \mathbb{N}}$  converges in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $\mathbf{S}(x) := S(q(x))$  as  $m \rightarrow \infty$ .
6.  $(X(p_{g, \mathcal{T}_m}))_{m \in \mathbb{N}}$  converges in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $\mathbf{X}(x) := X(p_g(x))$  as  $m \rightarrow \infty$ .
7.  $(K_{w, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $k_w(S(q))$  as  $m \rightarrow \infty$ .
8.  $(K_{g, \mathcal{T}_m})_{m \in \mathbb{N}}$  converges in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $k_g(S(q))$  as  $m \rightarrow \infty$ .
9.  $(\Psi_{\mathcal{T}_m})_{m \in \mathbb{N}}$  converges in  $\mathbf{L}^2((0, T) \times \Omega)$  to  $X(p_g)$  as  $m \rightarrow \infty$ .

and  $(p_g, p_w)$  is a weak solution of problem (1.2).

*Proof.* We use lemmas 3.2 and 3.4 to extract a subsequence of  $(\mathcal{T}_m)_{m \in \mathbb{N}}$  such that lemmas 3.5 to 3.10 and items 1. to 9. of the theorem hold. Note further that  $p_g$  and  $p_w$  are functions in  $\mathbf{L}^2((0, T); \mathbf{H}_0^1(\Omega))$ , since their extension by 0 outside of  $\Omega$  is in  $\mathbf{L}^2((0, T); \mathbf{H}^1(\mathbb{R}))$ .

Let now be  $\varphi \in \mathbf{C}_c^\infty([0, T])$  and  $\omega \in \mathbf{C}_c^\infty(\Omega)$  two test functions. Furthermore, writing  $\mathcal{T}$  instead of  $\mathcal{T}_m$  and dropping the index  $m$  wherever confusion is excluded, we define the approximation  $\omega_{\mathcal{T}}$  of  $\omega$  by

$$\begin{aligned} \omega_{\mathcal{T}} &= \sum_{K \in \mathcal{T}} \omega(x_K) \chi_K \\ \nabla_{\mathcal{T}} \omega_{\mathcal{T}} &= \left( \sum_{\mathcal{D}_{K|L}} \left( \frac{\omega(x_L) - \omega(x_K)}{d_{K|L}} n_{K|L} + \nabla_{K|L, t} \omega \right) \chi_{\mathcal{D}_{K|L}} + \sum_{\mathcal{D}_{K|\sigma}} \left( \frac{\omega(x_K)}{d_{K|\sigma}} n_{K|\sigma} + \nabla_{K|\sigma, t} \omega \right) \chi_{\mathcal{D}_{K|\sigma}} \right) \end{aligned}$$

where  $\nabla_{K|L,t}$  denotes the tangential part of the discrete gradient in the diamond  $\mathcal{D}_{K|L}$ . Analogously, we define the approximation  $\varphi_{\mathcal{T}}$  of  $\varphi$  by

$$\varphi_{\mathcal{T}} = \sum_{n=0}^{N-1} \varphi(t^n) \quad \delta_{\mathcal{T}}\varphi_{\mathcal{T}} = \sum_{n=0}^{N-1} \frac{\varphi(t^{n+1}) - \varphi(t^n)}{\delta t^{(n+\frac{1}{2})}}$$

Note that by consistency of the approximation  $\omega_{\mathcal{T}}$  converges in  $\mathbf{L}^2(\Omega)$  to  $\omega$  and  $\nabla_m \omega_{\mathcal{T}}$  converges in  $\mathbf{L}^2(\Omega)$  to  $\nabla \omega$ . Analogously,  $\varphi_{\mathcal{T}}$  converges in  $\mathbf{L}^2([0, T])$  to  $\varphi$  and  $\delta_{\mathcal{T}}\varphi_{\mathcal{T}}$  converges in  $\mathbf{L}^2([0, T])$  to  $\varphi'$ .

Multiplying (3.1a) by  $\delta t^{(n+\frac{1}{2})}\varphi(t^n)\omega(x_K)$  and summing over all  $K$  and  $n$ , we obtain

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} \delta t^{(n+\frac{1}{2})} \varphi(t^n) \omega(x_K) \quad (3.13a)$$

$$- \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^n) [p_w^{(n+1)}, \omega_{\mathcal{T}}]_{k_w^{(n+1)}, \mathcal{T}} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^n) f_{w,K}^{(n+1)} \omega(x_K). \quad (3.13b)$$

Reordering of the terms in (3.13a) yields

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} \delta t^{(n+\frac{1}{2})} \varphi(t^n) \omega(x_K) \\ &= - \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \delta_{\mathcal{T}}^{(n+\frac{1}{2})} \varphi^{(n+1)} \sum_{K \in \mathcal{T}} \Phi_K S_K^{(n+1)} \omega(x_K) - \varphi(0) \sum_{K \in \mathcal{T}} \Phi_K S_K^{(0)} \omega(x_K) \\ &= - \int_0^T \delta_{\mathcal{T}} \varphi_{\mathcal{T}} \int_{\Omega} \Phi(x) S_{\mathcal{T}} \omega_{\mathcal{T}} dx dt - \varphi(0) \int_{\Omega} \Phi(x) S(q_{\mathcal{T}}^{(0)}) \omega_{\mathcal{T}} dx. \end{aligned}$$

And therefore

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Phi_K \delta_{\mathcal{T}}^{(n+\frac{1}{2})} S_K^{(n+1)} \delta t^{(n+\frac{1}{2})} \varphi(t^n) \omega(x_K) \\ &= - \int_0^T \varphi'(t) \int_{\Omega} \Phi(x) S(q(x)) \omega(x) dx dt - \varphi(0) \int_{\Omega} \Phi(x) S(q^{(0)}(x)) \omega(x) dx. \end{aligned}$$

The term in (3.13b) can be rewritten as

$$\sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^n) [p_w^{(n+1)}, \omega_{\mathcal{T}}]_{k_w^{(n+1)}, \mathcal{T}} = \int_0^T \varphi_{\mathcal{T}}(t) \int_{\Omega} K_{w,\mathcal{T}} \nabla_m p_w \nabla_{\mathcal{T}} \omega_{\mathcal{T}} dx dt$$

We easily have

$$\lim_{m \rightarrow \infty} \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})} \varphi(t^n) f_{w,K}^{(n+1)} \omega(x_K) = \int_0^T \varphi(t) \int_{\Omega} f_w \omega dx dt,$$

and, due to strong / weak convergence, we have

$$\lim_{m \rightarrow \infty} \int_0^T \varphi_{\mathcal{T}}(t) \int_{\Omega} K_{w,\mathcal{T}} \nabla_m p_w \nabla_{\mathcal{T}} \omega_{\mathcal{T}} dx dt = \int_0^T \varphi(t) \int_{\Omega} k_w(S(q)) \nabla p_w \nabla \omega dx dt$$

which shows the convergence of (3.1a) to (1.2b). To show that (3.1b) converges to (1.2c), we follow the same steps as above.  $\square$

## 4 Numerical examples

To conclude the paper, we present two numerical example in one space dimension using the following parameter.

$$\begin{aligned} \Phi(x) &= 0.2 & \Lambda(x) &= 10^{-12} \\ \rho_w &= 10^3 \frac{\text{kg}}{\text{m}^3} & \rho_g &= 10 \frac{\text{kg}}{\text{m}^3} \\ f_w(t, x) &= 0 & f_g(t, x) &= 0 \\ D &= 0. \end{aligned}$$

This means that we examine the results obtained in the case where the gas diffusion in the water phase is negligible. Furthermore, we specify the functions  $S$ ,  $X$ ,  $k_g$  and  $k_w$  as follows.

$$\begin{aligned} S(q) &= \frac{1}{\left(\left(\frac{\max\{-q, 0\}}{p_0}\right)^n + 1\right)^m} & \text{with} & \quad n = 2, m = 1/3, p_0 = 10^5 \text{Pa} \\ X(p_g) &= \min\{c_p p_g, \bar{X}\} & \text{with} & \quad \bar{X} = 0.1 \\ k_w(S) &= \max\left\{\frac{S^2}{\mu_w}, 10^{-5}\right\} & \text{with} & \quad \mu_w = 10^{-3} \\ k_g(S) &= \max\left\{\frac{(1-S)^2}{\mu_g}, 10^{-5}\right\} & \text{with} & \quad \mu_g = 10^{-5}. \end{aligned}$$

Both test cases will be performed with the same initial and boundary data, given by

$$\begin{aligned} p_g(0, x) &= 10^5 \text{Pa} & p_w(0, x) &= 10^5 \text{Pa} \\ p_g(t, 0) &= 3 \cdot 10^5 \text{Pa} & p_g(t, L) &= 10^5 \text{Pa} \\ p_w(t, 0) &= 10^5 \text{Pa} & p_w(t, L) &= 10^5 \text{Pa}. \end{aligned}$$

To apply the algorithm (3.1), we define the values  $S_{K|L}$  and  $S_{K|\sigma}$  at the cell boundaries by

$$S_{K|L} = \frac{1}{2}(S_K + S_L) \qquad S_{K|\sigma} = \frac{1}{2}(S_K + S_\sigma),$$

with  $S_K = S(q_K) = S(p_{w,K} - p_{g,K})$  and  $S_\sigma = S(q_\sigma)$ . Hereby,  $q_\sigma$  denotes the value of  $q$  at the boundary, specified by  $q(t, 0)$  or  $q(t, L)$  respectively. Note that due to the initial conditions we have  $S^{(0)} := S(q^{(0)}) = 1$  in the whole computation domain. To take into account that this choice does not imply  $k_i(S) \geq k_i^{\min} > 0$  for the standard choices of  $k_i$ , we bound  $k_i$  artificially from below by  $10^{-5}$ . All convergence results continue to hold in this case, provided  $S > 0$  is satisfied during the computations.

For the first test case, we define the parameter  $c_p$  of the function  $X(p_g)$  by  $c_p = 3 \cdot 10^{-7}$  and assume a length of the spacial interval of  $L = 10^3 \text{m}$ . Figure 1 shows the distribution of  $S$ ,  $X(p_g)$ ,  $p_g$  and  $p_w$  at different times  $t$ . Each line corresponds to a time step of  $dt = 1 \text{month}$ , the simulation ends at time  $T = 2 \text{years}$ . In this example, the parameter  $c_p$  is chosen such that the maximum fraction of dissolved gas in the water phase is not attained,  $X(p_g)$  is therefore less than  $\bar{X}$  in the whole computational domain  $[0, L] \times [0, T]$ .

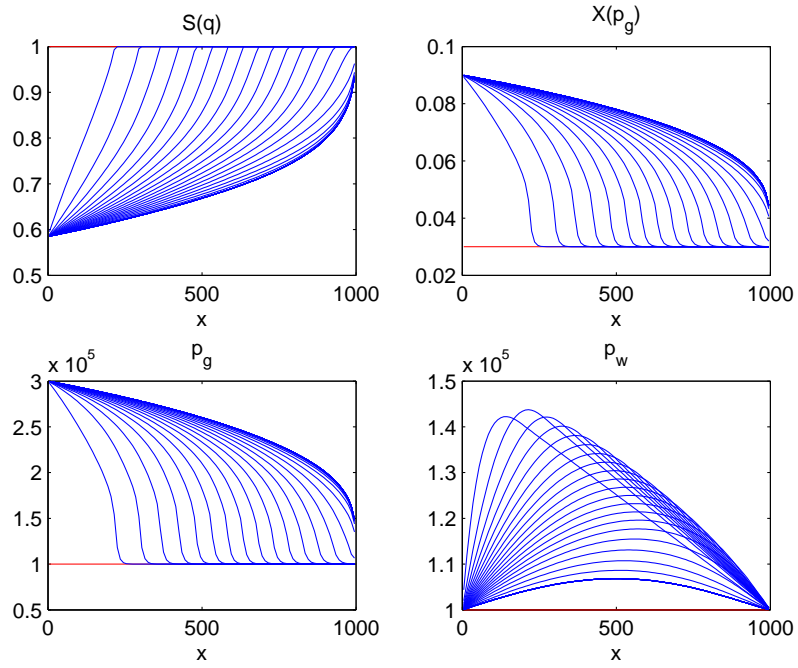


Figure 1: Test case 1:  $X$  does not attain its maximum value.

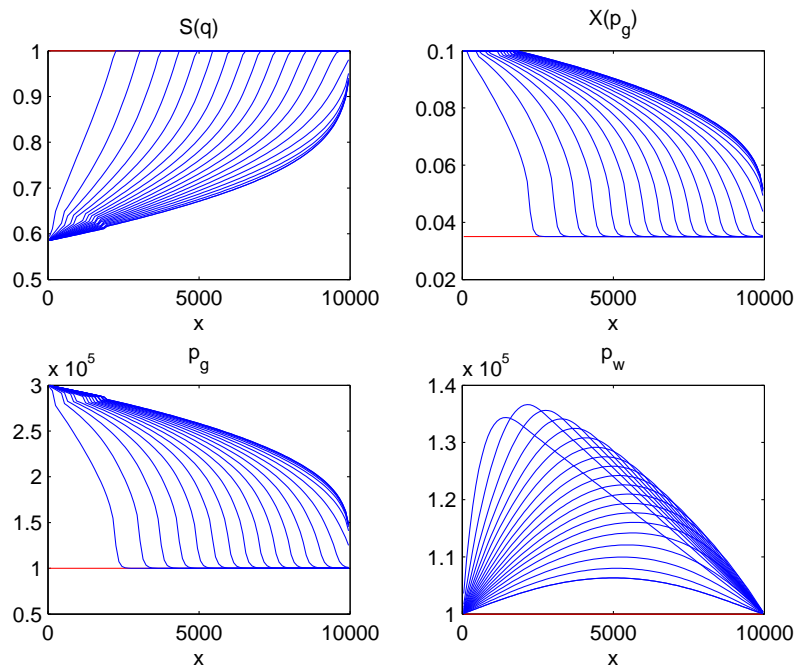


Figure 2: Test case 2: The maximum value of  $X$  is attained.

For the second test case, we chose  $c_p = 3.5 \cdot 10^{-7}$ , and a length of the domain of  $L = 10^4$ . In this case, the maximum value of  $X$  is attained. Figure 2 shows the distribution of  $S$ ,  $X(p_g)$ ,  $p_w$  and  $p_g$  for time steps of  $dt = 1year$ , the simulation ends after  $T = 20years$ .

In both test cases, we observe the profiles of the water pressure. They show a maximum value inside the domain, due to the fact that some water has to be removed (by flows at the two boundaries) in order to leave some porous volume for the apparition of the gas phase. As long as the gas phase does not appear, the gas pressure remains equal to its initial value, which simply corresponds to the initial gas concentration in the water phase (this remains exactly true since we performed the simulations with the value  $D = 0$ ). The gas pressure leaves its initial value mainly in the cells where the gas phase appears. We also remark, in the second case, that the moving front defined as the boundary of the domain  $X(p_g) < \bar{X}$  has some influence on the profile of the saturation and of the gas pressure.

## 5 Conclusion

In this paper we have studied the convergence of a finite volume scheme with two-point flux approximation, to a solution of two-phase flow in porous media in the case of the partial dissolution of one phase into the other one. It is interesting to notice that the scheme which has been chosen here was the only one on which we have been able to derive this study, due to the need of nonlinear expressions of the primary unknowns as test functions. Further works to overcome this difficulty seem to be necessary, although it is not clear whether they can be fruitful.

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