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Simplicial simple-homotopy of flag complexes in terms of graphs

Romain Boulet, Etienne Fieux and Bertrand Jouve

Institut de Mathématiques de Toulouse
Unité Mixte de Recherche
CNRS - UMR 5219

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Abstract

A flag complex can be defined as a simplicial complex whose simplices correspond to complete subgraphs of its 1-skeleton taken as a graph. In this article, by introducing the notion of s-dismantlability, we shall define the s-homotopy type of a graph and show in particular that two finite graphs have the same s-homotopy type if, and only if, the two flag complexes determined by these graphs have the same simplicial simple-homotopy type (Theorem 2.10, part 1). This result is closely related to similar results established by Barmak and Minian ([2]) in the framework of posets and we give the relation between the two approaches (theorems 3.5 and 3.7). We conclude with a question about the relation between the s-homotopy and the graph homotopy defined in [5].

Keywords : Barycentric subdivision, collapsibility, flag complexes, graphs, poset, simple-homotopy.

Introduction

Flag complexes are (abtract) simplicial complexes whose every minimal non simplex has two elements ([17],[16],[7]); this means that a flag complex is completely determined by its 1-skeleton (all necessary definitions are recalled below). They constitute an important subset of the set of simplicial complexes; in particular, the barycentric subdivision of any simplicial complex is a flag complex and we know that a simplicial complex and its barycentric subdivision have the same simple-homotopy type.

Flag complexes arise naturally from the graph point of view and are also sometimes called *clique complexes* ([5]). Indeed, the 1-skeleton of a simplicial complex can be considered as a graph and it is easy to see that a simplicial complex K is a flag complex if, and only if, we can write $K = \Delta_{\mathscr{G}}(G)$ for some graph G where, by definition, $\Delta_{\mathscr{G}}(G)$ is the simplicial complex whose simplices are given by the complete subgraphs of G (and this is sometimes taken as the definition of flag complexes, as in [12]).

In this paper, we are interested in the notion of simplicial simple-homotopy for flag complexes. We note that the determination of the simplicial simple-homotopy type is actually important not only for simplicial complexes but also for graphs because simplicial complexes arise in various constructions in graph theory. For example, this notion appears in the study of the clique graph ([14], [13]), in results about the polyhedral complex $\mathsf{Hom}(G, H)$ introduced by Lovasz ([1],[11]) or in relation to evasiveness ([10]).

Simplicial simple-homotopy is defined by formal deformations themselves defined by the notion of elementary collapses consisting in the deletion of certain pairs of simplices (see §2). As the set of simplices of a flag complex is determined by its 1-skeleton seen as a graph, the aim of this paper is to relate formal deformations on flag complexes to certain operations on graphs. The key notion will be the one of *s-dismantlability*: the deletion or the addition of *s-dismantlable* vertices in a graph will play the role of elementary reductions or expansions ([6]) in a simplicial complex. More precisely, a vertex g of a graph G will be called s-dismantlable if its open neighborhood is a dismantlable graph and the deletion of an s-dimantlable vertex g in G is equivalent to the deletion of all simplices which contain g in $\Delta_{\mathscr{G}}(G)$.

In Section 1, we introduce the notion of s-dismantlability which allows us to define an equivalence relation for graphs; the equivalence class $[G]_s$ of a graph G for this equivalence relation will be called the *s-homotopy type of* G and we give some properties related to these notions.

In Section 2, we study the correspondence between s-dismantlability in $\mathscr{G}$ (the set of finite undirected graphs, without multiple edges) and simplicial simple-homotopy in $\mathscr{K}$ (the set of finite simplicial complexes); we prove that two finite graphs G and H have the same s-homotopy type if, and only if, $\Delta_{\mathscr{G}}(G)$ and $\Delta_{\mathscr{G}}(H)$ have the same simple-homotopy type. Reciprocally, for a simplicial complex K, $\Gamma(K)$ is the graph whose vertices are the simplices of K and the edges are given by the inclusions; we show that two finite simplicial complexes K and L have the same simple-homotopy type if, and only if, $\Gamma(K)$ and $\Gamma(L)$ have the same s-homotopy type. We have to mention that these results need the introduction of barycentric subdivision in $\mathscr{G}$, defined, for a graph G, as the 1-skeleton of the usual barycentric subdivision (in $\mathscr{K}$) of $\Delta_{\mathscr{G}}(G)$.

In Section 3, we consider the important class of flag complexes which results from posets: if P is a poset, $\Delta_{\mathscr{P}}(P)$ is the flag complex whose simplices are given by chains of P. In ([2]), Barmak and Minian define a notion of *simple equivalence* in posets and show that there is a one-to-one correspondence between simple-homotopy types of finite simplicial complexes and simple equivalence classes of finite posets. As we have $\Delta_{\mathscr{P}}(P) = \Delta_{\mathscr{G}}(Comp(P))$ where $Comp(P)$ is the comparability graph of P, there is a close relation between this approach and our approach from graphs. We show that there is indeed a one-to-one correspondence between s-homotopy type in $\mathscr{G}$ and simple equivalence classes in the set $\mathscr{P}$ of finite posets. Finally, we consider a triangle between finite graphs, posets and simplicial complexes recapitulating the close relations between s-homotopy type (in $\mathscr{G}$), simple type (in $\mathscr{P}$) and simple homotopy type (in $\mathscr{K}$).

In Section 4 we describe a weaker version of s-dimantlability on graphs which provides a closer connection with simplicial collapse for flag complexes (Proposition 4.5).

Then we conclude in Section 5 with a question concerning the relation between s-homotopy and the graph homotopy defined in [5]

Some results have been set out in [4].

Definitions, notations

Let $\mathscr{G}$ be the set of finite undirected graphs, without multiple edges. If $G \in \mathscr{G}$, we have $G = (V(G), E(G))$ with $E(G) \subseteq \{ \{g, g'\}\ ,\ g, g' \in V(G)\}$. For brevity, we write $xy \in G$ or $x \sim y$ for $\{x, y\} \in E(G)$ and $x \in G$ for $x \in V(G)$. The closed neighborhood of g is $N_G[g] := \{h \in G,\, g \sim h\} \cup \{g\}$ and $N_G(g) := N_G[g] \setminus \{g\}$ is its open neighborhood. When no confusion is possible, a subset S of $V(G)$ will also denote the subgraph of G induced by S. We denote by $G \setminus S$ the graph obtained from G by deleting S and all the edges adjacent to a vertex of S. In particular, we use the notations $G \setminus x$ and $G \setminus xy$ to indicate the deletion of a vertex x or an edge xy. The notation $c = [g_1, \ldots, g_k]$ means that the the subset $\{g_1, \ldots, g_k\}$ of $V(G)$ induces a complete subgraph of G. We say that a graph G is a *cone* on a vertex $g \in G$ if $N_G[g] = G$. The notation *pt* will denote a graph reduced to a single vertex (looped or not looped).

Let $\mathscr{K}$ be the set of (abstract) finite simplicial complexes. A simplicial complex K is a family of subsets of a finite set $V(K)$ (the set of vertices of K) stable with respect to deletion of elements (if $\sigma \in K$ and $x \in \sigma$, then $\sigma \setminus \{x\} \in K$). An element $\{x_0, x_1, \ldots, x_k\}$ of K is called k-simplex and will be denoted by $< x_0, x_1, \ldots, x_k >$; the n-skeleton of K is the set K_n formed by all k-simplices of K with $k \leq n$. The 0-skeleton is identified with the set $V(K)$ of vertices of K. A face of $\sigma =< x_0, x_1, \ldots, x_k >$ is any simplex included in σ.

1 s-dismantlability and s-homotopy type in $\mathscr{G}$

1.1 Definitions

Let $G \in \mathscr{G}$. We recall ([14],[3],[8]) that a vertex g of G is called *dismantlable* if there is another vertex g' of G which dominates g (*i.e.*, $g \neq g'$ and $N_G[g] \subseteq N_G[g']$); we note that this implies $g \sim g'$. A graph G is called *dismantlable* if it is reduced to a single vertex or if we can write $V(G) = \{g_1, g_2, \ldots, g_n\}$ with g_i dismantlable in the subgraph induced by $\{g_1, \ldots, g_i\}$, for $2 \leq i \leq n$. In particular, a dismantlable graph is necessarily non empty.

Definition 1.1 *A vertex g of a graph G is called s-dismantlable in G if $N_G(g)$ is dismantlable.*

Let H a subgraph of a graph G. We shall say that G *is dismantlable on* H if we can go from G to H by successive deletions of dismantlable vertices.

Example 1.2 *No vertex of the graph G_1 of Fig. 1 is dismantlable but there are four s-dismantlable vertices, as the vertex a in the picture ($N_{G_1}(a)$ is a dismantlable path).*

Figure 1: The vertex a is s-dismantlable (and not dismantlable)

Clearly, a dismantlable vertex is s-dismantable (because the open neighborhood of a dismantlable vertex is a cone which is a dismantlable graph). When $g \in G$ is s-dismantlable, we shall write $G \overset{s}{\searrow} G \setminus \{g\}$ (elementary reduction); equivalently, $G \overset{s}{\nearrow} G \cup \{x\}$ (elementary expansion) indicates the addition to G of a vertex x such that $N_{G\cup\{x\}}(x)$ is dismantlable. By analogy with the usual situation in $\mathscr{K}$, $G \overset{s}{\searrow} H$ (resp. $G \overset{s}{\nearrow} H$) indicates that we can go from G to H, by deleting (resp. adding) successively s-dismantlable vertices.

Definition 1.3 *Two graphs G and H have the same s-homotopy type if there is a sequence $G = J_1, \ldots, J_k = H$ in $\mathscr{G}$ such that $G = J_1 \xrightarrow{s} J_2 \xrightarrow{s} \ldots \xrightarrow{s} J_{k-1} \xrightarrow{s} J_k = H$ where each arrow $\xrightarrow{s}$ represents the suppression or the addition of an s-dismantlable vertex.*

This defines an equivalence relation in $\mathscr{G}$ and we shall denote by $[G]_s$ the equivalence class representing the s-homotopy type of a graph G. A graph G will be called s-dismantlable if $[G]_s = [pt]_s$.

1.2 Properties

Lemma 1.4 *Let $G, H \in \mathscr{G}$ such that $[G]_s = [H]_s$; then, there is a graph W such that $G \overset{s}{\nearrow} W \overset{s}{\searrow} H$.*

proof : Let us suppose that G and H are two graphs such that $[G]_s = [H]_s$. It is sufficient to prove that an elementary reduction and an elementary expansion may be switched in the sequence of elementary operations from G to H. So, let us suppose that in a graph G', we have an elementary reduction followed by an elementary expansion:

$$(1) \qquad G' \overset{s}{\searrow} G' \setminus g_1 \overset{s}{\nearrow} (G' \setminus g_1) \cup \{g_2\}$$

This means that the graphs $N_{G'}(g_1)$ and $N_{G'\setminus g_1}(g_2)$ are dismantlable and, in particular, $g_2 \not\sim g_1$. So we can adjoin g_2 to G' by putting $N_{G'}(g_2) = N_{G'\setminus g_1}(g_2)$; of course, $N_{G'\cup\{g_2\}}(g_1) = N_{G'}(g_1)$ and the sequence (1) can be alternatively written

$$(2) \qquad G' \overset{s}{\nearrow} G' \cup \{g_2\} \overset{s}{\searrow} (G' \cup \{g_2\}) \setminus g_1$$

with isomorphic resultant graphs $(G' \setminus g_1) \cup \{g_2\}$ and $(G' \cup \{g_2\}) \setminus g_1$ and this proves that all the reductions can be pushed at the end of the sequence of elementary operations from G to H. □

We recall that the suspension SG of G is the graph whose vertex set is $V(G) \cup \{x, y\}$ where x and y are two distinct vertices which are not in $V(G)$ and whose edge set is $E(G) \cup \{xg, g \in V(G)\} \cup \{yg, g \in V(G)\}$; in the sequel, SG will be also denoted by $G \cup \{x, y\}$. We shall need the following result :

Proposition 1.5 *A graph G is dismantlable if, and only if, its suspension SG is dismantlable.*

proof : Let $SG = G \cup \{x, y\}$. If G is dismantlable, let $V(G) = \{g_1, g_2, \ldots, g_n\}$ with g_i dismantlable in the subgraph induced by $\{g_1, \ldots, g_i\}$, for $2 \leq i \leq n$. Then we can write $V(SG) = \{g_1, x, y, g_2, \ldots, g_n\}$ with g_i dismantlable in the subgraph induced by $\{g_1, x, y, \ldots, g_i\}$, for $2 \leq i \leq n$ and the subgraph of SG induced by $\{g_1, x, y\}$ is of course dismantlable (it is a path). Let us now suppose that SG is dismantlable and let g be a dismantlable vertex in SG. If $g = x$ or $g = y$, this means that G is a cone (because $N_{SG}(x) = N_{SG}(y) = G$ and a vertex g' which dominates g verifies $N_G[g'] = G$). If $g \neq x$ and $g \neq y$, g is also dismantlable in G (because we have $\{x, y\} \in N_{SG}(g)$ and a vertex which dominates g in SG is necessarily different from x and y and, consequently, dominates g in G). From this observation, it follows that when we delete a dismantlable vertex g in SG, either $g \in G$ and g is dismantlable in G, either $g \in \{x, y\}$ and this implies that G is dismantlable (because it is a cone). By iteration of this procedure, we get that G is a dismantlable graph. $\square$

Lemma 1.6 *(deletion of an edge) If g and g' are two distinct vertices of a graph G such that $g \sim g'$ and $N_G(g) \cap N_G(g')$ is nonempty and dismantlable, then $[G]_s = [G \setminus gg']_s$. In other words, we can s-delete the edge gg'.*

proof : We add to G a vertex x with edges xz for every z in $N_G[g] \setminus \{g'\}$ (it is an elementary expansion because $N_G[g] \setminus \{g'\}$ is a cone on g) and we write $G \cup x$ for the resulting graph. Let us verify that g is s-dismantlable in $G \cup x$. We have $N_{G\cup x}(g) = N_G(g) \cup \{x\}$. If $N_G(g) \subseteq N_G[g']$, we can write $N_{G\cup x}(g) = (N_G(g) \cap N_G(g')) \cup x \cup g'$. If $N_G(g) \not\subseteq N_G[g']$, every y in $N_G(g)$ which is not in $N_G[g']$ is dominated by x in $N_{G\cup x}(g)$; so, $N_{G\cup x}(g)$ is dismantlable on its subgraph induced by the set of vertices $(N_G(g) \cap N_G(g')) \cup x \cup g'$. But $(N_G(g) \cap N_G(g')) \cup x \cup g'$ is the suspension of $N_G(g) \cap N_G(g')$ and, by Proposition 1.5, it is a dismantlable graph because $N_G(g) \cap N_G(g')$ is dismantlable. Thus, g is s-dismantlable in $G \cup x$ and we can reduce $G \cup x$ on $(G \cup x) \setminus g$ which is clearly isomorphic to $G \setminus gg'$. $\square$

Proposition 1.7 *Let $G \in \mathscr{G}$ and $g \in V(G)$. If $N_G(g)$ is s-dismantlable, then $[G]_s = [G \setminus g]_s$.*

proof : Let $g \in V(G)$ such that the graph $N_G(g)$ is s-dismantlable. Let us suppose first that $N_G(g) \overset{s}{\searrow} pt$ and let $N_G(g) = \{y_1, y_2, \ldots, y_k\}$ with y_i s-dismantlable in the subgraph of $N_G(g)$ induced by $\{y_1, \ldots, y_i\}$, for $2 \leq i \leq k$. Then, $N_G(y_k) \cap N_G(g)$ is dismantlable (it is the open neighborhood of y_k in $N_G(g)$) and, by Lemma 1.6, G is s-dismantlable on the graph $H_k = G \setminus y_k g$. Now, $N_{H_k}(g) = \{y_1, y_2, \ldots, y_{k-1}\}$ and $N_{H_k}(y_{k-1}) \cap N_{H_k}(g) = N_G(y_{k-1}) \cap N_{H_k}(g)$ is dismantlable in H_k and, by the same argument, H_k is s-dismantlable on $H_{k-1} = G \setminus \{y_k g, y_{k-1} g\}$. The iteration of this procedure shows that G is s-dismantlable on $H_2 = G \setminus \{y_k g, y_{k-1} g, \ldots, y_3 g, y_2 g\}$. Of course, g is s-dismantlable (in fact, dismantlable) in H_2 and this proves that G is s-dismantlable on $G \setminus g$. If we don't have $N_G(g) \overset{s}{\searrow} pt$, we know by Lemma 1.4 that there is a graph W such that $N_G(g) \overset{s}{\nearrow} W \overset{s}{\searrow} pt$. Let $\{z_1, \ldots, z_m\}$ the set of vertices of W which are not in $N_G(g)$. We define $H = G \cup \{z_1, \ldots, z_m\}$ with $N_H(z_i) = N_W(z_i) \cup \{g\}$. It is clear that $G \overset{s}{\nearrow} H$ and that $N_H(g) = W \overset{s}{\searrow} pt$. Thus, by the previous discussion, we know that H is s-dismantlable on $H \setminus g$ but $H \setminus g \overset{s}{\searrow} G \setminus g$ (because of $W \overset{s}{\searrow} N_G(g)$) and we can conclude that $[G]_s = [G \setminus g]_s$ (by $[G]_s = [H]_s = [H \setminus g]_s = [G \setminus g]_s$). $\square$

This result implies that the procedure of s-dismantlability can be done more rapidly by deleting a vertex whose open neighborhood is s-dismantlable. For example, we immediately get the following result:

Corollary 1.8 *Let $G \in \mathscr{G}$. If G is s-dismantlable, then SG is also s-dismantlable.*

proof : As $N_{SG}(x) = N_{SG}(y) = G$ and G is s-dismantlable, we obtain by the previous proposition that $[SG]_s = [SG \setminus \{x, y\}]_s$, *i.e.* $[SG]_s = [G]_s$. $\square$

Now, by analogy with the notion of collapsibility in simplicial complexes (see below), we introduce the following definition:

Definition 1.9 *A graph G is called s-collapsible if $G \overset{s}{\searrow} pt$ (i.e. we can write $G = \{g_1, g_2, \dots, g_n\}$ with g_i s-dismantlable in the subgraph induced by $\{g_1, \dots, g_i\}$, for $2 \le i \le n$).*

Remark 1.10 *Of course, a dismantlable graph is s-collapsible but the inclusion (of the family of dismantlable graphs in the family of s-collapsible graphs) is strict; for instance, the graph G_1 of example 1.2 is not dismantlable (because there is no dismantlable vertex) and s-collapsible: $G_1 \overset{s}{\searrow} G_1 \setminus a$ because a is s-dismantlable and it is easy to see that the graph $G_1 \setminus a$ is s-collapsible (even more: $G_1 \setminus a$ is dismantlable). Furthermore, it can be proved that this graph G_1 is minimal (in terms of number of vertices) in the family of s-collapsible graphs and without any dismantlable vertex.*

Proposition 1.11 *Let $G \in \mathscr{G}$. The graph G is s-collapsible if, and only if, SG is s-collapsible.*

proof : Let $SG = G \cup \{x, y\}$. First, we observe that every s-dismantlable vertex g in G is also s-dismantlable in SG because $N_{SG}(g) = N_G(g) \cup \{x, y\} = SN_G(g)$ and the dismantlability of $SN_G(g)$ is a consequence of the dismantlability of $N_G(g)$ (Proposition 1.5). Suppose now that $G \overset{s}{\searrow} pt$ with $G = \{g_1, g_2, \dots, g_n\}$ and g_i s-dismantlable in the subgraph induced by $\{g_1, \dots, g_i\}$, for $2 \le i \le n$. Then, by the previous observation, g_i is s-dismantlable in the subgraph of SG induced by $\{g_1, x, y, g_2 \dots, g_i\}$, for $2 \le i \le n$. So, we have $SG \overset{s}{\searrow} \{g_1, x, y\} \overset{s}{\searrow} \{g_1\}$. Reciprocally, if SG is s-collapsible, we shall prove that G is also s-collapsible by induction on the number of vertices of G. If $|V(G)| = 1$, there is nothing to prove (G and SG are s-collapsible). Let us suppose that the s-collapsibility of SG implies the s-collapsibility of G if $|V(G)| = n$ for $n \ge 1$ and let G with $|V(G)| = n + 1$ such that SG is s-collapsible. Let $g \in V(SG)$ an s-dismantlable vertex. If $g \in \{x, y\}$, this means that G is dismantlable (because $N_{SG}(x) = N_{SG}(y) = G$) and thus, s-collapsible. So, we can assume that $g \ne x$ and $g \ne y$. We have $N_{SG}(g) = SN_G(g)$; so, by Proposition 1.5, the dismantlability of $N_{SG}(g)$ implies the dismantlability of $N_G(g)$ and we have $G \overset{s}{\searrow} G \setminus g$. Now, $SG \setminus g = S(G \setminus g)$ and we can apply the induction hypothesis to conclude that $G \setminus g$ is s-collapsible (and the same conclusion for G). □

2 Relation with simple homotopy in $\mathscr{K}$

2.1 From $\mathscr{G}$ to $\mathscr{K}$

Let $K \in \mathscr{K}$; let us recall ([6]) that an elementary simplicial reduction (or *collapse*) in K is the suppression of a pair of simplices (σ, τ) of K such that τ is a proper maximal face of σ and τ is not the face of another simplex (one says that τ is a free face of K).

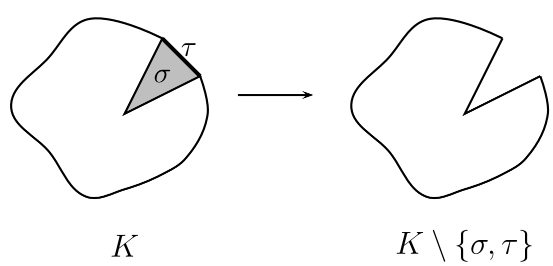

Figure 2: An elementary collapse

This is denoted by $K \overset{s}{\searrow} (K \setminus \{\sigma, \tau\})$ (and called *elementary collapse* or *elementary reduction*) or $(K \setminus \{\sigma, \tau\}) \overset{s}{\nearrow} K$ (*elementary anticollapse* or *elementary expansion*). More generally, a simplicial collapse $K \overset{s}{\searrow} L$ (resp. anticollapse $K \overset{s}{\nearrow} L$) is a succession of elementary simplicial collapses (resp. elementary simplicial anticollapses) which tranform K into L. Collapses or anticollapses are called *formal deformations*. A simplicial complex K is called *collapsible* if $K \overset{s}{\searrow} pt$. Two simplicial complexes K and L have the same *simple-homotopy type* if there is (in $\mathscr{K}$) a finite sequence $K = M_1 \overset{s}{\to} M_2 \overset{s}{\to} \dots \overset{s}{\to} M_{k-1} \overset{s}{\to} M_k = L$ where each arrow $\xrightarrow{s}$ represents a simplicial collapse or a simplicial anticollapse. We shall denote by $[K]_s$ the simple-homotopy type of K.

Let us recall that for a simplex $\sigma \in K$, $\mathrm{link}_K(\sigma) := \{\tau \in K\ , \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in K\}$ and $\mathrm{star}^o_K(\sigma)$ is the set of simplices of K containing σ. Let us recall :

Lemma 2.1 ([19, Lemma 2.7]) *Let σ be a simplex of $K \in \mathscr{K}$. If $\mathrm{link}_K(\sigma)$ is collapsible, then $K \searrow^s K \setminus \mathrm{star}^o_K(\sigma)$.*

We recall that the application $\Delta_{\mathscr{G}} : \mathscr{G} \to \mathscr{K}$ is defined in the following way : if $G \in \mathscr{G}$, $\Delta_{\mathscr{G}}(G)$ is the simplicial complex whose simplices are the complete subgraphs of G (so we have $V(\Delta_{\mathscr{G}}(G)) = V(G)$). We note that if a graph G does not contain any triangle (*i.e.* a complete subgraph with three vertices), we can identify G and $\Delta_{\mathscr{G}}(G)$ (we consider G either as a graph, or as a simplicial complex of dimension 1); it is the case of C_4 in examples of Fig. 3.

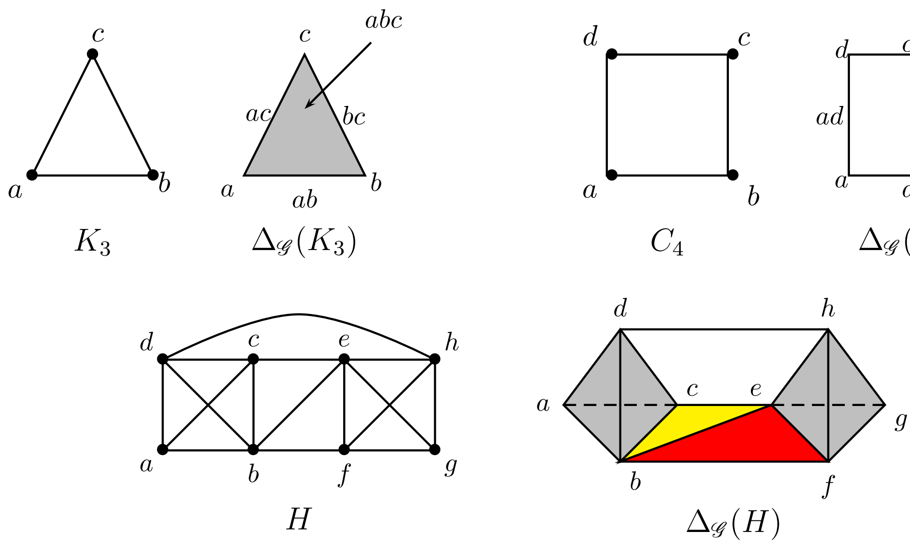

Figure 3: Illustration of $\Delta_{\mathscr{G}}$

Lemma 2.2 ([14, Proposition 3.2]) *Let G be a graph and g be a dismantlable vertex in G; then $\Delta_{\mathscr{G}}(G) \searrow^s \Delta_{\mathscr{G}}(G \setminus g)$.*

proof : Let a be a vertex which dominates g (*i.e.* $N_G[g] \subseteq N_G[a]$ with $a \neq g$); for every maximal complete subgraph c of G, we have $g \in c \Rightarrow a \in c$. So, let $c = [g, a, g_1, \dots, g_n]$ a maximal complete subgraph of G which contains g; then $c' = [g, g_1, \dots, g_n]$ is a free face of c (taken as a simplex of $\Delta_{\mathscr{G}}(G)$) and $\Delta_{\mathscr{G}}(G) \searrow^s \Delta_{\mathscr{G}}(G) \setminus \{c, c'\}$. By iteration, $\Delta_{\mathscr{G}}(G)$ collapses on the subcomplex formed by all simplices which do not contain g, *i.e.* $\Delta_{\mathscr{G}}(G) \searrow^s \Delta_{\mathscr{G}}(G \setminus g)$. □

Proposition 2.3 *Let $G, H \in \mathscr{G}$. Then, $G \searrow^s H \Longrightarrow \Delta_{\mathscr{G}}(G) \searrow^s \Delta_{\mathscr{G}}(H)$.*

proof : It suffices to prove that if g is s-dismantlable in G, then $\Delta_{\mathscr{G}}(G) \searrow^s \Delta_{\mathscr{G}}(G \setminus g)$. It follows from Lemma 2.2 that $\Delta_{\mathscr{G}}(H)$ is collapsible for every dismantlable graph H. Thus, by definition of s-dismantlability, $\mathrm{link}_{\Delta_{\mathscr{G}}(G)}(< g >) = \Delta_{\mathscr{G}}(N_G(g))$ is collapsible when g is s-dismantlable (where $< g >$ is the 0-simplex of $\Delta_{\mathscr{G}}(G)$ determined by g). As $\Delta_{\mathscr{G}}(G \setminus g) = \Delta_{\mathscr{G}}(G) \setminus \mathrm{star}^o_{\Delta_{\mathscr{G}}(G)}(< g >)$, the conclusion follows from Lemma 2.1. □

Remark 2.4 *The converse of Proposition 2.3 is not true; a counterexample is given by the graphs G and H of Fig. 4. Indeed, we have $K \searrow^s L = K \setminus \{< a, b, c >, < b, c >\}$ (the complex K collapses onto L) with $K = \Delta_{\mathscr{G}}(G)$, $L = \Delta_{\mathscr{G}}(H) \equiv H$ and we don't have $G \searrow^s H$ because there is no s-dismantlable vertex in G.*

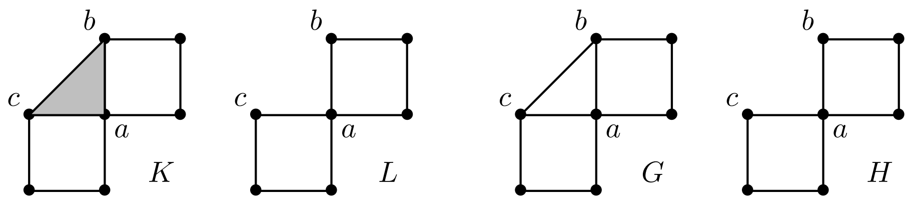

Figure 4: $\Delta_{\mathscr{G}}(G) \searrow^{s} \Delta_{\mathscr{G}}(H)$ but $G \not\searrow^{s} H$

2.2 From $\mathscr{K}$ to $\mathscr{G}$

We consider the application $\Gamma : \mathscr{K} \to \mathscr{G}$ whose definition is : if $K \in \mathscr{K}$, $\Gamma(K)$ is the graph whose vertices are the simplices of K with edges $\{\sigma, \sigma'\}$ when $\sigma \subset \sigma'$ or $\sigma' \subset \sigma$.

If σ is a simplex of $K \in \mathscr{K}$, we shall write $K[\sigma]$ for the simplicial subcomplex of K formed by all faces of σ ($K[\sigma] := \{\tau \in K, \tau \subset \sigma\}$); if τ is a maximal face of σ, $K[\sigma] \backslash \{\sigma, \tau\}$ is a simplicial complex. In order to understand the relation of formal deformations to s-dismantlability, we have the following results:

Lemma 2.5 *If τ is a maximal face of σ then $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable.*

proof : Let $\sigma =< a_0, a_1, \ldots, a_n >$ and $\tau =< a_1, \ldots, a_n >$. The vertices of $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ are the simplices of $K[\sigma] \backslash \{\sigma, \tau\}$. These vertices can be written $< a_{i_1}, \ldots, a_{i_k} >$ with $0 \leq i_1 < i_2 < \ldots < i_k \leq n$ excepting $< a_0, a_1, \ldots, a_n >$ and $< a_1, \ldots, a_n >$; we shall say that such a vertex $< a_{i_1}, \ldots, a_{i_k} >$ contains a if $a \in \{a_{i_1}, \ldots, a_{i_k}\}$. Every vertex $x =< a_{i_1}, \ldots, a_{i_{n-1}} >$ which does not contain a_0 is dismantlable (because it is dominated by $< a_0, a_{i_1}, \ldots, a_{i_{n-1}} >$ the unique $(n-1)$-simplex containing $< a_{i_1}, \ldots, a_{i_{n-1}} >$). Thus, $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ can be dismantled on Γ_{n-1} obtained by deleting all vertices corresponding to $(n-1)$-simplices which do not contain a_0. Next, every vertex $< a_{i_1}, \ldots, a_{i_{n-2}} >$ which does not contain a_0 is dismantlable in Γ_{n-1} (because it is dominated by $< a_0, a_{i_1}, \ldots, a_{i_{n-2}} >$ the unique $(n-2)$-simplex containing $< a_{i_1}, \ldots, a_{i_{n-2}} >$). Thus, Γ_{n-1} can be dismantled on Γ_{n-2} obtained by deleting all vertices corresponding to $(n-2)$-simplices which do not contain a_0. The iteration of this procedure shows that $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable on its subgraph induced by the vertices containing a_0. But this subgraph is a cone on $< a_0 >$ and this shows that $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable. $\square$

Proposition 2.6 *Let $K, L \in \mathscr{K}$. Then, $K \searrow^{s} L \Longrightarrow \Gamma(K) \searrow^{s} \Gamma(L)$.*

proof : It suffices to prove that if $\{\sigma, \tau\}$ is a collapsible pair in K, then $\Gamma(K) \searrow^{s} \Gamma(K \backslash \{\sigma, \tau\})$. We note that $\Gamma(K \setminus \{\sigma, \tau\}) = \Gamma(K) \setminus \{\sigma, \tau\}$ and that the vertex τ is dismantlable in $\Gamma(K)$ (because it is dominated by σ); so, we have the reduction $\Gamma(K) \searrow^{s} \Gamma(K) \setminus \tau$. Now, we have $N_{\Gamma(K)\backslash\tau}(\sigma) = \Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ (because σ is a maximal simplex), and we conclude that σ is s-dismantlable by the Lemma 2.5. $\square$

2.3 Barycentric subdivision

Let us recall the notion of barycentric subdivision in $\mathscr{K}$. If $K \in \mathscr{K}$, the n-simplices of the barycentric subdivision $Bd(K)$ (or K') of K are the $< \sigma_0, \sigma_1, \ldots, \sigma_n >$ composed of $n+1$ simplices of K such that $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_n$. Now, we define a similar notion in $\mathscr{G}$.

Definition 2.7 *If $G \in \mathscr{G}$, the barycentric subdivision $Bd(G)$ (or G') of G is the graph whose vertices are the complete subgraphs of G and there is an edge between two vertices if, and only if, there is an inclusion between the two corresponding complete subgraphs.*

Remark 2.8 *Each complete subgraph of cardinality at least two creates a new vertex in the barycentric subdivision (cf. Fig. 5). The equalities $\Gamma \circ \Delta_{\mathscr{G}} = Bd$ (in $\mathscr{G}$) and $\Delta_{\mathscr{G}} \circ \Gamma = Bd$ (in $\mathscr{K}$) follow directly from the definitions and will be useful in the following.*

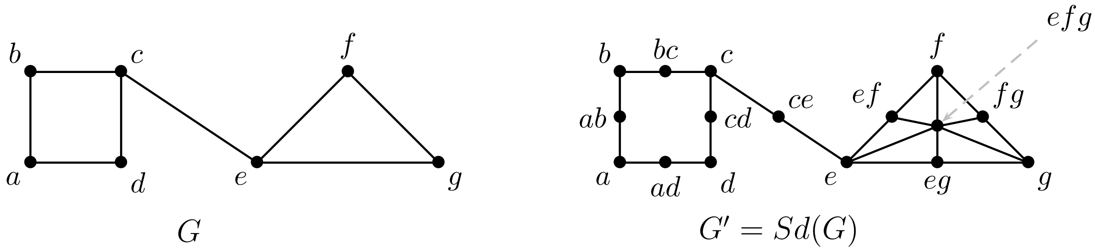

Figure 5: Example of barycentric subdivision

Proposition 2.9 *For every $G \in \mathscr{G}$, G and G' have the same s-homotopy type (i.e. $[G]_s = [G']_s$).*

proof : Let n be the cardinal of $V(G)$; we choose to number the vertices of G; thus, we have $V(G) = \{g_1, g_2, \dots, g_n\}$. Let us recall that $V(G') = \mathscr{C}(G)$, the set of complete subgraphs of G. In what follows, every complete subgraph c is considered under its unique expression $c = [g_{i_1}, \dots, g_{i_k}]$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$; we shall denote by $i_k = \max(c)$. If $g \in \{g_{i_1}, \dots, g_{i_k}\}$ and $c = [g_{i_1}, \dots, g_{i_k}] \in \mathscr{C}(G)$, we shall write $g \in c$ and for $c, d \in \mathscr{C}(G)$, we shall write $c \subset d$ when $c = [g_{i_1}, \dots, g_{i_k}]$, $d = [g_{j_1}, \dots, g_{j_m}]$ and $\{i_1, \dots, i_k\} \subset \{j_1, \dots, j_m\}$. We know that $E(G') = \{\, cd\,,\ c, d \in \mathscr{C}(G) \text{ with } c \subset d \text{ or } d \subset c\}$. To prove $[G]_s = [G']_s$, we go from G to G' in two steps (addition and suppression of s-dismantlable vertices).

First step: For every $c \in \mathscr{C}(G)$, we add a vertex $\widehat{c}$ to G. We begin with complete subgraphs of cardinal 1, we proceed with complete subgraphs of cardinal 2, next with complete subgraphs of cardinal 3... until we have reached all complete subgraphs. When we add a vertex $\widehat{c}$ corresponding to the complete subgraph $c = [g_{i_1}, \dots, g_{i_k}]$ of cardinal k, we add the edges $\widehat{c}\widehat{d}$ if $d \subset c$, $\widehat{c}g_{i_k}$ and $\widehat{c}g_j$ if $j > i_k$ and $c \cup g_j \in \mathscr{C}(G)$; this corresponds to the addition of an s-dismantlable vertex because the open neighborhood of $\widehat{c}$ (when we add it) is a cone on g_{i_k}.

The graph H obtained at the end of the first step is such that $V(H) = V(G) \cup V(G')$.

Second step: We note that g_1 is s-dismantlable in H (because $N_H(g_1) = \{\widehat{[g_1]}\} \cup N_G(g_1)$ is a cone on $\widehat{[g_1]}$) and, more generally, for $2 \leq i \leq n$, let us verify that the vertex g_i is s-dismantlable in $H_i = H \setminus \{g_1, \dots, g_{i-1}\}$. We have $W_i = N_{H_i}(g_i) = \{\widehat{c}\,,\ c \in \mathscr{C}(G), c \cup g_i \in \mathscr{C}(G) \text{ and } \max(c) \leq i\} \cup \{g_j\,,\ g_j \in N_G(g_i) \text{ and } j > i\}$. Let us denote $W'_i = \{\widehat{c}\,, c \in \mathscr{C}(G) \text{ and } \max(c) = i\} \cup \{g_j\,,\ g_j \in N_G(g_i) \text{ and } j > i\}$, the cone on $\widehat{[g_i]}$. We have either $W_i = W'_i$, or W_i is dismantlable on W'_i. Indeed, let us suppose $W_i \neq W'_i$ and let $\widehat{c}$ such that $c \cup g_i \in \mathscr{C}(G)$ and $\max(c) < i$. Then $\widehat{c \cup g_i} \in W_i$ and $N_{W_i}(\widehat{c}) \subseteq N_{W_i}(\widehat{c \cup g_i})$; so, $\widehat{c}$ is dismantlable in W_i and more generally W_i is dismantlable on W'_i. Consequently, W_i is dismantlable, *i.e.* g_i is s-dismantlable in H_i. Thus, in H, one can s-delete all vertices of G (in the following order: g_1, g_2, …, g_n); the resultant graph is G'. □

2.4 Correspondence of homotopy classes

Theorem 2.10 *1. Let $G, H \in \mathscr{G}$; G and H have the same s-homotopy type if, and only if, $\Delta_{\mathscr{G}}(G)$ and $\Delta_{\mathscr{G}}(H)$ have the same simple-homotopy type:*

$$[G]_s = [H]_s \Longleftrightarrow [\Delta_{\mathscr{G}}(G)]_s = [\Delta_{\mathscr{G}}(H)]_s$$

2. Let $K, L \in \mathscr{K}$; K and L have the same simple-homotopy type if, and only if, $\Gamma(K)$ and $\Gamma(L)$ have the same s-homotopy type:

$$[K]_s = [L]_s \Longleftrightarrow [\Gamma(K)]_s = [\Gamma(L)]_s$$

proof : 1. $\underline{\Longrightarrow :}$ corollary of Proposition 2.3. $\underline{\Longleftarrow :}$ By the Proposition 2.6, we get $[\Delta_{\mathscr{G}}(G)]_s = [\Delta_{\mathscr{G}}(H)]_s \Longrightarrow [G']_s = [\Gamma(\Delta_{\mathscr{G}}(G))]_s = [\Gamma(\Delta_{\mathscr{G}}(H))]_s = [H']_s$ and we conclude with the Proposition 2.9.

2. $\Longrightarrow$: corollary of Proposition 2.6. $\Longleftarrow$: By using assertion 1 of the theorem, we obtain $[\Gamma(K)]_s = [\Gamma(L)]_s \Longrightarrow [\Delta_{\mathscr{G}}(\Gamma(K))]_s = [\Delta_{\mathscr{G}}(\Gamma(L))]_s$. So, we get $[K']_s = [L']_s$ and we can conclude $[K]_s = [L]_s$ because it is well known that a simplicial complex and its barycentric subdivision have the same simple-homotopy type ([6]). $\square$

Remark 2.11 *It is clear that an s-collapsible graph is s-dismantlable (i.e., $G \searrow^s pt \Rightarrow [G]_s = [pt]_s$) and it follows from Proposition 2.3 and Theorem 2.10 that there exists s-dismantlable and not s-collapsible graphs. Indeed, there are well known examples of simplicial complexes K (triangulations of the dunce hat ([20]) or of the Bing's house ([5]), for instance) which are not collapsible ($K \not\searrow^s pt$) but have the same simple-homotopy type of a point ($[K]_s = [pt]_s$). So, for example, the graph D of Fig. 6 (with 17 vertices and 36 triangles) is s-dismantlable but not s-collapsible because $\Delta_{\mathscr{G}}(D)$ is a triangulation of the dunce hat. This graph D is actually the comparability graph of a poset given in [18, Figure 2, p.380] and named here P_d. An example of a graph B such that $\Delta_{\mathscr{G}}(B)$ is a triangulation of the Bing's house is given in [5, §5].*

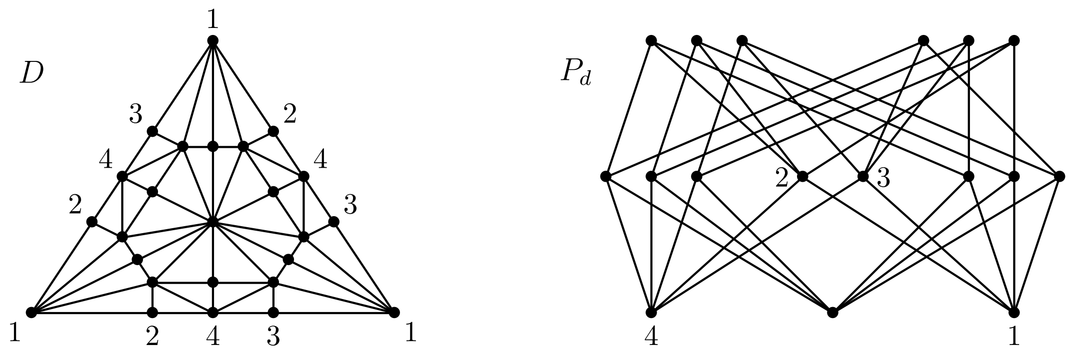

Figure 6: $\Delta_{\mathscr{G}}(D)$ is a triangulation of the dunce hat: $D = Comp(P_d)$

3 Relation with posets

3.1 From $\mathscr{P}$ to $\mathscr{G}$

Let $\mathscr{P}$ be the set of finite partially ordered sets or finite posets. In what follows, when $P \subseteq Q$ with $Q \in \mathscr{P}$, P will be called *subposet of* Q if, for every x, y in P, $x \leq_P y \Longleftrightarrow x \leq_Q y$. If $P \in \mathscr{P}$, $Comp(P) \in \mathscr{G}$ is the comparability graph of P (its vertices are the elements of P with an edge xy if, and only if, x and y are comparable).

Let $P \in \mathscr{P}$. For every x in P, we define $P_{<x} := \{y \in P\,,\, y < x\}$ and $P_{>x} := \{y \in P\,,\, y > x\}$. We recall that x is *irreducible*[1] either if $P_{<x}$ has a maximum, or if $P_{>x}$ has a minimum. The poset P is called *dismantlable* if we can write $P = \{x_1, x_2, \ldots, x_n\}$ with x_i irreducible in the subposet induced by $\{x_1, \ldots, x_i\}$, for $2 \leq i \leq n$. Let us recall that a *cone* is a poset having a maximum or a minimum; if we can write $P = P_{\geq x}$ or $P = P_{\leq x}$ for some x in P, P will be called a cone *on* x. Cones are examples of dismantlable posets.

If $P, Q \in \mathscr{P}$, $P * Q$ is the poset whose elements are those of P and Q and with the relations $p \leq_P p'$, $q \leq_Q q'$ and $p \leq q$ for all $p, p' \in P$ and $q, q' \in Q$. In particular, $P * \emptyset = \emptyset * P = P$ for all $P \in \mathscr{P}$.

Lemma 3.1 *Let $P, Q \in \mathscr{P}$; $P * Q$ is dismantlable if, and only if, P or Q is dismantlable.*

proof : Let us suppose that $P * Q$ is dismantlable with $P * Q = \{x_1, x_2, \ldots, x_N\}$ (where $N = |P| + |Q|$ and x_i is irreducible in the subposet of $P * Q$ induced by $\{x_1, \ldots, x_i\}$, for $2 \leq i \leq N$) and that Q is a non dismantlable poset. We can write $P = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ (where $k = |P|$) with $i_j < i_l$ for all $1 \leq j < l \leq k$. We shall verify that P is dismantlable with x_{i_l} irreducible in the subposet of P induced by $P \setminus \{x_{i_{l+1}}, \ldots, x_{i_k}\} = \{x_{i_1}, \ldots, x_{i_l}\}$, for $2 \leq l \leq k$. We know that x_{i_l} is irreducible in $(P * Q) \setminus \{x_{i_l+1}, x_{i_l+2}, \ldots, x_N\}$.

[1] It seems to be the most classical terminology ([15],[3],[8]); in [2], irreducible points are called *(up* or *down) beat points.*

- First case: a maximum of $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{<x_{i_l}}$ is also a maximum of $\big(P\setminus\{x_{i_{l+1}},x_{i_{l+2}},\dots,x_{i_k}\}\big)_{<x_{i_l}}$ (this follows from $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{<x_{i_l}}=\big(P\setminus\{x_{i_{l+1}},x_{i_{l+2}},\dots,x_{i_k}\}\big)_{<x_{i_l}}$).
- Second case: a minimum of $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{>x_{i_l}}$ is also a minimum of $\big(P\setminus\{x_{i_{l+1}},x_{i_{l+1}},\dots,x_{i_k}\}\big)_{>x_{i_l}}$. This follows from $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{>x_{i_l}}=\big(P\setminus\{x_{i_{l+1}},x_{i_{l+2}},\dots,x_{i_k}\}\big)_{>x_{i_l}}*Q'$ where Q' is a subposet of Q and the fact that $\big(P\setminus\{x_{i_{l+1}},x_{i_{l+1}},\dots,x_{i_k}\}\big)_{>x_{i_l}}\neq\emptyset$. Indeed, $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{>x_{i_l}}=Q'$ would mean that Q' is dismantlable (because $\big((P*Q)\setminus\{x_{i_l+1},x_{i_l+2},\dots,x_N\}\big)_{>x_{i_l}}$ has a minimum and, thus, is dismantlable) which contradicts the fact that Q is non dismantlable (because Q' is a subposet of Q obtained by suppression of irreducible elements).

In conclusion, x_{i_l} is irreducible in $\{x_{i_1},\dots,x_{i_l}\}$, for $2\leq l\leq k$ and P is dismantlable.

Reciprocally, let us suppose that P is a dismantlable poset and that we have $P=\{x_1,x_2,\dots,x_n\}$ with x_i irreducible in the subposet induced by $\{x_1,\dots,x_i\}$, for $2\leq i\leq n$. By the equalities $(P*Q)_{>p}=P_{>p}*Q$ and $(P*Q)_{<p}=P_{<p}$, we see that x_i is irreducible in the subposet of $P*Q$ induced by $\{x_1,\dots,x_i\}*Q$ if x_i is irreducible in the subposet of P induced by $\{x_1,\dots,x_i\}$. As a consequence, we can go from $P*Q$ to $\{x_1\}*Q$ by successive suppressions of irreducibles and this shows that $P*Q$ is dismantlable (because $\{x_1\}*Q$ is a cone). A similar argument yields that $P*Q$ is dismantlable if we suppose Q dismantlable. $\square$

In [2], Barmak and Minian introduce the notion of *weak points* in a poset: $x\in P$ is a *weak point* if $P_{<x}$ or $P_{>x}$ is dismantlable. So, by Lemma 3.1, x is a weak point if, and only if, $P_{>x}*P_{<x}$ is dismantlable. Now, it is well known ([3],[8]) that a poset P is a dismantlable poset if, and only if, $Comp(P)$ is a dismantlable graph. As we have $N_{Comp(P)}(x)=Comp(P_{>x}*P_{<x})$, we obtain the following result:

Proposition 3.2 *Let $P\in\mathscr{P}$ and $x\in P$. Then, x is a weak point of P if, and only if, x is s-dismantlable in $Comp(P)$.*

The notation $P\searrow\!\!\!{}^{s}\, P\setminus\{x\}$ will mean that x is a weak point of P and we shall write $P\searrow\!\!\!{}^{s}\, Q$ if Q is a subposet of P obtained by successive deletions of weak points.

3.2 From $\mathscr{G}$ to $\mathscr{P}$

If $G\in\mathscr{G}$, $C(G)\in\mathscr{P}$ is the poset whose elements are the complete subgraphs of G ordered by inclusion. Before establishing the relation between reduction by s-dismantlable vertices in $\mathscr{G}$ and deletion of weak points in $\mathscr{P}$, we recall that the poset product $P\times Q$ of two posets P and Q is the set $P\times Q$ ordered by $(p,q)\leq(p',q')$ if $p\leq_P p'$ and $q\leq_Q q'$ for all $(p,q),(p',q')\in P\times Q$. In particular, $P\times\{a,b,a<b\}$ is the poset formed by two copies of P (namely, $P_a:=P\times\{a\}=\{(p,a),p\in P\}$ and $P_b:=P\times\{b\}=\{(p,b),p\in P\}$) with relations of P in the two copies P_a and P_b and the additional relations $(p,a)\leq(p',b)$ if $p\leq_P p'$.

Lemma 3.3 *Let $P\in\mathscr{P}$ and $S\in\mathscr{P}$ such that S contains $W:=P\times\{a,b,a<b\}$ as a subposet with the two following properties:*

$$\forall p\in P\,,\qquad i)\ S_{<(p,b)}=W_{<(p,b)}\qquad\text{and}\qquad ii)\ S_{>(p,b)}=W_{>(p,b)}$$

Let Q be the poset obtained from S by adding an element x (not in S) with the only relations $x<(p,b)$ for all $p\in P$.

If P is a dismantlable poset, then $Q\searrow\!\!\!{}^{s}\, Q\setminus\{x,(p,b)\,;\,p\in P\}$.

proof : By definition of Q, we have $x\leq_Q y$ if, and only if, $y\in P_b=\{(p,b),p\in P\}$. Of course, P_b is isomorphic to P and, if we suppose that P is dismantlable, this means that x is a weak point in Q, *i.e.* $Q\searrow\!\!\!{}^{s}\, S=Q\setminus\{x\}$.

Now, let p be an irreducible element in P; we shall verify that (p,b) is a weak point in S.

- First case: we suppose that $P_{<p}$ has a maximum element M. We get

$$\begin{array}{rcl} S_{<(p,b)} \overset{i)}{=} W_{<(p,b)} & = & \{(p',b), p'<p\} \bigcup \{(p',a), p' \leq p\} \\ & = & \{(p',b), p'<p\} \bigcup \{(p',a), p' \leq M\} \bigcup \{(p,a)\} \end{array}$$

  In $S_{<(p,b)}$, we have $y < (p,a) \Leftrightarrow y \leq (M,a)$; in other words, (M,a) is a maximum element of $\left(S_{<(p,b)}\right)_{<(p,a)}$ and this shows that (p,a) is an irreducible point in $S_{<(p,b)}$. Now, $S_{<(p,b)} \setminus \{(p,a)\} = \{(p',b), p'<p\} \bigcup \{(p',a), p' \leq M\}$ is a cone on (M,b) (because $y \leq (M,b)$ for all y in $S_{<(p,b)} \setminus \{(p,a)\}$) and we can conclude that $S_{<(p,b)}$ is a dismantlable poset.

- Second case: we suppose that $P_{>p}$ has a minimum element m, then $S_{>(p,b)} \overset{ii)}{=} W_{>(p,b)}$ which is a poset isomorphic to $P_{>p}$; so, it is dismantlable (because it is a cone).

The conclusion of the two cases is that (p,b) is a weak point in S; so, we have $S \overset{s}{\searrow} S \setminus \{(p,b)\}$.

Now, let us suppose that P is dismantlable with $P = \{p_1, p_2, \ldots, p_n\}$ with p_i irreducible in the subposet induced by $\{p_1, \ldots, p_i\}$, for $2 \leq i \leq n$. By iterating the preceding discussion we get

$$\begin{array}{rcl} Q \overset{s}{\searrow} Q \setminus \{x\} \overset{s}{\searrow} Q \setminus \{x, (p_n,b)\} & \overset{s}{\searrow} & Q \setminus \{x, (p_n,b), (p_{n-1},b)\} \\ & \overset{s}{\searrow} & \ldots \overset{s}{\searrow} Q \setminus \{x, (p_n,b), (p_{n-1},b), \ldots, (p_2,b)\} \end{array}$$

By condition $i)$, we see that $y < (p_1,b)$ in $Q \setminus \{x, (p_n,b), (p_{n-1},b), \ldots, (p_2,b)\}$ if, and only if, $y \leq (p_1,a)$; so, (p_1,b) is a weak point in $Q \setminus \{x, (p_n,b), (p_{n-1},b), \ldots, (p_2,b)\}$ (in fact, it is irreducible) and we have proved

$$Q \overset{s}{\searrow} Q \setminus \{x, (p_n,b), \ldots, (p_2,b)\} \overset{s}{\searrow} Q \setminus \{x, (p_n,b), \ldots, (p_2,b), (p_1,b)\}.$$

$\square$

Proposition 3.4 *If g is s-dismantlable in G, then $C(G) \overset{s}{\searrow} C(G \setminus g)$.*

proof : We apply Lemma 3.3 with $Q = C(G)$ and $P = C(N_G(g))$. More precisely, with the notations of this lemma, we have $x = [g]$, $S = C(G) \setminus \{[g]\}$ and $W = C(N_G[g]) \setminus \{[g]\}$. If we denote by g_i the elements of $N_G(g)$, the isomorphism between $W = C(N_G[g]) \setminus \{[g]\}$ and $P \times \{a, b, a<b\} = C(N_G(g)) \times \{a, b, a<b\}$ is given by identifying $([g_1, \ldots, g_n], a) \in P \times \{a, b, a < b\}$ with $[g_1, \ldots, g_n] \in W$ and $([g_1, \ldots, g_n], b) \in P \times \{a, b, a<b\}$ with $[g, g_1, \ldots, g_n] \in W$. Conditions $i)$ and $ii)$ are clearly verified.

By supposing g s-dismantlable in G, we get that $N_G(g)$ is a dismantlable graph and $P = C(N_G(g))$ a dismantlable poset by [8, Lemma 2.2]. So, by Lemma 3.3, we obtain

$$C(G) \overset{s}{\searrow} C(G) \setminus \left(\{[g]\} \cup \{[g, g_1, \ldots, g_n], [g_1, \ldots, g_n] \in C(N_G(g))\}\right) = C(G \setminus g)$$

$\square$

3.3 Correspondence between s-homotopy and simple equivalence

A poset P is said ([2, Definition 3.4]) *simply equivalent* to the poset Q if we can transform P to Q by a finite sequence of additions or deletions of weak points. We denote by $[P]_s$ the equivalence class of P for this relation (and call it the *simple type* of P).

Theorem 3.5 *1. Let $P, Q \in \mathscr{P}$; $[P]_s = [Q]_s$ (in $\mathscr{P}$) $\Longleftrightarrow$ $[Comp(P)]_s = [Comp(Q)]_s$ (in $\mathscr{G}$).*
2. Let $G, H \in \mathscr{G}$; $[G]_s = [H]_s$ (in $\mathscr{G}$) $\Longleftrightarrow$ $[C(G)]_s = [C(H)]_s$ (in $\mathscr{P}$).

proof : 1. The equivalence is a direct consequence of Proposition 3.2.

2. $\underline{\Longrightarrow}$: Follows from Proposition 3.4.

$\underline{\Longleftarrow}$: If $[C(G)]_s = [C(H)]_s$, we get $[Comp(C(G))]_s = [Comp(C(H))]_s$ by the first assertion of the theorem. As $Comp \circ C = Bd$, we have $[G']_s = [H']_s$ and the conclusion follows from Proposition 2.9. $\square$

3.4 The triangle $(\mathscr{G}, \mathscr{P}, \mathscr{K})$

Let us recall that in $\mathscr{P}$ there is also a notion of barycentric subdivision $Bd : \mathscr{P} \to \mathscr{P}$ (for a poset P, $Bd(P) = P'$ is given by the chains of P ordered by inclusion of underlying sets). There is also two classical applications, $\Delta_{\mathscr{P}} : \mathscr{P} \to \mathscr{K}$ (the simplices of $\Delta_{\mathscr{P}}(P)$, the *order complex* of P, are the $< x_0, x_1, \ldots, x_n >$ for every chain $x_0 < x_1 < \ldots < x_n$ of P) and $\Pi : \mathscr{K} \to \mathscr{P}$ (the elements of $\Pi(K)$, the *face poset* of K, are the simplices of K ordered by inclusion). Thus, we get the triangle $(\mathscr{G}, \mathscr{P}, \mathscr{K})$ given in Fig. 7.

$$\begin{tikzcd}
\mathscr{P} \arrow[dr, shift left, "\Delta_{\mathscr{P}}"'] \arrow[dd, shift right, "Comp"'] & \\
& \mathscr{K} \arrow[ul, shift left, "\Pi"'] \arrow[dl, shift left, "\Gamma"] \\
\mathscr{G} \arrow[ur, shift left, "\Delta_{\mathscr{G}}"] \arrow[uu, shift right, "C"'] &
\end{tikzcd}$$

Figure 7: The triangle $(\mathscr{G}, \mathscr{P}, \mathscr{K})$

Let us list some easy properties of this triangle[2]:

Proposition 3.6 *1.* $\Pi \circ \Delta_{\mathscr{P}} = C \circ Comp = Bd$ *(in* $\mathscr{P}$*),* $\Delta_{\mathscr{P}} \circ \Pi = \Delta_{\mathscr{G}} \circ \Gamma = Bd$ *(in* $\mathscr{K}$*),* $Comp \circ C = \Gamma \circ \Delta_{\mathscr{G}} = Bd$ *(in* $\mathscr{G}$*).*

2. We have the " commutative triangles" : $\Delta_{\mathscr{P}} = \Delta_{\mathscr{G}} \circ Comp$*,* $C = \Pi \circ \Delta_{\mathscr{G}}$ *and* $\Gamma = Comp \circ \Pi$*.*

3. We have the " commutative triangles up to subdivision" : $\Gamma \circ \Delta_{\mathscr{P}} = Bd \circ Comp$*,* $\Delta_{\mathscr{P}} \circ C = Bd \circ \Delta_{\mathscr{G}}$ *et* $C \circ \Gamma = Bd \circ \Pi$*.*

Now from Theorem 2.10 and Theorem 3.5, we get another proof of the Theorem (part of [2, First main Theorem 3.9]):

Theorem 3.7 *1. Let* $P, Q \in \mathscr{P}$*. Then* P *and* Q *are simply equivalent if, and only if,* $\Delta_{\mathscr{P}}(P)$ *and* $\Delta_{\mathscr{P}}(Q)$ *have the same simple-homotopy type.*

2. Let $K, L \in \mathscr{K}$*. Then* K *and* L *have the same simple-homotopy type if, and only if,* $\Pi(K)$ *and* $\Pi(L)$ *are simply equivalent.*

Remark 3.8 *The image of* $\Delta_{\mathscr{G}}$ *is exactly the set of flag complexes but not all flag complexes are in the image of* $\Delta_{\mathscr{P}}$ *(for example, the cyclic graph* C_5 *with 5 vertices may be considered as a flag complex and is not in the image of* $\Delta_{\mathscr{P}}$*; equivalently,* C_5 *is not a comparability graph).*

4 The weak-s-dismantlability

Definition 4.1 *Let* $G \in \mathscr{G}$*. An edge* gg' *of* G *will be called* s-dismantlable *if* $N_G(g) \cap N_G(g')$ *is nonempty and dismantlable.*

We shall say that $G \searrow^{ws} H$ if we can go from G to H either by deleting s-dismantlable vertices or by deleting s-dismantlable edges.

Definition 4.2 *Two graphs* G *and* H *have the same ws-homotopy type if there is a sequence* $G = J_1, \ldots, J_k = H$ *in* $\mathscr{G}$ *such that* $G = J_1 \stackrel{ws}{\to} J_2 \stackrel{ws}{\to} \ldots \stackrel{ws}{\to} J_{k-1} \stackrel{ws}{\to} J_k = H$ *where each arrow* $\stackrel{ws}{\to}$ *represents either the suppression or the addition of an s-dismantlable vertex, or the suppression or the addition of an s-dismantlable edge.*

[2]In fact, we can consider $\mathscr{G}$, $\mathscr{K}$ and $\mathscr{P}$ as categories (with obvious morphisms) and it is easy to verify that all applications in the triangle $(\mathscr{G}, \mathscr{P}, \mathscr{K})$ are covariant functors. The reader not acquainted with the notions of functors or categories may refer to book [12].

This defines an equivalence relation in $\mathscr{G}$ and we shall denote by $[G]_{ws}$ the equivalence class representing the ws-homotopy type of a graph G. Of course, $G \searrow\!\!\!\!{}^{s} H$ implies $G \searrow^{ws} H$ and the example given in remark 2.4 shows that the reverse implication is false in general; nevertheless, s-homotopy type and ws-homotopy type are the same:

Proposition 4.3 *For every $G \in \mathscr{G}$, we have $[G]_s = [G]_{ws}$.*

proof : The inclusion $[G]_s \subseteq [G]_{ws}$ follows from $G \searrow^{s} H \Longrightarrow G \searrow^{ws} H$. Now, we have seen (Lemma 1.6) that the deletion of an s-dismantlable edge corresponds to the addition of an s-dismantlable vertex followed by the suppression of an s-dismantlable vertex. This means that a sequence $G = J_1 \stackrel{ws}{\to} J_2 \stackrel{ws}{\to} \dots \stackrel{ws}{\to} J_{k-1} \stackrel{ws}{\to} J_k = H$ can be rewritten as a sequence from G to H using only suppressions and additions of s-dismantlable vertices and proves that $[G]_s \supseteq [G]_{ws}$. $\square$

Actually, the weak-s-dismantlability behaves well with the map $\Delta_{\mathscr{G}}$. The 1-skeleton of a simplicial complex can be considered as a graph (whose vertices are given by the 0-simplices and the edges are given by the 1-simplices). Following the notation of [5], this defines a map $sk : \mathscr{K} \to \mathscr{G}$ (if K a simplicial complex, $sk(K)$ is its 1-skeleton taken as a graph). We note that $sk(\Delta_{\mathscr{G}}(G)) = G$ for all $G \in \mathscr{G}$. We have:

Lemma 4.4 *Let $K, L \in \mathscr{K}$. Then, $K \searrow^{s} L \Longrightarrow \big(sk(K) \searrow^{ws} sk(L)$ or $sk(K) = sk(L)\big)$.*

proof : The simplicial collapse $K \searrow^{s} L$ says that we obtain L by deleting successively various pairs of simplices $\{\sigma, \tau\}$ where τ is a free face of σ, so it is sufficient to prove that $sk(K) \searrow^{ws} sk(K \setminus \{\sigma, \tau\})$ for an elementary collapse $K \searrow^{s} (K \setminus \{\sigma, \tau\})$. Of course, if τ is a k-simplex, then σ is a $(k+1)$-simplex and we consider the three cases $k = 0$, $k = 1$ and $k \geq 2$. If $k = 0$, $\tau =< a >$ is a vertex (or 0-simplex) belonging to a unique 1-simplex $\sigma =< a, b >$. In this case, we have $sk(K) \searrow^{s} (sk(K) \setminus a)$ because a is a vertex dominated by the vertex b in $sk(K)$ and $sk(K) \searrow^{s} sk(K \setminus \{\sigma, \tau\}) = sk(K) \setminus a$ (and also $sk(K) \searrow^{ws} sk(K \setminus \{\sigma, \tau\})$. If $k = 1$, $\tau =< a, b >$ is a 1-simplex such that there is a unique vertex c such that $< a, b, c >$ is a 2-simplex (named σ). It follows that c is the unique vertex of $sk(K)$ adjacent to a and b; in other terms, $N_{sk(K)}(a) \cap N_{sk(K)}(b)$ is reduced to the vertex c and this shows that the edge ab is s-dismantlable in $sk(K)$. Now, it is clear that $sk(K \setminus \{\sigma, \tau\}) = sk(K \setminus \{< a, b, c >, < a, b >\}) = sk(K \setminus \{< a, b >\}) = sk(K) \setminus ab$, so we obtain $sk(K) \searrow^{ws} sk(K \setminus \{\sigma, \tau\}) = sk(K) \setminus ab$. Finally, if $k \geq 2$, the suppression of the pair $\{\sigma, \tau\}$ in K does not affect the 1-skeleton of K, *i.e.* $sk(K) = sk(K \setminus \{\sigma, \tau\})$. $\square$

Proposition 4.5 *Let $G, H \in \mathscr{G}$. Then, $G \searrow^{ws} H \Longleftrightarrow \Delta_{\mathscr{G}}(G) \searrow^{s} \Delta_{\mathscr{G}}(H)$.*

proof : By replacing the 0-simplex $< g >$ by the 1-simplex $< g, g' >$ in the proof of Proposition 2.3, we get $\Delta_{\mathscr{G}}(G) \searrow^{s} \Delta_{\mathscr{G}}(G \setminus gg')$ when the edge gg' is *s-dismantlable* in G. This shows that $G \searrow^{ws} H$ implies $\Delta_{\mathscr{G}}(G) \searrow^{s} \Delta_{\mathscr{G}}(H)$. The reverse inclusion follows from Lemma 4.4 and the fact that $sk(\Delta_{\mathscr{G}}(G)) = G$ for all $G \in \mathscr{G}$. $\square$

It is important to note that we can find a graph G whose vertices and edges are all non s-dismantlable and such that $\Delta_{\mathscr{G}}(G)$ collapses on a strict subcomplex which does not admit any collapsible pair and which is not a flag subcomplex; the 6-regular graph given in appendix provides such an example.

5 Relation with graph homotopy of Chen, Yau and Yeh

In [9], Ivashchenko introduces the notion of *contractible transformations* and calls *contractible* the trivial graph (the graph reduced to a point) and every graph obtained from the trivial graph by application of these contractible transformations. In what follows, to avoid any confusion, we call *I-contractibility* the contractibility in the sense of Ivashchenko. In a graph G, the contractible transformations are the deletion of a vertex g if $N_G(g)$ is I-contractible, the deletion of an edge gg' if $N_G(g) \cap N_G(g')$ is I-contractible, the addition of a vertex x if $N_{G \cup x}(x)$ is I-contractible and the addition of an edge between g and g' if $g \not\sim g'$ and $N_G(g) \cap N_G(g')$ is I-contractible. From these operations, in [5], the authors introduce the *graph homotopy type* of a graph G that we shall call here *I-homotopy type*. Let us say that a vertex g is I-dismantlable if $N_G(g)$ is I-contractible. The [5, Lemma 3.4] shows that we can reduce the four operations above to the two operations of deletion or addition of I-dismantlable vertices. Thus, one can define $G \searrow^{I} H$ as the passage from G to H by suppression of I-dismantlable vertices and $G \nearrow^{I} H$ as the passage from G to H by addition of I-dismantlable vertices. From this, we get the I-equivalence class of a graph (similarly to definition 1.3, we say that two graphs G and H have the same I-homotopy type if there is a sequence $G = J_1, \ldots, J_k = H$ in $\mathscr{G}$ such that $G = J_1 \xrightarrow{I} J_2 \xrightarrow{I} \ldots \xrightarrow{I} J_{k-1} \xrightarrow{I} J_k = H$ where each arrow $\xrightarrow{I}$ represents the suppression of a I-dismantlable vertex or the addition of a I-dismantlable vertex). So, $[G]_I$ denotes the I-homotopy type of a graph G, *i.e.* the graph homotopy type of G in the terminology of [5]. In that way, G is I-contractible if, and only if, $[G]_I = [pt]_I$.

Proposition 5.1 *Let $g \in G$.*

1. *If g is s-dismantlable, then g is I-dismantlable.*
2. *If G is s-dismantlable (i.e. $[G]_s = [pt]_s$), then G is I-contractible.*

proof : 1. Suppose that g is s-dismantlable in G; then $N_G(g)$ is dismantlable. But from the definition of I-contractibility, it is clear that "dismantlable $\Longrightarrow$ I-contractible" (the open neighborhood of a dismantlable vertex is a cone and this proves that a dismantlable vertex is I-dismantlable); thus $N_G(g)$ is I-contractible, i.e, g is I-dismantlable.

2. It is a consequence of the assertion 1. □

It also follows from the assertion 1 of Proposition 5.1 that if two graphs G and H have the same s-homotopy type, then they have the same I-homotopy type. We are unaware if the converse is true:

Question: *Let $G \in \mathscr{G}$. Are the s-homotopy type of G and the I-homotopy type of G identical ?*

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Appendix: A particular non ws-reducible graph

Some properties of the graph G of Fig. 8:

1. G is a 6-regular graph with $|V(G)| = 10$ and $|E(G)| = 30$.
2. For every vertex a of G, the subgraph $N_G(a)$ is not dismantlable because it is isomorphic to the graph shown in figure 8.b.
3. For every edge ab of G, the subgraph $N_G(a) \cap N_G(b)$ is not dismantlable because it is isomorphic to the (disconnected) graph shown in figure 8.c. So there is no s-dismantlable edge in G.
4. The cliques (or *maximal complete subgraphs*) of G are of order 4 or 3. The cliques of order 3 are

   $[1,2,3]$ $[1,5,6]$ $[2,4,6]$ $[1,8,9]$ $[2,7,9]$ $[3,4,5]$ $[3,7,8]$ $[x,5,8]$ $[x,6,9]$ $[x,4,7]$

   and the cliques of order 4 are:

   $c_1 = [1,3,5,8]$ $c_2 = [1,2,6,9]$ $c_3 = [2,3,4,7]$ $c_4 = [x,4,5,6]$ $c_5 = [x,7,8,9]$

5. In the simplicial complex $\Delta_{\mathscr{G}}(G)$, there are five tetrahedras σ_i corresponding to the five 4-cliques c_i. Corresponding to each tetrahedron σ_i, each pair (σ_i, τ) (with τ being any maximal proper face of σ_i) is a collapsible pair.
6. Let K be a subcomplex obtained from $\Delta_{\mathscr{G}}(G)$ after collapsing the five tetrahedras τ_i, $1 \le i \le 5$.

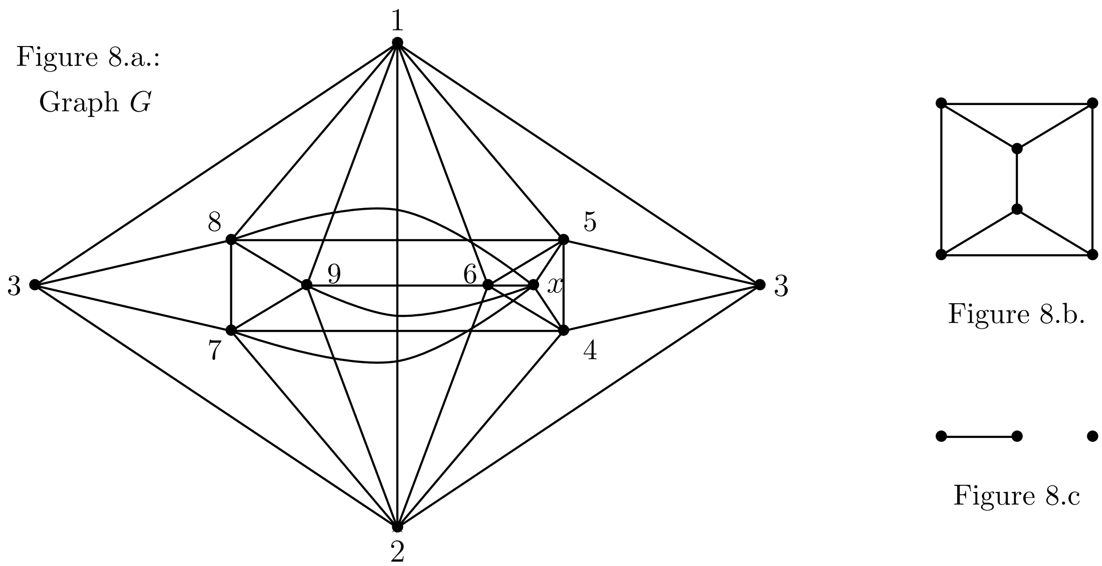

Figure 8.a.: Graph G

Figure 8.b.

Figure 8.c

Figure 8: $\Delta_{\mathscr{G}}(G)$ is non collapsible on a flag subcomplex

- There is no collapsible pair in K. Indeed, a collapsible pair in K must be of the form (σ', τ') with σ' a triangle (or 2-simplex) and τ' an edge of σ' which is not the edge of another triangle. But we see from the lists or 3-cliques and 4-cliques that every edge of G appears exactly once in the list of 3-cliques and exactly once in the list of 4-cliques; so, even after removing 5 collapsible pairs corresponding to the 5 tetrahedras, every edge appears in at least two triangles (and is not a free edge).
- K is not a flag complex. For example, let (σ, τ) be a pair which has been collapsed in $\Delta_{\mathscr{G}}(G)$: σ is a tetrahedron and τ, a maximal proper face of σ, is of the form $< a, b, c >$. So τ is a non-simplex of K with 3 vertices and every face of τ is a simplex of K.

In conclusion:

- G is a non ws-reducible graph (*i.e.*, there is not a strict subgraph H of G such that $G \searrow^{ws} H$).
- We can find a strict subcomplex K of $\Delta_{\mathscr{G}}(G)$ such that $\Delta_{\mathscr{G}}(G) \searrow^{s} K$.
- Every strict subcomplex K of $\Delta_{\mathscr{G}}(G)$ such that $\Delta_{\mathscr{G}}(G) \searrow^{s} K$ is not a flag complex.

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Romain Boulet, Université Toulouse Le Mirail, 5, Allées Antonio Machado, 31058 Toulouse Cedex 9; email:boulet@univ-tlse2.fr

Etienne Fieux, Université Paul Sabatier, 118, Route de Narbonne - F-31062 Toulouse cedex 9 ; email: fieux@math.univ-toulouse.fr

Bertrand Jouve, Université Toulouse Le Mirail, 5, Allées Antonio Machado, 31058 Toulouse Cedex 9; email: jouve@univ-tlse2.fr