Estimation of Multivariate Conditional Tail Expectation using Kendall's Process

Elena Di Bernardino, Clémentine Prieur

Estimation of Multivariate Conditional Tail Expectation using Kendall's Process

Elena Di Bernardino[1], Clémentine Prieur[2]

Abstract

This paper deals with the problem of estimating the Multivariate version of the *Conditional-Tail-Expectation*, proposed by Cousin and Di Bernardino (2012). We propose a new non-parametric estimator for this multivariate risk-measure, which is essentially based on the Kendall's process (see Genest and Rivest, 1993). Using the Central Limit Theorem for the Kendall's process, proved by Barbe *et al.* (1996), we provide a functional Central Limit Theorem for our estimator. We illustrate the practical properties of our estimator on simulations. A real case in environmental framework is also analyzed. The performances of our new estimator are compared to the ones of the level sets-based estimator, previously proposed in Di Bernardino *et al.* (2011).

Keywords: Multivariate Kendall distribution, Multidimensional risk measures, Kendall's process.

Introduction

Multivariate risk-measures

Traditionally, risk measures are thought of as mappings from a set of real-valued random variables to the real numbers. However, it is often insufficient to consider a single real measure to quantify risks, especially when the risk-problem is affected by other external risk factors whose sources cannot be controlled. Note that the evaluation of an individual risk may strongly be affected by the degree of dependence amongst all risks and these risks may also be strongly heterogeneous.

For instance, several hydrological phenomena are described by two or more correlated characteristics. These dependent characteristics should be considered jointly to be more representative of the multivariate nature of the phenomenon. Consequently, probabilities of occurrence of risks cannot be estimated on the basis of univariate analysis. The multivariate hydrological risks literature mainly treated one or more of the following three elements: (1) showing the importance and explaining the usefulness of the multivariate framework, (2) fitting the appropriate multivariate distribution (copula and marginal distributions) in order to model risks and (3) defining and studying multivariate return periods (see Chebana and Ouarda, 2011).

One of the most popular measures in hydrology and climate is undoubtedly the return period. A frequency analysis in hydrology focuses on the estimation of quantities (e.g., flows or annual rainfall) corresponding to a certain return period. It is closely related to the notion of quantile which has therefore been extensively studied in dimension one. For a random variable X that represents the

[1]CNAM, Paris, Département IMATH, 292 rue Saint-Martin, Paris Cedex 03, France. elena.di-bernardino@cnam.fr.
[2]Université Joseph Fourier, Tour IRMA, MOISE-LJK B.P. 53 38041 Grenoble, France. clementine.prieur@imag.fr.

magnitude of an event that occurs at a given time and at a given site, the quantile of order $1-\frac{1}{T}$ expresses the magnitude of the event which is exceeded with a probability equal to $\frac{1}{T}$. T is then called the return period. In finance, or more generally in univariate risk theory the quantile is known as the *Value-at Risk* (Var) and it is defined by

$$Q_X(\alpha) = \inf\{x \in R_+ : F_X(x) \geq \alpha\}, \quad \text{for } \alpha \in (0,1),$$

with F the univariate distribution of event X. A second important univariate risk measure, based on the quantile notion, is the *Conditional-Tail -Expectation* (CTE) defined by

$$\mathrm{CTE}_\alpha(X) = \mathbb{E}[\, X \,|\, X > Q_X(\alpha)\,], \quad \text{for } \alpha \in (0,1).$$

From the years 2000 onward, much research has been devoted to risk measures and many extension to multidimensional settings have been suggested (see, e.g., Jouini *et al.*, 2004; Bentahar, 2006; Embrechts and Puccetti, 2006; Nappo and Spizzichino, 2009; Cousin and Di Bernardino, 2012; Henry *et al.*, 2012).

In the following we deal with the multivariate version of *Conditional-Tail-Expectation*, proposed by Di Bernardino *et al.* (2011), Cousin and Di Bernardino (2012). It is constructed as the conditional expectation of a multivariate random vector given that the latter is located in a particular set corresponding to the α-upper level set of the associated multivariate distribution function (in a bivariate setting see Di Bernardino *et al.*, 2011). In this sense this measure is essentially based on a "multivariate distributional approach". More precisely they define, for $i = 1, \ldots, d$, and for $\alpha \in (0,1)$,

$$\mathrm{CTE}_\alpha(Z_i) = \mathbb{E}[\, Z_i \,|\, F_{\mathbf{Z}}(\mathbf{Z}) \geq \alpha\,] = \mathbb{E}[\, Z_i \,|\, \mathbf{Z} \in L(\alpha)\,], \tag{1}$$

where $\mathbf{Z} = (Z_1, \ldots, Z_d)$ is a non-negative multivariate random vector with distribution function F, $K(\alpha) = \mathbb{P}[F_{\mathbf{Z}}(\mathbf{Z}) \geq \alpha]$ is the associated multivariate Kendall distribution function and where the α-upper level set of F is defined by $L(\alpha) = \{x : F(x) \geq \alpha\}$. In particular, Cousin and Di Bernardino (2012) proved that properties of the Multivariate Conditional Tail-Expectation in (1) turn to be consistent with existing properties on univariate risk measures (positive homogeneity, translation invariance, increasing in risk-level α, *etc*).

Recently, level-curves $\partial L(\alpha)$ (where ∂A denotes the boundary of a set A) and associated level-sets $L(\alpha)$ have been proposed as risk measures in multivariate hydrological models because of their many advantages: they are simple, intuitive, interpretable and probability-based (see Chebana and Ouarda, 2011). de Haan and Huang (1995) model a risk-problem of flood in the bivariate setting using an estimator of level curves $\partial L(\alpha)$ of the bivariate distribution function. Furthermore, as noticed by Embrechts and Puccetti (2006) $\partial L(\alpha)$ can be viewed as a natural multivariate version of the univariate quantile. The interested reader is also referred to Tibiletti (1993), Belzunce *et al.* (2007), Nappo and Spizzichino (2009).

However the multivariate risk measure proposed in (1) can be seen as a more parsimonious and synthetic measure compared with Embrechts and Puccetti (2006)'s approach. Indeed, $\partial L(\alpha)$ is an hyperplane of dimension $d-1$. This choice can be unsuitable when we face real risk problems. Using measure in (1), instead of considering the whole geometric space $L(\alpha)$ corresponding to the α-level set of F, we only focus on the particular point in $\mathbb{R}_+^d$ that matches the conditional expectation of $\mathbf{Z}$ given that $\mathbf{Z}$ falls in $L(\alpha)$. This means that measure in (1) is a real-valued vector with the same dimension as the considered portfolio of risks. The latter feature could be relevant on practical grounds.

Estimating multivariate risk-measure in (1)

The problem of consistent estimation of the univariate quantile based risk-measures (VaR and CTE, see above) has received attention in literature essentially in the univariate case (e.g. see Brazauskas *et al.*, 2008; Necir *et al.*, 2010; Ahn and Shyamalkumar, 2011). The estimation of multivariate risk-measures has received some attention however, it is much less common, due to a number of theoretical and practical reasons. However in the last decade, several generalizations of the classical univariate CTE have been proposed, mainly using as conditioning events the total risk or some univariate extreme risk. This kind of measures are suitable to model risk problems with dependent and homogenous risks. Indeed these measures are based on an arbitrary real-valued aggregate transformation (sum, min, max,...) of risks. We remark that using an aggregate procedure between the risks can be inappropriate to measure risks with heterogeneous characteristics especially in an external risks problem. Some commonly used multivariate CTE measures are:

$$\mathrm{CTE}_{\alpha}^{\mathrm{sum}}(Z_i) = \mathbb{E}[\, Z_i \,|\, S > Q_S(\alpha)\,], \tag{2}$$

$$\mathrm{CTE}_{\alpha}^{\mathrm{min}}(Z_i) = \mathbb{E}[\, Z_i \,|\, Z_{(1)} > Q_{Z_{(1)}}(\alpha)\,], \tag{3}$$

$$\mathrm{CTE}_{\alpha}^{\mathrm{max}}(Z_i) = \mathbb{E}[\, Z_i \,|\, Z_{(d)} > Q_{Z_{(d)}}(\alpha)\,], \tag{4}$$

for $i = 1, \ldots, d$, with $S = Z_1 + \cdots + Z_d$ the total risk, $Z_{(1)} = \min\{Z_1, ..., Z_d\}$ and $Z_{(d)} = \max\{Z_1, ..., Z_d\}$ two extreme risks.

The interested reader is referred to Cai and Li (2005) for further details. For explicit formulas of $\mathrm{CTE}_{\alpha}^{\mathrm{sum}}(Z_i)$ in the case of Fairlie-Gumbel-Morgenstern family of copulas, see Bargès *et al.* (2009). Landsman and Valdez obtain an explicit formula for $\mathrm{CTE}_{\alpha}^{\mathrm{sum}}(Z_i)$ in the case of elliptic distribution functions (see Landsman and Valdez, 2003); Cai and Li in the case of *phase-type* distributions (see Cai and Li, 2005). Furthermore, for a comparison between measures in (2)-(4) and (1) we refer to Cousin and Di Bernardino (2012). In the recent literature, some efforts have been done to provide a consistent estimation of measures in (2)-(4). We refer the interested reader, for instance, to Hua and Joe (2011), Asimit *et al.* (2011).

In this paper we propose a new estimator for the multivariate risk measure defined by (1) above, and introduced by Di Bernardino *et al.* (2011), Cousin and Di Bernardino (2012).

A consistent estimator for $\mathrm{CTE}_{\alpha}(Z_i) = \mathbb{E}[\, Z_i \,|\, \mathbf{Z} \in L(\alpha)\,]$, has already been provided by Di Bernardino *et al.* (2011), who proposed a plug-in estimator based on the consistent estimation of the whole level sets $L(\alpha)$. As the level sets are not compact, their estimation procedure requires the choice of an increasing truncation sequence $(T_n)_{n\geq 1}$. The non-optimal rate of convergence provided by the authors depends on the rate of convergence of $(T_n)_{n\geq 1}$ to infinity. Making the "best choice" for $(T_n)_{n\geq 1}$ is not trivial, and requires the knowledge of the tail behavior of $\mathbf{Z}$, at least in its generic form. The interested reader is referred to Di Bernardino *et al.* (2011) for further details.

Contrarily to this approach, we propose in this paper a new non-parametric estimator for $\mathrm{CTE}_{\alpha}(Z_i) = \mathbb{E}[\, Z_i \,|\, F_{\mathbf{Z}}(\mathbf{Z}) \geq \alpha\,]$, based on the estimation of the Kendall's distribution, i.e. the distribution of the univariate random variable $F_{\mathbf{Z}}(\mathbf{Z})$. For this estimator we prove a functional central limit theorem. The main advantage of our new estimator is that it does not require the calibration of extra parameters or sequences.

Organization of the paper
The paper is organized as follows. In Section 1, we introduce some notation, tools and technical assumptions. In Section 2, we recall the Multivariate *Conditional Tail Expectation*, previously introduced by Cousin and Di Bernardino (2012), and we define our new non-parametric estimator for it. In Section 3 we state our main result, which is a functional central limit theorem for our estimator, and we give a sketch for its proof. The main lines of the proof follow then is in Sections 4 and 5. The practical properties of our estimator are further investigated on simulated data in Section 6; an hydrological real data-set is then analyzed. Finally, the proofs of auxiliary lemmas are postponed to Section 7 and to the Appendix.

1. A central tool: the Kendall's process

A central tool in this paper is the Kendall distribution which is a synthetical way to model dependence in multivariate problems. In this section we first introduce the Kendall's process and fix some notation. Then we introduce the Kendall empirical distribution function whose properties have been studied in Genest and Rivest (1993), Barbe *et al.* (1996).

1.1. Definitions and notation

Let $Z = (Z_1, \ldots, Z_d)$ be a $d-$dimensional random vector, $d \geq 2$. As we will see later on, our study of multivariate risk measures strongly relies on the key concept of Kendall distribution function (or multivariate probability integral transformation), that is, the distribution function of the random variable $F(Z)$, where F is the multivariate distribution of random vector Z. Let $\tilde{F}$ denote the copula associated to F through the relation $F(z) = \widetilde{F}(F_1(z_1), \ldots, F_d(z_d))$.

From now on, the Kendall distribution will be denoted by K, so that $K(t) = \mathbb{P}[F(Z) \leq t]$, for $t \in [0,1]$. We also denote by $\overline{K}(t)$ the survival distribution function of $F(Z)$, i.e., $\overline{K}(t) = \mathbb{P}[F(Z) > t]$. For more details on the multivariate probability integral transformation, the interested reader is referred to Capéraà *et al.*, (1997), Genest and Rivest (2001), Nelsen *et al.* (2003), Genest and Boies (2003), Genest *et al.* (2006) and Belzunce *et al.* (2007).

Remark 1 In contrast to the univariate case, it is not generally true that the distribution function K of $F(Z)$ is uniform on $[0,1]$, even when F is continuous. Note also that it is not possible to characterize the joint distribution F or reconstruct it from the knowledge of K alone, since the latter does not contain any information about the marginal distributions $F_1, \ldots, F_d$ (see Genest and Rivest, 2001). Indeed, as a consequence of Sklar's Theorem, the Kendall distribution only depends on the dependence structure or the copula function $\tilde{F}$ associated with Z (see Sklar, 1959). Thus, we also have $K(t) = \mathbb{P}[\tilde{F}(U) \leq t]$, where $U = (U_1, \ldots, U_d)$ and $U_1 = F_1(Z_1), \ldots, U_d = F_d(Z_d)$, for $t \in [0,1]$.

We now define the notion of *partially strictly increasing* as it will be needed further:

Definition 1.1 *A function $F(x_1, \ldots, x_d)$ is partially strictly increasing on $\mathbb{R}_+^d \setminus \{(0, \ldots, 0)\}$ if for any $i = 1, \ldots, d$, the function of one variable $g(\cdot) = F(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_d)$ is strictly increasing.*

1.2. The Kendall empirical distribution

We now recall the definition of the Kendall empirical distribution, which is a non parametric estimator of the Kendall distribution K, first introduced in Genest and Rivest (1993) (see also Barbe *et al.*, 1996). Let $\{Z_i\}_{i=1}^n$ be a random sample in $\mathbb{R}_+^d$ of size $n \geq 2$ and with joint d-variate distribution function F and marginals $F_1, \ldots, F_d$. Let $V_{i,n}$ stand for the proportion of observations Z_j, $j \neq i$, such that $Z_j \leq Z_i$ componentwise, i.e.,

$$V_{i,n} = \frac{1}{n-1} \sum_{j=1,\, j\neq i}^{n} \mathbf{1}_{\{Z_j \leq Z_i\}}. \tag{5}$$

Let also $F_n(x) = \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{Z_j \leq x\}}$ be the empirical distribution function associated to F. We then define K_n as the empirical distribution function based on the $V_{i,n}$s and K as the distribution function of the random variable $F(Z)$ taking values in $[0,1]$.

In Barbe *et al.* (1996) is proved that, under regularity conditions on K and F, the centered Kendall's process $\sqrt{n}\,(K_n(t) - K(t))$ is asymptotically Gaussian, considering the weak convergence in the space $\mathcal{D}$ of càdlàg functions from $[0,1]$ to $\mathbb{R}$ endowed with the Skorohod topology (see e.g., Billingsley, 1995), and an explicit expression for its limiting covariance function is given (see Theorem 1 in Barbe *et al.*, 1996). More precisely, let us assume that

I: the distribution function $K(t)$ of $F(Z)$ admits a continuous density $k(t)$ on $(0,1]$ that verifies $k(t) = o\left(t^{-1/2}\,\log^{-1/2-\epsilon}\left(\frac{1}{t}\right)\right)$, for some $\epsilon > 0$ as $t \to 0$,

and that

II: there exists a version of the conditional distribution of the vector $U := (F_1(Z_1), ..., F_d(Z_d))$ given $\widetilde{F}(U) = t$ and a countable family $\mathcal{P}$ of partitions $\mathcal{C}$ of $[0,1]^d$ into a finite number of Borel sets satisfying:

$$\inf_{\mathcal{C}\in\mathcal{P}} \max_{C\in\mathcal{C}} \operatorname{diam}(C) = 0,$$

such that for all $C \in \mathcal{C}$ the mapping

$$t \mapsto \eta_t(C) = k(t)\,\mathbb{P}[U \in C \,|\, \widetilde{F}(U) = t]$$

is continuous on $(0,1]$ with $\eta_1(C) = k(1)\mathbf{1}_{\{(1,\ldots,1)\in C\}}$.

Remark 2 Note that $\{\widetilde{F}(U) = t\}$ has Lebesgue-measure zero in $[0,1]$. Then we make sense of Assumption **II** using the limit procedure in Feller (1966), Section 3.2. Furthermore we remark that Assumption **II** is in fact a condition on the (unique) copula function $\tilde{F}$ (see Sklar, 1959).

Remark 3 In Barbe *et al.* (1996), Genest *et al.* (2006) can be found classes of multivariate copulas which satisfy Assumptions **I** and **II** (Archimedean copulas, bivariate extreme copulas, FGM, ...). However at the moment at our knowledge nothing is proved for elliptic or meta-elliptic copulas, even if numerical experiments in Genest *et al.* (2009) tend to show that the limit theorem for the Kendall's process holds true even in such cases.

Remark 4 In the following, *weak convergence* for processes will always be considered in the space $\mathcal{D}$ of càdlàg functions from $[0,1]$ to $\mathbb{R}^k$ endowed with the Skorohod topology, for some $k \in \mathbb{N}^*$.

In view of the definition of the $V_{i,n}$s (see (5)), and using Remark 1, one can write (see Ghoudi *et al.*, 1998):

$$K_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{\widetilde{F}_n(U_i) \leq t + \frac{1-t}{n}\}}.$$

We now state the main result in Barbe *et al.* (1996).

Theorem 1.1 (Theorem 1 in Barbe *et al.*, 1996) *Under Assumptions* **I** *and* **II** *above, the empirical process*

$$\alpha_n(t) \; = \; \sqrt{n}\,(K_n(t) - K(t)) \tag{6}$$

converges in distribution to a continuous Gaussian process α with zero mean and covariance function Γ.

Remark 5 A general formulation for Γ is given in Barbe *et al.* (1996). Its exact form can be provided for specific classes of copulas. Moreover, Barbe *et al.* notice that even for simple copula's structures such as Farlie-Gumbel-Morgenstern copula, it does not seem possible to derive an explicit analytical expression for the covariance $\Gamma(s,t)$.

2. A multivariate risk measure: the *t*-Conditional Tail Expectation

As said in the introduction, one wants to preserve the complete information about the multivariate dependence structure. To this end, we consider the Conditional Tail Expectation introduced in Di Bernardino *et al.* (2011) and Cousin and Di Bernardino (2012).

From now on, we consider non-negative absolutely-continuous random vector[3] $Z = (Z_1, \ldots, Z_d)$ (with respect to Lebesgue measure λ on $\mathbb{R}^d$). We assume moreover the two following conditions:

i) F is a partially strictly increasing multivariate distribution function,
ii) there exists $r > 2$ such that $\mathbb{E}(|Z_j|^r) < \infty$, for $j = 1, \ldots, d$.

These conditions will be called *regularity conditions.*

Remark 6 We remark that under these *regularity conditions* the copula $\widetilde{F}$, associated to the distribution function F, has continuous and strictly positive density function on $(0,1)^d$ (see Section 4 in Tibiletti, 1994).

Remark 7 From Theorem 2 in Barbe *et al.* (1996), the regularity conditions above imply Assumption **II** recalled in Section 1, and it also implies that for all $C \in \mathcal{C}$ (the notation is the same as in the statement of Assumption **II**)

$$t \mapsto \mu_t(C) = k(t)\; \mathbb{E}[Z_j\, \mathbf{1}_{\{U \in C\}} | \tilde{F}(U) = t]$$

is continuous on $(0,1]$ with $\mu_1(C) = k(1)\mathbf{1}_{\{(1,\ldots,1) \in C\}}$.

[3]We restrict ourselves to $\mathbb{R}_+^d$ because usually, in applications, components of $d-$dimensional vectors correspond to random losses and are then valued in $\mathbb{R}_+$. However extensions of our results in the case of multivariate distribution function on the entire space $\mathbb{R}^d$ are possible.

We can now define the Conditional Tail Expectation.

Definition 2.1 *Consider a d–dimensional random vector Z satisfying the regularity conditions with associated copula function $\widetilde{F}$. For $t \in (0,1)$, we define the Multivariate t-Conditional Tail Expectation by*

$$\mathrm{CTE}_t(Z) = \mathbb{E}[Z \,|\, \widetilde{F}(U) \geq t] = \begin{pmatrix} \mathbb{E}[\, Z_1 \,|\, \widetilde{F}(U) \geq t\,], \\ \mathbb{E}[\, Z_2 \,|\, \widetilde{F}(U) \geq t\,], \\ \vdots \\ \mathbb{E}[\, Z_d \,|\, \widetilde{F}(U) \geq t\,]. \end{pmatrix};$$

where $U := (F_1(Z_1), ..., F_d(Z_d))$.

Remark 8 Cousin and Di Bernardino (2012) derive several properties for the multivariate extension of the classical univariate *Conditional Tail Expectation* proposed in Definition 2.1. In particular, it satisfies the homogeneity and the translation invariance properties, as far as a weak version of the monotonicity property, which are required properties for coherent risk measures in the sense of Artzner *et al.* (1999). In the simulation study we will also illustrate two other properties:
- for some family of Archimedean copulas an increase of the dependence parameter θ yields a decrease in each component of $\mathrm{CTE}_t(\mathbf{Z})$,
- $\mathrm{CTE}_t^i(\mathbf{Z})$ is a non-decreasing function of risk level t, for $i \in 1, \ldots, d$.

For further details the interested reader is referred to Sections 3.1-3.5 in Cousin and Di Bernardino (2012).

We propose in this paper a new non parametric estimator for the Conditional Tail Expectation based on the Kendall empirical distribution.

Definition 2.2 *The Kendall-based estimator for the Multivariate t-Conditional Tail Expectation is defined by*

$$\widehat{\mathrm{CTE}}_t(Z) = \begin{pmatrix} \dfrac{\frac{1}{n}\sum_{i=1}^n Z_{1i} 1_{\{\widetilde{F}_n(U_i) \geq t\}}}{1 - K_n(t)} \\ \dfrac{\frac{1}{n}\sum_{i=1}^n Z_{2i} 1_{\{\widetilde{F}_n(U_i) \geq t\}}}{1 - K_n(t)} \\ \vdots \\ \dfrac{\frac{1}{n}\sum_{i=1}^n Z_{di} 1_{\{\widetilde{F}_n(U_i) \geq t\}}}{1 - K_n(t)} \end{pmatrix},$$

where K_n is the empirical Kendall estimator of K.

In the following, we use the following notation:

$$\mathrm{CTE}_t^j(Z) = \mathbb{E}[\, Z_j \,|\, \widetilde{F}(U) \geq t\,] \text{ and } \widehat{\mathrm{CTE}}_t^j(Z) = \frac{\frac{1}{n}\sum_{i=1}^n Z_{ji} 1_{\{\widetilde{F}_n(U_i) \geq t\}}}{1 - K_n(t)}, \text{ for } j = 1, \ldots, d.$$

3. Main results and sketch of the proof

Using Theorem 1.1 recalled in Section 1, we can prove the weak convergence of the Kendall-based process:

$$\alpha_n^{\mathrm{CTE}}(t) := \left(\alpha_{n,1}^{\mathrm{CTE}}(t), \dots \alpha_{n,d}^{\mathrm{CTE}}(t)\right)'$$

with

$$\alpha_{n,j}^{\mathrm{CTE}}(t) = \sqrt{n}\,(\widehat{\mathrm{CTE}}_t^j(Z) - \mathrm{CTE}_t^j(Z)), \qquad j = 1, \dots, d. \tag{7}$$

The first step is the coordinatewise convergence (see Theorem 3.1 below).

Theorem 3.1 *Under regularity conditions i) and ii) and Assumption I, the Kendall based process $\alpha_{n,j}^{\mathrm{CTE}}$, for $j = 1, \dots, d$, converges weakly to a continuous Gaussian process α_j^{CTE} with zero mean and covariance function Γ_{CTE}^j.*

Then, we deduce the main result (see Theorem 3.2 below).

Theorem 3.2 *Under regularity conditions i) and ii) and Assumption I, the Kendall based process α_n^{CTE} converges weakly to a continuous Gaussian process α^{CTE} with zero mean and (cross-)covariance function defined by $\Gamma_{\mathrm{CTE}}^{i,j}(s,t) = Cov(\alpha_i(s), \alpha_j(t))$, $(s,t) \in [0,1]^2$, $i = 1 \dots, d$, $j = 1, \dots, d$.*

Remark 9 Remark that the exact formulation for the (cross-)covariance function is complex. It was already discussed in Remark 5 for the limit covariance function for the centered Kendall's process $\{\alpha_n(t),\, t \in [0,1]\}$.

Proof of Theorem 3.2: Let $j_1 < \ldots < j_l \in \{1, \dots, d\}$, if one wants to prove the convergence of the finite-dimensional distributions of α_n^{CTE} we consider $\sum_{k=j_1}^{j_l} b_k \alpha_{n,k}^{CTE}$. Then the proof is similar to the proof of the convergence of the finite-dimensional distributions of $\alpha_{n,k}^{CTE}$ for some fixed $k \in \{1, \dots, d\}$ (see proof of Theorem 3.1). Now it remains to prove the tightness in $\mathcal{D}\left([0,1], \mathbb{R}^d\right)$ endowed with the Skorohod topology. As each component converges weakly in the Skorohod space $\mathcal{D}([0,1], \mathbb{R})$ to a continuous limit, we get the result. □

Sketch of the proof of Theorem 3.1:

We first write (7) as:

$$\alpha_{n,j}^{\mathrm{CTE}}(t) = \sqrt{n}\left(\frac{\frac{1}{n}\sum_{i=1}^n Z_{j_i}\mathbf{1}_{\{\widetilde{F}_n(U_i)\geq t\}}}{1-K_n(t)} - \frac{\mathbb{E}[\,Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]}{1-K(t)}\right) =$$
$$\sqrt{n}\left(\frac{(1-K(t))\left(\frac{1}{n}\sum_{i=1}^n Z_{j_i}\mathbf{1}_{\{\widetilde{F}_n(U_i)\geq t\}} - \mathbb{E}[\,Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\right) + \mathbb{E}[\,Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,](K_n(t)-K(t))}{(1-K_n(t))\,(1-K(t))}\right) \tag{8}$$

The denominator of (8) converges (in probability) to $(1-K(t))^2 = \overline{K}(t)^2 \in (0,1)$ (see Theorem 1.1). Thus we focus now our attention on the numerator of (8). We write it as $\vartheta_n(t) := \xi_n(t) + \psi_n(t)$, with

$$\xi_n(t) = \sqrt{n}\,\overline{K}(t)\left(\frac{1}{n}\sum_{i=1}^n Z_{j_i}\mathbf{1}_{\{\widetilde{F}_n(U_i)\geq t\}} - \mathbb{E}[\,Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\right)$$

and

$$\psi_n(t) = \sqrt{n}\ \mathbb{E}[\, Z_j\, \mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\,(K_n(t) - K(t)).$$

Notice that the dependence on j has been omitted for sake of simplicity in the notation. Furthermore the process $\xi_n(t)$ may be written as the sum of two subsidiary processes:

$$\zeta_n(t) = \sqrt{n}\,\overline{K}(t)\,\left(\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\left(\mathbf{1}_{\{\widetilde{F}_n(U_i)\geq t\}} - \mathbf{1}_{\{\widetilde{F}(U_i)\geq t\}}\right)\right),$$

and

$$\phi_n(t) = \sqrt{n}\,\overline{K}(t)\,\left(\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\,\mathbf{1}_{\{\widetilde{F}(U_i)\geq t\}} - \mathbb{E}[\, Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\right).$$

Then $\vartheta_n(t) := \zeta_n(t) + \phi_n(t) + \psi_n(t)$. In Section 4 we study separately the convergence of these three subsidiary processes $\psi_n(t)$ (see Section 4.1), $\zeta_n(t)$ (see Section 4.2) and $\phi_n(t)$ (see Section 4.3). We also introduce, following Barbe *et al.* (1996) , the empirical process ν_n defined by

$$\nu_n(A) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{U_i\in A\}} - \mathbb{P}[U\in A]\right),\ \text{with } A\in\mathcal{A},$$

with $\mathcal{A} := \left\{A_{1,z},\, z\in[0,1]^d\right\}\bigcup\{A_{2,t},\, t\in[0,1]\}$ where $A_{1,z} := \{z'\in[0,1]^d\,:\,z'\leq z\}$, and $A_{2,t} := \{z\in[0,1]^d\,:\,\widetilde{F}(z)\leq t\}$. Then $\mathcal{A}$ is a Vapnik-Cervonenkis class, and it implies that ν_n converges weakly to a centered Gaussian process ν over $\mathcal{A}$.

4. Convergence results for subsidiary processes

In this section we study the asymptotic behavior of each of the subsidiary processes. In the following we always assume that the regularity conditions i) and ii) as far as Assumption **I** are satisfied.

4.1. Asymptotic behavior of $\psi_n(t)$

Lemma 4.1 *Under regularity conditions i) and ii) and Assumption I, the empirical process*

$$\begin{aligned}\psi_n(t) &= \sqrt{n}\ \mathbb{E}[\, Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\,(K_n(t) - K(t))\\ &= \mathbb{E}[\, Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\ \alpha_n(t)\end{aligned}$$

converges weakly to a continuous Gaussian process with zero mean and covariance function

$$\Gamma_\psi(s,t) := M_s\, M_t\, \Gamma(s,t), \tag{9}$$

where $\Gamma(s,t)$ is as in Theorem 1.1 and $M_c = \mathbb{E}[\, Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq c\}}\,] < +\infty$, for $c\in(0,1)$.

Proof of Lemma 4.1: We just remark that $\psi_n(t) = \mathbb{E}[\, Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\,\alpha_n(t)$, where $\alpha_n(t)$ is the Kendall's empirical process introduced by Genest and Rivest (1993) and Barbe *et al.* (1996) (see (6)). Then the demonstration comes down trivially from Theorem 1.1. Hence the result. □

Remark 10 Note that in the following, the main element we will need is that

$$\sup_{t\in[0,1]}\left|\alpha_n(t) - \left(\nu_n(A_{2,t}) - \int_{[0,1]^d}\nu_n(A_{1,z})\,\eta_t(dz)\right)\right|$$

converges to zero in probability (see proof of Theorem 5 in Barbe *et al.*, 1996).

4.2. Asymptotic behavior of $\zeta_n(t)$

We now study $\zeta_n(t) = \overline{K}(t)\,\frac{1}{\sqrt{n}}\,\sum_{i=1}^{n} Z_{ji}\left(\mathbf{1}_{\{\widetilde{F}_n(U_i)\geq t\}} - \mathbf{1}_{\{\widetilde{F}(U_i)\geq t\}}\right)$.
The behavior of $\zeta_n(t)$ when t is bounded away from the origin is described in Lemmas 4.2 and 4.3. Its behavior for t in the neighborhood of the origin will be the object of Lemma 4.4.

Lemma 4.2 *Under regularity conditions i) and ii) and Assumption I, the following quantity converges in probability to* 0 *for any* $0 < t_0 \leq 1$*:*

$$\sup_{t_0\leq t\leq 1}\left|-\zeta_n(t) + \overline{K}(t)\int_{[0,1]^d}\nu_n(A_{1,\,z})\,\mu_t(\mathrm{d}z)\right|, \tag{10}$$

with $\mu_t(C) = k(t)\,\mathbb{E}[Z_j\mathbf{1}_{\{U\in C\}}|\widetilde{F}(U) = t]$ *for any rectangle* C *in* $[0,1]^d$*.*

Proof of Lemma 4.2: The proof is postponed to Section 7. □

Lemma 4.3 *Under regularity conditions i) and ii) and Assumption I, the restriction of the process* $\zeta_n(t)$ *to the interval* $[t_0, 1]$ *converges in law to a centered, continuous Gaussian process having the representation* $\overline{K}(t)\int_{[0,1]^d} B(z)\,\mu_t(\mathrm{d}z)$ *in terms of the weak limit* B *of* $\sqrt{n}\,(\widetilde{F}_n - \widetilde{F})(z)$*.*

Proof of Lemma 4.3: The proof is postponed to Section 7. □

The following result describes the behavior of $\zeta_n(t)$ for t in the neighborhood of the origin.

Lemma 4.4 *Define*

$$\delta_n(t) = \overline{K}(t)\,\sqrt{n}\,\frac{1}{n}\,\sum_{i=1}^{n} Z_{ji}\left(\mathbf{1}_{\{t<\widetilde{F}(U_i)\leq t+(\widetilde{F}_n-\widetilde{F})^-(U_i)\}}\right),$$

and

$$\epsilon_n(t) = \overline{K}(t)\,\sqrt{n}\,\frac{1}{n}\,\sum_{i=1}^{n} Z_{ji}\left(\mathbf{1}_{\{t-(\widetilde{F}_n-\widetilde{F})^+(U_i)<\widetilde{F}(U_i)\leq t\}}\right).$$

Under regularity conditions i) and ii) and Assumption I, for arbitrary $\lambda > 0$*, one has*

$$(i)\qquad \lim_{t_0\to 0}\limsup_{n\to\infty}\ \mathbb{P}\left[\sup_{0\leq t\leq t_0}\delta_n(t)\geq\lambda\right] = 0;$$

$$(ii)\qquad \lim_{t_0\to 0}\limsup_{n\to\infty}\ \mathbb{P}\left[\sup_{0\leq t\leq t_0}\epsilon_n(t)\geq\lambda\right] = 0;$$

$$(iii)\qquad \lim_{t_0\to 0}\limsup_{n\to\infty}\mathbb{P}\left[\sup_{0\leq t\leq t_0}|\zeta_n(t)|\geq\lambda\right] = 0;$$

Proof of Lemma 4.4:
The proof of (i) and (ii) is an adaptation of Lemma 3 in Barbe *et al.* (1996). Statement (iii) of this lemma is an immediate consequence of the first two parts. The detailed proof is postponed to Section 7. □

4.3. Asymptotic behavior of $\phi_n(t)$

Lemma 4.5 *Under regularity conditions i) and ii) and Assumption I, the empirical process*

$$\phi_n(t) = \sqrt{n}\,\overline{K}(t)\,\left(\frac{1}{n}\sum_{i=1}^{n} Z_{ji}\,\mathbf{1}_{\{\widetilde{F}(U_i)\geq t\}} - \mathbb{E}[\,Z_j\,\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}\,]\right)$$

converges weakly to a continuous Gaussian process ϕ with zero mean and covariance function

$$\Gamma_\phi(s,t) = \overline{K}(t)\,\overline{K}(s)\left(\mathbb{E}[Z_j^2\mathbf{1}_{\{\widetilde{F}(U)\geq t\vee s\}}] - \mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\geq t\}}]\;\mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\geq s\}}]\right). \tag{11}$$

Proof of Lemma 4.5: The proof is similar to the one of Lemma 7.1 (see Section 7). □

5. Proof of Theorem 3.1

We now prove the asymptotic normality stated in Theorem 3.1. First define

$$\chi_n(t) := \mathbb{E}[Z_j\mathbf{1}_{\tilde{F}(U)\geq t}]\left(\nu_n(A_{2,t}) - \int_{[0,1]^d}\nu_n(A_{1,z})\eta_t(dz)\right) + \overline{K}(t)\int_{[0,1]^d}\nu_n(A_{1,z})\mu_t(dz)\,.$$

We derive from Lemmas 4.1, 4.2 and 4.4 that $\sup_{t\in[0,1]}|\psi_n(t)+\zeta_n(t)-\chi_n(t)|$ converges to zero in probability.

We know that $\{\chi_n(t),\,t\in[0,1]\}$ converges in $\mathcal{D}([0,1],\mathbb{R})$ to some continuous process $\{\chi(t),\,t\in[0,1]\} := \left\{\mathbb{E}[Z_j\mathbf{1}_{\tilde{F}(U)\geq t}]\,\alpha(t) + \overline{K}(t)\int_{[0,1]^d}\nu(A_{1,z})\mu_t(dz),\,t\in[0,1]\right\}$. Thus, from the multivariate central limit theorem (see e.g., 2.18 in Van de Vaart, 1996), the finite-dimensional distributions of the empirical process $\begin{pmatrix}\phi_n\\ \chi_n\end{pmatrix}$ converge to those of $\begin{pmatrix}\phi\\ \chi\end{pmatrix}$. Furthermore the sequence $\left\{\begin{pmatrix}\phi_n(t)\\ \chi_n(t)\end{pmatrix},\,t\in[0,1]\right\}$ is tight in $\mathcal{D}\left([0,1],\mathbb{R}^2\right)$ as both $\{\phi_n(t),\,t\in[0,1]\}$ and $\{\chi_n(t),\,t\in[0,1]\}$ converge in $\mathcal{D}([0,1],\mathbb{R})$ to some continuous limit process.

Then using the continuous mapping theorem (see e.g. Theorem 2.3 in Van der Vaart, 1996) we obtain that $\alpha_{n,j}^{\mathrm{CTE}}$ converges weakly to a continuous Gaussian process α_j^{CTE} with zero mean and covariance function Γ_{CTE}^j. Moreover, the limiting process α_j^{CTE} has the following representation in terms of χ and ϕ:

$$\alpha_j^{\mathrm{CTE}}(t) = \frac{1}{\overline{K}(t)^2}\,(\phi(t)+\chi(t))\,.$$

Remark 11 We remark that the covariance function Γ_{CTE}^j can be derived by the explicit expressions of covariance function Γ_ψ (see (9)), Γ_ϕ (see (11)) and Γ (see Barbe *et al.*, 1996, proof of Theorem 5).

6. Numerical study

In this section we provide simulations which aim at studying the practical behavior of our estimator, as far as to compare its performances to the ones of the level-sets based estimator proposed in Di Bernardino *et al.* (2011). Simulations are performed in the 2-dimensional and 3-dimensional setting.

6.1. About asymptotic normality of $\sqrt{n}(\widehat{\mathrm{CTE}}_\alpha(X,Y) - \mathrm{CTE}_\alpha(X,Y))$

In this section we show the Q-Q plots obtained from 100 replications of our estimator computed with samples of size $n = 50, 250, 800$. The Q-Q plot draws the empirical quantiles against the gaussian theoretical quantiles.

We consider $Z = (X, Y)$ a random vector with independent and exponentially distributed components with parameter 2. The level α has been fixed to 0.38.

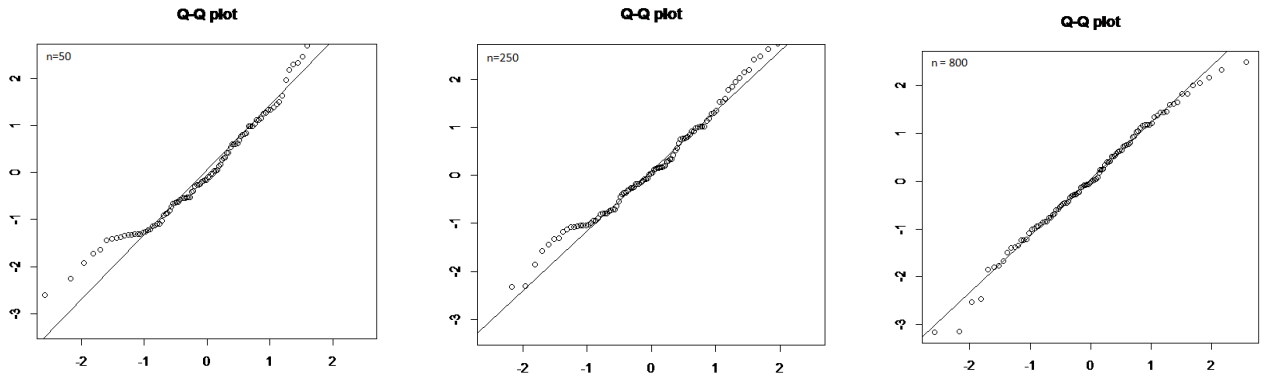

Figure 1: Q-Q plot for $\sqrt{n}\,(\widehat{\mathrm{CTE}}^1_\alpha(X,Y) - \mathrm{CTE}^1_\alpha(X,Y))$ on 100 simulations, with $\alpha = 0.38$, $n = 50, 250, 800$. (X, Y) with independent and exponentially distributed components with parameter 2.

We observe that increasing n leads to a better adequation of empirical quantiles with gaussian theoretical quantiles.

6.2. Comparison of our Kendall based estimator with the level-sets based one introduced in Di Bernardino et al. (2011)

Let us first recall the definition of the level-sets based estimator of the CTE_α, as it was introduced in Di Bernardino *et al.* (2011). To this aim, we define, for some $T > 0$,

$$L(\alpha)^T = \{x \in [0,T]^2 : F(x) \geq \alpha\}$$

and for $n \in \mathbb{N}^*$

$$L_n(\alpha)^T = \{x \in [0,T]^2 : F_n(x) \geq \alpha\},$$

with F_n the empirical distribution of F. Let T_n be an increasing positive sequence. Then we define the level-sets based estimator by

$$\widehat{\mathrm{CTE}}^{T_n}_\alpha(X,Y) = \begin{pmatrix} \dfrac{\sum_{i=1}^n X_i 1_{\{(X_i,Y_i)\in L_n(\alpha)^{T_n}\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in L_n(\alpha)^{T_n}\}}} \\ \dfrac{\sum_{i=1}^n Y_i 1_{\{(X_i,Y_i)\in L_n(\alpha)^{T_n}\}}}{\sum_{i=1}^n 1_{\{(X_i,Y_i)\in L_n(\alpha)^{T_n}\}}} \end{pmatrix}.$$

It could be interesting to consider the convergence $\left|\mathrm{CTE}_\alpha(X,Y) - \widehat{\mathrm{CTE}}_\alpha^{T_n}(X,Y)\right|$. Let $\mathrm{CTE}_\alpha^{T_n}(X,Y) = \mathbb{E}[(X,Y)|(X,Y) \in L(\alpha)^{T_n}]$. We remark that the speed of this convergence will also depend on the convergence rate to zero of $\left|\mathrm{CTE}_\alpha(X,Y) - \mathrm{CTE}_\alpha^{T_n}(X,Y)\right|$, then, in particular of $\mathbb{P}[(X,Y) \in L(\alpha) \setminus L(\alpha)^{T_n}]$ for $n \to \infty$. More precisely $\left|\mathrm{CTE}_\alpha(X,Y) - \mathrm{CTE}_\alpha^{T_n}(X,Y)\right|$ decays to zero at least with a convergence rate $(\mathbb{P}[X \geq T_n \text{ or } Y \geq T_n])^{-1}$. We remark that $(\mathbb{P}[X \geq T_n \text{ or } Y \geq T_n])^{-1}$ is increasing in T_n, whereas the speed of convergence of $\left|\mathrm{CTE}_\alpha^{T_n}(X,Y) - \widehat{\mathrm{CTE}}_\alpha^{T_n}(X,Y)\right|$ is decreasing in T_n (see Theorem 4.1. in Di Bernardino *et al.*, 2011). This kind of compromise provides an illustration on how to choose T_n in the estimator $\widehat{\mathrm{CTE}}_\alpha^{T_n}(X,Y)$. In this sense, running simulations with this estimator requires tuning the truncation sequence $(T_n)_{n\geq 1}$, which is not always easy. Conversely, an important advantage of our new estimator is that we do not have any parameter to tune.

In the following we denote our new estimator by $\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ and the level-sets based one by $\widehat{\mathrm{CTE}_\alpha}(X,Y)_{L_\alpha}$. The performances of both estimators will be evaluated by computing for each coordinate the mean, the empirical standard deviation and the relative mean squared error whose definitions are recalled below.

We denote $\overline{\widehat{\mathrm{CTE}_\alpha}}(X,Y) = \left(\overline{\widehat{\mathrm{CTE}_\alpha}^{1}}(X,Y), \overline{\widehat{\mathrm{CTE}_\alpha}^{2}}(X,Y)\right)$ the mean of $\widehat{\mathrm{CTE}}_\alpha(X,Y)$ on 100 simulations. We denote $\widehat{\sigma} = (\widehat{\sigma}_1, \widehat{\sigma}_2)$ the empirical standard deviation with

$$\widehat{\sigma}_1 = \sqrt{\frac{1}{99}\sum_{j=1}^{100}\left(\widehat{\mathrm{CTE}_\alpha}^{1}(X,Y)_j - \overline{\widehat{\mathrm{CTE}_\alpha}^{1}}(X,Y)\right)^2}$$

relative to the first coordinate (resp. $\widehat{\sigma}_2$ relative to the second one).

We denote $\mathrm{RMSE} = (\mathrm{RMSE}_1, \mathrm{RMSE}_2)$ the relative mean square error with

$$\mathrm{RMSE}_1 = \sqrt{\frac{1}{100}\sum_{j=1}^{100}\left(\frac{\widehat{\mathrm{CTE}}_\alpha^{1}(X,Y)_j - \mathrm{CTE}_\alpha^{1}(X,Y)}{\mathrm{CTE}_\alpha^{1}(X,Y)}\right)^2}$$

relative to the first coordinate of $\mathrm{CTE}_\alpha(X,Y)$ (resp. RMSE_2 relatives to the second one).

The explicit value of the theoretical $\mathrm{CTE}_\alpha(X,Y)$ was obtained with Maple.

In the following we consider: Independent copula with exponentially distributed marginals; Clayton copula with parameter 1, with exponential and Burr$(4,1)$ univariate marginals. The sample size is fixed to $n = 1000$ and $\alpha = 0.10$, 0.24, 0.38, 0.52, 0.66, 0.80. Results are gathered in Table 1 and 2.

The choice of the truncation sequence T_n is detailed in Section 5.2.1 in Di Bernardino *et al.* (2011) (see Tables 6-13).

We observe that the performances of our new estimator are always better in terms of relative mean squared error, and moreover as said before we did not have to tune a truncation sequence $(T_n)_{n\geq 1}$ which is a major advantage. Moreover the performances of both estimator decrease as the level α increases. In the next section we further investigate this phenomenon.

α	$\mathrm{CTE}_\alpha(X,Y)$	$\overline{\widehat{\mathrm{CTE}_\alpha}}(X,Y)_{L_\alpha}$	$\overline{\widehat{\mathrm{CTE}_\alpha}}(X,Y)_K$	$\widehat{\sigma}_{L_\alpha}$	$\widehat{\sigma}_K$	RMSE_{L_α}	RMSE_K
0.10	(1.255, 0.627)	(1.222, 0.638)	(1.259, 0.628)	(0.044, 0.022)	(0.039, 0.021)	(0.043, 0.039)	(0.032, 0.036)
0.24	(1.521, 0.761)	(1.488, 0.811)	(1.524, 0.761)	(0.069, 0.023)	(0.053, 0.023)	(0.051, 0.042)	(0.035, 0.037)
0.38	(1.792, 0.896)	(1.797, 0.911)	(1.791, 0.895)	(0.075, 0.038)	(0.068, 0.037)	(0.044, 0.046)	(0.037, 0.043)
0.52	(2.102, 1.051)	(2.082, 1.047)	(2.113, 1.056)	(0.104, 0.052)	(0.094, 0.045)	(0.052, 0.052)	(0.045, 0.044)
0.66	(2.492, 1.246)	(2.461, 1.255)	(2.507, 1.259)	(0.139, 0.071)	(0.137, 0.071)	(0.057, 0.056)	(0.056, 0.052)
0.80	(3.061, 1.531)	(3.011, 1.544)	(3.105, 1.535)	(0.251, 0.125)	(0.248, 0.122)	(0.084, 0.082)	(0.083, 0.081)

Table 1: (X,Y) with independent and exponentially distributed components with parameter 1 and 2 respectively.

α	$\mathrm{CTE}_\alpha(X,Y)$	$\overline{\widehat{\mathrm{CTE}_\alpha}}(X,Y)_{L_\alpha}$	$\overline{\widehat{\mathrm{CTE}_\alpha}}(X,Y)_K$	$\widehat{\sigma}_{L_\alpha}$	$\widehat{\sigma}_K$	RMSE_{L_α}	RMSE_K
0.10	(1.188, 1.229)	(1.049, 1.192)	(1.179, 1.231)	(0.032, 0.021)	(0.031, 0.021)	(0.019, 0.033)	(0.013, 0.018)
0.24	(1.448, 1.366)	(1.283, 1.379)	(1.442, 1.372)	(0.053, 0.224)	(0.039, 0.023)	(0.019, 0.063)	(0.014, 0.017)
0.38	(1.727, 1.505)	(1.525, 1.471)	(1.724, 1.506)	(0.046, 0.031)	(0.041, 0.029)	(0.019, 0.031)	(0.017, 0.022)
0.52	(2.049, 1.666)	(1.803, 1.625)	(2.065, 1.667)	(0.058, 0.041)	(0.048, 0.039)	(0.023, 0.034)	(0.021, 0.031)
0.66	(2.454, 1.875)	(2.129, 1.823)	(2.479, 1.873)	(0.071, 0.054)	(0.069, 0.046)	(0.035, 0.039)	(0.029, 0.033)
0.80	(3.039, 2.202)	(2.591, 2.144)	(3.029, 2.252)	(0.111, 0.105)	(0.103, 0.103)	(0.055, 0.054)	(0.041, 0.049)

Table 2: (X,Y) with Clayton copula with parameter 1, F_X exponential distribution with parameter 1, F_Y Burr$(4,1)$ distribution.

6.3. Deterioration of the performances of our estimator for high levels α

In this section, we first consider the case of independent and exponentially distributed marginals with *resp.* parameters 1 and 2, and we choose a level $\alpha = 0.9$. The theoretical value is then $\mathrm{CTE}_{0.9}(X,Y) = (3.78, 1.89)$. As illustrated in Table 3, we need in this case between 2000 and 2500 data to get the same performances as for lower level (see Table 1).

n	1000	1500	2000	2500
$\widehat{\sigma}_K$	(0.416, 0.299)	(0.411, 0.256)	(0.368, 0.155)	(0.221, 0.113)
$\widehat{\sigma}_{L_\alpha}$	(0.444, 0.308)	(0.431, 0.295)	(0.377, 0.168)	(0.241, 0.123)
RMSE_K	(0.113, 0.158)	(0.111, 0.135)	(0.095, 0.087)	(0.072, 0.063)
RMSE_{L_α}	(0.123, 0.163)	(0.115, 0.161)	(0.099, 0.089)	(0.077, 0.079)

Table 3: Comparison of the evolution of $\widehat{\sigma}_K$ and RMSE_K with respect to $\widehat{\sigma}_{L_\alpha}$ and RMSE_{L_α}, in terms of sample size n for $\alpha = 0.9$; (X,Y) independent and exponentially distributed components with parameter 1 and 2 respectively.

Let (X,Y) a random vector with independent and exponentially distributed marginals with *resp.* parameters 1 and 2. Now we compute 100 replications of our estimate with a sample-size $n = 1000$.

In Figure 2, we have drawn the empirical confidence intervals for $\widehat{\mathrm{CTE}_\alpha}^2(X,Y)_K$:

$$\left[\overline{\widehat{\mathrm{CTE}^2_\alpha}}(X,Y)_K - u_{0.95}\,\frac{\widehat{\sigma}_K}{\sqrt{n}}\,,\ \overline{\widehat{\mathrm{CTE}^2_\alpha}}(X,Y)_K + u_{0.95}\,\frac{\widehat{\sigma}_K}{\sqrt{n}}\right]$$

with $u_{0.95}$ the quantile of order 0.95 of the standard gaussian distribution, for various values of α.

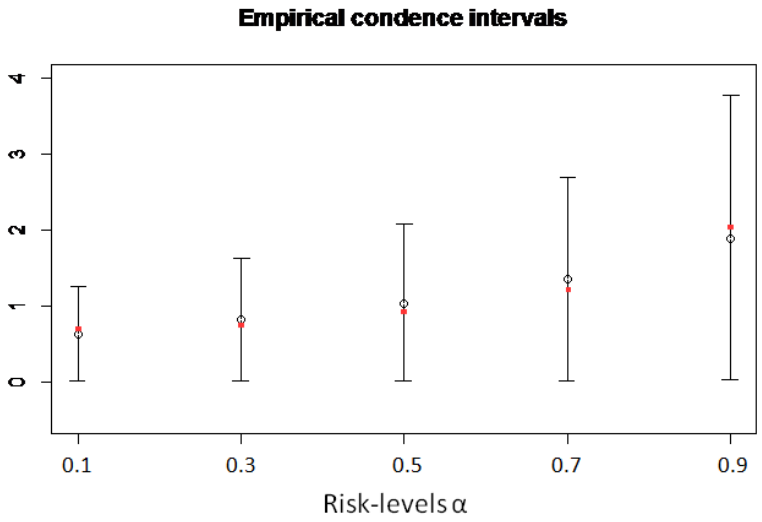

Figure 2: Empirical confidence intervals for $\widehat{\mathrm{CTE}_\alpha}^2(X,Y)_K$ of order 0.95, for different values of α level. Red square are the theoretical values of $\mathrm{CTE}^2_\alpha(X,Y)$. X and Y are independent and exponentially distributed components with parameter 1 and 2 respectively.

We observe that the length of the empirical confidence interval increases with α. It seems that the unknown limit variance Γ^2_{CTE} in Theorem 3.1 explodes as α tends to one.

6.4. Influence of structure of dependence on CTE_α

From theoretical results in Cousin and Di Bernardino (2012), we know that for a large class of parametric families of copulas (indexed by a parameter θ of dependence), the Multivariate Condition Tail Expectation is a decreasing function of the parameter of dependence θ. For instance, it is true for Clayton, Gumbel, Frank or Ali-Mikhail-Haq families.

In the following we illustrate this result for the comprehensive[4] Clayton family (see e.g., 2.4 in Nelsen, 2009). We define for u, $v \in [0,1]$

$$C(u,v) = \Big(\max(u^{-\theta} + v^{-\theta} - 1\,,\, 0)\Big)^{-1/\theta}$$

with $-1 \leq \theta \leq +\infty$. The case $\theta = -1$ corresponds to the perfect negative dependence, $\theta = 0$ to the independence, $\theta = +\infty$ to the comonotonicity. As the behavior of $\mathrm{CTE}_\alpha(X,Y)$ in terms of α and parameter dependence θ only depends on copula structure (and not on marginal distributions) we consider, in the Table 4, a Clayton copula with parameter θ and uniform marginal distributions. In particular this choice implies that $\mathrm{CTE}^1_\alpha(X,Y)$=$\mathrm{CTE}^2_\alpha(X,Y)$. In Table 4 we compute 100 replications of our estimator with a sample-size $n = 1000$.

[4] A family of copulas that includes comonotonicity, counter-monotonicity and independence dependence structures is called comprehensive (see Nelsen, 1999).

α \ θ	-0.95	0	1	10^4
0.10	$\mathrm{CTE}_\alpha(X,Y)$ 0.6419	$\mathrm{CTE}_\alpha(X,Y)$ 0.6047	$\mathrm{CTE}_\alpha(X,Y)$ 0.5827	$\mathrm{CTE}_\alpha(X,Y)$ 0.5500
	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.6397	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.6062	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.5831	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.5495
	$\widehat{\sigma}_K$ 0.0337	$\widehat{\sigma}_K$ 0.0106	$\widehat{\sigma}_K$ 0.0102	$\widehat{\sigma}_K$ 0.0091
	RMSE_K 0.0538	RMSE_K 0.0177	RMSE_K 0.0176	RMSE_K 0.0165
0.38	$\mathrm{CTE}_\alpha(X,Y)$ 0.7757	$\mathrm{CTE}_\alpha(X,Y)$ 0.7617	$\mathrm{CTE}_\alpha(X,Y)$ 0.7494	$\mathrm{CTE}_\alpha(X,Y)$ 0.6900
	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.7723	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.7644	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.7521	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.6903
	$\widehat{\sigma}_K$ 0.0475	$\widehat{\sigma}_K$ 0.0127	$\widehat{\sigma}_K$ 0.0108	$\widehat{\sigma}_K$ 0.0105
	RMSE_K 0.0611	RMSE_K 0.0179	RMSE_K 0.0178	RMSE_K 0.0171
0.66	$\mathrm{CTE}_\alpha(X,Y)$ 0.8825	$\mathrm{CTE}_\alpha(X,Y)$ 0.8789	$\mathrm{CTE}_\alpha(X,Y)$ 0.8754	$\mathrm{CTE}_\alpha(X,Y)$ 0.8300
	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.8936	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.8848	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.8799	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.8305
	$\widehat{\sigma}_K$ 0.1261	$\widehat{\sigma}_K$ 0.0181	$\widehat{\sigma}_K$ 0.0119	$\widehat{\sigma}_K$ 0.0117
	RMSE_K 0.1442	RMSE_K 0.0184	RMSE_K 0.0182	RMSE_K 0.0176
0.80	$\mathrm{CTE}_\alpha(X,Y)$ 0.9321	$\mathrm{CTE}_\alpha(X,Y)$ 0.9308	$\mathrm{CTE}_\alpha(X,Y)$ 0.9297	$\mathrm{CTE}_\alpha(X,Y)$ 0.9000
	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.9213	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.9416	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.9385	$\widehat{\mathrm{CTE}_\alpha}(X,Y)_K$ 0.9041
	$\widehat{\sigma}_K$ 0.1422	$\widehat{\sigma}_K$ 0.0264	$\widehat{\sigma}_K$ 0.0192	$\widehat{\sigma}_K$ 0.0187
	RMSE_K 0.1448	RMSE_K 0.0342	RMSE_K 0.0229	RMSE_K 0.0183

Table 4: (X,Y) with Clayton copula with parameter $\theta = -0.95,\ 0,\ 1,\ 10^4$ and $\alpha = 0.1,\ 0.38,\ 0.66,\ 0.8$, F_X and F_Y uniform marginals.

6.5. Real data

We consider here the estimation of CTE_α in a real case: **river flow data-set**. The data-set comes from the National River Flow Archive of the Center for Ecology & Hydrology in UK, (see http://www.ceh.ac.uk/index.html). We consider an hydrological data-set recorded in the uplands of mid-Wales. This data-set represents the river flow data measured at the Hore site (for X random variable) at the Tanllwyth site (for Y random variable), and at the Wye-Gwy site (for Z random variable), from 1985 to 2003. The unit of measurements of river-flows is m^3/s. The river flows are recorded each 15 minutes. In order to make the temporal data independent, we keep only 9 days spaced measurements from August the 11th 1985 to October the 20th 2003. The data-set size is now $n = 2134$.

The multivariate dependence structure that we have fitted on the data is a 3-dimensional Gumbel copula with parameter $\theta = 1.19$, thus Assumptions **I** and **II** in Section 1 seems to be satisfied. In Figure 3 below, we represent the estimated Kendall distribution function (solid line), the comonotonic Kendall distribution (bold line), the independent Kendall distribution (dotted line) and the Gumbel Kendall distribution with $\theta = 1.19$ (dashed line).

In Figure 4 we represent data-set and estimated 3-variate Conditional Tail Expectation, for several values of risk-level α. Table 5 contains the estimated components of the vectorial CTE for three different values of α.

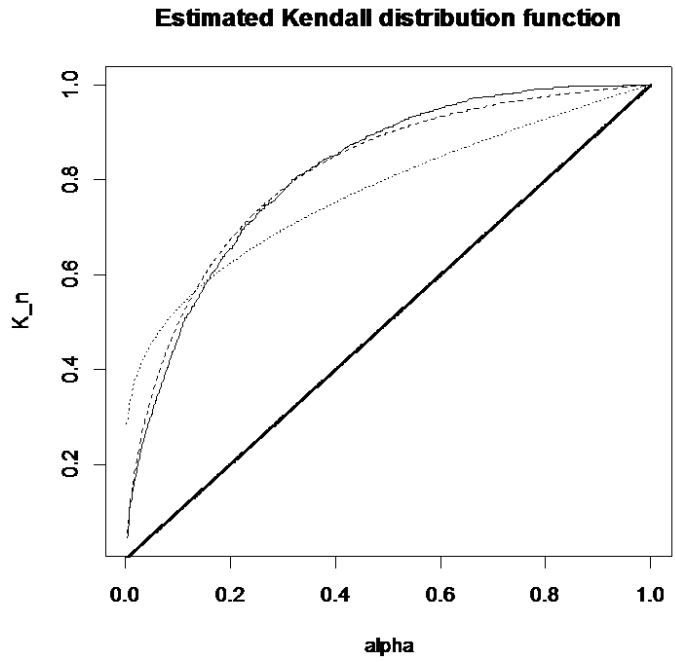

Figure 3: Estimated Kendall distribution function (solid line), comonotonic Kendall distribution (bold line), independent Kendall distribution (dotted line) and Gumbel Kendall distribution with $\theta = 1.19$ (dashed line).

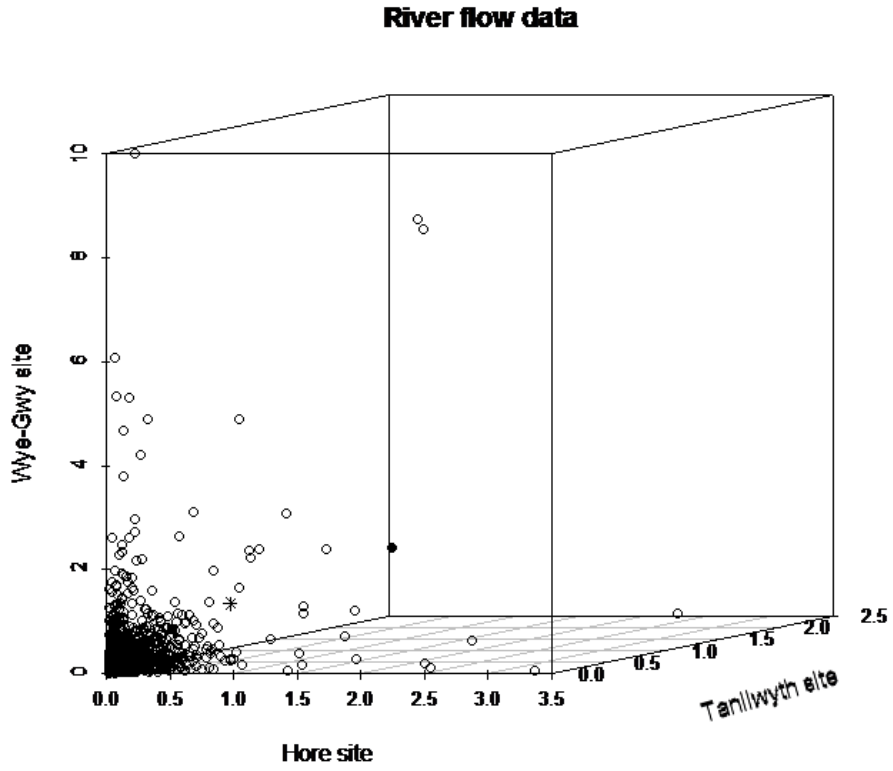

Figure 4: River flow data; $\widehat{\mathrm{CTE}}_{\alpha,K}$ for different values of α: $\alpha = 0.6$ (black square), $\alpha = 0.8$ (star), $\alpha = 0.95$ (black dot).

α	0.6	0.8	0.95
$\widehat{\mathrm{CTE}}_{\alpha,K}$	(0.333, 0.207, 0.756)	(0.677, 0.339, 1.201)	(1.031, 1.363, 1.807)

Table 5: $\widehat{\mathrm{CTE}}_{\alpha,K}$ for 3-dimensional river flow data-set, for different values of level α.

In this real setting the estimation of CTE_α can be used in order to quantify the mean of the river flow in the Hore site (resp. in the Tanllwyth site or Wye-Gwy site) conditionally to the fact that the data belong jointly to the specific risk's area $L(\alpha)$. It seems that data on the Wye-Gwy site have an heavier tail than data on both other sites. For the three different values of α, the estimated value for the third component of the vectorial CTE is indeed greatest than the estimated values for the first and the second components. Among these three sites we may thus consider on this study that the Wye-Gwy is the most dangerous, and maybe infrastructure efforts should be focused on this area.

7. Technical proofs

7.1. Proof of Lemma 4.2

The proof of this result follows the proof of Lemma 1 in Barbe *et al.* (1996). Let $(\widetilde{F}_n - \widetilde{F})^+$ and $(\widetilde{F}_n - \widetilde{F})^-$ respectively denote the positive and negative parts of $\widetilde{F}_n - \widetilde{F}$. Observe that $\nu_n(A_{1,z}) = \sqrt{n}\,(\widetilde{F}_n - \widetilde{F})(z)$ for all $z \in [0,1]^d$. Since $-\zeta_n(t) = \overline{K}(t)\,\frac{1}{\sqrt{n}}\,\sum_{i=1}^n Z_{ji}\left(\mathbf{1}_{\{\widetilde{F}_n(U_i)\le t\}} - \mathbf{1}_{\{\widetilde{F}(U_i)\le t\}}\right)$, then $-\zeta_n(t) = \delta_n(t) - \epsilon_n(t)$ with

$$\delta_n(t) = \overline{K}(t)\,\sqrt{n}\,\frac{1}{n}\,\sum_{i=1}^n Z_{ji}\left(\mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+(\widetilde{F}_n-\widetilde{F})^-(U_i)\}}\right),$$

and

$$\epsilon_n(t) = \overline{K}(t)\,\sqrt{n}\,\frac{1}{n}\,\sum_{i=1}^n Z_{ji}\left(\mathbf{1}_{\{t-(\widetilde{F}_n-\widetilde{F})^+(U_i)<\widetilde{F}(U_i)\le t\}}\right).$$

In the following we prove:

$$\sup_{t_0\le t\le 1}\left|\delta_n(t) - \overline{K}(t)\int_{[0,1]^d}\sqrt{n}\,(\widetilde{F}_n - \widetilde{F})^-(z)\,\mu_t(\mathrm{d}z)\right| \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0. \tag{12}$$

Convergence in (12) is established below. The convergence

$$\sup_{t_0\le t\le 1}\left|\epsilon_n(t) - \overline{K}(t)\int_{[0,1]^d}\sqrt{n}\,(\widetilde{F}_n - \widetilde{F})^+(z)\,\mu_t(\mathrm{d}z)\right| \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} 0, \tag{13}$$

is analogous and left to the reader. Convergence in (10) follows immediately from (12) and (13), since $\widetilde{F}_n - \widetilde{F} = (\widetilde{F}_n - \widetilde{F})^+ - (\widetilde{F}_n - \widetilde{F})^-$.
For any elements C of a partition $\mathcal{C} = (C_l)_{l=1}^m \in \mathcal{P}$, let $I_{n,l} = \inf_{z\in C_l}\sqrt{n}(\widetilde{F}_n - \widetilde{F})^-(z)$, $S_{n,l} = \sup_{z\in C_l}\sqrt{n}(\widetilde{F}_n - \widetilde{F})^-(z)$ and

$$\rho_{n,C}(t) = \overline{K}(t)\,\sqrt{n}\left[\frac{1}{n}\,\sum_{i=1}^n Z_{ji}\mathbf{1}_{\{\widetilde{F}(U_i)\le t, U_i\in C\}} - \mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\le t, U\in C\}}]\right].$$

Let also

$$\delta_{n,l}(t) = \overline{K}(t)\,\sqrt{n}\,\left[\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\left(\mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+(\widetilde{F}_n-\widetilde{F})^-(U_i)\}}\right)\,\mathbf{1}_{\{U_i\in C_l\}}\right],$$

so that $\delta_n(t) = \sum_{l=1}^{m}\delta_{n,l}(t)$. For arbitrary integers $1 \le l \le m$, one may write

$$\begin{aligned}
\delta_{n,l}(t) &\le \overline{K}(t)\,\sqrt{n}\,\left[\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+\frac{S_{n,l}}{\sqrt{n}}\}}\,\mathbf{1}_{\{U_i\in C_l\}}\right]\\
&\le \left[\rho_{n,C_l}(t+S_{n,l}/\sqrt{n})-\rho_{n,C_l}(t)\right] + \overline{K}(t)\,\sqrt{n}\int_t^{t+\frac{S_{n,l}}{\sqrt{n}}}\mu_s(C_l)\mathrm{d}s\\
&= \left[\rho_{n,C_l}(t+S_{n,l}/\sqrt{n})-\rho_{n,C_l}(t)\right] + \overline{K}(t)\,\sqrt{n}\int_t^{t+\frac{S_{n,l}}{\sqrt{n}}}[\mu_s(C_l)-\mu_t(C_l)]\mathrm{d}s + \mu_t(C_l)\,S_{n,l}\,\overline{K}(t)\\
&= \left[\rho_{n,C_l}(t+S_{n,l}/\sqrt{n})-\rho_{n,C_l}(t)\right] + \overline{K}(t)\,\sqrt{n}\int_t^{t+\frac{S_{n,l}}{\sqrt{n}}}[\mu_s(C_l)-\mu_t(C_l)]\mathrm{d}s\\
&+ \left[\mu_t(C_l)S_{n,l}\,\overline{K}(t) - \overline{K}(t)\int_{C_l}\sqrt{n}(\widetilde{F}_n-\widetilde{F})^-(z)\,\mu_t(\mathrm{d}z)\right] + \overline{K}(t)\int_{C_l}\sqrt{n}(\widetilde{F}_n-\widetilde{F})^-(z)\,\mu_t(\mathrm{d}z)\\
&\le \left[\rho_{n,C_l}(t+S_{n,l}/\sqrt{n})-\rho_{n,C_l}(t)\right] + \overline{K}(t)\,\sqrt{n}\int_t^{t+\frac{S_{n,l}}{\sqrt{n}}}[\mu_s(C_l)-\mu_t(C_l)]\mathrm{d}s\\
&+ \mu_t(C_l)\,\overline{K}(t)\,(S_{n,l}-I_{n,l}) + \overline{K}(t)\int_{C_l}\sqrt{n}(\widetilde{F}_n-\widetilde{F})^-(z)\,\mu_t(\mathrm{d}z).
\end{aligned}$$

An analogous argument yields

$$\begin{aligned}
\delta_{n,l}(t) &\ge \overline{K}(t)\,\sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+\frac{I_{n,l}}{\sqrt{n}}\}}\,\mathbf{1}_{\{U_i\in C_l\}}\\
&\ge \left[\rho_{n,C_l}(t+I_{n,l}/\sqrt{n})-\rho_{n,C_l}(t)\right] - \overline{K}(t)\,\sqrt{n}\int_t^{t+\frac{I_{n,l}}{\sqrt{n}}}[\mu_s(C_l)-\mu_t(C_l)]\,\mathrm{d}s\\
&- \mu_t(C_l)\,\overline{K}(t)\,(S_{n,l}-I_{n,l}) + \overline{K}(t)\int_{C_l}\sqrt{n}(\widetilde{F}_n-\widetilde{F})^-(z)\,\mu_t(\mathrm{d}z).
\end{aligned}$$

Lemma 7.1 *Under regularity conditions i) and ii) and Assumption I, the empirical process $\rho_{n,C}$ converges weakly to a continuous Gaussian process with zero mean and covariance function*

$$\overline{K}(t)\,\overline{K}(s)\left(\mathbb{E}[Z_j^2\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le t\wedge s\}}] - \mathbb{E}[Z_j\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le t\}}]\,\mathbb{E}[Z_j\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le s\}}]\right).$$

Proof of Lemma 7.1:
We first observe that for arbitrary $C\in\mathcal{C}$, the finite-dimensional distributions of the empirical process $\rho_{n,C}$ converges weakly to those of a centered Gaussian process with covariance function

$$\overline{K}(t)\,\overline{K}(s)\left(\mathbb{E}[Z_j^2\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le t\wedge s\}}] - \mathbb{E}[Z_j\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le t\}}]\,\mathbb{E}[Z_j\mathbf{1}_{\{U\in C,\widetilde{F}(U)\le s\}}]\right).$$

It remains to prove the tightness of $\{\rho_{n,C}(t),t\}$. The proof of the tightness is postponed to the Appendix. □

Furthermore, it follows from Theorem 2.1.3 in Gaenssler and Stute (1979) that the multivariate empirical process $\sqrt{n}(\widetilde{F}_n - \widetilde{F})$ converges, in $\mathcal{D}_d = D[0,1]^d$, to a continuous Gaussian process $B = B^+ - B^-$, where the space of càdlàg functions $\mathcal{D}_d$ is equipped with the Skorohod topology. As a result, $f\{\sqrt{n}(\widetilde{F}_n - \widetilde{F})\}$ converges weakly to $f\{B\}$ for any $\mathcal{D}_d$-measurable function that is continuous at every point of $C[0,1]^d$. In particular, $I_{n,l}$ and $S_{n,l}$ converge in distribution to $\inf_{z\in C_j} B^-(z)$ and $\sup_{z\in C_j} B^-(z)$ respectively. As a result, the quantities $I_{n,l}/\sqrt{n}$ and $S_{n,l}/\sqrt{n}$ both converge in probability to 0 as $n \to \infty$, and hence the same must be true of

$$R_{n,1} = \sum_{l=1}^{m} \sup_{0\le t\le 1} \left|\rho_{n,C_l}(t + I_{n,l}/\sqrt{n}) - \rho_{n,C_l}(t)\right|,$$

and

$$R_{n,2} = \sum_{l=1}^{m} \sup_{0\le t\le 1} \left|\rho_{n,C_l}(t + S_{n,l}/\sqrt{n}) - \rho_{n,C_l}(t)\right|,$$

because m is fixed and the processes ρ_{n,C_l} are tight for all $1 \le l \le m$. The convergence of $I_{n,l}/\sqrt{n}$ and $S_{n,l}/\sqrt{n}$ to zero further implies that for arbitrary $0 < t_0 \le 1$, the quantities

$$R_{n,3} = \sqrt{n} \sum_{l=1}^{m} \sup_{t_0\le t\le 1} \left|\overline{K}(t) \int_t^{t+\frac{S_{n,l}}{\sqrt{n}}} [\mu_s(C_l) - \mu_t(C_l)]\mathrm{d}s\right|,$$

and

$$R_{n,4} = \sqrt{n} \sum_{l=1}^{m} \sup_{t_0\le t\le 1} \left|\overline{K}(t) \int_t^{t+\frac{I_{n,l}}{\sqrt{n}}} [\mu_s(C_l) - \mu_t(C_l)]\mathrm{d}s\right|,$$

converge to 0 in probability, because $\mu_s(C_l)$ is continuous for all $s \in [t_0, 1]$. Finally, note that

$$\begin{aligned}
R_{n,5}(t) &= \overline{K}(t) \sum_{l=1}^{m} \mu_t(C_l)\,(S_{n,l} - I_{n,l}) \\
&\le \sum_{l=1}^{m} \mu_t(C_l) \max_{1\le l\le m} \sup_{z_1,z_2\in C_l} \sqrt{n}\,\left|(\widetilde{F}_n - \widetilde{F})^-(z_1) - (\widetilde{F}_n - \widetilde{F})^-(z_2)\right| \\
&\le k(t)\,\mathbb{E}[Z_j] \sum_{l=1}^{m} \max_{1\le l\le m} \sup_{z_1,z_2\in C_l} \sqrt{n}\,\left|(\widetilde{F}_n - \widetilde{F})^-(z_1) - (\widetilde{F}_n - \widetilde{F})^-(z_2)\right| \\
&\le k(t)\,\mathbb{E}[Z_j]\,\omega\{\sqrt{n}\,(\widetilde{F}_n - \widetilde{F})^-, \max_{1\le l\le m} \mathrm{diam}(C_l)\},
\end{aligned}$$

where

$$\omega\{f, r\} = \sup_{z_1,z_2\in[0,1]^d, d(z_1,z_2)\le z} |f(z_1) - f(z_2)|$$

is the modulus of continuity of f. By choosing a partition $\mathcal{C} \in \mathcal{P}$ with an appropriate mesh, it is thus possible to make $R_{n,5} = \sup_{t_0\le t\le 1} R_{n,5}(t)$ arbitrarily small with high probability when n is large enough. Collecting terms, one may then conclude that

$$\sup_{t_0\le t\le 1} \left|\delta_n(t) - \overline{K}(t) \int_{[0,1]^d} \sqrt{n}\,(\widetilde{F}_n - \widetilde{F})^-(z)\,\mu_t(\mathrm{d}z)\right| \le \max(R_{n,1}, R_{n,2}) + \max(R_{n,3}, R_{n,4}) + R_{n,5}.$$

Since the left-hand side does not depend on the choice of the partition, the proof is complete. □

7.2. Proof of Lemma 4.3

First observe that there exists a continuous version $\widetilde{F}_n^*$ of $\widetilde{F}_n$ with the property that $\sup_{z\in[0,1]^d}\left|\widetilde{F}_n^*(z)-\widetilde{F}_n(z)\right| \leq \frac{1}{n}$ and $\sqrt{n}\,(\widetilde{F}_n^*-\widetilde{F})(z)$ converges weakly to B in $C[0,1]^d$. Note also that

$$\sup_{t_0\leq t\leq 1}\left|\int_{[0,1]^d}\sqrt{n}\,(\widetilde{F}_n^*-\widetilde{F})(z)\,\mu_t(\mathrm{d}z)-\int_{[0,1]^d}\sqrt{n}\,(\widetilde{F}_n-\widetilde{F})(z)\,\mu_t(\mathrm{d}z)\right|\leq \sup_{t_0\leq t\leq 1}k(t)\;\mathbb{E}[Z_j]/\sqrt{n}.$$

Thus, in view of Lemma 4.2, it suffices to show that for any $f\in C[0,1]^d$, the function

$$t\mapsto\int_{[0,1]^d}f(z)\,\mu_t(\mathrm{d}z)$$

belongs to $C[t_0,1]$. For if the latter is true, then the mapping

$$f\mapsto\int_{[0,1]^d}f(z)\,\mu_t(\mathrm{d}z)$$

will be a bounded linear (and hence continuous) functional from $C[0,1]^d$ to $C[t_0,1]$. Given a partition $\mathcal{C}=(C_l)_{l=1}^m\in\mathcal{P}$, it is known by hypothesis that the function $t\mapsto\mu_t(C_l)$ is continuous on $[t_0,1]$ for any $1\leq l\leq m$. Thus, for any sequence (t_l) in $[t_0,1]$ converging to t, one has

$$\overline{L}=\limsup_{l\to\infty}\int f(z)\mu_{t_l}(\mathrm{d}z)\leq\sum_{l=1}^m\mu_t(C_l)\sup_{z\in C_l}f(z)<+\infty,$$

and

$$\underline{L}=\liminf_{l\to\infty}\int f(z)\mu_{t_l}(\mathrm{d}z)\geq\sum_{l=1}^m\mu_t(C_l)\inf_{z\in C_l}f(z)>-\infty,$$

Consequently,

$$\begin{aligned}0\leq\overline{L}-\underline{L}&\leq\sum_{l=1}^m\mu_t(C_l)\{\sup_{z\in C_l}f(z)-\inf_{z\in C_l}f(z)\}\\&\leq\sum_{l=1}^m\mu_t(C_l)\sup_{z,w\in C_l}|f(z)-f(w)|\\&\leq k(t)\,\omega\{f,\max_{1\leq l\leq m}\mathrm{diam}(C_l)\}.\end{aligned}$$

Since this string of inequalities must hold whatever the choice of the partition $\mathcal{C}$, one may conclude that $\overline{L}=\underline{L}$, hence the result. □

7.3. Proof of Lemma 4.4

The proof of (i) (*resp.* (ii)) in Lemma 4.4 is similar to the one of (i) (*resp.* (ii)) of Lemma 3 in Barbe *et al.* (1996). We detail below the few changes induced by the fact that we study the CTE and not the Kendall's process itself.

Proof of (i)*:*

We refer the interested reader to pages 217 to 219 of the paper by Barbe *et al.* (1996). We start from the corollary which follows Theorem 4 in Barbe *et al.* (1996). Nothing is changed in the proof until the beginning of page 218. Then we write the following chain of inequalities on $F_{n,}$:

$$\begin{aligned}
\delta_n(t) &= \tfrac{\overline{K}(t)}{\sqrt{n}} \textstyle\sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+(\widetilde{F}_n-\widetilde{F})^-(U_i)\}} \\
&\le \tfrac{\overline{K}(t)}{\sqrt{n}} \textstyle\sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+(\widetilde{F}_n-\widetilde{F})^-(U_i)\}} \mathbf{1}_{\{\widetilde{F}(U_i)\ge t_n\}} + \tfrac{\overline{K}(t)}{\sqrt{n}} \textstyle\sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{\widetilde{F}(U_i)\le t_n\}} \\
&\le \tfrac{\overline{K}(t)}{\sqrt{n}} \textstyle\sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{t<\widetilde{F}(U_i)\le t+\frac{M}{\sqrt{n}}q(2t)\}} + \tfrac{\overline{K}(t)}{\sqrt{n}} \textstyle\sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{\widetilde{F}(U_i)\le t_n\}} \\
&= \overline{K}(t) \left(g_n(t_n) + \sqrt{n}\, G(t_n) + \left[g_n\left(t+\tfrac{M}{\sqrt{n}}q(2t)\right) - g_n(t) \right] + \sqrt{n} \left[G\left(t+\tfrac{M}{\sqrt{n}}q(2t)\right) - G(t) \right] \right)
\end{aligned}$$

where $G(t) = \mathbb{E}[Z_j \mathbf{1}_{\{\widetilde{F}(U_i)\le t\}}]$, $g_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_{j_i} \mathbf{1}_{\{\widetilde{F}(U_i)\le t\}} - \sqrt{n}\, G(t)$, $M>0$, $t_n = \log^r(n)/n$, $q(t) = \sqrt{t}\,\log^p(1/t)$, for $r > 2p > 1$.

We then get

$$\mathbb{P}\Big\{ \sup_{0\le t\le t_0} \delta_n(t) \ge \lambda,\, F_{n,M} \Big\} \le \mathbb{P}\big\{ \overline{K}(t)\,|g_n(t_n)| \ge \frac{\lambda}{4} \big\} + \mathbb{P}\big\{ \sqrt{n}\,\overline{K}(t)\,G(t_n) \ge \frac{\lambda}{4} \big\} + \mathbb{P}\big\{ R_{n,6} \ge \frac{\lambda}{4} \big\} + \mathbb{P}\big\{ R_{n,7} \ge \frac{\lambda}{4} \big\},$$

with

$$R_{n,6} = \sup_{0\le t\le t_0} \overline{K}(t) \left| g_n\left(t+\frac{M}{\sqrt{n}}q(2t)\right) - g_n(t) \right|$$

and

$$R_{n,7} = \sup_{0\le t\le t_0} \sqrt{n}\,\overline{K}(t) \left[G\left(t+\frac{M}{\sqrt{n}}q(2t)\right) - G(t) \right].$$

The first and the third terms are handled as in Barbe *et al.* (1996), page 219. The second term writes

$$\sqrt{n}\,\overline{K}(t)\,\mathbb{E}[Z_j \mathbf{1}_{\{\widetilde{F}(U_i)\le t_n\}}].$$

From Assumption **II** we know that G is continuously differentiable on $[0,1]$ with derivative $t \mapsto \eta_t([0,1])$ which is continuous thus bounded on $[0,1]$. Thus

$$\left| \sqrt{n}\,\overline{K}(t)\,\mathbb{E}[Z_j \mathbf{1}_{\{\widetilde{F}(U_i)\le t_n\}}] \right| \le \sqrt{n} \sup_{s\in[0,1]} |G'(s)|\, t_n \le \frac{\log^r(n)}{\sqrt{n}} \sup_{s\in[0,1]} |G'(s)|,$$

which tends to zero as n tends to infinity. It remains to handle $R_{n,7}$.

We use once more that G is continuously differentiable on $[0,1]$ with derivative $t \mapsto \eta_t([0,1])$ which is continuous thus bounded on $[0,1]$. It implies that

$$R_{n,7} \le \sqrt{n} \sup_{s\in[0,1]} |G'(s)|\, \frac{M}{\sqrt{n}} \sup_{0\le t\le t_0} q(2t)$$

which tends to zero as t_0 tends to zero. $\square$

Proof of (ii)*:*
We have to adapt the proof of (ii) of Lemma 3 in Barbe *et al.* (1996). The procedure to adapt the proof is similar as for (i) and thus will be omitted here. $\square$

Acknowledgements: This work has been partially supported by the French research national agency (ANR) under the reference ANR-08BLAN-0314-01. The authors thank Philippe Berthet for having drawn their attention to the Kendall's process. They also thank Stéphane Girard for fruitful discussions.

References:

[1] J. Y. Ahn and N. D. Shyamalkumar. Large sample behavior of the CTE and VaR estimators under importance sampling. *N. Am. Actuar. J.*, 15(3):393–416, 2011.

[2] P. Artzner, F. Delbaen, J. M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9(3):203–228, 1999.

[3] A. V. Asimit, E. Furman, Q. Tang, and R. Vernic. Asymptotics for risk capital allocations based on conditional tail expectation. *Insurance Math. Econom.*, 49(3):310–324, 2011.

[4] P. Barbe, C. Genest, K. Ghoudi, and B. Rémillard. On Kendall's process. *J. Multivariate Anal.*, 58(2):197–229, 1996.

[5] M. Bargès, H. Cossette, and É. Marceau. TVaR-based capital allocation with copulas. *Insurance Math. Econom.*, 45(3):348–361, 2009.

[6] F. Belzunce, A. Castaño, A. Olvera-Cervantes, and A. Suárez-Llorens. Quantile curves and dependence structure for bivariate distributions. *Comput. Statist. Data Anal.*, 51(10):5112–5129, 2007.

[7] I. Bentahar. Tail conditional expectation for vector-valued risks. SFB 649 Discussion Papers SFB649DP2006-029, Humboldt University, Collaborative Research Center 649, 2006.

[8] P. Billingsley. *Probability and Measure (Third ed.)*. Wiley, 1995.

[9] V. Brazauskas, B. L. Jones, M. L. Puri, and R. Zitikis. Estimating conditional tail expectation with actuarial applications in view. *J. Statist. Plann. Inference*, 138(11):3590–3604, 2008.

[10] J. Cai and H. Li. Conditional tail expectations for multivariate phase-type distributions. *Journal of Applied Probability*, 42(3):810–825, 2005.

[11] P. Capéraà, A.-L. Fougères, and C. Genest. A stochastic ordering based on a decomposition of Kendall's tau. In *Distributions with given marginals and moment problems (Prague, 1996)*, pages 81–86. Kluwer Acad. Publ., Dordrecht, 1997.

[12] F. Chebana and T. Ouarda. Multivariate quantiles in hydrological frequency analysis. *Environmetrics*, 22(1):63–78, 2011.

[13] A. Cousin and E. Di Bernardino. A multivariate extension of value-at-risk and conditional-tail-expectation. *Submited*, 2012.

[14] L. de Haan and X. Huang. Large quantile estimation in a multivariate setting. *J. Multivariate Anal.*, 53(2):247–263, 1995.

[15] E. Di Bernardino, T. Laloë, V. Maume-Deschamps, and C. Prieur. Plug-in estimation of level sets in a non-compact setting with applications in multivariable risk theory. *ESAIM: Probability and Statistics, http://dx.doi.org/10.1051/ps/2011161*, 2011.

[16] P. Embrechts and G. Puccetti. Bounds for functions of multivariate risks. *Journal of Multivariate Analysis*, 97(2):526–547, 2006.

[17] W. Feller. *An introduction to probability theory and its applications. Vol. II.* John Wiley & Sons Inc., New York, 1966.

[18] P. Gaenssler and W. Stute. Empirical processes: A survey of results for independent and identically distributed random variables. *Ann. Probab.*, 7:193–243, 1979.

[19] C. Genest and J.-C. Boies. Detecting dependence with kendall plots. *The American Statistician*, 57:275–284, 2003.

[20] C. Genest, J.-F. Quessy, and B. Rémillard. Goodness-of-fit procedures for copula models based on the probability integral transformation. *Scandinavian Journal of Statistics*, 33:337–366, 2006.

[21] C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: a review and a power study. *Insurance Math. Econom.*, 44(2):199–213, 2009.

[22] C. Genest and L.-P. Rivest. Statistical inference procedures for bivariate Archimedean copulas. *J. Amer. Statist. Assoc.*, 88(423):1034–1043, 1993.

[23] C. Genest and L.-P. Rivest. On the multivariate probability integral transformation. *Statist. Probab. Lett.*, 53(4):391–399, 2001.

[24] K. Ghoudi and B. Remillard. Empirical processes based on pseudo-observations. In *Asymptotic methods in probability and statistics (Ottawa, ON, 1997)*, pages 171–197. North-Holland, Amsterdam, 1998.

[25] M. Henry, A. Galichon, and I. Ekeland. Comonotonic measures of multivariate risks. Technical report, 2012.

[26] L. Hua and H. Joe. Second order regular variation and conditional tail expectation of multiple risks. *Insurance Math. Econom.*, 49(3):537–546, 2011.

[27] E. Jouini, M. Meddeb, and N. Touzi. Vector-valued coherent risk measures. *Finance Stoch.*, 8(4):531–552, 2004.

[28] Z. M. Landsman and E. A. Valdez. Tail conditional expectations for elliptical distributions. *N. Am. Actuar. J.*, 7(4):55–71, 2003.

[29] G. Nappo and F. Spizzichino. Kendall distributions and level sets in bivariate exchangeable survival models. *Information Sciences*, 179:2878–2890, August 2009.

[30] A. Necir, A. Rassoul, and R. Zitikis. Estimating the conditional tail expectation in the case of heady-tailed losses. *J. Probab. Stat.*, 2010.

[31] R. B. Nelsen. *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1999.

[32] R. B. Nelsen, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, and M. Úbeda-Flores. Kendall distribution functions and associative copulas. *Fuzzy Sets and Systems*, 160(1):52–57, 2009.

[33] R. B. Nelsen, J. J. Quesada-Molinab, J. A. Rodrguez-Lallenac, and M. Úbeda-Floresc. Kendall distribution functions. *Statistics and Probability Letters*, 65:263–268, 2003.

[34] M. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.

[35] L. Tibiletti. On a new notion of multidimensional quantile. *Metron. International Journal of Statistics*, 51(3-4):77–83, 1993.

[36] L. Tibiletti. Connectedness, arcwise-connectedness and convexity for levels sets of multidimensional distribution functions. *Ratio Mathematica*, 8:21–27, 1994.

[37] A. W. van der Vaart and J. A. Wellner. *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York, 1996. With applications to statistics.

Appendix: Proof of the tightness of $\{\rho_{n,C}(t), t\}$:

To prove the tightness, we apply Theorem 12.3 in Billingsley (1995). Recall that

$$\rho_{n,C}(t) = \overline{K}(t)\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\mathbf{1}_{\{\widetilde{F}(U_i)\leq t, U_i\in C\}} - \mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\leq t, U\in C\}}]\right].$$

It is sufficient to prove the tightness of

$$\tilde{\rho}_{n,C}(t) = \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} Z_{j_i}\mathbf{1}_{\{\widetilde{F}(U_i)\leq t, U_i\in C\}} - \mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\leq t, U\in C\}}]\right].$$

Let us denote, for any $0 < t < 1$,

$$B_i(t) := Z_{j_i}\mathbf{1}_{\{\widetilde{F}(U_i)\leq t, U_i\in C\}} - \mathbb{E}[Z_j\mathbf{1}_{\{\widetilde{F}(U)\leq t, U\in C\}}],$$

and for any $0 < s < t < 1$

$$L_i(s,t) := Z_{j_i}\mathbf{1}_{\{s<\widetilde{F}(U_i)\leq t, U_i\in C\}},$$
$$L_i^c(s,t) := Z_{j_i}\mathbf{1}_{\{s<\widetilde{F}(U_i)\leq t, U_i\in C\}} - \mathbb{E}[Z_j\mathbf{1}_{\{s<\widetilde{F}(U)\leq t, U\in C\}}].$$

For any $0 < u < s < t < 1$ the $L_i^c(s,t)$, $i = 1,\ldots,n$ are i.i.d. centered random variables and for any $i \neq j \in \{1,\ldots,n\}$, $L_i^c(s,t)$ is independent of $L_j^c(u,s)$.
Let us compute, for arbitrary $0 < u < s < t < 1$,

$$\mathbb{E}\left[\{\tilde{\rho}_{n,C}(t) - \tilde{\rho}_{n,C}(s)\}^2\{\tilde{\rho}_{n,C}(s) - \tilde{\rho}_{n,C}(u)\}^2\right]$$

One has, for $0 < s < t < 1$,

$$\tilde{\rho}_{n,C}(t) - \tilde{\rho}_{n,C}(s) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} L_i^c(s,t).$$

Thus

$$\mathbb{E}\left[\{\tilde{\rho}_{n,C}(t) - \tilde{\rho}_{n,C}(s)\}^2\{\tilde{\rho}_{n,C}(s) - \tilde{\rho}_{n,C}(u)\}^2\right]$$
$$= \frac{1}{n^2}\mathbb{E}\left[\{\sum_{i=1}^{n}(L_i^c)^2(s,t) + \sum_{1\leq i<j\leq n} L_i^c(s,t)L_j^c(s,t)\}\{\sum_{k=1}^{n}(L_k^c)^2(u,s) + \sum_{1\leq k<l\leq n} L_k^c(u,s)L_l^c(u,s)\}\right]$$
$$= \frac{1}{n^2}\,\mathbb{E}\left(n(n-1)(L_1^c)^2(s,t)(L_2^c)^2(u,s) + n(L_1^c)^2(s,t)(L_1^c)^2(u,s) + \frac{n(n-1)}{2}L_1^c(s,t)L_1^c(u,s)L_2^c(s,t)L_2^c(u,s)\right)$$

Study of $\mathbb{E}[(L_1^c)^2(s,t)(L_2^c)^2(u,s)]$:

Let $1 < p < 2$ and $q > 2$ be such that $1/p + 1/q = 1$. From independence on has

$$\mathbb{E}[(L_1^c)^2(s,t)(L_2^c)^2(u,s)] = \mathbb{E}[(L_1^c)^2(s,t)]\mathbb{E}[(L_2^c)^2(u,s)]$$
$$\leq \left(\mathbb{E}[{Z_j}^2\mathbf{1}_{\{u<\widetilde{F}(U)\leq t, U\in C\}}]\right)^2$$
$$\leq \left(\mathbb{E}[{Z_j}^{2q}]\right)^{2/q}\left(\mathbb{P}[u < \widetilde{F}(U) \leq t, U\in C]\right)^{2/p}.$$

Study of $\mathbb{E}[(L_1^c)^2(s,t)(L_1^c)^2(u,s)]$:

One has

$$L_1^c(s,t)L_1^c(u,s) = -L_1(s,t)\mathbb{E}[L_1(u,s)] - L_1(u,s)\mathbb{E}[L_1(s,t)] + \mathbb{E}[L_1(u,s)]\,\mathbb{E}[L_1(s,t)].$$

Then

$$0 \le H := L_1(s,t)\mathbb{E}[L_1(u,s)] + L_1(u,s)\mathbb{E}[L_1(s,t)] \le L_1(u,t)\mathbb{E}[L_1(u,t)].$$

Define $J := \mathbb{E}[L_1(u,s)]\,\mathbb{E}[L_1(s,t)]$.
Then

$$\mathbb{E}[(J-H)^2] \le \mathbb{E}[J^2] + \mathbb{E}[H^2]$$

$$\le \left(\mathbb{E}[Z_j^q]\right)^{2/q} \left(\mathbb{P}[u < \widetilde{F}(U) \le t, U \in C]\right)^{2/p} + \left(\mathbb{E}[L_1(u,t)]\right)^2$$

$$\le \left(\mathbb{E}[Z_j^q]\right)^{2/q} \left(\mathbb{P}[u < \widetilde{F}(U) \le t, U \in C]\right)^{2/p} + \mathbb{E}[Z_j^2]\,\left(\mathbb{E}[{Z_j}^q]\right)^{2/q} \left(\mathbb{P}[u < \widetilde{F}(U) \le t, U \in C]\right)^{2/p}$$

$$\le \left(\mathbb{E}[Z_j^q]\right)^{2/q} \left(\mathbb{P}[u < \widetilde{F}(U) \le t, U \in C]\right)^{2/p} \left(1 + \mathbb{E}[Z_j^2]\right).$$

Study of $\mathbb{E}[L_1^c(s,t)L_1^c(u,s)L_2^c(s,t)L_2^c(u,s)]$:

One has by independence

$$\mathbb{E}[L_1^c(s,t)L_1^c(u,s)L_2^c(s,t)L_2^c(u,s)] = \left(\mathbb{E}[L_1^c(s,t)L_1^c(u,s)]\right)^2 \le \mathbb{E}[L_1^2(s,t)]\,\mathbb{E}[L_1^2(u,s)]$$

$$\le \left(\mathbb{E}[L_1^2(s,t)]\right)^2 \le \left(\mathbb{E}[Z_j^{2q}]\right)^{2/q} \left(\mathbb{P}[u < \widetilde{F}(U) \le t, U \in C]\right)^{2/p}.$$

Using the bounds of the three terms studied above, we get

$$\mathbb{E}\left[\{\tilde{\rho}_{n,C}(t) - \tilde{\rho}_{n,C}(s)\}^2\{\tilde{\rho}_{n,C}(s) - \tilde{\rho}_{n,C}(u)\}^2\right] \le 2\,\left(\mathbb{E}[{Z_j}^{2q}]\right)^{2/q} + \left(\mathbb{E}[Z_j^q]\right)^{2/q} \left(1 + \mathbb{E}[Z_j^2]\right)(G(t) - G(u))^{2/p}$$

with $G(s) := \mathbb{P}[\widetilde{F}(U) \le s, U \in C]$.

Thus applying Theorem 12.3 in Billingsley (1995), with $\gamma = 2$, $\alpha = 2/p > 1$ and $F(s) = G(s)$ which is nondecreasing continuous function on $[0,1]$, we get the tightness. □