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ON CAUCHY-STIELTJES KERNEL FAMILIES

WŁODEK BRYC, RAOUF FAKHFAKH, AND ABDELHAMID HASSAIRI

ABSTRACT. We explore properties of Cauchy-Stieltjes families that have no counterpart in exponential families. We relate the variance function of the iterated Cauchy-Stieltjes family to the pseudo-variance function of the initial Cauchy-Stieltjes family. We also investigate when the domain of means can be extended beyond the "natural domain".

1. INTRODUCTION

This paper is a continuation of the study of Cauchy-Stieltjes Kernel (CSK) families. Our goal here is to advance the understanding of two phenomena that have no known analogues for the classical exponential families. Firstly, a typical member of a given CSK family generates a different CSK family, so one can construct new CSK families by the iteration process. Secondly, in the natural parametrization of a CSK family by the mean, one can sometimes extend the family beyond the natural domain of means, preserving the variance function when the variance exists.

The notations used in what follows are the ones used in Bryc-Hassairi [BH11]. Throughout the paper ν is a non-degenerate probability measure with support bounded from above. Then

$$M(\theta) = \int \frac{1}{1 - \theta x} \nu(dx) \quad (1.1)$$

is well defined for all $\theta \in [0, \theta_+)$ with $1/\theta_+ = \max\{0, \sup \text{supp}(\nu)\}$ and

$$\mathcal{K}_+(\nu) = \{P_\theta(dx); \theta \in (0, \theta_+)\} = \{Q_m(dx), m \in (m_0, m_+)\} \quad (1.2)$$

is the CSK family generated by ν . That is,

$$P_\theta(dx) = \frac{1}{M(\theta)(1 - \theta x)} \nu(dx)$$

and $Q_m(dx)$ is the corresponding parametrization by the mean, which for $m \neq 0$ is given by

$$Q_m(dx) = \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m - x)} \nu(dx), \quad (1.3)$$

with

$$Q_0(dx) = \frac{\mathbb{V}'(0)}{\mathbb{V}'(0) - x} \nu(dx)$$

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if $m_0 < 0 < m_+$, and which involves the *pseudo-variance function* $\mathbb{V}(m)$. The interval (m_0, m_+) is called the (one sided) domain of means, and is determined as the image of $(0, \theta_+)$ under the strictly increasing function $k(\theta) = \int xP_\theta(dx)$ which is given by the formula

$$k(\theta) = \frac{M(\theta) - 1}{\theta M(\theta)}. \quad (1.4)$$

The pseudo-variance function has straightforward probabilistic interpretation if $m_0 = \int x d\nu$ is finite. Then, see [BH11, Proposition 3.2] we know that

$$\frac{\mathbb{V}(m)}{m} = \frac{v(m)}{m - m_0}. \quad (1.5)$$

where

$$v(m) = \int (x - m)^2 Q_m(dx) \quad (1.6)$$

is the so called variance function of the family $\{Q_m\}$. In particular, $\mathbb{V} = v$ when $m_0 = 0$.

In general,

$$\frac{\mathbb{V}(m)}{m} = \frac{1}{\psi(m)} - m, \quad (1.7)$$

where $\psi : (m_0, m_+) \rightarrow (0, \theta_+)$ is the inverse of the function $k(\cdot)$. From (1.7) it is clear that when $0 \in (m_0, m_+)$, we must have $\mathbb{V}(0) = 0$. In this case, we assign the value $1/\psi(0)$ to the undefined expression $\mathbb{V}(m)/m$ at $m = 0$.

The generating measure ν is determined uniquely by the pseudo-variance function \mathbb{V} through the following identities (for technical details, see [BH11]): if

$$z = z(m) = m + \frac{\mathbb{V}(m)}{m} \quad (1.8)$$

then the Cauchy transform

$$G_\nu(z) = \int \frac{1}{z - x} \nu(dx). \quad (1.9)$$

satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}(m)}. \quad (1.10)$$

Let

$$A = A(\nu) = \sup \text{supp}(\nu), \quad B = B(\nu) = \max\{0, A(\nu)\}. \quad (1.11)$$

We note that $B(\nu) = 1/\theta_+ \in [0, \infty)$.

From [BH11, Remark 3.3] we read out the following.

Proposition 1.1 ([BH11]). *For a non-degenerate probability measure ν with support bounded from above, the one-sided domain of means (m_0, m_+) of is determined from the following formulas*

$$m_0 = \lim_{\theta \rightarrow 0^+} k(\theta) \quad (1.12)$$

and with $B = B(\nu)$,

$$m_+ = B - \lim_{b \rightarrow B^+} \frac{1}{G(b)}. \quad (1.13)$$

Remark 1.2. We list some additional properties relevant to this paper.

- (i) $m_0 < m_+$

(ii) Since $1/\psi(m) = m + \mathbb{V}(m)/m$, we know that

$$m + \mathbb{V}(m)/m > m_+ + \mathbb{V}(m_+)/m_+ = 1/\theta_+ = B(\nu) \geq 0.$$

In particular, $G(m + \mathbb{V}(m)/m)$ is well defined, and non-negative.

(iii) Since $v(m) \geq 0$, from (1.5) we see that $\mathbb{V}(m)/m > 0$ for $m \in (m_0, m_+)$. This holds also when the variance is infinite – just apply Remark 1.2 and (1.10).

2. ITERATED CSK FAMILIES

One difference between the exponential and CSK families is that one can build nontrivial iterated CSK families. That is, each member of an exponential family generates the same exponential family so it does not matter which of them we use for the generating measure. But this is not so for CSK families: each member of a CSK family generates something different than the original family, so the construction can be iterated.

Suppose Q_m is in the CSK family generated by a probability measure ν with support bounded, say, from above, as given by (1.3). Then necessarily Q_m has the support bounded from above, but it also has one more moment than ν . Consider now a new CSK family generated by Q_m . Then, as long as $m \neq m_0$, the variance function of this new family necessarily exists. Our goal is to relate the variance function of this new family to the pseudo-variance function of the initial family.

2.1. Example: iterated semicircle CSK families. Here we use integral identities related to the semicircle law to construct iterated CSK families by elementary means. The iterations get progressively more cumbersome, and illustrate the need for the general theory.

For complex a_1, a_2, a_3, a_4 let

$$\tilde{f}(x; a_1, a_2, a_3, a_4) = \sqrt{4 - x^2} \prod_{j=1}^4 (1 + a_j^2 - a_j x)^{-1}$$

Our starting point is the following formula.

Lemma 2.1. *If $|a_1|, \dots, |a_4| < 1$, then*

$$\int_{-2}^2 \tilde{f}(x; a_1, a_2, a_3, a_4) dx = K(a_1, a_2, a_3, a_4), \quad (2.1)$$

where

$$K(a_1, a_2, a_3, a_4) = 2\pi(1 - a_1 a_2 a_3 a_4) \prod_{1 \leq i < j \leq 4} (1 - a_i a_j)^{-1}. \quad (2.2)$$

Applying this formula with $a_1 = a_2 = a_3 = 0$ and $a_4 = m \in (0, 1)$ (recall that $m_0 = 0$ and we are in one-sided setting as in (1.2)), we get

$$\int_{-2}^2 \frac{\sqrt{4 - x^2}}{1 + m(m - x)} dx = 2\pi,$$

so

$$Q_m = \frac{\sqrt{4 - x^2}}{2\pi(1 + m(m - x))} 1_{|x| < 2} dx.$$

Now we fix $Q_{m_1} \in \mathcal{K}_+(\nu)$ with mean m_1 . Applying (2.1) with $a_1 = a_2 = 0$, $a_3 = a$ and $a_4 = m_1$, we get

$$\int_{-2}^2 \frac{\sqrt{4-x^2}}{(1+a(a-x))((1+m_1(m_1-x)))} dx = \frac{2\pi}{1-am_1},$$

i.e.

$$\int_{-2}^2 \frac{1-am_1}{1+a(a-x)} Q_{m_1}(dx) = 1.$$

Rewriting this into the form suggested by (1.3), we see that

$$\begin{aligned} \frac{1-am_1}{1+a(a-x)} &= \frac{1}{\frac{1+a^2}{1-am_1} - \frac{a}{1-am_1}x} = \frac{1}{1 + \frac{a(a+m_1)}{1-am_1} - \frac{a}{1-am_1}x} \\ &= \frac{1}{1 + \frac{a}{1-am_1}(a+m_1-x)} = \frac{1}{1 + \frac{a(a+m_1)}{(1-am_1)(a+m_1)}(a+m_1-x)} \end{aligned} \quad (2.3)$$

Taking $m = a + m_1$, from (1.3) we see that the pseudo-variance function of the CSK family $\mathcal{K}_+(Q_{m_1})$ is

$$\mathbb{V}_1(m) = \frac{(1-am_1)(a+m_1)}{a} = (1-(m-m_1)m_1) \frac{m}{m-m_1},$$

i.e. the corresponding variance function

$$v_1(m) = \frac{m-m_1}{m} \mathbb{V}_1(m) = 1 + m_1^2 - m_1 m$$

is an affine function of m . It is clear that the formula works for all $m \in (m_1, m_1 + 1)$ (again, we use the one-sided setup as in (1.2)). This variance function corresponds to an affine transformation of the Marchenko-Pastur law, see [Bry09, Example 4.1]. We will see that the same result will follow from general theory, see (2.12).

Now we iterate this procedure. Fix

$$Q_{m_2, m_1} = \frac{1-m_1(m_2-m_1)}{1+(m_2-m_1)(m_2-m_1-x)} Q_{m_1}(dx) \in \mathcal{K}_+(Q_{m_1})$$

with mean m_2 . Applying (2.1) again, with $a_1 = 0$, $a_2 = a$, $a_3 = m_2 - m_1$ and $a_4 = m_1$, we get

$$\begin{aligned} \int_{-2}^2 \frac{\sqrt{4-x^2}}{(1+a(a-x))((1+(m_2-m_1)(m_2-m_1-x))((1+m_1(m_1-x))))} dx \\ = \frac{2\pi}{(1-(m_2-m_1)m_1)(1-am_1)(1-a(m_2-m_1))}, \end{aligned} \quad (2.4)$$

i.e.

$$\int_{-2}^2 \frac{(1-am_1)(1-a(m_2-m_1))}{1+a(a-x)} Q_{m_2, m_1}(dx) = 1.$$

As previously, after the appropriate choice of a we want to represent

$$\frac{(1-am_1)(1-a(m_2-m_1))}{1+a(a-x)}$$

as (1.3). From

$$\begin{aligned} \frac{(1 - am_1)(1 - a(m_2 - m_1))}{1 + a(a - x)} &= \frac{(1 - am_1)(1 - a(m_2 - m_1))}{1 + a^2 - ax} \\ &= \frac{1}{1 + \frac{a}{(1-am_1)(1-a(m_2-m_1))}(a(m_1^2 - m_2m_1 + 1) + m_2 - x)} \end{aligned} \quad (2.5)$$

we read out that with

$$a = \frac{m - m_2}{m_1^2 - m_2m_1 + 1}$$

the pseudo-variance function of the CSK family $\mathcal{K}_+(Q_{m_2, m_1})$ is

$$\begin{aligned} \mathbb{V}_2(m) &= \frac{(1 - am_1)(1 - a(m_2 - m_1))(am_1^2 - am_2m_1 + a + m_2)}{a} \\ &= \frac{m(1 - (m - m_1)m_1)((m_1 - m_2)(m + m_1 - m_2) + 1)}{(m - m_2)(m_1^2 - m_2m_1 + 1)}. \end{aligned} \quad (2.6)$$

So by (1.5) the corresponding variance function

$$v_2(m) = \frac{(1 - (m - m_1)m_1)((m_1 - m_2)(m + m_1 - m_2) + 1)}{(m_1^2 - m_2m_1 + 1)}$$

is a quadratic polynomial in m . The above argument works when $|a| < 1$, i.e. since we are in one-sided setting as in (1.2), for all $m_2 < m < m_2 + m_1^2 - m_2m_1 + 1$. (This variance function corresponds to an affine transformation of the free Meixner law.)

The calculations for the next iteration that would start with $Q_{m_3, m_2, m_1} \in \mathcal{K}_+(Q_{m_2, m_1})$ with mean m_3 , seems to be too cumbersome.

2.2. General approach. In this section, we show how to relate to the domains of means and the pseudo-variance functions of the original family $\mathcal{K}_+(\nu)$ and the new family $\mathcal{K}_+(Q_{m_1})$.

Fix $m_1 \in (m_0, m_+)$, and consider $Q_{m_1} = P_{\theta_1} \in \mathcal{K}_+(\nu)$, with $\theta_1 \in (0, \theta_+)$. Define

$$M_1(\theta) = \int \frac{1}{1 - \theta x} P_{\theta_1}(dx),$$

for $\theta \in \Theta = \{\theta \geq 0; M_1(\theta) < \infty\}$.

The CSK family generated by $Q_{m_1} = P_{\theta_1}$ is

$$\mathcal{K}_+(P_{\theta_1}) = \{\bar{P}_\theta(dx)\} = \left\{ \frac{1}{M_1(\theta)(1 - \theta x)} P_{\theta_1}(dx), \theta \in \Theta \right\}.$$

Proposition 2.2. (i) $\Theta = (0, \theta_+)$

(ii) For $\theta \in \Theta$, we have

$$M_1(\theta) = \begin{cases} \frac{\theta M(\theta) - \theta_1 M(\theta_1)}{M(\theta_1)(\theta - \theta_1)} & \text{if } \theta \neq \theta_1; \\ \frac{M(\theta_1) + \theta_1 M'(\theta_1)}{M(\theta_1)} & \text{if } \theta = \theta_1. \end{cases} \quad (2.7)$$

(iii) For $\theta \in \Theta$, we set $k(\theta) = \int xP_\theta(dx)$ the mean of P_θ , and $k_1(\theta) = \int x\bar{P}_\theta(dx)$, the mean of \bar{P}_θ . Then

$$k_1(\theta) = \begin{cases} \frac{\theta k(\theta) - \theta_1 k(\theta_1)}{(\theta - \theta_1) + \theta\theta_1(k(\theta) - k(\theta_1))} & \text{if } \theta \neq \theta_1; \\ \frac{k(\theta_1) + \theta_1 k'(\theta_1)}{1 + \theta_1^2 k'(\theta_1)} & \text{if } \theta = \theta_1. \end{cases} \quad (2.8)$$

Proof. (i) We have that

$$M_1(\theta) = \int \frac{1}{1 - \theta x} P_{\theta_1}(dx) = \int \frac{1}{M(\theta_1)(1 - \theta_1 x)(1 - \theta x)} \nu(dx).$$

As $\theta_1 \in (0, \theta_+)$, the function $x \mapsto \frac{1}{1 - \theta_1 x}$ is bounded on the support of ν , so that $M_1(\theta)$ exists for θ such that the integral $\int \frac{1}{1 - \theta x} \nu(dx)$ converges that is for θ in $(0, \theta_+)$.

(ii)

$$M_1(\theta) = \int \frac{1}{1 - \theta x} P_{\theta_1}(dx) = \int \frac{1}{M_1(\theta_1)(1 - \theta_1 x)(1 - \theta x)} \nu(dx),$$

If $\theta \neq \theta_1$, then,

$$\frac{1}{(1 - \theta x)(1 - \theta_1 x)} = \frac{\theta}{(\theta - \theta_1)(1 - \theta x)} - \frac{\theta_1}{(\theta - \theta_1)(1 - \theta_1 x)}.$$

It follows that

$$\begin{aligned} M_1(\theta) &= \frac{\theta}{M(\theta_1)(\theta - \theta_1)} \int \frac{1}{1 - \theta x} \nu(dx) - \frac{\theta_1}{M(\theta_1)(\theta - \theta_1)} \int \frac{1}{1 - \theta_1 x} \nu(dx) \\ &= \frac{\theta M(\theta) - \theta_1 M(\theta_1)}{M(\theta_1)(\theta - \theta_1)}. \end{aligned}$$

If $\theta = \theta_1$, then

$$M_1(\theta_1) = \int \frac{1}{M(\theta_1)(1 - \theta_1 x)^2} \nu(dx) = \frac{M(\theta_1) + \theta_1 M'(\theta_1)}{M(\theta_1)}.$$

(iii) We have that

$$k_1(\theta) = \frac{M_1(\theta) - 1}{\theta M_1(\theta)}.$$

If $\theta \neq \theta_1$, then

$$k_1(\theta) = \frac{\frac{\theta M(\theta) - \theta_1 M(\theta_1)}{M(\theta_1)(\theta - \theta_1)} - 1}{\theta \frac{\theta M(\theta) - \theta_1 M(\theta_1)}{M(\theta_1)(\theta - \theta_1)}} = \frac{M(\theta) - M(\theta_1)}{\theta M(\theta) - \theta_1 M(\theta_1)}.$$

As $M(\theta) = \frac{1}{1 - \theta k(\theta)}$, we obtain that

$$k_1(\theta) = \frac{\theta k(\theta) - \theta_1 k(\theta_1)}{(\theta - \theta_1) + \theta\theta_1(k(\theta) - k(\theta_1))}.$$

If $\theta = \theta_1$, then

$$k_1(\theta_1) = \frac{M_1(\theta_1) - 1}{\theta_1 M_1(\theta_1)} = \frac{\frac{(M(\theta_1) + \theta_1 M'(\theta_1))}{M(\theta_1)} - 1}{\theta_1 \frac{(M(\theta_1) + \theta_1 M'(\theta_1))}{M(\theta_1)}} = \frac{M'(\theta_1)}{M(\theta_1) + \theta_1 M'(\theta_1)}.$$

Given that

$$M'(\theta_1) = \left(\frac{1}{1 - \theta k(\theta)} \right)' \Big|_{\theta=\theta_1} = \frac{k(\theta_1) + \theta_1 k'(\theta_1)}{(1 - \theta_1 k(\theta_1))^2},$$

we obtain

$$k_1(\theta_1) = \frac{k(\theta_1) + \theta_1 k'(\theta_1)}{1 + \theta_1^2 k'(\theta_1)}.$$

□

Next we denote by $D_+(\nu)$ and \mathbb{V} the domain of the means and the pseudo-variance function of the family $\mathcal{K}_+(\nu)$, and by $D_+(Q_{m_1})$ and \mathbb{V}_1 the domain of the means and the pseudo-variance function of $\mathcal{K}_+(Q_{m_1})$. Recall that $D_+(\nu) = k((0, \theta_+))$ and $D_+(Q_{m_1}) = k_1((0, \theta_+))$. We set

$$m = k(\theta) \quad \text{and} \quad \bar{m} = k_1(\theta).$$

We will also use the inverse ψ of the function $\theta \mapsto k(\theta)$ from $(0, \theta_+)$ into (m_0, m_+) , and the inverse ψ_1 of the function $\theta \mapsto k_1(\theta)$ from $(0, \theta_+)$ into its image (\bar{m}_0, \bar{m}_+) .

Theorem 2.3. *Let ν be a probability measure with support bounded from above, and let $\mathcal{K}_+(\nu)$ be the CSK family generated by ν . Fix $m_1 \in (m_0, m_+)$ and let $B = B(\nu)$ be given by (1.11). With the notations introduced above, we have*

(i)

$$\bar{m} = k_1(\psi(m)) = \begin{cases} \frac{m^2 \mathbb{V}(m_1) - m_1^2 \mathbb{V}(m)}{m \mathbb{V}(m_1) - m_1 \mathbb{V}(m)} & \text{if } m \neq m_1 \\ \frac{2m_1 \mathbb{V}(m_1) - m_1^2 \mathbb{V}'(m_1)}{\mathbb{V}(m_1) - m_1 \mathbb{V}'(m_1)} & \text{if } m = m_1. \end{cases} \quad (2.9)$$

(ii) the (one sided) domain of means is

$$D_+(Q_{m_1}) = (\bar{m}_0, \bar{m}_+) = \left(m_1, \frac{m_+ G_\nu(B) - m_1^2 / \mathbb{V}(m_1)}{G_\nu(B) - m_1 / \mathbb{V}(m_1)} \right).$$

(interpreted as the limit $b \rightarrow B^+$.)

(iii)

$$\frac{\mathbb{V}_1(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m. \quad (2.10)$$

Note that the function $m \mapsto \bar{m}$ is a bijection from $D_+(\nu)$ into $D_+(Q_{m_1})$, so that to get explicitly the pseudo-variance function of the CSK family $\mathcal{K}_+(Q_{m_1})$, we need to express m in terms of \bar{m} from (2.9) and insert it in (2.10).

Proof. (i) Suppose that $m \neq m_1$.

$$\begin{aligned}
\bar{m} &= k_1(\psi(m)) \\
&= \frac{m\psi(m) - m_1\psi(m_1)}{(\psi(m) - \psi(m_1)) + \psi(m)\psi(m_1)(m - m_1)} \\
&= \frac{m^2(\mathbb{V}(m_1) + m_1^2) - m_1^2(\mathbb{V}(m) + m^2)}{m(\mathbb{V}(m_1) + m_1^2) - m_1(\mathbb{V}(m) + m^2) + mm_1(m - m_1)} \\
&= \frac{m^2\mathbb{V}(m_1) - m_1^2\mathbb{V}(m)}{m\mathbb{V}(m_1) - m_1\mathbb{V}(m)}.
\end{aligned}$$

For $m = m_1$, we have

$$\begin{aligned}
\bar{m}_1 = k_1(\psi(m_1)) &= k_1(\theta_1) = \frac{k(\theta_1) + \theta_1 k'(\theta_1)}{1 + \theta_1^2 k'(\theta_1)} = \lim_{\theta \rightarrow \theta_1} \frac{\theta k(\theta) - \theta_1 k(\theta_1)}{(\theta - \theta_1) + \theta\theta_1(k(\theta) - k(\theta_1))} \\
&= \lim_{m \rightarrow m_1} \frac{m^2\mathbb{V}(m_1) - m_1^2\mathbb{V}(m)}{m\mathbb{V}(m_1) - m_1\mathbb{V}(m)} \\
&= \lim_{m \rightarrow m_1} \frac{\mathbb{V}(m_1)\mathbb{V}(m)\left(\frac{m^2}{\mathbb{V}(m)} - \frac{m_1^2}{\mathbb{V}(m_1)}\right)}{\mathbb{V}(m_1)\mathbb{V}(m)\left(\frac{m}{\mathbb{V}(m)} - \frac{m_1}{\mathbb{V}(m_1)}\right)} \\
&= \frac{(m^2/\mathbb{V}(m))'}{(m/\mathbb{V}(m))'} \Big|_{m=m_1} \\
&= \frac{2m_1\mathbb{V}(m_1) - m_1^2\mathbb{V}'(m_1)}{\mathbb{V}(m_1) - m_1\mathbb{V}'(m_1)}.
\end{aligned}$$

(ii) Using the definition of the domain of means,

$$\bar{m}_0 = \lim_{\theta \rightarrow 0} k_1(\theta) = \lim_{m \rightarrow m_0} \frac{m^2\mathbb{V}(m_1) - m_1^2\mathbb{V}(m)}{m\mathbb{V}(m_1) - m_1\mathbb{V}(m)} = \lim_{m \rightarrow m_0} \frac{\frac{m^2}{\mathbb{V}(m)} - \frac{m_1^2}{\mathbb{V}(m_1)}}{\frac{m}{\mathbb{V}(m)} - \frac{m_1}{\mathbb{V}(m_1)}} = m_1.$$

$$\begin{aligned}
\bar{m}_+ &= \lim_{\theta \rightarrow \theta_+} k_1(\theta) = \lim_{m \rightarrow m_+} \frac{m^2\mathbb{V}(m_1) - m_1^2\mathbb{V}(m)}{m\mathbb{V}(m_1) - m_1\mathbb{V}(m)} = \lim_{m \rightarrow m_+} \frac{\frac{m^2}{\mathbb{V}(m)} - \frac{m_1^2}{\mathbb{V}(m_1)}}{\frac{m}{\mathbb{V}(m)} - \frac{m_1}{\mathbb{V}(m_1)}} \\
&= \lim_{b \rightarrow B^+} \frac{m_+ G_\nu(b) - \frac{m_1^2}{\mathbb{V}(m_1)}}{G_\nu(b) - \frac{m_1}{\mathbb{V}(m_1)}} = \frac{m_+ G_\nu(B) - \frac{m_1^2}{\mathbb{V}(m_1)}}{G_\nu(B) - \frac{m_1}{\mathbb{V}(m_1)}}.
\end{aligned}$$

(This is m_+ when $\lim_{b \rightarrow B^+} G(b) = \infty$.)

(iii) For $\theta \in (0, \theta_+)$ we have $\theta = \psi(k(\theta)) = \psi_1(k_1(\theta))$ so that $\psi(m) = \psi_1(\bar{m})$. By (1.7), this implies (2.10). \square

Note that as the probability measure Q_{m_1} has a finite first moment $\bar{m}_0 = m_1$, the variance function $v_1(\cdot)$ of the CSK family $\mathcal{K}_+(Q_{m_1})$ exists and from (1.5) we have

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{\bar{m} - m_1} v_1(\bar{m}).$$

2.3. Applications. The following examples illustrate the usefulness of Theorem 2.3, and provide examples of CSK families with rational variance functions.

2.3.1. CSK families with quadratic variance function. The CSK families with quadratic variance function have

$$v(m) = 1 + am + bm^2 = \mathbb{V}(m), \quad (2.11)$$

(we consider centered case here, with $m_0 = 0$).

Formula (2.9) gives that $m = \frac{\bar{m} - m_1}{1 + am_1 + bm_1\bar{m}}$.

Formula (2.10) gives that

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{(\bar{m} - m_1)(1 + am_1 + bm_1\bar{m})} P(\bar{m}),$$

where $P(\bar{m}) = (1 + \bar{m}(a + b\bar{m})) (1 + m_1(a - \bar{m} + (b + 1)m_1))$.

The corresponding variance function is

$$v_1(\bar{m}) = \frac{1}{1 + am_1 + bm_1\bar{m}} P(\bar{m}),$$

The following two special cases are of interest.

Example 2.4. The Wigner's semicircle (free Gaussian) law

$$\nu(dx) = \frac{\sqrt{4 - x^2}}{2\pi} 1_{(-2,2)}(x) dx,$$

has a constant variance function i.e. (2.11) holds with $a = b = 0$: $v(m) = 1 = \mathbb{V}(m)$ and the (one-sided) domain of means is $D_+(\nu) = (0, 1)$. (The full two-sided domain of means is of course $(-1, 1)$.) For $m_1 \in D_+(\nu)$, the probability measure

$$Q_{m_1}(dx) = \frac{\sqrt{4 - x^2}}{2\pi(1 + m_1(m_1 - x))} 1_{(-2,2)}(x) dx,$$

generates CSK family with pseudo-variance function $\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{\bar{m} - m_1} (-m_1\bar{m} + m_1^2 + 1)$, and domain of means $D_+(Q_{m_1}) = (m_1, 1 + m_1)$. The corresponding variance function is

$$v_1(\bar{m}) = -m_1\bar{m} + m_1^2 + 1. \quad (2.12)$$

Up to affine transformation, this is in fact the Marchenko-Pastur law, see next example.

Example 2.5. The (absolutely continuous) Marchenko-Pastur (free Poisson) law

$$\nu(dx) = \frac{\sqrt{4 - (x - a)^2}}{2\pi(1 + ax)} 1_{(a-2, a+2)}(x) dx$$

corresponds to (2.11) with $b = 0$ and $0 < a^2 < 1$. The variance function is $v(m) = 1 + am = \mathbb{V}(m)$, and the domain of means is $D_+(\nu) = (0, 1)$.

For $m_1 \in D_+(\nu)$, the probability measure

$$Q_{m_1}(dx) = \frac{(1 + am_1)\sqrt{4 - (x - a)^2}}{2\pi(1 + m_1(a + m_1 - x))(1 + ax)} 1_{(a-2, a+2)}(x) dx$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{(1 + am_1)(\bar{m} - m_1)} (1 + a\bar{m}) (1 + m_1(a + m_1 - \bar{m})).$$

The domain of means is

$$D_+(Q_{m_1}) = (m_1, 1 + (a + 1)m_1).$$

The variance function is

$$v_1(\bar{m}) = \frac{(1 + a\bar{m})(1 + m_1(a + m_1 - \bar{m}))}{1 + am_1}.$$

Example 2.6. For $a^2 > 1$, the Marchenko Pastur law is

$$\nu(dx) = \frac{\sqrt{4 - (x - a)^2}}{2\pi(1 + ax)} 1_{(a-2, a+2)}(x) dx + (1 - 1/a^2)\delta_{-1/a}(dx)$$

If $a > 1$, $B(\nu) = a + 2$ and the upper endpoint of the domain of means is $m_+ = 1$. In this case, $D_+(\nu) = (0, 1)$, and for $m_1 \in D_+(\nu)$, we have

$$Q_{m_1}(dx) = \frac{(1 + am_1)\sqrt{4 - (x - a)^2}}{2\pi(1 + m_1(a + m_1 - x))(1 + ax)} 1_{(a-2, a+2)}(x) dx + \frac{1 + am_1}{1 + m_1(a + m_1 + 1/a)} (1 - 1/a^2)\delta_{-1/a}(dx). \quad (2.13)$$

This distribution generates the CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{(1 + am_1)(\bar{m} - m_1)} (1 + a\bar{m}) (1 + m_1(a + m_1 - \bar{m})),$$

and domain of means

$$D_+(Q_{m_1}) = (m_1, 1 + (a + 1)m_1).$$

The variance function is

$$v_1(\bar{m}) = \frac{(1 + a\bar{m})(1 + m_1(a + m_1 - \bar{m}))}{1 + am_1}.$$

If $a < -1$, then $B(\nu) = -1/a$, and the domain of means is $D_+(\nu) = (0, -1/a)$. For $m_1 \in D_+(\nu)$, we have that

$$Q_{m_1}(dx) = \frac{(1 + am_1)\sqrt{4 - (x - a)^2}}{2\pi(1 + m_1(a + m_1 - x))(1 + ax)} 1_{(a-2, a+2)}(x) dx + \frac{1 + am_1}{1 + m_1(a + m_1 - x)} (1 - 1/a^2)\delta_{-1/a}(dx). \quad (2.14)$$

It generates the CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{(1 + am_1)(\bar{m} - m_1)} (1 + a\bar{m}) (1 + m_1(a + m_1 - \bar{m})),$$

with domain of means

$$D_+(Q_{m_1}) = (m_1, -1/a).$$

The variance function is, in this case,

$$v_1(\bar{m}) = \frac{(1 + a\bar{m})(1 + m_1(a + m_1 - \bar{m}))}{1 + am_1}.$$

2.3.2. *CSK families with cubic pseudo-variance function* : For $a > 0$, the cubic pseudo-variance function

$$\mathbb{V}(m) = m(am^2 + bm + c) \tag{2.15}$$

corresponds to CSK families without variance. Formula (2.9) gives that $m = -\frac{\bar{m}(b + am_1) + c}{a(\bar{m} - m_1)}$.

Formula (2.10) gives $\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{a(\bar{m} - m_1)^2}Q(\bar{m})$, so the corresponding variance function is

$$v_1(\bar{m}) = \frac{1}{a(\bar{m} - m_1)}Q(\bar{m}),$$

with $Q(\bar{m}) = (c + \bar{m}(b + a\bar{m}))(c - \bar{m} + m_1(b + am_1 + 1))$.

The following special cases are of interest

Example 2.7. The Free Abel (or Free Borel-Tanner) law

$$\nu(dx) = \frac{1}{\pi(1-x)\sqrt{-x}}1_{(-\infty,0)}(x)dx$$

has domain of means $D_+(\nu) = (-\infty, 0)$ and pseudo-variance function $\mathbb{V}(m) = m^2(m - 1)$. For $m_1 \in D_+(\nu)$, probability measure

$$Q_{m_1}(dx) = \frac{m_1(m_1 - 1)}{\pi(m_1^2 - x)(1-x)\sqrt{-x}}1_{(-\infty,0)}(x)dx,$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}^2}{(\bar{m} - m_1)^2}(1 - \bar{m})(\bar{m} - m_1^2),$$

and the domain of means $D_+(Q_{m_1}) = (m_1, 0)$. The corresponding variance function is

$$v_1(\bar{m}) = \frac{\bar{m}}{\bar{m} - m_1}(1 - \bar{m})(\bar{m} - m_1^2).$$

Example 2.8. The free Ressel (or free Kendall) law

$$\nu(dx) = \frac{-1}{\pi x \sqrt{-1-x}}1_{(-\infty,-1)}(x)dx$$

has domain of means $D_+(\nu) = (-\infty, -2)$ and the pseudo-variance function $\mathbb{V}(m) = m^2(m + 1)$. For $m_1 \in D_+(\nu)$, the probability measure

$$Q_{m_1}(dx) = \frac{-m_1(1 + m_1)}{\pi x(m_1^2 + 2m_1 - x)\sqrt{-1-x}}1_{(-\infty,-1)}(x)dx,$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}^2}{(\bar{m} - m_1)^2}(-\bar{m}^2 + (m_1^2 + 2m_1 - 1)\bar{m} + m_1^2 + 2m_1).$$

and the domain of means

$$D(Q_{m_1}) = \left(m_1, \frac{2m_1}{1 - m_1} \right).$$

The corresponding variance function is

$$v_1(\bar{m}) = \frac{\bar{m}}{\bar{m} - m_1} (\bar{m} + 1)(m_1^2 + 2m_1 - \bar{m}).$$

Example 2.9. The free strict arcsine law

$$\nu(dx) = \frac{\sqrt{3 - 4x}}{2\pi(1 + x^2)} 1_{(-\infty, 3/4)}(x) dx$$

has pseudo-variance function $\mathbb{V}(m) = m(1 + m^2)$, and the domain of means $D_+(\nu) = (-\infty, -1/2)$. For $m_1 \in D_+(\nu)$, probability measure

$$Q_{m_1}(dx) = \frac{(m_1^2 + 1)\sqrt{3 - 4x}}{2\pi(m_1^2 + m_1 + 1 - x)(1 + x^2)} 1_{(-\infty, 3/4)}(x) dx$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}}{(\bar{m} - m_1)^2} (-\bar{m}^3 + (m_1^2 + m_1 + 1)\bar{m}^2 - \bar{m} + (m_1^2 + m_1 + 1))$$

and with domain of means

$$D_+(Q_{m_1}) = \left(m_1, \frac{2 + m_1}{1 - 2m_1} \right).$$

The corresponding variance function is

$$v_1(\bar{m}) = \frac{1 + \bar{m}^2}{\bar{m} - m_1} (m_1^2 + m_1 + 1 - \bar{m}).$$

Example 2.10. The inverse semicircle law

$$\nu(dx) = \frac{p\sqrt{-p^2 - 4x}}{2\pi x^2} 1_{(-\infty, -p^2/4)}(x) dx,$$

corresponds to (2.15) with $a = 1/p^2$, $b = c = 0$. The pseudo-variance function is $\mathbb{V}(m) = m^3/p^2$, and the domain of means is $D_+(\nu) = (-\infty, -p^2)$.

For $m_1 \in D_+(\nu)$, probability measure

$$Q_{m_1}(dx) = \frac{pm_1^2\sqrt{-p^2 - 4x}}{2\pi x^2(m_1^2 + p^2(m_1 - x))} 1_{(-\infty, -p^2/4)}(x) dx$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_1(\bar{m}) = \frac{\bar{m}^3}{(\bar{m} - m_1)^2} (m_1^2/p^2 + m_1 - \bar{m}),$$

and with domain of means

$$D_+(Q_{m_1}) = \left(m_1, \frac{p^2 m_1}{p^2 - m_1} \right).$$

The corresponding variance function is

$$v_1(\bar{m}) = \frac{\bar{m}^2}{(\bar{m} - m_1)} (m_1^2/p^2 + m_1 - \bar{m}).$$

3. EXTENDING THE DOMAIN FOR PARAMETRIZATION BY THE MEAN

Given a compactly supported measure ν , Proposition 1.1 tells us how to determine the one-sided domain of means (m_0, m_+) and how to compute the pseudo-variance function $\mathbb{V}(m)$ for $m \in (m_0, m_+)$. (There is a similar result for the two-sided domain of means, see [BH11, Remark 3.3].) But the pseudo-variance function is often well defined for other values of m , too. So it is natural to ask whether the corresponding "family of measures" can also be enlarged. The following example illustrates the idea, drawing on well known properties of the Marchenko-Pastur law.

Example 3.1. Consider the (two-sided) CSK family generated by the semicircle law $\nu = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|<2}dx$ with the variance function $v(m) = \mathbb{V}(m) = 1$, the domain of means $(-1, 1)$ and

$$\mathcal{K}(\nu) = \left\{ \frac{\sqrt{4-x^2}}{2\pi(1+m(m-x))}1_{|x|<2}dx : m \in (-1, 1) \right\}.$$

This is a family of atomless Marchenko-Pastur laws, which can be naturally enlarged to include all Marchenko-Pastur laws:

$$\overline{\mathcal{K}}(\nu) = \left\{ \pi_m(dx) = \frac{\sqrt{4-x^2}}{2\pi(1+m(m-x))}1_{|x|<2}dx + (1-1/m^2)^+\delta_{m+1/m} : m \in (-\infty, \infty) \right\}$$

Noting that

$$\int \pi_m(dx) = 1, \quad \int x\pi_m(dx) = m, \quad \int (x-m)^2\pi_m(dx) = 1,$$

we see that $v(m) = 1$ is the variance function of this enlarged family.

Of course, it may also happen that the extension beyond the natural domain of means is not possible. family is full.

Example 3.2. Let $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution. Then $M(\theta) = \frac{1}{1-\theta^2}$ and $m(\theta) = \theta$. The (two-sided) range of parameter is $\Theta = (-1, 1)$. So the domain of means here is $(-1, 1)$, and with $m_0 = 0$ the pseudo-variance function is equal to the variance function,

$$v(m) = \mathbb{V}(m) = 1 - m^2.$$

In this case, the variance function is negative outside the domain of means, so we cannot extend the family $\{Q_m : m \in (-1, 1)\}$ beyond the original domain of means while preserving the variance function $v(m)$, and the relation between $v(m)$ and the Cauchy-Stieltjes transform.

Our next example shows that the extension sometimes may proceed in two separate steps.

Example 3.3. Consider the inverse semicircle law from Example 2.10 with $p = 1$. Since $m^2 + m \geq -1/4$, it is clear that measure Q_m is non-negative and well defined for all m . Since the integral $\int Q_m(dx)$ is an analytic function of $m < -1/2$, it must be 1, so Q_m is a probability measure for all $m < -1/2$. This is the "first part" of the extension, from $(-\infty, -1)$ to a larger interval $(-\infty, -1/2)$.

At $m = -1/2$ the integrand has singularity at $x = -1/4$ but the integral is still 1, see the calculation below. For $m > -1/2$, the mass becomes less than one, as $\int Q_m(dx) =$

$m^2/(1+m)^2$. So for $m > -1/2$ we can define a new probability measure

$$\bar{Q}_m(dx) = Q_m(dx) + \left(1 - \frac{m^2}{(1+m)^2}\right) \delta_{m+m^2}(dx) = Q_m(dx) + \frac{(1+2m)_+}{(1+m)^2} \delta_{m+m^2}(dx) \quad (3.1)$$

with extra mass in the atomic part.

The definition (1.7) of pseudo-variance is not directly applicable beyond $m > -1$. However, if we use relation (1.10), then $\mathbb{V}(m) = m^3$ also for $m > -1$. Thus we may claim that the family $\{\bar{Q}_m(dx)\}$ extends the domain of means for $\mathbb{V}(m) = m^3$ to $(-\infty, \infty)$.

We now prove the above two claims.

Proof of the claims in Example 3.3. By the change of variable $t = \sqrt{-1-4x}$ in

$$\int Q_m(dx) = \int_{-\infty}^{-1/4} \frac{m^2 \sqrt{-1-4x}}{2\pi x^2 (m^2 + m - x)} dx,$$

we obtain

$$\int Q_m(dx) = \frac{16m^2}{\pi} \int_0^{+\infty} \frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)} dt.$$

The integrand can be decomposed as follows

$$\begin{aligned} \frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)} &= \frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+1)} - \frac{1}{((2m+1)^2-1)(t^2+1)^2} \\ &\quad - \frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+(2m+1)^2)}. \end{aligned}$$

For real numbers $a, b, r \neq 0$, we denote $J_n = \int_a^b \frac{dx}{(x^2+r^2)^n}$. Then we have

$$J_{n+1} = \frac{1}{2nr^2} \left((2n-1)J_n + \left[\frac{x}{(x^2+r^2)^n} \right]_a^b \right).$$

Using this, we get:

For $m = -1/2$,

$$\begin{aligned} \int Q_{-1/2}(dx) &= \frac{4}{\pi} \int_0^{+\infty} \frac{1}{(t^2+1)^2} dt \\ &= \frac{4}{\pi} \left(\frac{1}{2} \left(\frac{\pi}{2} + \left[\frac{x}{1+x^2} \right]_0^{+\infty} \right) \right) = 1. \end{aligned}$$

For $m \neq -1/2$

$$\begin{aligned}
\int Q_m(dx) &= \frac{16m^2}{\pi} \int_0^{+\infty} \frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)} dt \\
&= \frac{16m^2}{\pi} \left(\int \frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+1)} dt - \int \frac{1}{((2m+1)^2-1)(t^2+1)^2} dt \right. \\
&\quad \left. - \int \frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+(2m+1)^2)} dt \right) \\
&= \frac{16m^2}{\pi} \left(\frac{(2m+1)^2}{((2m+1)^2-1)^2} [\arctan(t)]_0^{+\infty} - \frac{1/2}{(2m+1)^2-1} \left(\frac{\pi}{2} + \left[\frac{t}{(t^2+1)^2} \right]_0^{+\infty} \right) \right. \\
&\quad \left. - \frac{(2m+1)}{((2m+1)^2-1)^2} \left[\arctan\left(\frac{t}{2m+1}\right) \right]_0^{+\infty} \right).
\end{aligned}$$

If $m < -1/2$

$$\begin{aligned}
\int Q_m(dx) &= \frac{16m^2}{\pi} \left(\frac{(2m+1)^2}{((2m+1)^2-1)^2} \frac{\pi}{2} - \frac{1}{((2m+1)^2-1)} \frac{\pi}{4} \right. \\
&\quad \left. - \frac{(2m+1)}{((2m+1)-1)^2} \left(-\frac{\pi}{2}\right) \right) = 1.
\end{aligned}$$

If $m > -1/2$

$$\begin{aligned}
\int Q_m(dx) &= \frac{16m^2}{\pi} \left(\frac{(2m+1)^2}{((2m+1)^2-1)^2} \frac{\pi}{2} - \frac{1}{((2m+1)^2-1)} \frac{\pi}{4} \right. \\
&\quad \left. - \frac{(2m+1)}{((2m+1)-1)^2} \frac{\pi}{2} \right) = \frac{m^2}{(1+m)^2}.
\end{aligned}$$

We now verify that the atomic part works as needed.

By the change of variable $t = \sqrt{-1-4x}$ from (3.1) we get

$$\begin{aligned}
\int xQ_m(dx) &= \int_{-\infty}^{-1/4} \frac{m^2\sqrt{-1-4x}}{2\pi x(m^2+m-x)} dx \\
&= -\frac{4m^2}{\pi} \int_0^{+\infty} \frac{t^2}{(t^2+1)((2m+1)^2+t^2)} dt \\
&= -\frac{4m^2}{\pi} \left(-\int_0^{+\infty} \frac{1}{4m(1+m)(t^2+1)} dt + \int_0^{+\infty} \frac{(2m+1)^2}{4(m^2+m)((2m+1)^2+t^2)} dt \right) \\
&= -\frac{4m^2}{\pi} \left(\frac{-1}{4m(1+m)} [\arctan(t)]_0^{+\infty} + \frac{(2m+1)}{4m(1+m)} \left[\arctan\left(\frac{t}{2m+1}\right) \right]_0^{+\infty} \right) \\
&= -\frac{4m^2}{\pi} \left(\frac{-1}{4m(1+m)} \frac{\pi}{2} + \frac{(2m+1)}{4m(1+m)} \frac{\pi}{2} \right) = -\frac{m^2}{1+m}.
\end{aligned}$$

So

$$\int x\bar{Q}_m(dx) = -\frac{m^2}{1+m} + \frac{1-2m}{(1+m)^2} m(1+m) = m$$

as expected. □

We now give a general theory that shows how the two-step extension works.

3.1. The first extension. Suppose that the pseudo-variance function \mathbb{V} extends as a real analytic function to $(m_0, +\infty)$. Recall notation (1.11) and define

$$\mathbf{m}_+(\nu) = \inf\left\{m > m_0 : m + \frac{\mathbb{V}(m)}{m} = A(\nu)\right\}. \quad (3.2)$$

From Remark 1.2 we know that $\mathbf{m}_+(\nu) \geq m_+$ is well defined. We will verify that one can use (1.3) to extend the domain of means to $(m_0, \mathbf{m}_+(\nu))$, preserving the pseudo-variance function. (The definition (1.7) of pseudo-variance is not directly applicable beyond $m > m_+$, so we use an equivalent definition).

Theorem 3.4. *Formula (1.3) defines the family of probability measures $\{Q_m(dx) : m \in (m_0, \mathbf{m}_+)\}$, parametrized by the mean $m = \int xQ_m(dx)$. The Cauchy-Stieltjes transform of the generating measure ν satisfies (1.10) with z given by (1.8) for all $m \in (m_0, \mathbf{m}_+)$. In particular, if ν has finite first moment m_0 then for $m \in (m_0, \mathbf{m}_+)$ the variance of $Q_m(dx)$ is given by (1.5).*

The rest of this section contains proof of Theorem 3.4.

We consider the set Θ for which the transform (1.1) exists.

In fact, if $A(\nu) \geq 0$, then $\Theta = (0, \theta_+)$ with $\theta_+ = \frac{1}{B}$, and if $A(\nu) < 0$, then

$$\Theta = \left(-\infty, \frac{1}{A(\nu)}\right) \cup (0, \infty). \quad (3.3)$$

One can always write

$$\Theta = \left(0, \frac{1}{B}\right) \cup \left(\frac{\text{sign}(A(\nu))}{B}, \frac{1}{A(\nu)}\right)$$

with

$$\text{sign}(A(\nu)) = \begin{cases} 1, & \text{if } A(\nu) \geq 0 \\ -1, & \text{if } A(\nu) < 0 \end{cases}.$$

One can then define the first extension of $\mathcal{K}_+(\nu)$ as

$$\bar{\mathcal{K}}_+(\nu) = \{P_\theta(dx) = \frac{1}{M(\theta)(1-\theta x)}\nu(dx) ; \theta \in \left(\frac{\text{sign}(A(\nu))}{B}, \frac{1}{A(\nu)}\right) \cup (0, \frac{1}{B})\}.$$

Note that $\bar{\mathcal{K}}_+(\nu) = \mathcal{K}_+(\nu)$ when $A(\nu) \geq 0$, because in this case $\left(\frac{\text{sign}(A(\nu))}{B}, \frac{1}{A(\nu)}\right) = \emptyset$.

Therefore, the first extension is non-trivial only when $A(\nu) < 0$.

Proposition 3.5. *Suppose $A(\nu) < 0$. For $\theta \in \Theta = \left(-\infty, \frac{1}{A(\nu)}\right) \cup (0, \infty)$ the mean*

$$k(\theta) = \int xP_\theta(dx) = \frac{M(\theta) - 1}{\theta M(\theta)}, \quad (3.4)$$

is strictly increasing on $(0, \infty)$ and on $\left(-\infty, \frac{1}{A(\nu)}\right)$

Proof. It is known ([BH11]) that the function $k(\cdot)$ is strictly increasing on $(0, \infty)$, we will use the same reasoning to show that it is also increasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$. We first observe

that for $\theta \in \left(-\infty, \frac{1}{A(\nu)}\right)$, the expression $(1 - \theta x)$ is negative for all x in the support of ν . In fact, $x < A(\nu)$ implies that $\theta x > \theta A(\nu) > 1$, that is $1 - \theta x < 1 - \theta A(\nu) < 0$. Hence

$$\begin{aligned} \int \frac{|x|}{(1 - \theta x)^2} \nu(dx) &= \frac{1}{|\theta|} \int \frac{|\theta x - 1 + 1|}{(1 - \theta x)^2} \nu(dx) \\ &\leq \left(-\frac{1}{\theta}\right) \int \frac{|\theta x - 1|}{(1 - \theta x)^2} \nu(dx) + \left(-\frac{1}{\theta}\right) \int \frac{1}{(1 - \theta x)^2} \nu(dx) \\ &\leq \frac{M(\theta)}{\theta} + \left(-\frac{1}{\theta}\right) \frac{M(\theta)}{1 - \theta A(\nu)} < \infty. \end{aligned}$$

Now fix $-\infty < \alpha < \beta < \frac{1}{A(\nu)}$. For $x \in \text{supp}(\nu) \subset (-\infty, 0)$, the function

$$\theta \mapsto \frac{\partial}{\partial \theta} \left(\frac{1}{1 - \theta x} \right) = \frac{x}{(1 - \theta x)^2}$$

is decreasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$, so for all $\theta \in [\alpha, \beta]$,

$$\frac{x}{(1 - \beta x)^2} \leq \frac{x}{(1 - \theta x)^2} \leq \frac{x}{(1 - \alpha x)^2}.$$

We define for $x \in \text{supp}(\nu)$

$$g(x) = \frac{|x|}{(1 - \alpha x)^2} + \frac{|x|}{(1 - \beta x)^2}.$$

Then $g \geq 0$, and g is ν -integrable, because α and β are in $\left(-\infty, \frac{1}{A(\nu)}\right)$, and $\frac{\partial}{\partial \theta} \left(\frac{1}{1 - \theta x} \right) = \frac{x}{(1 - \theta x)^2} \leq g(x)$, for all $\theta \in [\alpha, \beta]$. Thus, one can differentiate $M(\theta)$ under the integral sign and formula (1.4) gives

$$k'(\theta) = \frac{M(\theta) + \theta M'(\theta) - M(\theta)^2}{(\theta M(\theta))^2}.$$

The fact that

$$M(\theta) + \theta M'(\theta) - M(\theta)^2 = \int \frac{1}{(1 - \theta x)^2} \nu(dx) - \left(\int \frac{1}{1 - \theta x} \nu(dx) \right)^2 \geq 0$$

implies that the function $\theta \mapsto k(\theta)$ is increasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$. □

We have that

$$\begin{aligned} \lim_{\theta \rightarrow -\infty} k(\theta) &= \lim_{\theta \rightarrow -\infty} \frac{M(\theta) - 1}{\theta M(\theta)} \\ &= \lim_{\theta \rightarrow -\infty} \frac{\frac{1}{\theta} G_\nu\left(\frac{1}{\theta}\right) - 1}{G_\nu\left(\frac{1}{\theta}\right)} \\ &= 0 - \frac{1}{G_\nu(0)} = B - \frac{1}{G_\nu(B)} = m_+. \end{aligned}$$

For the proof of Theorem 3.4 instead of using (3.2), we define

$$\mathbf{m}_+(\nu) = \lim_{\theta \rightarrow \frac{1}{A(\nu)}} k(\theta). \quad (3.5)$$

(We will later verify that this coincides with (3.2) when $A(\nu) < 0$.) Then, the function $k(\cdot)$ realizes a bijection from $(-\infty, \frac{1}{A(\nu)})$ onto its image $(m_+, \mathbf{m}_+(\nu))$. We then define the function ψ on (m_0, m_+) as the inverse of the restriction of $k(\cdot)$ to $(0, \infty)$, and on $(m_+, \mathbf{m}_+(\nu))$ as the inverse of the restriction of $k(\cdot)$ to $(-\infty, \frac{1}{A(\nu)})$. This leads to the parametrization by the mean $m \in (m_0, m_+) \cup (m_+, \mathbf{m}_+(\nu))$ of the family $\bar{\mathcal{K}}_+(\nu)$. The definition of the pseudo-variance function can also be extended using the function ψ . Following (1.7), we define $\mathbb{V}(\cdot)$ for $m \in (m_0, m_+) \cup (m_+, \mathbf{m}_+(\nu))$ as

$$\mathbb{V}(m) = m \left(\frac{1}{\psi(m)} - m \right).$$

We have that

$$\lim_{m \rightarrow (m_+)^-} \frac{1}{\psi(m)} = 0 = \lim_{m \rightarrow (m_+)^+} \frac{1}{\psi(m)},$$

so that we define $\mathbb{V}(\cdot)$ at m_+ by $\mathbb{V}(m_+) = -m_+^2$. Note that $Q_{m_+}(dx) = \frac{m_+}{x} \nu(dx)$ is well defined for $A(\nu) < 0$.

The explicit parametrization by the means of the enlarged family can then be given by

$$\bar{\mathcal{K}}_+(\nu) = \{Q_m(dx) = \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)} \nu(dx) ; m \in (m_0, \mathbf{m}_+(\nu))\}.$$

The function $m \mapsto \psi(m) = \frac{1}{\mathbb{V}(m)/m + m}$ is increasing on $(m_+, \mathbf{m}_+(\nu))$, so the function $m \mapsto \mathbb{V}(m)/m + m$ is decreasing on $(m_+, \mathbf{m}_+(\nu))$ and

$$\lim_{m \rightarrow \mathbf{m}_+(\nu)} \mathbb{V}(m)/m + m = A(\nu).$$

This implies that (3.2) holds when $A(\nu) < 0$.

If $A(\nu) \geq 0$, then (3.2) gives $\mathbf{m}_+(\nu) = m_+$ because $m_+ + \frac{\mathbb{V}(m_+)}{m_+} = \frac{1}{\theta_+} = B = A(\nu)$, and then $\bar{\mathcal{K}}_+(\nu) = \mathcal{K}_+(\nu)$. This ends the proof of Theorem 3.4.

4. THE SECOND EXTENSION

As indicated by Examples 3.1 and 3.3, family $\bar{\mathcal{K}}_+(\nu)$ may have a further extension. Define

$$\mathbf{M}_+ = \inf\{m > m_0 : \mathbb{V}(m)/m < 0\}. \quad (4.1)$$

From Remark 1.2(iii) it is clear that $\mathbf{M}_+ \geq m_+$. In fact, $\mathbf{M}_+ \geq \mathbf{m}_+$. This can be seen from (3.2): since the mean must be smaller than $A(\nu)$ we have $\mathbf{m}_+ \leq A(\nu)$, so $\mathbb{V}(m)/m \geq 0$ for all $m < \mathbf{m}_+$.

It is easy to see that $\mathbf{M}_+ = \infty > \mathbf{m}_+$ in Example 3.1 and in Example 3.3 while $\mathbf{M}_+ = \mathbf{m}_+ = m_+$ in Example 3.2.

We now introduce the second extension of the family $\mathcal{K}_+(\nu)$ as the family of measures

$$\bar{\bar{\mathcal{K}}}_+(\nu) = \{\bar{Q}_m(dx) : m_0 < m < \mathbf{M}_+(\nu)\},$$

with \bar{Q}_m given by

$$\bar{Q}_m(dx) = \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)}\nu(dx) + p(m)\delta_{m+\mathbb{V}(m)/m}, \quad (4.2)$$

where the weight of the atom is

$$p(m) = \begin{cases} 0 & \text{if } m < m_+ := B - \frac{1}{G_\nu(B)} \\ 1 - \frac{\mathbb{V}(m)}{m}G_\nu\left(m + \frac{\mathbb{V}(m)}{m}\right) & \text{if } m > m_+ \text{ and } \mathbb{V}(m)/m \geq 0 \end{cases}$$

Since formula (1.10) holds for all $m \in (m_0, \mathbf{m}_+)$, it is clear that $\bar{\mathcal{K}}_+(\nu) \subset \bar{\bar{\mathcal{K}}}_+(\nu)$. We now verify that the extension satisfies desired conditions.

Theorem 4.1. *Let $\mathbf{m}_+ < m < \mathbf{M}_+$. Then (4.2) defines a probability measure $\bar{Q}_m(dx)$ with mean m , and if ν has finite first moment m_0 then the variance of \bar{Q}_m is*

$$\int (x-m)^2 \bar{Q}_m(dx) = \frac{(m-m_0)\mathbb{V}(m)}{m}. \quad (4.3)$$

Here the use of $\mathbb{V}(m)$ is based on the assumption the pseudo-variance function \mathbb{V} extends as a real analytic function to $(m_0, +\infty)$. We will show later the definition of the pseudo-variance function \mathbb{V} may extended to $\mathbf{m}_+ < m < \mathbf{M}_+$.

Since Marchenko-Pastur law is free-infinitely divisible, from [Bry09, Example 4.1] one can see that there is no "one simple formula" for $m_+(\nu)$ under the free convolution power. On the other hand, the domain of means for exponential families scales nicely under classical convolution power, and it is satisfying to note that the extended domain of means lead to the analogous formula:

$$\mathbf{M}_+(\nu^{\boxplus\alpha}) = \alpha\mathbf{M}_+(\nu). \quad (4.4)$$

Indeed, since $\mathbb{V}_{\nu^{\boxplus\alpha}}(m) = \alpha\mathbb{V}_\nu(m/\alpha)$, see [BH11, (3.17)], the result follows from (4.1).

The rest of this section contains proof of Theorem 4.1. In the proof, we focus on the behavior of the function

$$h(m) = \frac{\mathbb{V}(m)}{m} + m, \text{ for } m > \mathbf{m}_+(\nu), \quad (4.5)$$

where $\mathbf{m}_+(\nu)$ is defined by (3.2). In order to make clear the idea, we first study some examples,

Example 4.2. The Wigner's semicircle (free Gaussian) law.

$$\nu(dx) = \frac{\sqrt{4-x^2}}{2\pi} 1_{(-2,2)}(x)dx,$$

has a constant variance function $v(m) = 1 = \mathbb{V}(m)$ and the (one-sided) domain of means is $D_+(\nu) = (0, 1)$, and $\Theta = (0, \theta_+) = (0, 1/2)$.

$\mathbf{m}_+(\nu) = m_+ = 1$. We observe that, for $m < \mathbf{m}_+(\nu)$, there exists a unique $\bar{m} \geq \mathbf{m}_+(\nu)$, such that

$$\frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m.$$

In fact, $\bar{m} = \frac{1}{m} = \frac{\mathbf{m}_+^2(\nu)}{m}$.

Example 4.3. The (absolutely continuous) Marchenko-Pastur law

$$\nu(dx) = \frac{\sqrt{4 - (x - a)^2}}{2\pi(1 + ax)} 1_{(a-2, a+2)}(x) dx$$

with $0 < a^2 < 1$. The variance function is $v(m) = 1 + am = \mathbb{V}(m)$, and the domain of means is $D_+(\nu) = (0, 1)$, and $\Theta = (0, \theta_+) = (0, 1/2)$.

$\mathbf{m}_+(\nu) = m_+ = 1$. We also observe that for $m < \mathbf{m}_+(\nu)$, there exists a unique $\bar{m} = \frac{1}{m} = \frac{\mathbf{m}_+^2(\nu)}{m} \geq \mathbf{m}_+(\nu)$, such that

$$\frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m.$$

Example 4.4. The free strict arcsine law

$$\nu(dx) = \frac{\sqrt{3 - 4x}}{2\pi(1 + x^2)} 1_{(-\infty, 3/4)}(x) dx$$

has pseudo-variance function $\mathbb{V}(m) = m(1 + m^2)$, the domain of means $D_+(\nu) = (-\infty, -1/2)$ and $\Theta = (0, \theta_+) = (0, 4/3)$.

$\mathbf{m}_+(\nu) = m_+ = -1/2$, and for $m < \mathbf{m}_+(\nu)$, there exists a unique $\bar{m} = -m - 1 = -m + 2\mathbf{m}_+(\nu) \geq \mathbf{m}_+(\nu)$, such that

$$\frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m.$$

Example 4.5 (compare Example 3.3). The inverse semicircle law

$$\nu(dx) = \frac{p\sqrt{-p^2 - 4x}}{2\pi x^2} 1_{(-\infty, -p^2/4)}(x) dx,$$

has the pseudo-variance function $\mathbb{V}(m) = m^3/p^2$, and the domain of means is $D_+(\nu) = (-\infty, -p^2)$. Consider the inverse semicircle law with $p = 1$. $\Theta = (-\infty, -4) \cup (0, +\infty) = (-\infty, 1/A(\nu)) \cup (0, +\infty)$.

$\mathbf{m}_+(\nu) = -1/2 > m_+ = -1$. For $m \leq \mathbf{m}_+(\nu)$, there exists a unique $\bar{m} = -m - 1 = -m + 2\mathbf{m}_+(\nu) \geq \mathbf{m}_+(\nu)$, such that

$$\frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m.$$

Example 4.6. The free Ressel law

$$\nu(dx) = \frac{-1}{\pi x \sqrt{-1 - x}} 1_{(-\infty, -1)}(x) dx$$

has domain of means $D_+(\nu) = (-\infty, -2)$, the pseudo-variance function $\mathbb{V}(m) = m^2(m + 1)$, and $\Theta = (-\infty, -1) \cup (0, +\infty) = (-\infty, 1/A(\nu)) \cup (0, +\infty)$.

$\mathbf{m}_+(\nu) = -1 > m_+ = -2$. For $m \leq \mathbf{m}_+(\nu)$, there exists a unique $\bar{m} = -m - 2 = -m + 2\mathbf{m}_+(\nu) \geq \mathbf{m}_+(\nu)$, such that

$$\frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m} = \frac{\mathbb{V}(m)}{m} + m.$$

4.1. **Proof of Theorem 4.1.** Without loss of generality we suppose that $\mathbf{m}_+(\nu) < +\infty$.

Definition 4.7. For $m_1 \in (m_0, \mathbf{m}_+(\nu))$, we define the set

$$\mathcal{V}_{m_1} = \left\{ m \geq \mathbf{m}_+(\nu) ; \frac{\mathbb{V}(m)}{m} + m = \frac{\mathbb{V}(m_1)}{m_1} + m_1 \right\}.$$

Since \mathbb{V} is assumed analytic, \mathcal{V}_m is a (possibly empty) countable set with no accumulation points.

Proposition 4.8. *If for $m_1 \in (m_0, \mathbf{m}_+(\nu))$, $\mathcal{V}_{m_1} \neq \emptyset$, then for m such that $m_1 \leq m \leq \mathbf{m}_+(\nu)$, $\mathcal{V}_m \neq \emptyset$.*

Proof. Consider the function $h : m \mapsto \mathbb{V}(m)/m + m$ and suppose that for $m_1 \in (m_0, \mathbf{m}_+(\nu))$, $\mathcal{V}_{m_1} \neq \emptyset$, then there exists $m'_1 \geq \mathbf{m}_+(\nu)$ such that

$$\mathbb{V}(m_1)/m_1 + m_1 = \mathbb{V}(m'_1)/m'_1 + m'_1.$$

We have that

$$h(\mathbf{m}_+(\nu)) = A(\nu) \quad \text{and} \quad h(m'_1) = \mathbb{V}(m_1)/m_1 + m_1.$$

For $y \in (A(\nu), \mathbb{V}(m_1)/m_1 + m_1)$, by continuity of h , there exists $m' \in (\mathbf{m}_+(\nu), m'_1)$ such that $y = h(m') = \mathbb{V}(m')/m' + m'$.

In other words, for all $m \in (m_1, \mathbf{m}_+(\nu))$, there exists $m' \in (\mathbf{m}_+(\nu), m'_1)$ such that $h(m) = y = h(m')$, then $\mathcal{V}_m \neq \emptyset$. \square

Remark 4.9. From this proposition, it follows that the set of m belonging to $(m_0, \mathbf{m}_+(\nu))$ such that $\mathcal{V}_m \neq \emptyset$ is an interval.

Define

$$\tilde{\mathbf{m}} = \inf\{m \in (m_0, \mathbf{m}_+(\nu)), \text{ such that } \mathcal{V}_m \neq \emptyset\}.$$

For $m \in (\tilde{\mathbf{m}}, \mathbf{m}_+(\nu))$ let

$$\bar{m} = \inf\{\mathcal{V}_m\}.$$

Note that it may happen that $\mathcal{V}_m = \emptyset$ for all $m \in (m_0, \mathbf{m}_+(\nu))$, see Example 3.2. However, when $\tilde{\mathbf{m}} < \mathbf{m}_+(\nu)$ we have the following.

Proposition 4.10. *The function $g : m \mapsto \bar{m}$ is (strictly) decreasing on $(\tilde{\mathbf{m}}, \mathbf{m}_+(\nu))$.*

Proof. Let $m_1, m_2 \in (\tilde{\mathbf{m}}, \mathbf{m}_+(\nu))$ such that $m_1 < m_2$, the fact that the function h from (4.5) is decreasing on $(m_0, \mathbf{m}_+(\nu))$ implies that

$$\frac{\mathbb{V}(m_1)}{m_1} + m_1 > \frac{\mathbb{V}(m_2)}{m_2} + m_2.$$

As $h(m_1) = h(\bar{m}_1)$ and $h(m_2) = h(\bar{m}_2)$, we have that

$$\frac{\mathbb{V}(\bar{m}_1)}{\bar{m}_1} + \bar{m}_1 > \frac{\mathbb{V}(\bar{m}_2)}{\bar{m}_2} + \bar{m}_2 \tag{4.6}$$

and necessarily, we have $\bar{m}_1 > \bar{m}_2$.

Indeed, $\bar{m}_1 = \bar{m}_2$ is not possible, and if $\bar{m}_1 < \bar{m}_2$, then the inequality (4.6) and the continuity of the function h implies that there exists $y < \bar{m}_1$ such that $\frac{\mathbb{V}(\bar{m}_2)}{\bar{m}_2} + \bar{m}_2 =$

$h(y) = \frac{\mathbb{V}(m_2)}{m_2} + m_2$, which is in contradiction with the fact that

$$\bar{m}_2 = \inf \left\{ m \geq \mathbf{m}_+(\nu) : \frac{\mathbb{V}(m)}{m} + m = \frac{\mathbb{V}(m_2)}{m_2} + m_2 \right\}.$$

□

We have $\overline{\mathbf{m}_+(\nu)} = \mathbf{m}_+(\nu)$, and set $\widetilde{\mathbf{M}} = \lim_{m \rightarrow \bar{\mathbf{m}}} \bar{m}$.

Proposition 4.11. *The function $h : \bar{m} \mapsto \frac{\mathbb{V}(\bar{m})}{\bar{m}} + \bar{m}$ is increasing on $(\mathbf{m}_+(\nu), \widetilde{\mathbf{M}})$.*

Proof. Let $\bar{m}_1, \bar{m}_2 \in (\mathbf{m}_+(\nu), \widetilde{\mathbf{M}})$ such that $\bar{m}_1 \leq \bar{m}_2$. Then there exists $m_1, m_2 \in (\widetilde{\mathbf{m}}, \mathbf{m}_+(\nu))$ such that $m_1 \geq m_2$ and $\frac{\mathbb{V}(\bar{m}_1)}{\bar{m}_1} + \bar{m}_1 = \frac{\mathbb{V}(m_1)}{m_1} + m_1$ and $\frac{\mathbb{V}(\bar{m}_2)}{\bar{m}_2} + \bar{m}_2 = \frac{\mathbb{V}(m_2)}{m_2} + m_2$. This implies that

$$\frac{\mathbb{V}(\bar{m}_1)}{\bar{m}_1} + \bar{m}_1 \leq \frac{\mathbb{V}(\bar{m}_2)}{\bar{m}_2} + \bar{m}_2.$$

□

We are now in position to show that one can extend the definition of the pseudo-variance function $\mathbb{V}(m)$ to $m \in (\mathbf{m}_+, \mathbf{M}_+)$. We first use the mean function $k(\cdot)$ given in (1.4), to define a new mean function $\bar{k}(\cdot)$. In fact, we know that the function $g : m \mapsto \bar{m}$ realizes a bijection from $(\widehat{\mathbf{m}}, \mathbf{m}_+)$ into $(\mathbf{m}_+, \mathbf{M}_+)$ with $\mathbf{M}_+ = g(\widehat{\mathbf{m}})$. We then define $\bar{k}(\cdot)$ on $\bar{\Theta} = k^{-1}((\widehat{\mathbf{m}}, \mathbf{m}_+)) \subset \Theta$, by $\bar{k}(\theta) = g(k(\theta))$. If $m = k(\theta)$, then $\bar{m} = g(k(\theta)) = \bar{k}(\theta)$. We define $\bar{\psi}$ as the inverse of $\bar{k}(\cdot)$ from $(\mathbf{m}_+, \mathbf{M}_+)$ into $\bar{\Theta}$. The pseudo-variance function is then defined $\forall m \in (\mathbf{m}_+, \mathbf{M}_+)$, as in (1.7), that is

$$\mathbb{V}(m) = m \left(\frac{1}{\bar{\psi}(m)} - m \right), \quad \forall m \in (\mathbf{m}_+, \mathbf{M}_+).$$

Conclusion of proof of Theorem 4.1. Note that for $m > \mathbf{m}_+(\nu)$ formula (1.3) defines $\bar{Q}_m(dx)$ which is not a probability measure, and it may be negative. Our restriction to $m < \mathbf{M}_+(\nu)$ given by (4.1) guarantees its positivity, so measure \bar{Q}_m is also non-negative.

We now verify that \bar{Q}_m is a probability measure with required properties for $m \in (m_0, \mathbf{M}_+(\nu))$. We write (4.2) explicitly

$$\bar{Q}_m(dx) = \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)} \nu(dx) + \left(1 - \frac{\mathbb{V}(m)}{m} G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right)\right) \delta_{\frac{\mathbb{V}(m)}{m} + m}.$$

Note that when $m \in (m_0, \mathbf{m}_+(\nu))$ we have that

$$1 - \frac{\mathbb{V}(m)}{m} G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right) = 0,$$

so that \bar{Q}_m reduces to the distribution given in (1.3), and has desired properties. Therefore, without loss of generality we restrict ourselves to $m > \mathbf{m}_+(\nu)$ such that $\mathbb{V}(m)/m \geq 0$.

From (1.3) we have

$$\begin{aligned} \int Q_m(dx) &= \int \frac{\mathbb{V}(m)}{\mathbb{V}(m) + m(m-x)} \nu(dx) \\ &= \frac{\mathbb{V}(m)}{m} \int \frac{1}{\frac{\mathbb{V}(m)}{m} + m - x} \nu(dx) \\ &= \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \int \bar{Q}_m(dx) &= \int Q_m(dx) + p(m) \\ &= \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) + 1 - \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \\ &= 1, \quad \forall m \in (m_0, \mathbf{M}_+). \end{aligned}$$

We now verify that $\int \bar{Q}_m(dx) = m$. We have

$$\begin{aligned} \int x \bar{Q}_m(dx) &= \int x Q_m(dx) + \int x p(m) \delta_{\frac{\mathbb{V}(m)}{m} + m}(x) dx \\ &= \int \frac{\mathbb{V}(m)x}{\mathbb{V}(m) + m(m-x)} \nu(dx) + \left(\frac{\mathbb{V}(m)}{m} + m \right) p(m). \end{aligned}$$

The integral is

$$\begin{aligned} \int \frac{\mathbb{V}(m)x}{\mathbb{V}(m) + m(m-x)} \nu(dx) &= \frac{-\mathbb{V}(m)}{m} \int \frac{-x}{\mathbb{V}(m)/m + m - x} \nu(dx) \\ &= \frac{-\mathbb{V}(m)}{m} \int \frac{(\mathbb{V}(m)/m + m) - x - (\mathbb{V}(m)/m + m)}{\mathbb{V}(m)/m + m - x} \nu(dx) \\ &= \frac{-\mathbb{V}(m)}{m} \left[1 - (\mathbb{V}(m)/m + m) G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \right] \\ &= \frac{-\mathbb{V}(m)}{m} \left[1 - \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) - m G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \right] \\ &= \frac{-\mathbb{V}(m)}{m} \left[1 - \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \right] \\ &\quad + \mathbb{V}(m) G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right). \end{aligned}$$

On the other hand we have

$$\begin{aligned} \left(\frac{\mathbb{V}(m)}{m} + m \right) p(m) &= \left(\frac{\mathbb{V}(m)}{m} + m \right) \left[1 - \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \right] \\ &= m - \mathbb{V}(m) G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) + \frac{\mathbb{V}(m)}{m} \left[1 - \frac{\mathbb{V}(m)}{m} G_\nu \left(\frac{\mathbb{V}(m)}{m} + m \right) \right]. \end{aligned}$$

Thus

$$\int x\bar{Q}_m(dx) = \int xQ_m(dx) + \int xp(m)\delta_{\frac{\mathbb{V}(m)}{m}+m}(x)dx = m.$$

Next we verify formula (4.3). We have

$$\begin{aligned} \int x(x-m)\bar{Q}_m(dx) &= \int x(x-m)Q_m(dx) + \int x(x-m)p(m)\delta_{m+\frac{\mathbb{V}(m)}{m}} \\ &= \int x(x-m)Q_m(dx) + \left(m + \frac{\mathbb{V}(m)}{m}\right) \frac{\mathbb{V}(m)}{m} p(m). \end{aligned}$$

The integral is

$$\begin{aligned} \int x(x-m)Q_m(dx) &= \frac{\mathbb{V}(m)}{m} \left[\int xQ_m(dx) - \int x\nu(dx) \right] \\ &= \frac{\mathbb{V}(m)}{m} \left[\left(\frac{-\mathbb{V}(m)}{m}\right) \left(1 - \frac{\mathbb{V}(m)}{m} G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right)\right) \right. \\ &\quad \left. + \mathbb{V}(m)G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right) - m_0 \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(m + \frac{\mathbb{V}(m)}{m}\right) \frac{\mathbb{V}(m)}{m} p(m) &= \left(m + \frac{\mathbb{V}(m)}{m}\right) \frac{\mathbb{V}(m)}{m} \left[1 - \frac{\mathbb{V}(m)}{m} G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right)\right] \\ &= \frac{\mathbb{V}(m)}{m} \left[m - \mathbb{V}(m)G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right) + \frac{\mathbb{V}(m)}{m} \left(1 - \frac{\mathbb{V}(m)}{m} G_\nu\left(\frac{\mathbb{V}(m)}{m} + m\right)\right) \right]. \end{aligned}$$

Thus

$$\int x(x-m)\bar{Q}_m(dx) = \frac{\mathbb{V}(m)}{m} (m - m_0).$$

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