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Asymptotics of empirical distribution function for Gaussian subordinated arrays with an application to multiple testing

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Abstract: This paper introduces a new framework to study the asymptotical behavior of the empirical distribution function of Gaussian vector components, whose correlation matrix $\Gamma^{(m)}$ is dimension-dependent. Hence, by contrast with the existing literature, the vector is not assumed to be stationary. Rather, the covariance matrix $\Gamma^{(m)}$ should be close enough to the identity matrix as m grows to infinity. Markedly, under this assumption, the convergence result depends on $\Gamma^{(m)}$ only through the sequence $\gamma_m = m^{-2} \sum_{i \neq j} \Gamma_{i,j}^{(m)}$. This result recovers some of the previous results for stationary long-range dependencies while it also applies to various non-stationary cases, for which the most correlated variables are not necessarily next to each other. Finally, we present an application of this work to the multiple testing problem, which was the initial statistical motivation for developing such a methodology.

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1. Introduction

1.1. Presentation of the main result

Let us consider $\{Y^{(m)}, m \geq 1\}$ a triangular array for which each $Y^{(m)} = (Y_1^{(m)}, \dots, Y_m^{(m)})$ is a m -dimensional Gaussian vector, defined on $(\Omega_m, \mathcal{F}_m, \mathbb{P}_m)$, with zero mean and covariance matrix $\Gamma^{(m)}$. For the sake of simplicity, assume that each $Y_i^{(m)}$ is of variance 1, that is, $\Gamma_{i,i}^{(m)} = 1$ for all i . Denote $\Phi(z) = \mathbb{P}(Z \geq z)$, $z \in \mathbb{R}$, $Z \sim \mathcal{N}(0, 1)$, the upper tail distribution function of a standard Gaussian variable, and consider the empirical cumulative distribution function:

$$\widehat{\mathbb{F}}_m(t) = m^{-1} \sum_{i=1}^m \mathbf{1} \left\{ \Phi(Y_i^{(m)}) \leq t \right\}, \quad t \in [0, 1]. \quad (1)$$

In this paper, the motivation for studying the asymptotic properties of (1) in a possibly non-stationary setting is the multiple testing problem, which pertains to the recent and flourishing

field of statistics for high-dimensional data. In such data, while the dimension m can be very large (typically, several thousands), the matrix $\Gamma^{(m)}$ is generally complex and not-necessarily locally structured (e.g., spiked correlation matrices, see [Johnstone, 2001](#); [Friguet et al., 2009](#)).

To study (1), let us introduce the following quantities:

$$\gamma_m = m^{-2} \sum_{i \neq j} \Gamma_{i,j}^{(m)}; \quad (2)$$

$$r_m = (m^{-1} + |\gamma_m|)^{-1/2}. \quad (3)$$

In a nutshell, our main result is as follows: by assuming, when $m \rightarrow \infty$,

$$\frac{r_m^2}{m^2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^2 \rightarrow 0; \quad (\text{vanish-secondorder})$$

$$\frac{r_m^{4+\varepsilon_0}}{m^2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^4 \rightarrow 0, \quad \text{for some } \varepsilon_0 > 0; \quad (H_1)$$

$$m\gamma_m \rightarrow \theta, \quad \text{for some } \theta \in [-1, +\infty]; \quad (H_2)$$

the following weak convergence holds (in the Skorokhod topology):

$$r_m(\widehat{\mathbb{F}}_m - I) \rightsquigarrow \mathbb{Z}, \quad \text{as } m \rightarrow \infty, \quad (4)$$

where $I(t) = t$ and \mathbb{Z} is some continuous Gaussian process on $[0, 1]$ with a distribution only function of θ . Therefore, under our assumptions, the asymptotic behavior of $\widehat{\mathbb{F}}_m$ solely depends on the sequence $(\gamma_m)_m$. In particular, the convergence rate r_m is of order $m^{1/2}$ when $m\gamma_m$ is bounded (i.e., $\theta < +\infty$) and is of smaller order otherwise.

1.2. Organization of the paper

In Section 2, we derive an expansion of the covariance function of $\widehat{\mathbb{F}}_m$ by using the Hermite polynomials and we show that (vanish-secondorder) ensures that this expansion “stops at order 1 asymptotically”. The main theorem is precisely stated in Section 3 together with many illustrative examples, showing that our assumptions cover many standard models of correlation matrices (e.g., equi-correlated, long-range, spiked correlations, sample correlations). This new methodology is then applied to the multiple testing problem in Section 4. The proof of the main result is presented in Section 5; it mainly relies on central limit theorems for martingale arrays and on a suitable tightness criterion. To make the proof as clear as possible, some technical and auxiliary results are deferred to appendices.

1.3. Relation to existing literature

A colossal number of work aimed at extending Donsker’s theorem ([Doob, 1949](#); [Donsker, 1952](#); [Dudley, 1966](#)) to a more relaxed setup. Among them, a particularly prospering research field deals with the introduction of weak dependence between the original variables, mainly by using mixing conditions. Here, we do not attempt to provide an exhaustive list for such results and we refer the reader to, e.g., [Dedecker and Prieur \(2007\)](#); [Doukhan et al. \(2010\)](#) for a detailed review.

When restricted to the Gaussian subordinated setting, asymptotics for the e.c.d.f. (1) are described in the two well-known papers of [Dehling and Taqqu \(1989\)](#) (long-range) and [Csörgő and Mielniczuk \(1996\)](#) (short-range). However, both studies make a *stationarity* assumption, that is, assume that the covariance matrix is of the form $\Gamma_{i,j}^{(m)} = r(|i - j|)$, $1 \leq i, j \leq m$, for some function $r(\cdot)$ vanishing at infinity and not depending on m . Here, the setting is markedly different: first, the covariance matrix $\Gamma^{(m)}$ is allowed to depend on m , that is, the $Y^{(m)}$'s form a triangular array of Gaussian variables. Second, $\Gamma^{(m)}$ need not be locally structured, that is, $\Gamma_{i,j}^{(m)}$ is not necessarily related to the distance between i and j . Instead, our conditions are permutation invariant, that is, are unchanged when permuting the columns of the triangular array. This is quite natural because the e.c.d.f. is itself permutation invariant. Third, the correlations considered in our framework are possibly negative (although in a reasonable amount to keep the overall covariance matrix nonnegative) and our approach shows that the negative correlations can decrease the asymptotic covariance or even increase the convergence rate.

Compared to the existing literature, our framework covers a part of the previous results (e.g., includes long-range of [Dehling and Taqqu \(1989\)](#) but excludes short range of [Csörgő and Mielniczuk \(1996\)](#), see Section 3.2) while it opens a window for other dependence models, as, for instance, the spiked correlation model introduced by [Johnstone \(2001\)](#). The latter is of primary interest for modeling microarrays data, for which few external variables (factors) are able to have a small influences on all the measured Y_i 's, see [Friguet et al. \(2009\)](#) and [Fan et al. \(2010\)](#). Typically, a “benchmark” sub-model for our approach is the equi-correlated case where $\Gamma_{i,j}^{(m)} = \rho_m$, $i \neq j$, for some correlation ρ_m tending to zero (at some arbitrary rate). Let us note that we have studied this special case in an earlier work, see [Delattre and Roquain \(2011\)](#).

Finally, let us mention that recently, [Bardet and Surgailis \(2011\)](#) have also removed the stationarity assumption, by establishing central limit theorems (CLT) for Gaussian subordinated arrays. There are two major differences with our work: first, they deal with a CLT for the partial-sum process and not with a functional CLT for the e.c.d.f. Second, their assumptions are not of the same nature, because they require that $|\Gamma_{i,j}^{(m)}| \leq r(|i - j|)$ for all i, j , for some function $r(\cdot)$ vanishing at infinity.

2. Preliminaries: first order approximation for the covariance of $\widehat{\mathbb{F}}_m$

Throughout the paper, to alleviate the notation, we will often denote \mathbb{P}_m by \mathbb{P} , $Y^{(m)}$ by Y and $\Gamma^{(m)}$ by Γ when not ambiguous.

Let us consider the sequence of Hermite polynomials $H_\ell(x)$, $\ell \geq 0$, $x \in \mathbb{R}$ (see Appendix B). By using Melher's formula, the covariance function of the process $\widehat{\mathbb{F}}_m(\cdot)$ can be described as a function of the correlation matrix Γ of Y .

Proposition 2.1. *Consider $\widehat{\mathbb{F}}_m(\cdot)$ the process defined by (1) and the function family $\{c_\ell(\cdot), \ell \geq 1\}$ defined by*

$$c_\ell(t) = H_{\ell-1}(\Phi^{-1}(t))\phi(\Phi^{-1}(t)), \quad t \in [0, 1], \ell = 1, 2, \dots, \quad (5)$$

where $\phi = -\Phi'$ is the standard Gaussian density. Then for all $t, s \in [0, 1]$, we have

$$\text{Cov}(\widehat{\mathbb{F}}_m(t), \widehat{\mathbb{F}}_m(s)) = \sum_{\ell \geq 1} \frac{c_\ell(t)c_\ell(s)}{\ell!} \left(m^{-2} \sum_{i,j} (\Gamma_{i,j})^\ell \right). \quad (6)$$

This result can be found, e.g., in Proposition 1 of [Schwartzman \(2010\)](#) (itself referred to Theorem 1 of [Efron \(2010\)](#)). We provide a proof in Appendix B for completeness. While (6) is an exact expression, we can try to approximate the covariance $\text{Cov}(\widehat{\mathbb{F}}_m(t), \widehat{\mathbb{F}}_m(s))$ when m grows to infinity, while making some assumption on the matrix $\Gamma = \Gamma^{(m)}$.

Firstly, let us note the following: since $m^{-2} \sum_{i,j} (\Gamma_{i,j})^\ell = (\ell!)^{-1} \text{Var} (m^{-1} \sum_{i=1}^m H_\ell(Y_i)) \geq 0$ (by using (72) in Appendix B), expression (6) shows that the following conditions are equivalent as m tends to infinity,

$$\forall t \in [0, 1], \text{Var}(\widehat{\mathbb{F}}_m(t)) = o(1) \quad (7)$$

$$\forall \ell \geq 1, m^{-2} \sum_{i,j} (\Gamma_{i,j})^\ell = o(1) \quad (8)$$

$$m^{-2} \sum_{i,j} (\Gamma_{i,j})^2 = o(1) \quad (\text{A})$$

As a consequence, condition (A) is required as soon as a convergence result of the form (4) holds. Note that the rate r_m defined by (3) satisfies $1 \leq \left(m^{-2} \sum_{i,j} (\Gamma_{i,j})^2\right)^{-1/4} \leq r_m \leq \sqrt{m}$. Hence r_m tends to infinity under (A) but not faster than \sqrt{m} .

Secondly, let us rewrite (6) as follows:

$$\begin{aligned} \text{Cov}(\widehat{\mathbb{F}}_m(t), \widehat{\mathbb{F}}_m(s)) &= m^{-1}(t \wedge s - ts) + \gamma_m c_1(t) c_1(s) \\ &+ \sum_{\ell \geq 2} \left(m^{-2} \sum_{i \neq j} (\Gamma_{i,j})^\ell \right) c_\ell(t) c_\ell(s) (\ell!)^{-1}, \end{aligned} \quad (9)$$

where γ_m is defined by (2). The latter holds because, for two independent $\mathcal{N}(0, 1)$ variables U and V , we have $m^{-1} \sum_{\ell \geq 1} c_\ell(t) c_\ell(s) (\ell!)^{-1} = \text{Cov}(\mathbf{1}\{\Phi(U) \leq t\}, \mathbf{1}\{\Phi(V) \leq s\})$. In expansion (9), the second order term (i.e., the sum over $\ell \geq 2$) is negligible w.r.t. the other terms if ([vanish-secondorder](#)) holds. Hence, assuming now ([vanish-secondorder](#)), we obtain that the rescaled covariance $\text{Cov}(r_m \widehat{\mathbb{F}}_m(t), r_m \widehat{\mathbb{F}}_m(s))$ of $r_m \widehat{\mathbb{F}}_m$ converges to the following covariance function

$$K(t, s) = \frac{1}{1 + |\theta|} (t \wedge s - ts) + \frac{\theta}{1 + |\theta|} c_1(t) c_1(s), \quad (10)$$

where θ is defined in ([H₂](#)) and where we use the conventions $\theta/(1 + |\theta|) = 1$ and $1/(1 + |\theta|) = 0$ when $\theta = +\infty$. Note that ([H₂](#)) always holds up to consider a subsequence, because $m\gamma_m \geq -1$ from the nonnegativeness of $\Gamma^{(m)}$.

Remark 2.2. *One should note that, in the RHS of expression (10), the second term is not necessarily a covariance function because θ can be negative. Nevertheless, K can be written as $K(t, s) = \frac{1}{1 + |\theta|} \widetilde{K}(t, s) + \frac{1 + \theta}{1 + |\theta|} c_1(t) c_1(s)$, where*

$$\widetilde{K}(t, s) = t \wedge s - ts - c_1(t) c_1(s) \quad (11)$$

turns out to be a covariance function; considering a Wiener process $(W_t)_{t \in [0, 1]}$, \widetilde{K} is the covariance function of the process $W_t - tW_1 - c_1(t) \int_0^1 \Phi^{-1}(s) dW_s$, which is the orthogonal projection in L^2 of W_t onto the orthogonal of the linear space spanned by W_1 and $\int_0^1 \Phi^{-1}(s) dW_s$. Interestingly, the latter also shows that the original covariance K given by (10) can be seen as the covariance function of $\mathbb{Z}_t = (1 + |\theta|)^{-1/2} (W_t - tW_1) + (1 + |\theta|)^{-1/2} ((1 + \theta)^{1/2} - 1) c_1(t) \int_0^1 \Phi^{-1}(s) dW_s$.

3. Main result

3.1. Statement

Our main result establishes that the convergence of the covariance functions investigated in Section 2 can be extended to the case of a weak convergence of process. For this, we should additionally assume (H_1) (which is related to the tightness of the process).

Theorem 3.1. *Let us consider the empirical distribution function $\widehat{\mathbb{F}}_m$ defined by (1). Assume that the covariance matrix $\Gamma^{(m)}$ depends on m in such a way that (vanish-secondorder) and (H_1) hold with r_m defined by (3) and assume (H_2) . Consider $(Z_t)_{t \in [0,1]}$ a continuous process with covariance function K defined by (10). Then we have the convergence (in the Skorokhod topology)*

$$r_m(\widehat{\mathbb{F}}_m - I) \rightsquigarrow \mathbb{Z}, \text{ as } m \rightarrow \infty, \quad (12)$$

where $I(t) = t$ denotes the identity function.

A striking point is that (12) depends on Γ only through the sequence γ_m . In regards to the limiting covariance function (10), there are two possible regimes:

- (i) if $m\gamma_m \rightarrow \theta < +\infty$, the process $m^{1/2}(\widehat{\mathbb{F}}_m - I)$ converge to a (continuous Gaussian) process with covariance function given by $(t, s) \mapsto t \wedge s - ts + \theta \phi(\Phi^{-1}(t))\phi(\Phi^{-1}(s))$. Hence, the limit process is a standard Brownian bridge when $\theta = 0$, but has a covariance function smaller (resp. larger) if $\theta < 0$ (resp. $\theta > 0$).
- (ii) if $m\gamma_m \rightarrow \theta = +\infty$, the ‘‘Brownian’’ part asymptotically disappears and $(\gamma_m)^{-1/2}(\widehat{\mathbb{F}}_m - I)$ converge to the process $\phi(\Phi^{-1}(\cdot))Z$ for $Z \sim \mathcal{N}(0, 1)$.

The above behaviors are illustrated in Figure 1: as $m\gamma_m$ grows, the ‘‘Brownian’’ part disappears and the process looks like function $\phi(\Phi^{-1}(\cdot))$ (randomly rescaled). Also, the Y -axis indicates that the scaling \sqrt{m} is not suitable for large values of $m\gamma_m$.

Remark 3.2. *Assumptions (vanish-secondorder) and (H_1) always hold under the following condition*

$$|\Gamma_{i,j}^{(m)}| \leq a_m \text{ for all } i \neq j \text{ and } a_m \text{ satisfies } m^{1+\delta} a_m^2 \rightarrow 0 \text{ for some } \delta > 0. \quad (13)$$

Remark 3.3. *In Theorem 3.1, the result holds by replacing the set of assumptions (vanish-secondorder), (H_1) and (H_2) by the two following one:*

$$\frac{r_m^{2+\varepsilon_0}}{m^2} \sum_{i \neq j} (\Gamma_{i,j})^2 = o(1), \quad \text{with } \varepsilon_0 > 0; \quad (H_3)$$

$$m\gamma_m^{1+\varepsilon_0} \rightarrow +\infty, \quad \text{with } \varepsilon_0 > 0. \quad (H_4)$$

This is proved in Section 5.5. However, note that the latter can only be useful in regime (ii) since (H_4) implies $m\gamma_m \rightarrow +\infty$.

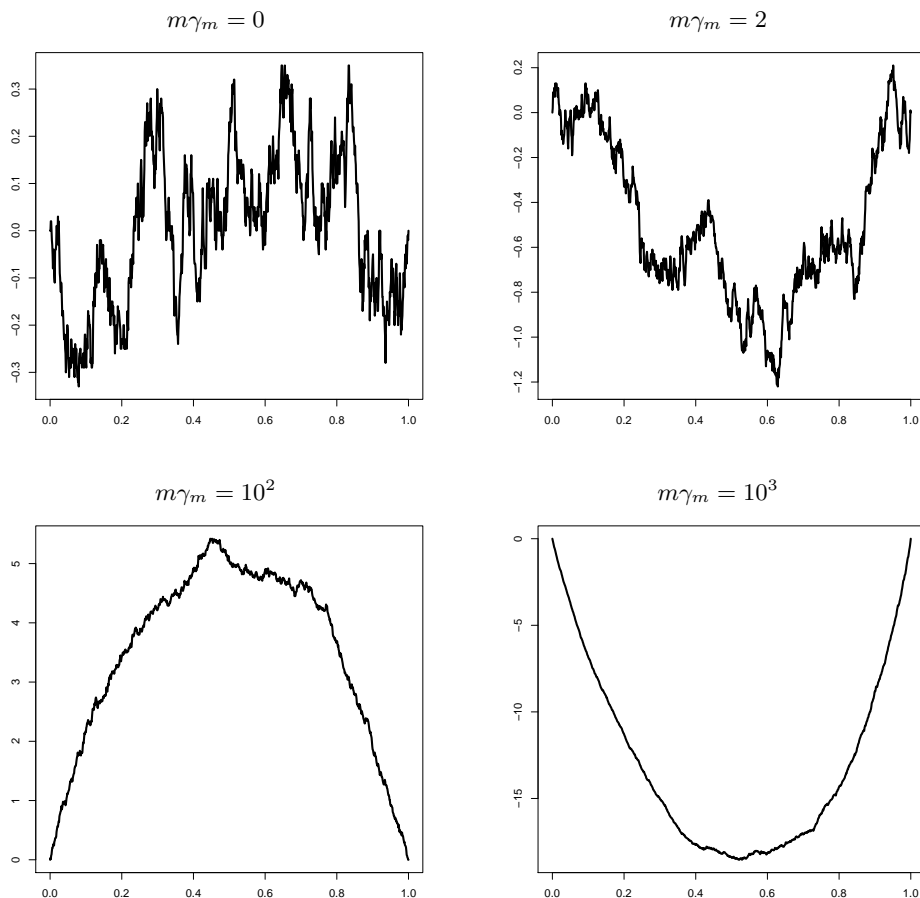


FIG 1. Plot of $t \mapsto m^{1/2}(\widehat{\mathbb{F}}_m(t) - t)$ for some observed $Y(\omega)$. These realizations have been generated in the equi-correlated model (14) and for $m = 10^4$.

3.2. Examples

Equi-correlation Let us start with the following simple example:

$$\Gamma^{(m)} = \begin{pmatrix} 1 & \rho_m & \dots & \rho_m \\ \rho_m & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_m \\ \rho_m & \dots & \rho_m & 1 \end{pmatrix} = (1 - \rho_m)I_m + \rho_m \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}^T, \quad (14)$$

where $\rho_m \in [-(m-1)^{-1}, 1]$ is some parameter. We easily check that γ_m defined by (2) is given by $m\gamma_m = (m-1)\rho_m$ and that the assumptions of Theorem 3.1 are all satisfied if $\rho_m \rightarrow 0$ and $m\rho_m$ converges to some $\theta \in [-1, +\infty]$, which yields convergence (12). This is in accordance with Lemma 3.3 of Delattre and Roquain (2011).

This simple example already shows that, following the choice of the sequence $(\rho_m)_m$, the empirical distribution function can have various asymptotic behaviors. For instance, taking $\rho_m = -(m-1)^{-1}$ gives a process in regime (i) with a minimal asymptotic covariance function

($\theta = -1$, see (11)), while taking $\rho_m \sim m^{-2/3}$ leads to a rate $r_m \sim m^{1/3} \ll m^{1/2}$ and thus a process converging in regime (ii).

Alternate equi-correlation Let us consider the covariance matrix:

$$\Gamma^{(m)} = \begin{pmatrix} 1 & -\rho_m & \rho_m & \dots & \\ -\rho_m & 1 & -\rho_m & \ddots & \vdots \\ \rho_m & \ddots & \ddots & \ddots & \rho_m \\ \vdots & \ddots & -\rho_m & 1 & -\rho_m \\ \dots & \rho_m & -\rho_m & 1 & \end{pmatrix} = (1 - \rho_m)I_m + \rho_m \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}^T, \quad (15)$$

where $\rho_m \in [-(m-1)^{-1}, 1]$ is a given parameter. Clearly, γ_m is such that

$$m\gamma_m = 2\rho_m m^{-1} \sum_{i=1}^{m-1} \sum_{k=1}^{m-i} (-1)^k = \rho_m [m/2]/(m/2)$$

Hence the rate r_m defined by (3) is $r_m \sim \sqrt{m}$ and assumptions of Theorem 3.1 are fulfilled (with $\theta = 0$) by assuming that $m^{1+\delta}\rho_m^2 \rightarrow 0$, with $\delta > 0$ (because (13) holds). Hence, under that assumption, $\sqrt{m}(\mathbb{F}_m - I)$ converges to a standard Brownian bridge.

Maybe surprisingly, this example shows that, even if the correlations are “strong” (e.g., $\rho_m \sim m^{-2/3}$, to be compared with the equi-correlated case), positive and negative correlations can exactly compensate each other to provide the same convergence result as under independence.

Long-range stationary correlations Let us consider the correlation matrix of the following form:

$$\Gamma_{i,j}^{(m)} = r(|j-i|), \quad \text{for } r(0) = 1, \quad r(k) = k^{-D}L(k), \quad 0 < D < 1, \quad (16)$$

where $L : (0, +\infty) \rightarrow (0, +\infty)$ is slowly varying at infinity ($\forall t > 0, L(tx) \sim L(x)$ as $x \rightarrow +\infty$). This framework is often referred to as “long-range dependence” in literature dealing with a stationary setup (see, e.g., [Dehling and Taqqu \(1989\)](#); [Doukhan et al. \(2002\)](#)). First, standard calculations easily show that for all $\nu \geq 0$,

$$m^{-1} \sum_{i \neq j} |j-i|^{-\nu} = 2m^{-1} \sum_{i=1}^{m-1} \sum_{k=1}^i k^{-\nu} \sim \begin{cases} 2 \frac{m^{1-\nu}}{(1-\nu)(2-\nu)} & \text{if } \nu \in [0, 1) \\ 2 \log m & \text{if } \nu = 1 \\ 2 \sum_{k \geq 1} k^{-\nu} & \text{if } \nu > 1 \end{cases}. \quad (17)$$

Thus, for any $\nu_1 \in (D, 1)$, since L is slowly varying,

$$m\gamma_m \gtrsim m^{-1} \sum_{i \neq j} |j-i|^{-\nu_1} \gtrsim m^{1-\nu_1},$$

by applying (17), where the “ $u_m \lesssim v_m$ ” means $u_m = O(v_m)$. This entails $m\gamma_m^{1+(1-\nu_1)/(2\nu_1)} \gtrsim m^{(1-\nu_1)/2}$ and thus Assumption (H₄) holds. In particular, $r_m \sim \gamma_m^{-1/2}$. Additionally, for any $\nu_2 \in (D, 1)$ and $\nu_3 \in (0, 2D)$ such that $\nu_3/\nu_2 > 1$, by applying again (17),

$$\gamma_m^{-\delta} m^{-2} \sum_{i \neq j} (\Gamma_{i,j})^2 \lesssim m^{\nu_1 \delta} m^{-2} \sum_{i \neq j} |j-i|^{-\nu_3} \lesssim m^{\nu_2 \delta - \nu_3} \vee (m^{\nu_2 \delta - 1} \log m)$$

for any $\delta > 0$. We derive (H₃) because taking $\delta > 1$ such that $\delta < \nu_3/\nu_2$ and $\delta < 1/\nu_2$ is possible. By using Theorem 3.1 under the conditions of Remark 3.3, we derive

$$\gamma_m^{-1/2}(\widehat{\mathbb{F}}_m - I) \rightsquigarrow c_1(\cdot)Z, \text{ as } m \rightarrow \infty,$$

for $Z \sim \mathcal{N}(0, 1)$. This is in accordance with Theorem 1.1 of Dehling and Taqqu (1989) (see in particular Example 1 therein).

Finally, let us note that Assumption (vanish-secondorder) of Theorem 3.1 is not satisfied for a covariance matrix of the type (16) taken with $D \geq 1$ (short-range) (the other terms in the covariance expansion (9) are required in the limit, see Csörgő and Mielniczuk (1996)).

Weak short/long range correlations Let us modify slightly the matrix (16), by letting:

$$\Gamma_{i,j}^{(m)} = \rho_m r(|j - i|), \quad \text{for } r(0) = 1, \quad r(k) = k^{-D}, \quad D > 0, \quad (18)$$

where ρ_m is some nonnegative parameter (we removed the slowly varying function for the sake of simplicity). When ρ_m varies in function of m , note that the latter is not of the stationary type. From (17), we have

$$m\gamma_m \sim \begin{cases} 2\rho_m \frac{m^{1-D}}{(1-D)(2-D)} & \text{if } D \in [0, 1) \\ 2\rho_m \log m & \text{if } D = 1 \\ 2\rho_m \sum_{k \geq 1} k^{-D} & \text{if } D > 1 \end{cases} \quad (19)$$

Assuming that the quantity (19) as a limit (denoted θ) and that $\rho_m \rightarrow 0$ as m grows to infinity, (vanish-secondorder) and (H₁) hold if $m^{1+\delta}\rho_m^2 \rightarrow 0$ with $\delta > 0$ (because (13) holds). The resulting rate of convergence r_m is given as a function of D and ρ_m in Table 1. Markedly, weak short-range correlations ($D > 1$) always yields $r_m \sim m^{1/2}$ while weak long-range correlations ($D < 1$) can give both regimes. For instance, taking $\rho_m \sim m^{-2/3}$ yields $r_m \sim m^{D/2+1/3}$ for $D < 1/3$ and $r_m \sim m^{1/2}$ otherwise. Overall, the convergence rate increases with D .

	$D \in [0, 1)$	$D \geq 1$
$\theta < \infty$	$\rho_m m^{1-D} = O(1)$ $r_m \sim \sqrt{m}$	$\theta = 0$ $r_m \sim \sqrt{m}$
$\theta = \infty$	$\rho_m m^{1-D} \rightarrow \infty$ $r_m \sim \rho_m^{-1/2} m^{D/2}$	not possible

TABLE 1

Rate r_m defined by (3) in function of $D \geq 0$ and ρ_m such that $\rho_m = o(m^{-(1/2+\delta)})$ for some $\delta > 0$, for the particular covariance (18).

Spiked correlation matrix model The “spiked” covariance model has been introduced in Johnstone (2001). It assumes that the k -first eigenvalues of the covariance matrix are greater than 1 (for some fixed value of k) while the other are all equal to 1. In our setting where we

consider only *correlation* matrices, we assume that the sequence of eigenvalues is constant after some fixed rank k . Precisely, let us consider a matrix $\Gamma^{(m)}$ of the following form:

$$\Gamma^{(m)} = (1 - \rho_m)I_m + \rho_m P H P^T, \quad (20)$$

where H is a $k \times k$ diagonal matrix with diagonal entries $h_1^{(m)}, \dots, h_k^{(m)} \in (1, \infty)$, where $P = (p_{i,r}^{(m)})_{1 \leq i \leq m, 1 \leq r \leq k}$ is an $m \times k$ matrix such that $P^T P = I_k$ and where $\rho_m \in [-1, 1]$ is some parameter. Importantly, k is taken fixed with m . The k first eigenvalues of $\Gamma^{(m)}$ are thus given by $1 - \rho_m + \rho_m h_r^{(m)}$, $r = 1, \dots, k$, while the remaining eigenvalues are all equal to $1 - \rho_m$. Hence, to ensure that $\Gamma^{(m)}$ given by (20) is a well defined correlation matrix, we should additionally assume that for all $r = 1, \dots, k$, $1 - \rho_m + \rho_m h_r^{(m)} \geq 0$, and that $P H P^T$ has diagonal entries equal to 1, that is, for all $i = 1, \dots, m$, $\sum_{r=1}^k h_r^{(m)} (p_{i,r}^{(m)})^2 = 1$. Note that the latter requires $h_1^{(m)} + \dots + h_k^{(m)} = m$ and thus $\max_r \{h_r^{(m)}\} \geq m/k$.

Next, by using (20), the conditions above and some properties of the Frobenius norm, we can derive the following:

$$m\gamma_m = \rho_m \sum_{r=1}^k h_r^{(m)} \left(m^{-1/2} \sum_{i=1}^m p_{i,r}^{(m)} \right)^2 - \rho_m; \quad (21)$$

$$m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^2 = \rho_m^2 \left(\sum_{r=1}^k (h_r^{(m)}/m)^2 - 1/m \right). \quad (22)$$

Since the RHS of (22) is upper-bounded by $\rho_m^2 (k - m^{-1})$ and lower-bounded by $\rho_m^2 (k^{-2} - m^{-1})$ and since k is taken fixed with m , condition (A) is satisfied if and only if $\rho_m \rightarrow 0$ while (**vanish-secondorder**) holds if and only if $r_m \rho_m \rightarrow 0$. Additionally, we have

$$\begin{aligned} m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^4 &= \rho_m^4 \left(m^{-2} \sum_{r_1, \dots, r_4} h_{r_1} \dots h_{r_4} \left(\sum_{i=1}^m p_{i,r_1} \dots p_{i,r_4} \right)^2 - 1/m \right) \\ &\leq \rho_m^4 \left(m^{-2} \left(\sum_{r_1, r_2} h_{r_1} h_{r_2} \sum_{i=1}^m p_{i,r_1}^2 p_{i,r_2}^2 \right)^2 - 1/m \right) \\ &= \rho_m^4 (1 - 1/m), \end{aligned}$$

where we used the Cauchy-Schwartz inequality (we dropped the dependence in m in the notation for short). Finally, the assumption of Theorem 3.1 are all fulfilled provided that

$$r_m^{2+\delta} \rho_m^2 \rightarrow 0 \text{ with } \delta > 0 \quad (23)$$

(up to consider a subsequence making the quantity into (21) converges to some θ). In (23), the rate r_m can be computed by using the definition, see (3), or expression (21). The rate of convergence thus intrinsically depends on the asymptotic behavior of the coordinate-wise mean of each eigenvector $(p_{i,r}^{(m)})_{1 \leq i \leq m}$.

To further illustrate this example, we can focus on the particular case where $k = 1$. In that case, the model can be equivalently written as

$$\Gamma^{(m)} = (1 - \rho_m)I_m + \rho_m \xi \xi^T, \quad (24)$$

where $\xi = \xi^{(m)}$ is a $m \times 1$ vector in $\{-1, 1\}^m$ and where $\rho_m \in [-(m-1)^{-1}, 1]$. The model (24) contains as particular instances the equicorrelated matrix ($\xi^{(m)} = (1 \ 1 \cdots 1)^T$) and the alternate equicorrelated matrix ($\xi^{(m)} = (1 \ -1 \ 1 \ \cdots)^T$) that we have studied above. We easily check that condition (23) recovers the conditions that we obtained in each of these particular cases. In general, for an arbitrary $\xi^{(m)} \in \{-1, 1\}^m$, since the quantity in (21) is equal to

$$\rho_m \left(m^{-1/2} \sum_{i=1}^m \xi_i^{(m)} \right)^2 - \rho_m, \quad (25)$$

the rate r_m is directly related to the number of -1 and $+1$ into $\xi^{(m)}$. For instance, if $\xi^{(m)} = (U_1, \dots, U_m)$ where U_1, U_2, \dots are i.i.d. random signs, we have by the central limit theorem that the quantity (25) tends to 0 (in probability) whenever $\rho_m \rightarrow 0$, which gives a rate $r_m \sim \sqrt{m}$ (in probability). Hence, we obtain the convergence (12) with the same rate and asymptotic variance as in the independent case whenever $m^{1+\delta} \rho_m^2 \rightarrow 0$ with $\delta > 0$.

Sample correlation matrix We consider the model where the correlation matrix is generated *a priori* as a Gaussian empirical correlation matrix. Namely, let us assume that

$$\Gamma^{(m)} = D^{-1} S D^{-1}, \text{ for } S = n_m^{-1} X^T X \text{ and } D = \text{diag}(S_{1,1}, \dots, S_{m,m})^{1/2} \quad (26)$$

where X is a $n_m \times m$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries. Assume $m/n_m \rightarrow 0$ as m tends to infinity, which, in a statistical setup, corresponds to assume that the number m of variables (columns of X) is of smaller order than the sample size n_m .

A by-product of Theorem 2 in Bai and Yin (1993) (adding a number of variables which is a vanishing small proportion of the sample size) is that,

$$\|S - I_m\|_2 \xrightarrow{P} 0,$$

where $\|\cdot\|_2$ denotes the Euclidian-operator norm, that is, $\|S - I_m\|_2 = \max_{1 \leq i \leq m} |\lambda_i^{(m)} - 1|$ and $\lambda_1^{(m)}, \dots, \lambda_m^{(m)}$ denote the eigenvalues of S . Hence, $\max_{1 \leq i \leq m} |S_{i,i} - 1| \xrightarrow{P} 0$, which in turn implies $\|\Gamma^{(m)} - I_m\|_2 \xrightarrow{P} 0$. Next, simple arguments entail the following inequalities:

$$\begin{aligned} \left| m^{-1} \sum_{i \neq j} \Gamma_{i,j}^{(m)} \right| &= m^{-1} | \langle (1 \cdots 1)^T, (\Gamma^{(m)} - I_m)(1 \cdots 1)^T \rangle | \leq \|\Gamma^{(m)} - I_m\|_2; \\ r_m^2 m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^2 &\leq m^{-1} \sum_{i=1}^m (\lambda_i^{(m)} - 1)^2 \leq \|\Gamma^{(m)} - I_m\|_2^2; \\ r_m^{4+\varepsilon_0} m^{-2} \sum_{i \neq j} \left(\Gamma_{i,j}^{(m)} \right)^4 &\leq \left\{ \min_{1 \leq i \leq n_m} |S_{i,i}| \right\}^{-4} m^{\varepsilon_0/2} \sum_{i \neq j} (S_{i,j})^4. \end{aligned}$$

Moreover, we easily check that $\mathbb{E}(n_m^{1/2} S_{i,j})^4 = \mathbb{E} \left(n_m^{-1/2} \sum_{k=1}^{n_m} X_{k,i} X_{k,j} \right)^4$ is upper bounded by some positive constant. Hence, by assuming that the sequence n_m satisfies

$$m^{1+\delta}/n_m \rightarrow 0 \text{ for some } \delta > 0,$$

the above inequalities implies that the rate is $r_m \sim \sqrt{m}$, that (H₂) holds with $\theta = 0$ and that (vanish-secondorder) and (H₁) are satisfied (all these convergences holding in probability). Hence, Theorem 3.1 can be applied and this shows that the asymptotic of the empirical distribution function is the same as under independence.

4. Statistical application: asymptotic study of Benjamini-Hochberg's procedure under dependence

4.1. Two-group model and background

Let us add to the original vector $Y \sim \mathcal{N}(0, \Gamma)$ an unknown vector $H = (H_i)_{1 \leq i \leq m} \in \{0, 1\}^m$ as follows: for $1 \leq i \leq m$,

$$X_i = \mu H_i + Y_i, \quad (27)$$

for some positive number μ (assumed to be fixed with m). Hence $X \sim \mathcal{N}(\mu H, \Gamma)$. Now consider the statistical problem of finding H from the observation of $X = (X_i)_{1 \leq i \leq m}$. From an intuitive point of view, H is the “signal” (unknown parameter of interest), Y is the “noise” (unobserved) while Γ and μ are “nuisance” parameters, generally assumed to be unknown. Obviously, because of the stochastic variations, the signal cannot be recovered exactly, hence we should define an *error criterion* that supervises the statistical decision.

In this context, a statistical procedure is a (measurable) function with a 0/1 output for each $1 \leq i \leq m$, that is, declaring some of the H_i 's as 0 and the other as 1. To evaluate the quality of such procedure, the false discovery rate (FDR) is a criterion coming from the multiple testing research field that has received a major and growing attention in the two last decades since its introduction by [Benjamini and Hochberg \(1995\)](#). The FDR is the expected value of the false discovery proportion (FDP), itself defined as the proportion of H_i equal to zero among the H_i deemed to be 1 by the procedure. The procedure that came up with — often called the “BH procedure” — suitably controls the FDR under independence of the X_i 's (and positive dependence, see [Benjamini and Yekutieli \(2001\)](#)) and is nowadays commonly used to find signal in massive data generated from high-throughput devices (e.g., in genomics or neuroimaging). However, a challenging issue often rose in recent statistical literature is the precise mathematical study of the BH procedure when the observed X_i 's are *dependent*. While the FDR of the BH procedure is reported to be essentially unaffected by dependencies, see [Farcomeni \(2006\)](#); [Kim and van de Wiel \(2008\)](#), some authors showed with simulations that this should not be the case for the FDP, see, e.g., [Korn et al. \(2004\)](#). The present work brings a broad theoretical support for this, by showing that the length of the asymptotic confidence interval for FDP is widening as the quantity γ_m defined by [\(2\)](#) grows.

The formal link between the FDP, the BH procedure and the e.c.d.f. has been delineated in [Genovese and Wasserman \(2004\)](#); [Farcomeni \(2007\)](#) (FDP at a fixed threshold) and consolidated later in [Neuvial \(2008\)](#) (FDP at BH threshold). Here, we follow the approach of [Neuvial \(2008\)](#), by using that the FDP of BH procedure is a Hadamard differentiable function of (rescaled) empirical distribution functions. Convergence results are thus simply derived from [Theorem 3.1](#) by applying the functional delta method. Nevertheless, let us underline that we have made the following new interesting finding that complements the methodology of [Neuvial \(2008\)](#): when one wants to study the FDP of BH procedure (and not another procedure), only the convergence of the e.c.d.f. *in the one-class model* of [Section 1.1](#) is needed (and not the joint convergence in the two-class model). This relies on the “partial functional delta method on $D(0, 1)$ ”, see [Proposition C.2](#).

4.2. Link with Theorem 3.1

Let us first define the following e.c.d.f.'s: for $t \in [0, 1]$,

$$\widehat{\mathbb{F}}_{0,m}(t) = m_0^{-1} \sum_{i=1}^m (1 - H_i) \mathbf{1} \{ \Phi(X_i) \leq t \}; \quad (28)$$

$$\widehat{\mathbb{F}}_{1,m}(t) = m_1^{-1} \sum_{i=1}^m H_i \mathbf{1} \{ \Phi(X_i) \leq t \}; \quad (29)$$

$$\widehat{\mathbb{G}}_m(t) = m^{-1} \sum_{i=1}^m \mathbf{1} \{ \Phi(X_i) \leq t \} = \frac{m_0}{m} \widehat{\mathbb{F}}_{0,m} + \frac{m_1}{m} \widehat{\mathbb{F}}_{1,m}, \quad (30)$$

where $m_0 = \sum_{i=1}^m (1 - H_i)$ and $m_1 = \sum_{i=1}^m H_i$. The proportions m_0/m and m_1/m are supposed to converge when m grows to infinity and the limits are denoted by $\pi_0 \in (0, 1)$ and $\pi_1 \in (0, 1)$, respectively. From Section 2, when Γ satisfies (A), the e.c.d.f.'s $\widehat{\mathbb{F}}_{0,m}(t)$, $\widehat{\mathbb{F}}_{1,m}(t)$ and $\widehat{\mathbb{G}}_m(t)$ converge in probability and we denote in what follows the limiting c.d.f.'s by $F_0(t) = t$, $F_1(t) = \Phi(\Phi^{-1}(t) - \mu)$ and $G(t) = \pi_0 F_0(t) + \pi_1 F_1(t)$, respectively.

In a nutshell, the FDP of BH procedure (at level $\alpha \in (0, 1)$) corresponds to the random variable

$$\text{FDP}_m = \alpha \frac{\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}(\mathcal{T}(\widehat{\mathbb{G}}_m))}{\mathcal{T}(\widehat{\mathbb{G}}_m)} = \Psi \left(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m} \widehat{\mathbb{F}}_{1,m} \right), \quad (31)$$

where we used the following functionals:

$$\mathcal{T}(H) = \sup\{t \in [0, 1] : H(t) \geq t/\alpha\} \text{ for } H \in D(0, 1); \quad (32)$$

$$\Psi(H_0, H_1) = \alpha H_0(\mathcal{T}(H_0 + H_1)) / \mathcal{T}(H_0 + H_1) \text{ for } (H_0, H_1) \in D(0, 1)^2, \quad (33)$$

and where we used the conventions $\sup\{\emptyset\} = 0$ and $0/0 = 0$. By Corollary 7.12 in [Neuivial \(2008\)](#), \mathcal{T} is Hadamard differentiable at function G , tangentially to the set $C(0, 1)$ of continuous functions on $(0, 1)$ and w.r.t. the supremum norm (we refer to Section 20.2 in [van der Vaart \(1998\)](#) for a formal definition of Hadamard differentiable functions). This holds because G is strictly concave and $\lim_{t \rightarrow 0} G(t)/t = +\infty$, which yields in particular $\mathcal{T}(G) \in (0, 1)$. As a consequence, standard calculations show that Ψ is Hadamard differentiable at $(\pi_0 F_0, \pi_1 F_1)$ tangentially to $C(0, 1)$, with derivative

$$\dot{\Psi}_{(\pi_0 F_0, \pi_1 F_1)}(H_0, H_1) = \alpha \frac{H_0(\mathcal{T}(G))}{\mathcal{T}(G)}, \text{ for } (H_0, H_1) \in C(0, 1)^2. \quad (34)$$

Now, by using (31), the functional delta method provides the asymptotic behavior of FDP_m from the one of $(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m} \widehat{\mathbb{F}}_{1,m})$. As a matter of fact, since the derivative $\dot{\Psi}_{(F_0, F_1)}(H_0, H_1)$ only depends on H_0 while the limit processes are (a.s.) continuous, establishing convergence results separately for $\widehat{\mathbb{F}}_{0,m}$ and $\widehat{\mathbb{F}}_{1,m}$ is sufficient (we do not need to consider the joint process $(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}, \frac{m_1}{m} \widehat{\mathbb{F}}_{1,m})$). We have precisely formulated this argument in Proposition C.2. Hence, applying twice Theorem 3.1 (for $\widehat{\mathbb{F}}_{0,m} \sim \widehat{\mathbb{F}}_{m_0}$ and $\widehat{\mathbb{F}}_{1,m} \sim \widehat{\mathbb{F}}_{m_1} \circ F_1$), we are able to derive a convergence result for FDP_m , as we establish in the next section.

4.3. Results

First, let us introduce the following additional quantities:

$$r_{0,m} = \left(m_0^{-1} + \left| m_0^{-2} \sum_{i \neq j} (1 - H_i)(1 - H_j) \Gamma_{i,j} \right| \right)^{-1/2}; \quad (35)$$

$$r_{1,m} = \left(m_1^{-1} + \left| m_1^{-2} \sum_{i \neq j} H_i H_j \Gamma_{i,j} \right| \right)^{-1/2}. \quad (36)$$

Corollary 4.1. *Consider the two-group model (27), generated from parameters μ , $H = H^{(m)}$ and a correlation matrix $\Gamma = \Gamma^{(m)}$. Assume that m_0 (depending on H) is such that $\sqrt{m}(m_0/m - \pi_0) \rightarrow 0$. Assume that Γ satisfies (vanish-secondorder), (H_1) and that the rates r_m , $r_{0,m}$ and $r_{1,m}$, respectively defined by (3), (35) and (36), grow proportionally to infinity as m tends to infinity. Let $\alpha \in (0, 1)$ and $t^* = t^*(\mu, \alpha)$ be the unique $t \in (0, 1)$ such that $G(t) = t/\alpha$. Let $h(t^*) = (\phi(\Phi^{-1}(t^*))/t^*)^2$. Then the sequence of r.v. FDP_m defined by (31) enjoys the following convergence:*

$$\frac{FDP_m - \pi_0 \alpha}{\pi_0 \alpha \{(1/t^* - 1)/m_0 + h(t^*)\gamma_{0,m}\}^{1/2}} \rightsquigarrow \mathcal{N}(0, 1), \quad (37)$$

where $\gamma_{0,m} = m_0^{-2} \sum_{i \neq j} (1 - H_i)(1 - H_j) \Gamma_{i,j}$.

Proof. First, classically, it is sufficient to prove that the convergence (37) holds up to consider a subsequence. Hence, we can assume that (H_2) and the convergences

$$\begin{aligned} m_0^{-1} \sum_{i \neq j} (1 - H_i)(1 - H_j) \Gamma_{i,j} &\rightarrow \theta_0; \\ m_1^{-1} \sum_{i \neq j} H_i H_j \Gamma_{i,j} &\rightarrow \theta_1; \end{aligned} \quad (38)$$

hold, with θ , θ_0 and θ_1 valued in $[-1, +\infty]$. Also note that since $r_m \propto r_{0,m}$ (resp. $r_m \propto r_{1,m}$), condition (vanish-secondorder) and (H_1) also holds for the submatrix $(\Gamma_{i,j})_{i,j:H_i=H_j=0}$ (resp. $(\Gamma_{i,j})_{i,j:H_i=H_j=1}$). Now, let us write

$$r_{0,m} \left(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}(t) - \pi_0 F_0(t) \right) = r_{0,m} m^{-1} \sum_{i=1}^m (1 - H_i) (\mathbf{1}\{\Phi(X_i) \leq t\} - t) + r_{0,m} t (m_0/m - \pi_0). \quad (39)$$

In the RHS of (39), while the second term converges to 0 by assumption, a consequence of Theorem 3.1 is that the first term converges to a process with covariance function

$$\pi_0^2 \left[\frac{1}{1 + |\theta_0|} (t \wedge s - ts) + \frac{\theta_0}{1 + |\theta_0|} c_1(t) c_1(s) \right], \quad \text{for all } t, s \in [0, 1].$$

Obviously, a similar result holds for the process $r_{1,m} (\frac{m_1}{m} \widehat{\mathbb{F}}_{1,m} - \pi_1 F_1)$.

Applying the (partial) functional delta method as explained in Proposition C.2 (by using $r_{0,m} \propto r_{1,m}$ and (34)), we obtain

$$r_{0,m}(\text{FDP}_m - \pi_0\alpha) \rightsquigarrow \mathcal{N}\left(0, (\alpha\pi_0)^2 \left[\frac{1/t^* - 1}{1 + |\theta_0|} + \frac{\theta_0}{1 + |\theta_0|} (c_1(t^*)/t^*)^2 \right] \right). \quad (40)$$

Finally, we easily derive (37) by separating the cases $\theta_0 < +\infty$ and $\theta_0 = +\infty$. \square

As an illustration, Corollary 4.1 can be used in the independent case ($\gamma_{0,m} = 0$) or ρ_m -equi-correlated case ($\gamma_{0,m} = \rho_m$), so recovering the previous results of Neuvial (2008, 2009) (in the Gaussian case) and Delattre and Roquain (2011), respectively. This holds for any H satisfying $\sqrt{m}(m_0/m - \pi_0) \rightarrow 0$. However, for an arbitrary covariance Γ , the quantity $\gamma_{0,m}$ unfortunately depends in general on the unknown H and thus cannot be computed in a practical situation. This fact is inherent to the multiple testing setting considered here, because the dependencies accounting in the FDP of BH's procedure are related to the submatrix $(\Gamma_{i,j})_{i,j:H_i=H_j=0}$ and thus are directly linked to the location of the null hypotheses.

A convenient way to circumvent this problem is to add *prior random effects*, by assuming that, previously and independently to the model (27), we have drawn $H = (H_1, \dots, H_m)$ for H_1, H_2, \dots i.i.d. Bernoulli variables of parameter $\pi_1 = 1 - \pi_0$, for some $\pi_0 \in (0, 1)$. Thus X follows the distribution $\mathcal{N}(\mu H, \Gamma)$ conditionally on H . The corresponding global (unconditional) model, often referred to as the *two-group mixture model* has been widely used in the multiple testing literature, see, e.g. Efron et al. (2001); Storey (2003); Genovese and Wasserman (2004); Roquain and Villers (2011). By contrast with the previous model, H is now random. In particular, $m_0 = \sum_{i=1}^m (1 - H_i) \sim \mathcal{B}(m, \pi_0)$ and $\sqrt{m}(m_0/m - \pi_0)$ does not degenerate at the limit, which adds some extra variance in the FDP convergence result. The counterpart is that the statement is substantially simplified, as we can see below.

Corollary 4.2. *Consider the two-group mixture model defined above, generated from parameters $\mu > 0$, $\pi_0 \in (0, 1)$ and a correlation matrix $\Gamma = \Gamma^{(m)}$. Assume that Γ satisfies (vanish-secondorder) and (H1). Let $\alpha \in (0, 1)$ and $t^* = t^*(\mu, \alpha)$ be the unique $t \in (0, 1)$ such that $G(t) = t/\alpha$. Let $h(t^*) = (\phi(\Phi^{-1}(t^*))/t^*)^2$. Then the sequence of r.v. FDP_m defined by (31) enjoys the following convergence:*

$$\frac{\text{FDP}_m - \pi_0\alpha}{\pi_0\alpha \left\{ (1/t^* - \pi_0)/(\pi_0 m) + h(t^*)\gamma_m \right\}^{1/2}} \rightsquigarrow \mathcal{N}(0, 1), \quad (41)$$

where γ_m is defined by (2).

Proof. Again, it is sufficient to state the result up to consider a subsequence. Thus (H2) holds without loss of generality. First check that (vanish-secondorder) entails

$$\frac{r_m^2}{m^2} \left(\sum_{i \neq j} (1 - H_i)(1 - H_j)\Gamma_{i,j} - \pi_0^2 \sum_{i \neq j} \Gamma_{i,j} \right) = o_P(1), \quad (42)$$

(computing, e.g., the variance of the latter) and this convergence can be made a.s. by taking a suitable subsequence. A consequence of (42) is that $\gamma_{0,m} \sim \gamma_m$ a.s. (in particular, θ_0 defined by (38) equals $\pi_0\theta$.) This implies $r_m \propto r_{0,m}$ (a.s.), which provides (vanish-secondorder) and (H1) for the submatrix $(\Gamma_{i,j})_{i,j:H_i=H_j=0}$ (a.s.). Now, by using (39), we obtain that

$r_{0,m} \left(\frac{m_0}{m} \widehat{\mathbb{F}}_{0,m}(t) - \pi_0 F_0(t) \right)$ converges (unconditionally) to a process with covariance function defined by: for all $t, s \in [0, 1]$,

$$\begin{aligned} & \pi_0^2 \left[\frac{1}{1 + \pi_0 |\theta|} (t \wedge s - ts) + \frac{\pi_0 \theta}{1 + \pi_0 |\theta|} c_1(t) c_1(s) \right] + \frac{\pi_0 (1 - \pi_0)}{1/\pi_0 + |\theta|} ts \\ & = \pi_0^2 \left[\frac{1}{1 + \pi_0 |\theta|} (t \wedge s - \pi_0 ts) + \frac{\pi_0 \theta}{1 + \pi_0 |\theta|} c_1(t) c_1(s) \right] \end{aligned}$$

Obviously, a similar result holds for the process $r_{1,m}(\frac{m_1}{m} \widehat{\mathbb{F}}_{1,m} - \pi_1 F_1)$. We finish the proof by applying the (partial) functional delta method, see Proposition C.2. \square

Corollary 4.2 clearly shows that the concentration of FDP_m around $\pi_0 \alpha$ deteriorates when γ_m increases, so when positive correlations appear between the individual statistical tests. However, notice that, perhaps surprisingly, negative correlations help to decrease γ_m and can yields to a concentration even better than under independence when γ_m is negative (although this phenomenon is necessary of limited amplitude because $\gamma_m \geq -1/m$).

From a practical point of view, (41) still contains the unknown quantities π_0 and t^* . Fortunately, t^* can be easily estimated because $\mathcal{T}(\widehat{\mathbb{G}}_m) \xrightarrow{P} \mathcal{T}(G) = t^*$ (by the Hadamard differentiability of \mathcal{T} at G and by using the same arguments as in the proof of Proposition C.2). As for π_0 , this parameter seems hard to estimate in our dependent context, but we can use a conservative uniform upper-bound. Finally, under the assumption of Corollary 4.2, we obtain the following asymptotic $(1 - \zeta)$ -confidence upper-bound for FDP_m :

$$\sup_{\pi_0 \in [0,1]} \left\{ \pi_0 \alpha \left(1 + \Phi^{-1}(\zeta) \left\{ (1/\hat{t} - \pi_0)/(\pi_0 m) + h(\hat{t}) \gamma_m \right\}^{1/2} \right) \right\}, \quad (43)$$

by denoting $\hat{t} = \mathcal{T}(\widehat{\mathbb{G}}_m) \vee (\alpha/m)$ and $h(t) = (\phi(\Phi^{-1}(t))/t)^2$ for $t \in (0, 1]$.

5. Proof of Theorem 3.1

5.1. A related result and additional notation

Let us define the “modified” empirical distribution function $\widetilde{\mathbb{F}}_m$ by the following relation: for $t \in [0, 1]$,

$$r_m(\widehat{\mathbb{F}}_m(t) - t) = r_m(\widetilde{\mathbb{F}}_m(t) - t) + c_1(t) r_m \bar{Y}_m. \quad (44)$$

The convergence of the two processes $r_m(\widehat{\mathbb{F}}_m - I)$ and $r_m(\widetilde{\mathbb{F}}_m - I)$ are strongly related by (44). The main idea of our proof is to deduce the convergence of $r_m(\widehat{\mathbb{F}}_m - I)$ from the one of $r_m(\widetilde{\mathbb{F}}_m - I)$. Precisely, the following result will be proved together with Theorem 3.1 in the sequel.

Proposition 5.1. *Under the assumptions of Theorem 3.1, let us consider the corrected empirical distribution function $\widetilde{\mathbb{F}}_m$ defined by (44) and a continuous process $(\widetilde{\mathbb{Z}}_t)_{t \in [0,1]}$ with covariance function \widetilde{K} defined by (11). Then we have the convergence (in the Skorokhod topology)*

$$r_m(\widetilde{\mathbb{F}}_m - I) \rightsquigarrow \widetilde{\mathbb{Z}}/(1 + |\theta|)^{1/2}, \text{ as } m \rightarrow \infty, \quad (45)$$

where $I(t) = t$ denotes the identity function.

Additionally, throughout the section, we use the following notation

$$h_t(x) = \mathbf{1}\{\Phi(x) \leq t\} - t - c_1(t)x, \quad (46)$$

so that $\tilde{\mathbb{F}}_m(t) - t = m^{-1} \sum_{i=1}^m h_t(Y_i)$. Finally, we will sometimes use the following assumption:

there exists $\eta > 0$ (independent on m) lower bounding the m eigenvalues of $\Gamma^{(m)}$.
(eigenvalues-away0)

5.2. Convergence of finite dimensional laws for $\tilde{\mathbb{F}}_m$

Let us prove the following result.

Proposition 5.2. *Assume that the covariance matrix Γ depends on m in such a way that (vanish-secondorder) holds with r_m defined by (3) and assume (H_2) . Consider a continuous process $(\tilde{\mathbb{Z}}_t)_{t \in [0,1]}$ with covariance function \tilde{K} defined by (11). Then, the process $(r_m(\tilde{\mathbb{F}}_m - I), Y_1^{(m)})$ (jointly) converges to $\mathcal{L}(\tilde{\mathbb{Z}}/(1+|\theta|)^{1/2}) \otimes \mathcal{N}(0, 1)$ in the sense of the finite dimensional convergence. In particular, the convergence (45) holds in the sense of the finite dimensional convergence.*

Proof. The proof is based on central limit theorems for martingale arrays as presented, e.g., in Chapter 3 of [Hall and Heyde \(1980\)](#).

First, since we aim at obtaining a convergence jointly with $Y_1^{(m)}$, a (somewhat technical) but useful task is to define the array of random variables $(Y_i^{(m)}, 1 \leq i \leq m, m \geq 1)$ in such a way that $Y_1^{(m)}$ is fixed with m . This is possible by first considering some variable $Z \sim \mathcal{N}(0, 1)$, by letting $Y_1^{(m)} = Z$ for all $m \geq 1$, and then by choosing for each $m \geq 2$, the variables $Y_i^{(m)}, 2 \leq i \leq m$, such that

- $(Z, Y_i^{(m)}, 2 \leq i \leq m) \sim \mathcal{N}(0, \Gamma^{(m)})$;
- $\{(Y_i^{(m)})_{2 \leq i \leq m}, m \geq 2\}$ is a family of mutually independent vectors conditionally on Z .

This also define a common underlying space $(\Omega, \mathcal{F}, \mathbb{P})$ for the array of random variables.

Now, define the following nested array of σ -field: for $m \geq 1$, $\mathcal{G}_{m,0} = \sigma(\emptyset)$ and for $1 \leq i \leq m$,

$$\mathcal{G}_{m,i} = \sigma(Y_j^{(\ell)}, 1 \leq j \leq i \wedge \ell, 1 \leq \ell \leq m).$$

Next, let us consider for each $t \in [0, 1]$, the martingale array $(M_{m,i}(t), \mathcal{G}_{m,i}, 1 \leq i \leq m, m \geq 1)$ defined as follows:

$$M_{m,i}(t) = \sum_{j=1}^i X_{m,j}(t) \quad \text{for } X_{m,j}(t) = \frac{r_m}{m} \left(h_t(Y_j^{(m)}) - \mathbb{E} \left(h_t(Y_j^{(m)}) \mid \mathcal{G}_{m,j-1} \right) \right). \quad (47)$$

Clearly,

$$r_m(\tilde{\mathbb{F}}_m(t) - t) = M_{m,m}(t) + \frac{r_m}{m} \sum_{i=1}^m \mathbb{E} \left(h_t(Y_i^{(m)}) \mid \mathcal{G}_{m,i-1} \right). \quad (48)$$

Also note that we can replace each $\mathcal{G}_{m,i}$ by $\mathcal{F}_{m,i} = \sigma(Y_1^{(m)}, \dots, Y_i^{(m)})$ ($\mathcal{F}_{m,0} = \sigma(\emptyset)$) in the above expression, because $(Y_i^{(m)}, 2 \leq i \leq m)$ is independent of $(Y_j^{(\ell)}, 2 \leq j \leq i \wedge \ell, 2 \leq \ell < m)$, conditionally on $Y_1^{(m)}$.

Case 1: (eigenvalues-away0) is assumed We show in Lemma A.1 expression (68) that the second term in the RHS of (48) has a vanishing variance as m tends to infinity. Therefore, it remains to show that the conclusion of Proposition 5.2 holds for the process $M_{m,m}$, which we prove by using Lindeberg's theorem. We use Corollary 3.1 page 58 in Hall and Heyde (1980) (or more precisely its generalization to the multidimensional case). The conditions are as follows:

- (i) for all $t \in [0, 1]$, for all $\varepsilon > 0$, $\sum_{i=1}^m \mathbb{E}((X_{m,i}(t))^2 \mathbf{1}\{|X_{m,i}(t)| > \varepsilon\} | \mathcal{F}_{m,i-1}) \xrightarrow{P} 0$;
- (ii) for all $t, s \in [0, 1]$, $\sum_{i=1}^m \mathbb{E}(X_{m,i}(t)X_{m,i}(s) | \mathcal{F}_{m,i-1}) \xrightarrow{P} \tilde{K}(t, s)$.

To check (i), let us fix $t \in [0, 1]$ and prove $\sum_{i=1}^m \mathbb{E}(X_{m,i}(t))^4 = o(1)$. By definition, we have

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}(X_{m,i}(t))^4 &= \frac{r_m^4}{m^4} \sum_{i=1}^m \mathbb{E} \left(h_t(Y_i^{(m)}) - \mathbb{E}(h_t(Y_i^{(m)}) | \mathcal{F}_{m,i-1}) \right)^4 \\ &\leq 2^4 \left(\frac{r_m^4}{m^3} m^{-1} \sum_{i=1}^m \mathbb{E} \left(h_t(Y_i^{(m)}) \right)^4 + \frac{r_m^4}{m^4} \sum_{i=1}^m \mathbb{E} \left(\mathbb{E}(h_t(Y_i^{(m)}) | \mathcal{F}_{m,i-1}) \right)^4 \right) \\ &\leq 2^5 \frac{r_m^4}{m^3} m^{-1} \sum_{i=1}^m \mathbb{E} \left(h_t(Y_i^{(m)}) \right)^4. \end{aligned}$$

Now, the RHS of the previous display converges to zero because $r_m \leq \sqrt{m}$ and $\mathbb{E}(h_t(Y_i^{(m)}))^4 < \infty$. This proves condition (i) of Lindeberg's theorem.

Let us now turn to condition (ii). For $t, s \in [0, 1]$, we obviously obtain

$$\begin{aligned} \sum_{i=1}^m \mathbb{E}(X_{m,i}(t)X_{m,i}(s) | \mathcal{F}_{m,i-1}) &= \frac{r_m^2}{m^2} \sum_{i=1}^m \mathbb{E}(h_t(Y_i^{(m)})h_s(Y_i^{(m)}) | \mathcal{F}_{m,i-1}) \\ &\quad - \frac{r_m^2}{m^2} \sum_{i=1}^m \mathbb{E}(h_t(Y_i^{(m)}) | \mathcal{F}_{m,i-1})\mathbb{E}(h_s(Y_i^{(m)}) | \mathcal{F}_{m,i-1}). \end{aligned} \quad (49)$$

Next, by using $ab \leq 2(a^2 + b^2)$ for all $a, b \in \mathbb{R}$ together with (67), the second term in the RHS of (49) tends to zero in probability. Moreover, we have

$$\begin{aligned} &\text{Var} \left(\frac{r_m^2}{m^2} \sum_{i=1}^m \left((h_t(Y_i^{(m)})h_s(Y_i^{(m)}) - \mathbb{E}(h_t(Y_i^{(m)})h_s(Y_i^{(m)}) | \mathcal{F}_{m,i-1})) \right) \right) \\ &= \frac{r_m^4}{m^4} \sum_{i=1}^m \text{Var} \left((h_t(Y_i^{(m)})h_s(Y_i^{(m)}) - \mathbb{E}(h_t(Y_i^{(m)})h_s(Y_i^{(m)}) | \mathcal{F}_{m,i-1})) \right), \end{aligned}$$

because the elements inside the sum are martingale increments. Hence, the quantity inside the above display tends to zero. Combining the latter with (49) establishes condition (ii) of Lindeberg's theorem provided that the following holds:

$$\frac{r_m^2}{m^2} \sum_{i=1}^m h_t(Y_i^{(m)})h_s(Y_i^{(m)}) \xrightarrow{P} (1 + |\theta|)^{-1} \tilde{K}(t, s).$$

This comes directly from the law of large number stated in Lemma C.3, because $r_m^2/m \rightarrow (1 + |\theta|)^{-1}$ by (3) and (H₂).

Applying Lindeberg's theorem (in the underlying space described above), for any $t_1, \dots, t_k \in [0, 1]$, the random vector

$$Z_m = (M_{m,m}(t_1), \dots, M_{m,m}(t_k))$$

converges stably in the following sense (see, e.g., [Jacod and Shiryaev \(2003\)](#) Definition 5.28): for all (fixed) bounded random variable U and continuous bounded function f in \mathbb{R}^k ,

$$\mathbb{E}(Uf(Z_m)) \rightarrow \mathbb{E}(U)\mathbb{E}(f(Z)) \text{ as } m \rightarrow \infty,$$

where Z is a centered multivariate Gaussian vector with covariance $(1+|\theta|)^{-1}(\tilde{K}(t_i, t_j))_{1 \leq i, j \leq k}$. This implies that (Z_m, Y_1) converges (jointly) in distribution to $\mathcal{L}(Z) \otimes \mathcal{N}(0, 1)$. This finishes the proof of Proposition 5.2 in the case where ([eigenvalues-away0](#)) is assumed to hold.

Case 2: ([eigenvalues-away0](#)) is not assumed The strategy is to apply Lemma C.4 in order to reduce the study to “Case 1” above. For any $\varepsilon > 0$, let

$$Y_i^\varepsilon = \frac{Y_i + \varepsilon \xi_i}{(1 + \varepsilon^2)^{1/2}},$$

where ξ_1, ξ_2, \dots are i.i.d. $\mathcal{N}(0, 1)$ variables, independent of all the Y_i 's. The covariance matrix of $(Y_1^\varepsilon, \dots, Y_m^\varepsilon)$ is obviously

$$\Gamma^\varepsilon = \frac{\varepsilon^2}{1 + \varepsilon^2} I_m + \frac{1}{1 + \varepsilon^2} \Gamma.$$

Clearly, the corresponding rate (3) is $r_m^\varepsilon = (m^{-1} + (1 + \varepsilon^2)^{-1} |\gamma_m|)^{-1/2}$. It is related to r_m via the following inequalities: $r_m \leq r_m^\varepsilon \leq (1 + \varepsilon^2)^{1/2} r_m$. Hence, Γ^ε satisfies ([vanish-secondorder](#)) and (H_2) with θ replaced by $\theta^\varepsilon = \frac{1}{1 + \varepsilon^2} \theta$. Since it also satisfies ([eigenvalues-away0](#)), by using Proposition 5.2 in the “Case 1” above, it satisfies for any $t_1, \dots, t_k \in [0, 1]$,

$$(a) \quad (r_m^\varepsilon(\tilde{\mathbb{F}}_m^\varepsilon(t_1) - t_1), \dots, r_m^\varepsilon(\tilde{\mathbb{F}}_m^\varepsilon(t_k) - t_k), Y_1^\varepsilon) \rightsquigarrow \mathcal{L}\left(\frac{(\tilde{\mathbb{Z}}(t_1), \dots, \tilde{\mathbb{Z}}(t_k))}{(1 + |\theta^\varepsilon|)^{1/2}}\right) \otimes \mathcal{N}(0, 1),$$

where $\tilde{\mathbb{F}}_m^\varepsilon(t) - t = m^{-1} \sum_{i=1}^m h_t(Y_i^\varepsilon)$ for all t . Next, we clearly have,

$$(b) \quad \frac{(\tilde{\mathbb{Z}}(t_1), \dots, \tilde{\mathbb{Z}}(t_k))}{(1 + |\theta^\varepsilon|)^{1/2}} \rightsquigarrow \frac{(\tilde{\mathbb{Z}}(t_1), \dots, \tilde{\mathbb{Z}}(t_k))}{(1 + |\theta|)^{1/2}} \text{ as } \varepsilon \rightarrow 0.$$

Let us now prove that for any $t \in [0, 1]$,

$$\limsup_m \left\{ \mathbb{E} \left| r_m(\tilde{\mathbb{F}}_m(t) - t) - r_m^\varepsilon(\tilde{\mathbb{F}}_m^\varepsilon(t) - t) \right| \right\} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (50)$$

This will conclude the proof by applying Lemma C.4. First, we write

$$\begin{aligned} & \mathbb{E} \left| r_m(\tilde{\mathbb{F}}_m(t) - t) - r_m^\varepsilon(\tilde{\mathbb{F}}_m^\varepsilon(t) - t) \right| \\ & \leq \mathbb{E} \left| r_m/m \sum_{i=1}^m (h_t(Y_i) - h_t(Y_i^\varepsilon)) \right| + (r_m^\varepsilon - r_m) \mathbb{E} \left| m^{-1} \sum_{i=1}^m h_t(Y_i^\varepsilon) \right| \\ & \leq \left\{ (r_m/m)^2 \mathbb{E} \left(\sum_{i=1}^m (h_t(Y_i) - h_t(Y_i^\varepsilon)) \right)^2 \right\}^{1/2} + ((1 + \varepsilon^2)^{1/2} - 1) \mathbb{E} \left| r_m^\varepsilon(\tilde{\mathbb{F}}_m^\varepsilon(t) - t) \right|. \end{aligned}$$

By taking the lim sup in the above display, it only remains to show

$$\limsup_m \left\{ (r_m/m)^2 \mathbb{E} \left(\sum_{i=1}^m (h_t(Y_i) - h_t(Y_i^\varepsilon)) \right)^2 \right\} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (51)$$

This can be proved by using Lemma B.3 (80) as follows:

$$\begin{aligned} & (r_m/m)^2 \mathbb{E} \left(\sum_{i=1}^m (h_t(Y_i) - h_t(Y_i^\varepsilon)) \right)^2 \\ &= (r_m/m)^2 \sum_{i,j=1}^m \mathbb{E} ((h_t(Y_i) - h_t(Y_i^\varepsilon))(h_t(Y_j) - h_t(Y_j^\varepsilon))) \\ &= (r_m/m)^2 \sum_{i,j=1}^m (\mathbb{E}(h_t(Y_i)h_t(Y_j)) - \mathbb{E}(h_t(Y_i)h_t(Y_j^\varepsilon)) - \mathbb{E}(h_t(Y_i^\varepsilon)h_t(Y_j)) + \mathbb{E}(h_t(Y_i^\varepsilon)h_t(Y_j^\varepsilon))) \\ &= (r_m/m)^2 \sum_{i,j=1}^m \sum_{\ell \geq 2} \frac{(c_\ell(t))^2}{\ell!} (\Gamma_{i,j})^\ell \left(1 + (1 + \varepsilon^2)^{-\ell} - 2(1 + \varepsilon^2)^{-\ell/2} \right) \end{aligned}$$

because $\text{Cov}(Y_i, Y_j) = \Gamma_{i,j}$, $\text{Cov}(Y_i^\varepsilon, Y_j) = \text{Cov}(Y_i, Y_j^\varepsilon) = \Gamma_{i,j}/(1 + \varepsilon^2)^{1/2}$ and $\text{Cov}(Y_i^\varepsilon, Y_j^\varepsilon) = \Gamma_{i,j}/(1 + \varepsilon^2)$. Next, by separating the case $i = j$ and $i \neq j$, the previous display can be upper bounded by

$$\sum_{\ell \geq 2} \frac{(c_\ell(t))^2}{\ell!} \left| 1 + (1 + \varepsilon^2)^{-\ell} - 2(1 + \varepsilon^2)^{-\ell/2} \right| + (r_m/m)^2 \sum_{i \neq j} (\Gamma_{i,j})^2 \times 4 \sum_{\ell \geq 2} \frac{(c_\ell(t))^2}{\ell!}.$$

While the first term above does not depend on m and converges to zero as $\varepsilon \rightarrow 0$, the second term above as a \limsup_m equal to zero by (vanish-secondorder). This implies (51) and finishes the proof. \square

5.3. Convergence of finite dimensional laws for $\widehat{\mathbb{F}}_m$

In this section, we aim at proving the following result:

Proposition 5.3. *Consider the assumptions of Proposition 5.2. Then, (12) holds in the sense of the finite dimensional convergence.*

Proof. From expression (44), we investigate the (joint) convergence of $(r_m(\widehat{\mathbb{F}}_m - I), r_m \bar{Y}_m)$.

Case 1: $\theta = -1$ In that case, $r_m^2 \text{Var}(\bar{Y}_m) \rightarrow 0$. Hence, we can directly use Proposition 5.2 to state that $(r_m(\widehat{\mathbb{F}}_m - I), r_m \bar{Y}_m)$ converges to $\mathcal{L}(\tilde{\mathbb{Z}}/(1 + |\theta|)^{1/2}) \otimes \delta_0$ in the sense of the finite dimensional convergence. This establishes Proposition 5.3 in that case.

Case 2: $\theta > -1$ Now, $r_m^2 \text{Var}(\bar{Y}_m)$ is converging to some positive real number, namely $(1 + \theta)/(1 + |\theta|) > 0$. In particular, $\text{Var}(\bar{Y}_m) > 0$ for m large enough. Let us define the random variable

$$Y_0 = \bar{Y}_m (\text{Var} \bar{Y}_m)^{-1/2}.$$

We now consider the $(m + 1)$ -dimensional random vector $(Y_i)_{0 \leq i \leq m}$, which is centered, with a covariance matrix denoted $\Lambda^{(m+1)} = (\Lambda_{i,j}^{(m+1)})_{0 \leq i,j \leq m}$ and such that $\Lambda_{0,0}^{(m+1)} = 1$, $\Lambda_{i,j}^{(m+1)} = \Gamma_{i,j}^{(m)}$ for $1 \leq i, j \leq m$. We easily check that $\Lambda^{(m+1)}$ satisfies (**vanish-secondorder**) and (**H₂**) with the same value of θ and a rate asymptotically equivalent to the original r_m , see Lemma A.2. Hence, Proposition 5.2 shows that (by using notation therein),

$$\left(r_m \left((m + 1)^{-1} \sum_{i=0}^m h_t(Y_i) \right), Y_0 \right) \rightsquigarrow \mathcal{L}(\tilde{\mathbb{Z}}/(1 + |\theta|)^{1/2}) \otimes \mathcal{N}(0, 1),$$

in the sense of the finite dimensional convergence. Since $r_m h_t(Y_0)/m$ tends to zero in probability, the last display can be rewritten as

$$\left(r_m(\tilde{\mathbb{F}}_m - I), \bar{Y}_m(\text{Var } \bar{Y}_m)^{-1/2} \right) \rightsquigarrow \mathcal{L}(\tilde{\mathbb{Z}}/(1 + |\theta|)^{1/2}) \otimes \mathcal{N}(0, 1).$$

Finally, since $r_m^2 \text{Var}(\bar{Y}_m) \rightarrow (1 + \theta)/(1 + |\theta|)$, we finish the proof by applying (44). \square

5.4. Tightness

To complete the proof of Proposition 5.1, we now prove that the process $X_m = r_m(\tilde{\mathbb{F}}_m - I)$ is tight in the Skorokhod space, under the assumption of Theorem 3.1. This will also imply tightness for $r_m(\hat{\mathbb{F}}_m - I)$ by (44) because c_1 is a continuous function on $[0, 1]$. Hence Theorem 3.1 will be proved.

For proving the tightness of X_m , we use Proposition C.1. This is possible because $|c_1(t) - c_1(s)| \leq L|t - s|^{1/2}$, $0 \leq s, t \leq 1$ for some constant $L > 1$ (see Lemma B.4). Below, we prove that (82) holds in the following way: for large m ,

$$\mathbb{E}|X_m(t) - X_m(s)|^4 \leq C(|t - s|^{3/2} + (r_m)^{-\varepsilon_0}|t - s|), \text{ for all } t, s \in [0, 1], \quad (52)$$

for some constant $C > 0$ and for a constant $\varepsilon_0 > 0$ such that (**H₁**) holds.

To establish (52), fix $t, s \in [0, 1]$, $s \leq t$ and write

$$\mathbb{E}|X_m(t) - X_m(s)|^4 = \frac{r_m^4}{m^4} \sum_{i,j,k,\ell} \mathbb{E}(\bar{h}(Y_i)\bar{h}(Y_j)\bar{h}(Y_k)\bar{h}(Y_\ell)), \quad (53)$$

where we let $\bar{h}(x) = \mathbf{1}\{s < \Phi(x) \leq t\} - (t - s) - (c_1(t) - c_1(s))x = h_t(x) - h_s(x)$. Now, we split the sum in the RHS of (53) following the value of the cardinal of $\{i, j, k, \ell\}$.

Sum over $\#\{i, j, k, \ell\} = 1$ The corresponding summation is $\frac{r_m^4}{m^4} \sum_{i=1}^m \mathbb{E}((\bar{h}(Y_i))^4)$. We have

$$\mathbb{E}((\bar{h}(Y_i))^4) \leq 3^4 \left(|t - s| + |t - s|^4 + \mathbb{E}(|Y_1|^4)L^4|t - s|^{4/2} \right) \leq C_1|t - s|, \quad (54)$$

for $C_1 = 5 \cdot 3^4 L^4 > 0$. Since $r_m^2 \leq m$, we obtain

$$\frac{r_m^4}{m^4} \sum_{i=1}^m \mathbb{E}((\bar{h}(Y_i))^4) \leq \frac{C_1}{m} |t - s|. \quad (55)$$

Sum over $\#\{i, j, k, \ell\} = 2$ Up to a multiplicative constant, we should consider the sum

$$\frac{r_m^4}{m^4} \sum_{i \neq j} \mathbb{E} ((\bar{h}(Y_i))^2 (\bar{h}(Y_j))^2) = T_1^{(1)} + T_2^{(1)},$$

where, for an arbitrary $\eta_1 > 0$, $T_1^{(1)}$ and $T_2^{(1)}$ are defined by

$$T_1^{(1)} = \frac{r_m^4}{m^4} \sum_{i \neq j} \mathbf{1}_{\{|\Gamma_{i,j}| > \eta_1\}} \mathbb{E} ((\bar{h}(Y_i))^2 (\bar{h}(Y_j))^2); \quad (56)$$

$$T_2^{(1)} = \frac{r_m^4}{m^4} \sum_{i \neq j} \mathbf{1}_{\{|\Gamma_{i,j}| \leq \eta_1\}} \mathbb{E} ((\bar{h}(Y_i))^2 (\bar{h}(Y_j))^2). \quad (57)$$

On the one hand, by using (54),

$$T_1^{(1)} \leq \frac{r_m^4}{\eta_1^2 m^4} \sum_{i \neq j} |\Gamma_{i,j}|^2 \mathbb{E} ((\bar{h}(Y_i))^2 (\bar{h}(Y_j))^2) \leq \frac{C_1}{\eta_1^2 m} \left(\frac{r_m^2}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^2 \right) |t - s|. \quad (58)$$

On the other hand, by using (76) in Proposition B.1 (with $g_1 = g_2 = (\bar{h})^2$ and $d = 2$), we obtain that for any $i \neq j$ such that $|\Gamma_{i,j}| \leq \eta_1$ (choosing $\eta_1 > 0$ such that $2\sqrt{3}\eta_1 < 1$),

$$\mathbb{E} ((\bar{h}(Y_i))^2 (\bar{h}(Y_j))^2) \leq \frac{1}{(1 - 2\sqrt{3}\eta_1)^2} \left(\mathbb{E} (|\bar{h}(Z)|^{8/3}) \right)^{3/2} \leq \frac{C_2}{(1 - 2\sqrt{3}\eta_1)^2} |t - s|^{3/2},$$

for $C_2 = 3^4 L^4 (\mathbb{E} (|Z|^{8/3}))^{3/2} \in (0, \infty)$. Hence, we get

$$T_2^{(1)} \leq \frac{C_2}{(1 - 2\sqrt{3}\eta_1)^2} |t - s|^{3/2}. \quad (59)$$

Sum over $\#\{i, j, k, \ell\} = 3$ Up to a multiplicative constant, we should consider the sum

$$\frac{r_m^4}{m^4} \sum_{i, j, k \neq} \mathbb{E} (\bar{h}(Y_i) \bar{h}(Y_j) (\bar{h}(Y_k))^2) = T_1^{(2)} + T_2^{(2)},$$

where, for an arbitrary $\eta_2 > 0$, $T_1^{(2)}$ and $T_2^{(2)}$ are defined similarly to (56) and (57), by separating the case where $\max_{e_1 \neq e_2 \in \{i, j, k\}} |\Gamma_{e_1, e_2}|$ is above or below η_2 .

On the one hand, by using (54), we have

$$\begin{aligned} T_1^{(2)} &\leq \frac{C_1 r_m^4}{m^4} \sum_{i, j, k \neq} \mathbf{1}_{\left\{ \max_{e_1 \neq e_2 \in \{i, j, k\}} |\Gamma_{e_1, e_2}| > \eta_2 \right\}} |t - s| \\ &\leq 3 \frac{C_1 / \eta_2^4}{m} \left(\frac{r_m^4}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^4 \right) |t - s|. \end{aligned} \quad (60)$$

On the other hand, by using (77) in Proposition B.1 (with $g_1 = g_2 = \bar{h}$, $g_3 = (\bar{h})^2$, $f_1 = f_2 = \mathbf{1}_{\{s < \Phi(\cdot) \leq t\}}$, $d = 3$ and $d' = 2$), we obtain that for any distinct i, j, k such that $\max_{e_1 \neq e_2 \in \{i, j, k\}} |\Gamma_{e_1, e_2}| \leq \eta_2$ (choosing $\eta_2 > 0$ such that $3\sqrt{3}\eta_2 < 1$),

$$\mathbb{E} (\bar{h}(Y_i) \bar{h}(Y_j) (\bar{h}(Y_k))^2) \leq \max_{e_1 \neq e_2 \in \{i, j, k\}} |\Gamma_{e_1, e_2}|^2 \frac{27^2}{(1 - 3\sqrt{3}\eta_2)^3} |t - s|^{3/2} \times \sqrt{C_2}.$$

This yields

$$T_2^{(2)} \leq 3 \frac{\sqrt{C_2} 27^2}{(1 - 3\sqrt{3}\eta_2)^3} \left(\frac{r_m^2}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^2 \right) |t - s|^{3/2}. \quad (61)$$

Sum over $\#\{i, j, k, \ell\} = 4$ The last sum to be considered is

$$\frac{r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \mathbb{E}(\bar{h}(Y_i)\bar{h}(Y_j)\bar{h}(Y_k)\bar{h}(Y_\ell)) = T_1^{(3)} + T_2^{(3)},$$

where, for an arbitrary $\eta_3 > 0$, $T_1^{(3)}$ and $T_2^{(3)}$ are defined similarly to (56) and (57), by separating the case where $\max_{e_1 \neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}|$ is above or below η_3 . As before,

$$\begin{aligned} T_1^{(3)} &\leq \frac{C_1 r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \mathbf{1} \left\{ \max_{e_1 \neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}| > \eta_3 \right\} |t - s| \\ &\leq 6 \frac{C_1}{\eta_3^4} \left(\frac{r_m^4}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^4 \right) |t - s|. \end{aligned} \quad (62)$$

Next, by using (77) in Proposition B.1 (with $g_i = \bar{h}$, $f_i = \mathbf{1}\{s < \Phi(\cdot) \leq t\}$ and $d' = d = 4$), we obtain that (choosing $\eta_3 > 0$ such that $4\sqrt{3}\eta_3 < 1$),

$$\begin{aligned} T_1^{(3)} &\leq \frac{r_m^4}{m^4} \sum_{i,j,k,\ell \neq} \max_{e_1 \neq e_2 \in \{i,j,k,\ell\}} |\Gamma_{e_1,e_2}|^4 \frac{48^4}{(1 - 4\sqrt{3}\eta_3)^4} |t - s|^3 \\ &\leq 6 \frac{48^4}{(1 - 4\sqrt{3}\eta_3)^4} \left(\frac{r_m^4}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^4 \right) |t - s|^3. \end{aligned} \quad (63)$$

Finally, we obtain (52) by combining the bounds (55), (58), (59), (60), (61), (62), (63) and by using the assumptions (vanish-secondorder) and (H₁).

5.5. Proof for Remark 3.3

Obviously, (H₃) and (H₄) imply (vanish-secondorder), (H₂) with $\theta = +\infty$, and $r_m \sim \gamma_m^{-1/2}$. Hence, Proposition 5.2 entails that the finite dimensional laws of $X_m = r_m(\mathbb{F}_m - I)$ converge to 0 and it only remains to prove that X_m is tight. This can be done as in the previous section, except that we use $\kappa = 2$ in Proposition C.1. Namely, we prove that, for large m ,

$$\mathbb{E}|X_m(t) - X_m(s)|^2 \leq C \gamma_m^{\delta_0} |t - s|, \text{ for all } t, s \in [0, 1], \quad (64)$$

for some constants $C > 0$, $\delta_0 > 0$. To prove (64), we write (by using the same notation as in the previous section)

$$\begin{aligned} \mathbb{E}|X_m(t) - X_m(s)|^2 &= \frac{r_m^2}{m^2} \sum_{i,j} \mathbb{E}(\bar{h}(Y_i)\bar{h}(Y_j)) \\ &\leq C_3 \frac{r_m^2}{m} |t - s| + C_3 |t - s| \frac{r_m^2}{\eta^2 m^2} \sum_{i \neq j} |\Gamma_{i,j}|^2 \\ &\quad + \frac{r_m^2}{m^2} \sum_{i \neq j} \mathbf{1}\{|\Gamma_{i,j}| \leq \eta\} \mathbb{E}(\bar{h}(Y_i)\bar{h}(Y_j)) \end{aligned}$$

for some $\eta > 0$ and by letting $C_3 = 4 \cdot 3^2 L^2 > 0$. Applying now (77) in Proposition B.1 (with $g_i = \bar{h}$, $f_i = \mathbf{1}\{s < \Phi(\cdot) \leq t\}$ for $i = 1, 2$ and $d' = d = 2$), we obtain that (choosing $\eta > 0$ such that $2\sqrt{3\eta} < 1$),

$$\frac{r_m^2}{m^2} \sum_{i \neq j} \mathbf{1}\{|\Gamma_{i,j}| \leq \eta\} \mathbb{E}(\bar{h}(Y_i)\bar{h}(Y_j)) \leq \frac{(12)^2}{(1 - 2\sqrt{3\eta})^2} |t - s|^{3/2} \left(\frac{r_m^2}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^2 \right).$$

Finally, since (H₃) and (H₄) provide $\frac{r_m^2}{m^2} \sum_{i \neq j} |\Gamma_{i,j}|^2 = O(\gamma_m^{\varepsilon_1})$ and $r_m^2/m = O(\gamma_m^{\varepsilon_2})$ for some $\varepsilon_1, \varepsilon_2 > 0$, the criterion (64) is proved with $\delta_0 = \varepsilon_1 \wedge \varepsilon_2$ and the proof is finished.

Appendix A: Technical results for proving the main theorem

Lemma A.1. *Assume that $\Gamma^{(m)}$ satisfies (vanish-secondorder) and (eigenvalues-away0). For $1 \leq i \leq m$, let us consider the filtration $\{\mathcal{F}_i\}_{0 \leq i \leq m}$ defined by $\mathcal{F}_0 = \sigma(\emptyset)$ and $\mathcal{F}_i = \sigma(Y_1, \dots, Y_i)$, and denote $\sigma_i^2 = \text{Var}[\mathbb{E}(Y_i | \mathcal{F}_{i-1})]$. Consider the function $h_t(\cdot)$ defined by (46), the Hermite polynomials $H_\ell(\cdot)$ defined by (71) and the coordinates $c_\ell(\cdot)$ defined by (5). Then the following holds:*

$$\frac{r_m^2}{m^2} \sum_{i=1}^m \sigma_i^2 \rightarrow 0; \quad (65)$$

$$\frac{r_m^2}{m^2} \sum_{i,j} (\mathbb{E}[\mathbb{E}(Y_i | \mathcal{F}_{i-1}) \mathbb{E}(Y_j | \mathcal{F}_{j-1})])^2 \rightarrow 0; \quad (66)$$

$$\frac{r_m^2}{m^2} \sum_{i=1}^m \mathbb{E} \left[(\mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}))^2 \right] \rightarrow 0, \quad \text{for any } t \in [0, 1]; \quad (67)$$

$$\mathbb{E} \left[\left(\frac{r_m}{m} \sum_{i=1}^m \mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) \right)^2 \right] \rightarrow 0, \quad \text{for any } t \in [0, 1]. \quad (68)$$

Proof. By using Cholesky's decomposition, we can write $\Gamma = RR^T$ where R is $m \times m$ a lower triangular matrix. Hence, denoting by $R_{1,\cdot}, \dots, R_{m,\cdot}$ the lines of R , we have $\langle R_{i,\cdot}, R_{j,\cdot} \rangle = \Gamma_{i,j}$ for all i, j . Moreover, since we can write $Y_i = \sum_{j=1}^i R_{i,j} Z_j$ for some Z_1, \dots, Z_m i.i.d. $\mathcal{N}(0, 1)$, we have $R_{i,i}^2 = \text{Var}(Y_i | \mathcal{F}_{i-1}) = 1 - \sigma_i^2$ and $\sigma_i^2 = \sum_{j=1}^{i-1} R_{i,j}^2$ for all i .

Let us now prove (65). From (eigenvalues-away0), we have for all $x \in \mathbb{R}^m$, $\|R^T x\|^2 = x^T \Gamma x \geq \eta \|x\|^2$. Hence for all $x \in \mathbb{R}^m$, $\|Rx\|^2 \geq \eta \|x\|^2$. Thus, we have

$$\sum_{i < j} \Gamma_{i,j}^2 = \sum_{j=1}^m \sum_{i=1}^{j-1} ([RR_j^T]_i)^2 \geq \eta \sum_{j=1}^m \sum_{i=1}^{j-1} R_{j,i}^2 = \eta \sum_{i=1}^m \sigma_i^2,$$

which proves (65) by (vanish-secondorder). As for (66), we have for $i < j$,

$$\mathbb{E}[\mathbb{E}(Y_i | \mathcal{F}_{i-1}) \mathbb{E}(Y_j | \mathcal{F}_{j-1})] = \mathbb{E} \left[\sum_{k=1}^{i-1} R_{i,k} Z_k \sum_{\ell=1}^{j-1} R_{j,\ell} Z_\ell \right] = \sum_{k=1}^{i-1} R_{i,k} R_{j,k} = \Gamma_{i,j} - R_{i,i} R_{j,i}.$$

Hence, we obtain

$$\sum_{i < j} (\mathbb{E} [\mathbb{E}(Y_i | \mathcal{F}_{i-1}) \mathbb{E}(Y_j | \mathcal{F}_{j-1})])^2 = \sum_{i < j} (\Gamma_{i,j} - R_{i,i}R_{j,i})^2 \leq 2 \left(\sum_{i < j} \Gamma_{i,j}^2 + \sum_{i < j} R_{j,i}^2 \right),$$

which establishes (66) by (65) and (**vanish-secondorder**).

Next, let us establish the following equality in $L^2(\mathbb{P}_m)$: for any $i = 1, \dots, m$ and $t \in [0, 1]$,

$$\mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) = \sum_{\ell \geq 2} \frac{c_\ell(t)}{\ell!} \sigma_i^\ell H_\ell \left(\frac{\mathbb{E}(Y_i | \mathcal{F}_{i-1})}{\sigma_i} \right), \quad (69)$$

where the RHS of (69) is 0 if $\sigma_i = 0$. For this, consider some $1 \leq i \leq m$ and assume $\sigma_i > 0$ (otherwise the result is obvious). Let $\tilde{Y}_i = \frac{\mathbb{E}(Y_i | \mathcal{F}_{i-1})}{\sigma_i} \sim \mathcal{N}(0, 1)$. By using the multivariate Gaussian structure of Y , the distribution of Y_i conditionally on \mathcal{F}_{i-1} only depends on \tilde{Y}_i . Hence, we can write $\mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) = g(\tilde{Y}_i)$ for a (unique) function g in $L^2(\mathbb{R}, \mathcal{N}(0, 1))$. We now consider the expansion of g w.r.t. the Hermite polynomials in that space:

$$g(\cdot) = \sum_{\ell \geq 0} \frac{\mathbb{E}(g(\tilde{Y}_i) H_\ell(\tilde{Y}_i))}{\ell!} H_\ell(\cdot),$$

and we can compute each coordinate $\mathbb{E}(g(\tilde{Y}_i) H_\ell(\tilde{Y}_i))$ in the following way: for any $\ell \geq 0$,

$$\begin{aligned} \mathbb{E} \left[H_\ell(\tilde{Y}_i) \mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) \right] &= \mathbb{E} \left[H_\ell(\tilde{Y}_i) h_t(Y_i) \right] \\ &= \sum_{\ell' \geq 2} \frac{c_{\ell'}(t)}{(\ell')!} \mathbb{E} \left[H_\ell(\tilde{Y}_i) H_{\ell'}(Y_i) \right] \\ &= \frac{c_\ell(t)}{\ell!} \sigma_i^\ell \ell! \mathbf{1} \{ \ell \geq 2 \}, \end{aligned}$$

by using Fubini's theorem (because $\sum_{\ell' \geq 2} \frac{|c_{\ell'}(t)|}{(\ell')!} \mathbb{E} \left[|H_\ell(\tilde{Y}_i) H_{\ell'}(Y_i)| \right] \leq (\ell!)^{1/2} \sum_{\ell' \geq 2} \frac{|c_{\ell'}(t)|}{(\ell')^{1/2}} < \infty$), and by applying (72) with $\text{Cov}(Y_i, \tilde{Y}_i) = \sigma_i$. This proves (69).

Finally, by using (69), (72) and notation above, we have

$$\begin{aligned} \mathbb{E} \left[\left(\frac{r_m}{m} \sum_{i=1}^m \mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) \right)^2 \right] &= \frac{r_m^2}{m^2} \sum_{i,j} \mathbb{E} [\mathbb{E}(h_t(Y_i) | \mathcal{F}_{i-1}) \mathbb{E}(h_t(Y_j) | \mathcal{F}_{j-1})] \\ &= \frac{r_m^2}{m^2} \sum_{i,j} \sum_{\ell \geq 2} \sum_{\ell' \geq 2} \frac{c_\ell(t)}{\ell!} \frac{c_{\ell'}(t)}{(\ell')!} \sigma_i^\ell \sigma_j^{\ell'} \mathbb{E} \left[H_\ell(\tilde{Y}_i) H_{\ell'}(\tilde{Y}_j) \right] \\ &= \frac{r_m^2}{m^2} \sum_{i,j} \sum_{\ell \geq 2} \frac{c_\ell(t)^2}{\ell!} \sigma_i^\ell \sigma_j^\ell \left(\mathbb{E} [\tilde{Y}_i \tilde{Y}_j] \right)^\ell \\ &\leq \left(\sum_{\ell \geq 2} \frac{c_\ell(t)^2}{\ell!} \right) \left(\frac{r_m^2}{m^2} \sum_{i,j} (\mathbb{E} [\mathbb{E}(Y_i | \mathcal{F}_{i-1}) \mathbb{E}(Y_j | \mathcal{F}_{j-1})])^2 \right), \end{aligned}$$

which proves (68) by using (66). Exactly the same calculation with “ $i = j$ ” shows (67) from (65). \square

Lemma A.2. Assume that $\Gamma^{(m)}$ satisfies (**vanish-secondorder**) and that $r_m^2 \text{Var}(\bar{Y}_m)$ converges to some positive real number. Consider the $(m+1) \times (m+1)$ covariance matrix $\Lambda^{(m+1)}$ of $(Y_i)_{0 \leq i \leq m}$ defined in Section 5.3. Then the rate

$$r_{m+1}(\Lambda^{(m+1)}) = \left((m+1)^{-1} + \left| (m+1)^{-2} \sum_{0 \leq i \neq j \leq m} \Lambda_{i,j}^{(m+1)} \right| \right)^{-1/2}$$

satisfies $r_{m+1}(\Lambda^{(m+1)}) \sim r_m$ and moreover

$$(m+1)^{-2} r_m^2 \sum_{0 \leq i \neq j \leq m} \left(\Lambda_{i,j}^{(m+1)} \right)^2 = o(1). \quad (70)$$

In particular, $\Lambda^{(m+1)}$ satisfies (**vanish-secondorder**). Finally, when (**H₂**) holds for $\Gamma^{(m)}$, it also holds for $\Lambda^{(m+1)}$, with the same value of θ .

Proof. By definition,

$$m^{-2} \sum_{0 \leq i \neq j \leq m} \Lambda_{i,j} = m^{-2} \sum_{1 \leq i \neq j \leq m} \Gamma_{i,j} + 2m^{-2} \sum_{1 \leq j \leq m} \Lambda_{0,j}.$$

Since $\Lambda_{0,j} = (\text{Var} \bar{Y}_m)^{-1/2} m^{-1} \sum_{i=1}^m \Gamma_{i,j}$, we have

$$m^{-2} \sum_{1 \leq j \leq m} \Lambda_{0,j} = (\text{Var} \bar{Y}_m)^{-1/2} m^{-1} m^{-2} \sum_{1 \leq i,j \leq m} \Gamma_{i,j} = m^{-1} \left(m^{-2} \sum_{1 \leq i,j \leq m} \Gamma_{i,j} \right)^{1/2},$$

which is $o(1/m)$ because Γ satisfies (**vanish-secondorder**) and thus (**A**). This implies $r_{m+1}(\Lambda) \sim r_m$. Next, we establish (70). Let us write

$$(m+1)^{-2} r_m^2 \sum_{0 \leq i \neq j \leq m} (\Lambda_{i,j})^2 = (m+1)^{-2} r_m^2 \left(\sum_{1 \leq i \neq j \leq m} (\Gamma_{i,j})^2 + 2 \sum_{1 \leq j \leq m} (\Lambda_{0,j})^2 \right).$$

Furthermore, we have

$$\begin{aligned} \sum_{1 \leq j \leq m} (\Lambda_{0,j})^2 &= (\text{Var} \bar{Y}_m)^{-1} \sum_{1 \leq j \leq m} \left(m^{-1} \sum_{i=1}^m \Gamma_{i,j} \right)^2 \\ &\leq (\text{Var} \bar{Y}_m)^{-1} m^{-2} \sum_{1 \leq i,i' \leq m} \left(2\Gamma_{i,i'} + \sum_{j \notin \{i,i'\}} \Gamma_{i,j} \Gamma_{i',j} \right) \\ &\leq 2 + (m \text{Var} \bar{Y}_m)^{-1} \sum_{1 \leq i \neq j \leq m} (\Gamma_{i,j})^2. \end{aligned}$$

This implies the result, because $m \text{Var} \bar{Y}_m \geq r_m^2 \text{Var} \bar{Y}_m$, which is bounded away from 0 by assumption. \square

Appendix B: Results related to Hermite polynomials

Let us first recall that the sequence of Hermite polynomials $H_\ell(x)$, $\ell \geq 0$, $x \in \mathbb{R}$, is defined by the expression: for all $\ell \geq 0$,

$$\forall x \in \mathbb{R}, \phi^{(\ell)}(x) = (-1)^\ell H_\ell(x)\phi(x), \quad (71)$$

where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the density of a Gaussian standard variable and $\phi^{(\ell)}$ denotes its ℓ -th derivative (by convention, $\phi^{(0)} = \phi$). For instance, we have $H_0(x) = 1$, $H_1(x) = x$ and $H_2(x) = x^2 - 1$.

A well known fact is that $\{H_\ell(\cdot)/(\ell!)^{1/2}, \ell \geq 0\}$ is an Hilbert basis in $L^2(\mathbb{R}, \mathcal{N}(0, 1))$, the Hilbert space composed by square integrable functions w.r.t. the standard Gaussian measure. Moreover, the following property holds: for any centered 2-dimensional Gaussian vector (U, V) with $\mathbb{E}U^2 = \mathbb{E}V^2 = 1$,

$$\forall \ell, \ell' \geq 0, \ell \neq \ell', \mathbb{E}(H_\ell(U)H_{\ell'}(V)) = (\text{Cov}(U, V))^\ell \ell! \delta_{\ell, \ell'}. \quad (72)$$

The latter can be seen as a consequence of Mehler's formula, itself being nicely presented in [Foata \(1981\)](#) (1.4) (see also references therein).

Proof of Proposition 2.1 Let us start by expanding, for any $t \in [0, 1]$, the function $\mathbf{1}\{\Phi(\cdot) \leq t\}$ w.r.t. the Hermite polynomial basis in $L^2(\mathbb{R}, \mathcal{N}(0, 1))$:

$$\mathbf{1}\{\Phi(\cdot) \leq t\} = \sum_{\ell \geq 0} c_\ell(t) H_\ell(\cdot)/(\ell!). \quad (73)$$

By applying (73) at Y_i , we obtain the following expansion in $L^2(\mathbb{P}_m)$: for all $i = 1, \dots, m$,

$$\mathbf{1}\{\Phi(Y_i) \leq t\} = \sum_{\ell \geq 0} c_\ell(t) H_\ell(Y_i)/(\ell!). \quad (74)$$

By averaging w.r.t. i , we obtain

$$\widehat{\mathbb{F}}_m(t) - t = \sum_{\ell \geq 1} \frac{c_\ell(t)}{\ell!} m^{-1} \sum_{i=1}^m H_\ell(Y_i). \quad (75)$$

where the series in the RHS of (75) converges in $L^2(\mathbb{P}_m)$ (by using the triangle inequality). The proof is finished by combining (75) with (72).

Next, the following proposition shares some similarities with Lemma 4.5 of [Taqqu \(1977\)](#) and Lemma 3 of [Csörgő and Mielniczuk \(1996\)](#).

Proposition B.1. *Consider an integer $d \geq 2$, a positive number ρ such that $\sqrt{3\rho d} < 1$ and $Z \sim \mathcal{N}(0, 1)$. Let g_1, \dots, g_d be d measurable real functions defined on \mathbb{R} such that $\mathbb{E}(|g_i(Z)|^{4/3}) < +\infty$, $1 \leq i \leq d$. Let (U_1, \dots, U_d) be d -dimensional centered Gaussian vector with $\mathbb{E}U_i^2 = 1$, $1 \leq i \leq d$, and $|\mathbb{E}(U_i U_j)| \leq \rho$, $1 \leq i \neq j \leq d$. Then the following holds:*

$$\mathbb{E} \left[\prod_{i=1}^d |g_i(U_i)| \right] \leq \frac{1}{(1 - \sqrt{3\rho d})^d} \prod_{i=1}^d \left(\mathbb{E} \left(|g_i(Z)|^{4/3} \right) \right)^{3/4}; \quad (76)$$

Furthermore, if $\mathbb{E}(g_i(Z)) = 0$ and $\mathbb{E}(Zg_i(Z)) = 0$ for $1 \leq i \leq d'$ for an integer d' , $1 \leq d' \leq d$, we have

$$\left| \mathbb{E} \left[\prod_{i=1}^d g_i(U_i) \right] \right| \leq \rho^{d'} \frac{(3d^2)^{d'}}{(1 - \sqrt{3\rho d})^d} \prod_{i=1}^d \left(\mathbb{E} \left(|f_i(Z)|^{4/3} \right) \right)^{3/4}, \quad (77)$$

where f_i is any function such that $f_i(x) = g_i(x) - \alpha_i - \beta_i x$, $x \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{R}$, for $1 \leq i \leq d'$ and $f_i = g_i$ otherwise.

Proof. The Kibble-Slepian formula [Kibble \(1945\)](#); [Slepian \(1972\)](#) (given, e.g., in expression (2.2) of [Foata, 1981](#)) provides that

$$\begin{aligned} \mathbb{E} \left[\prod_{i=1}^d g_i(U_i) \right] &= \mathbb{E} \left(\sum_{\nu} \prod_{i < j} \frac{(\mathbb{E}(U_i U_j))^{\nu_{ij}}}{\nu_{ij}!} \cdot \prod_{i=1}^d g_i(Z) H_{\nu_i}(Z) \right) \\ &= \sum_{\nu} \prod_{i < j} \frac{(\mathbb{E}(U_i U_j))^{\nu_{ij}}}{\nu_{ij}!} \cdot \prod_{i=1}^d \mathbb{E}(g_i(Z) H_{\nu_i}(Z)), \end{aligned} \quad (78)$$

where the summation is over all the $d \times d$ symmetric matrix $\nu = (\nu_{ij})_{1 \leq i, j \leq d}$ with nonnegative integral entries and with diagonal entries equal to zero, while ν_i denotes $\nu_{i1} + \dots + \nu_{id}$. Above, we have implicitly used Fubini's theorem (the summation over ν is infinite). The next calculations show that this is indeed valid: by using the assumptions, we have

$$\begin{aligned} &\sum_{\nu} \prod_{i < j} \frac{|\mathbb{E}(U_i U_j)|^{\nu_{ij}}}{\nu_{ij}!} \cdot \prod_{i=1}^d \mathbb{E} |g_i(Z) H_{\nu_i}(Z)| \\ &\leq \sum_{\nu} \prod_{i=1}^d \left(\frac{\rho^{\nu_i}}{\prod_j \nu_{ij}!} \right)^{1/2} \mathbb{E} |g_i(Z) H_{\nu_i}(Z)| \\ &\leq \sum_{x_1, \dots, x_d \in \mathbb{N}^d} \prod_{i=1}^d \left(\frac{\rho^{x_i}}{\prod_j x_{ij}!} \right)^{1/2} \mathbb{E} |g_i(Z) H_{x_i}(Z)| \\ &= \prod_{i=1}^d \left[\sum_{y \in \mathbb{N}^d} \left(\frac{\rho^{y_1 + \dots + y_d}}{\prod_j y_j!} \right)^{1/2} \mathbb{E} |g_i(Z) H_{y_1 + \dots + y_d}(Z)| \right] \\ &= \prod_{i=1}^d \left[\sum_{\ell \geq 0} \rho^{\ell/2} \mathbb{E} \left| g_i(Z) H_{\ell}(Z) / (\ell!)^{1/2} \right| \sum_{\substack{y \in \mathbb{N}^d \\ y_1 + \dots + y_d = \ell}} \left(\frac{\ell!}{\prod_j y_j!} \right)^{1/2} \right]. \end{aligned} \quad (79)$$

Now, in the latter display, the sum over y is upper bounded by d^{ℓ} , which gives that the RHS of (79) is upper bounded by

$$\sum_{\ell \geq 0} (\rho d^2)^{\ell/2} \mathbb{E} \left| g_i(Z) H_{\ell}(Z) / (\ell!)^{1/2} \right| \leq \left(\sum_{\ell \geq 0} (3\rho d^2)^{\ell/2} \right) \left(\mathbb{E} \left(|g_i(Z)|^{4/3} \right) \right)^{3/4},$$

where the latter combines Hölder's inequality with Lemma [B.2](#) (used with $p = 4$). This proves (76) and shows that Fubini's theorem can be applied to get (78).

Finally, we prove (77) by using (78) and the same calculations as above, except that the absolute values should be kept outside the expectations. As a result, for $1 \leq i \leq d'$, since $\mathbb{E}(g_i(Z)H_\ell(Z)) = 0$ for $\ell = 0, 1$ by assumption, the corresponding sums over ℓ start at $\ell = 2$. This establishes (77), because for all $\ell \geq 2$ and $1 \leq i \leq d'$, $\mathbb{E}(g_i(Z)H_\ell(Z)) = \mathbb{E}(f_i(Z)H_\ell(Z))$. \square

The following result was obtained in the proof of Lemma 3.1 in Taqqu (1977). We provide an elementary proof below. Also, let us mention that there are more accurate such results when ℓ grows to infinity, see Theorem 2.1 in Larsson-Cohn (2002).

Lemma B.2. *For all even integer $p \geq 2$ and $\ell \geq 0$, we have $\left[\mathbb{E}\left(H_\ell(Z)/\sqrt{\ell!}\right)^p\right]^{1/p} \leq (p-1)^{\ell/2}$, for $Z \sim \mathcal{N}(0, 1)$.*

Proof. For some $\ell \geq 1$, by using $H'_\ell = \ell H_{\ell-1}$ and (71), we obtain

$$\begin{aligned} \int [H_\ell(x)]^p \phi(x) dx &= (-1)^\ell \int [H_\ell(x)]^{p-1} \phi^{(\ell)}(x) dx, \\ &= \ell(p-1) \int [H_\ell(x)]^{p-2} [H_{\ell-1}(x)]^2 \phi(x) dx. \end{aligned}$$

Next, by using Hölder's inequality, we get $(\int [H_\ell(x)]^p \phi(x) dx)^{2/p} \leq \ell(p-1) (\int |H_{\ell-1}(x)|^p \phi(x) dx)^{2/p}$, and the result is obtained by induction on ℓ . \square

Lemma B.3. *Consider the function $h_t(\cdot)$ defined by (46) and $c_\ell(\cdot)$ defined by (5). Let us consider a two-dimensional centered Gaussian vector (U, V) with $\mathbb{E}U^2 = \mathbb{E}V^2 = 1$. Then for any $t, s \in [0, 1]$, the following holds:*

$$\mathbb{E}(h_t(U)h_s(V)) = \sum_{\ell \geq 2} \frac{c_\ell(t)c_\ell(s)}{\ell!} (\text{Cov}(U, V))^\ell. \quad (80)$$

Proof. Expression (80) is a direct consequence of (72) and of Fubini's theorem. \square

Lemma B.4. *The function $c_1(\cdot) = \phi(\Phi^{-1}(\cdot))$ satisfies the following: for all $\nu \in (0, 1)$, there exists some constant $C_\nu > 0$ such that for all $s, t \in [0, 1]$,*

$$|c_1(t) - c_1(s)| \leq C_\nu |t - s|^{1-\nu}. \quad (81)$$

Proof. First note that the derivative of c_1 on $(0, 1)$ is Φ^{-1} . Classically (see, e.g., Lemma 12.3 of Abramovich et al. (2006)), there is some $x_0 \in (0, 1/2)$ such that for any $u \in (0, x_0)$, $\Phi^{-1}(u) \leq \sqrt{2 \log(1/u)}$. Also, obviously, for some fixed $\nu > 0$, there is some $C'_\nu > 0$ such that for any $u \in (0, x_0)$, $\sqrt{2 \log(1/u)} \leq C'_\nu u^{-\nu}$. As a consequence, since $|\Phi^{-1}|$ is bounded on $[x_0, 1 - x_0]$, there exists some constant $C''_\nu > 0$ such that for all $u \in (0, 1)$, $|\Phi^{-1}(u)| \leq C''_\nu u^{-\nu}$. This entails that for all $0 < s \leq t < 1$,

$$|c_1(t) - c_1(s)| \leq \int_s^t |\Phi^{-1}(u)| du \leq \frac{C''_\nu}{1-\nu} (t^{1-\nu} - s^{1-\nu}) \leq C_\nu (t - s)^{1-\nu}$$

by letting $C_\nu = C''_\nu/(1-\nu) > 0$ and because $(x+y)^\delta \leq x^\delta + y^\delta$ for any $x, y \geq 0$ and any $\delta \in (0, 1)$. \square

Appendix C: Useful auxiliary results

The following result can certainly be considered as well known, although we failed to find a precise reference for it. It can be seen as a reformulation in our framework of classical tightness results as given, e.g., in Lemma 2 of Csörgő and Mielniczuk (1996), in Remark 2.1 of Shao and Yu (1996) and Proposition 6 of Dedecker and Prieur (2007).

Proposition C.1 (Tightness criterion for empirical distribution function with non-standard scaling parameters). *Consider ξ_1, \dots, ξ_m real random variables (that need not to be independent or identically distributed) such that $\bar{\xi}_m \xrightarrow{P} 0$ as m tends to infinity, for $\bar{\xi}_m = m^{-1} \sum_{i=1}^m \xi_m$, and consider the process*

$$Z_m(t) = (a_m/m) \sum_{i=1}^m g_t(\xi_i), \text{ for } t \in [0, 1],$$

where $(a_m)_m$ is some positive sequence tending to infinity as m tends to infinity and where $g_t(x) = \mathbf{1}\{\Phi(x) \leq t\} - f_0(t) - f_1(t)x$ for functions f_0, f_1 on $[0, 1]$ such that $|f_0(t) - f_0(s)| \vee |f_1(t) - f_1(s)| \leq L|t - s|^q$, $0 \leq s, t \leq 1$, for some $q \in (0, 1]$ and $L > 0$. Assume that the following holds: for large m ,

$$\mathbb{E}|Z_m(t) - Z_m(s)|^\kappa \leq C(|t - s|^{\delta_1} + (a_m)^{-\delta_2/q}|t - s|^{q'}), \text{ for all } t, s \in [0, 1], \quad (82)$$

for constants $\kappa > 0$, $C > 0$, $\delta_1 > 1$, $q' \in (0, 1]$ and $\delta_2 > 1 - q'$. Then, as m grows to infinity, the sequence of processes $(Z_m)_m$ is tight in $D(0, 1)$ (endowed with the Skorokhod topology and the corresponding Borel σ -field) and any limit is a.s. a continuous process.

Proof. The proof is based on standard arguments and is similar to the proof of Theorem 22.1 in Billingsley (1968). Fix $\varepsilon \in (0, 1)$ and $\eta > 0$. Following Theorem 15.5 in Billingsley (1968), it is sufficient to prove that there exists a $\delta \in (0, 1)$ such that for large m ,

$$\mathbb{P} \left(\sup_{\substack{0 \leq s, t \leq 1 \\ |s - t| \leq \delta}} |Z_m(t) - Z_m(s)| > \varepsilon \right) < \eta.$$

We merely check (see, e.g., the proof of Theorem 8.3 in Billingsley (1968)) that the latter holds if there exists $\delta \in (0, 1)$ such that for large m ,

$$\forall s \in [0, 1], \mathbb{P} \left(\sup_{t: s \leq t \leq (s + \delta) \wedge 1} |Z_m(t) - Z_m(s)| > \varepsilon \right) < \eta \delta. \quad (83)$$

Let us now prove (83). Fix $s \in [0, 1]$. Assumption (82) entails that for all $u, v \in [0, 1]$ such that $(v - u)^q \geq \varepsilon/a_m$, we have

$$\mathbb{E}|Z_m(v) - Z_m(u)|^\kappa \leq \frac{2C}{\varepsilon^{\delta_2/q}} |v - u|^{\delta_3}$$

for $\delta_3 = \delta_1 \wedge (q' + \delta_2) > 1$. Hence, if $p > 0$ is such that $p^q \geq \varepsilon/a_m$, applying Theorem 12.2 of Billingsley (1968) we have for all integer M such that $s + Mp \leq 1$ and for all $\lambda > 0$,

$$\mathbb{P} \left(\max_{1 \leq i \leq M} |Z_m(s + ip) - Z_m(s)| > \lambda \right) \leq \frac{K}{\lambda^\kappa \varepsilon^{\delta_2/q}} (Mp)^{\delta_3} \quad (84)$$

for some positive constant $K > 0$ (only depending on δ_3 , κ and C). Next, we use the following inequality: for all $0 \leq u, v \leq 1$, $u \leq v \leq u + p$,

$$|Z_m(v) - Z_m(u)| \leq |Z_m(u + p) - Z_m(u)| + 2La_m p^q (1 + |\bar{\xi}_m|). \quad (85)$$

The latter holds because we have

$$\begin{aligned} Z_m(v) - Z_m(u) &= (a_m/m) \sum_{i=1}^m \mathbf{1}\{u < \xi_i \leq v\} - a_m(f_0(v) - f_0(u)) - a_m(f_1(v) - f_1(u))\bar{\xi}_m \\ &\leq Z_m(u + p) - Z_m(u) + 2La_m p^q (1 + |\bar{\xi}_m|) \end{aligned}$$

and $Z_m(u) - Z_m(v) \leq a_m(f_0(v) - f_0(u)) + a_m(f_1(v) - f_1(u))\bar{\xi}_m \leq La_m p^q (1 + |\bar{\xi}_m|)$.

Now, by using (85), we obtain

$$\sup_{t:s \leq t \leq s+Mp} |Z_m(t) - Z_m(s)| \leq 3 \max_{1 \leq i \leq M} |Z_m(s + ip) - Z_m(s)| + 2La_m p^q (1 + |\bar{\xi}_m|). \quad (86)$$

Furthermore, provided that $a_m p^q \leq 2\varepsilon$, we have $\mathbb{P}(2La_m p^q (1 + |\bar{\xi}_m|) > 5L\varepsilon) \leq \mathbb{P}(|\bar{\xi}_m| > 1/4)$. Hence, combining (84) and (86), by taking $\delta \in (0, 1)$ such that $K\delta^{\delta_3-1}/\varepsilon^{\kappa+\delta_2/q} < \eta/2$, we will obtain that for all $s \in [0, 1 - \delta]$, for large m ,

$$\mathbb{P}\left(\sup_{t:s \leq t \leq s+\delta} |Z_m(t) - Z_m(s)| > (3 + 5L)\varepsilon\right) \leq \frac{K}{\varepsilon^{\kappa+\delta_2/q}} \delta^{\delta_3} + \mathbb{P}(|\bar{\xi}_m| > 1/4) < \eta\delta,$$

as soon as we can choose $p > 0$ and an integer M such that $Mp = \delta$ and $\varepsilon/a_m \leq p^q \leq 2\varepsilon/a_m$. This holds if there exists an integer into the interval $[\delta(a_m/\varepsilon)^{1/q}, \delta(a_m/(2\varepsilon))^{1/q}]$, which is true for large m because a_m tends to infinity. This entails (83) with ε replaced by $(3 + 5L)\varepsilon$ and the proof is finished. \square

Proposition C.2 (Partial functional delta method on $D(0, 1)$). *Consider the linear space $D(0, 1)$ of càd-làg function on $[0, 1]$ and the linear space $C(0, 1)$ of continuous functions on $[0, 1]$. Let $\theta = (\theta_0, \theta_1) \in D(0, 1)^2$. Let $\phi : D(0, 1)^2 \mapsto \mathbb{R}$ be Hadamard differentiable at θ tangentially to $C(0, 1)$, w.r.t. the supremum norm, and such that the derivative is of the form*

$$\dot{\phi}_\theta(H_0, H_1) = g_\theta(H_0), \text{ for any } (H_0, H_1) \in C(0, 1)^2,$$

for a continuous linear mapping $g_\theta : C(0, 1) \mapsto \mathbb{R}$. Consider $\mathbb{Z}_{0,m}, \mathbb{Z}_{1,m}$, $m \geq 1$, processes valued in $D(0, 1)$ and $\mathbb{Z}_0, \mathbb{Z}_1$ two processes valued a.s. in $C(0, 1)$. Assume that the two following distribution convergences hold (w.r.t. the Skorokhod topology and the corresponding Borel σ -field), for some positive sequence $(a_m)_m$ tending to infinity:

$$\begin{aligned} a_m(\mathbb{Z}_{0,m} - \theta_0) &\rightsquigarrow \mathbb{Z}_0; \\ a_m(\mathbb{Z}_{1,m} - \theta_1) &\rightsquigarrow \mathbb{Z}_1. \end{aligned}$$

Then we have

$$a_m(\phi(\mathbb{Z}_{0,m}, \mathbb{Z}_{1,m}) - \phi(\theta)) \rightsquigarrow g_\theta(\mathbb{Z}_0). \quad (87)$$

Proof. Classically, let us show that for any subsequence $\{n\}$ there exists a further subsequence $\{\ell\}$ such that (87) holds along this subsequence. For any $\{n\}$, since both processes $a_n(\mathbb{Z}_{0,n} - \theta_0)$ and $a_n(\mathbb{Z}_{1,n} - \theta_1)$ are (Skorokhod-)tight, the joint process $(a_n(\mathbb{Z}_{0,n} - \theta_0), a_n(\mathbb{Z}_{1,n} - \theta_1))$ also is. Hence, by Prohorov's theorem, there exists a further subsequence $\{\ell\}$ such that $(a_\ell(\mathbb{Z}_{0,\ell} - \theta_0), a_\ell(\mathbb{Z}_{1,\ell} - \theta_1))$ converges in distribution. Now applying the Skorokhod's representation theorem (see, e.g., Theorem 6.7 page 70 in Billingsley (1999)), there exists random elements $T_\ell = (T_{0,\ell}, T_{1,\ell})$, $\ell \geq 1$, $T = (T_0, T_1)$, defined on a common probability space, such that $\mathcal{L}(T_\ell) = \mathcal{L}(a_\ell(\mathbb{Z}_{0,\ell} - \theta_0), a_\ell(\mathbb{Z}_{1,\ell} - \theta_1))$, $\mathcal{L}(T_0) = \mathcal{L}(\mathbb{Z}_0)$, $\mathcal{L}(T_1) = \mathcal{L}(\mathbb{Z}_1)$ and T_ℓ converges a.s. to T . Since both T_0 and T_1 belong to $C(0, 1)$ (a.s.) and since any sequence of càd-làg functions converging (w.r.t. to the Skorokhod distance) to a continuous function also converges uniformly, we obtain

$$\|T_{0,\ell} - T_0\|_\infty + \|T_{1,\ell} - T_1\|_\infty \rightarrow 0 \quad \text{a.s.}$$

Hence, the Hadamard differentiability of ϕ entails:

$$\frac{\phi(\theta + t_\ell T_\ell) - \phi(\theta)}{t_\ell} \rightarrow g_\theta(T_0) \quad \text{a.s.},$$

for any sequence $t_\ell \rightarrow 0$. By taking $t_\ell = 1/a_\ell$, we derive (87) along the subsequence $\{\ell\}$, which proves the result. \square

Lemma C.3. *Assume that Γ satisfies (A). Then for any $h : \mathbb{R} \rightarrow \mathbb{R}$ measurable such that $\mathbb{E}|h(Z)| < \infty$, we have*

$$m^{-1} \sum_{i=1}^m h(Y_i) \xrightarrow{P} \mathbb{E}[h(Z)], \quad \text{for } Z \sim \mathcal{N}(0, 1). \quad (88)$$

Proof. By Section 2, Assumption (A) implies that $\forall t \in [0, 1]$, $\widehat{\mathbb{F}}_m(t) \xrightarrow{P} t$. Since $h \in L^1(\mathbb{R}, \mathcal{N}(0, 1))$, for any $\varepsilon > 0$, there is a continuous bounded function h_ε such that $\mathbb{E}|h(Z) - h_\varepsilon(Z)| \leq \varepsilon$. Moreover, by definition of the weak convergence, (88) holds for $h = h_\varepsilon$ (for instance, the convergence in probability can be seen as an a.s. convergence up to consider subsequence). Since we have

$$\sup_{m \geq 1} \left\{ \mathbb{E} \left| m^{-1} \sum_{i=1}^m (h(Y_i) - h_\varepsilon(Y_i)) \right| \right\} \leq \sup_{m \geq 1} \left\{ m^{-1} \sum_{i=1}^m \mathbb{E} |h(Y_i) - h_\varepsilon(Y_i)| \right\} \leq \varepsilon,$$

we can conclude by using Lemma C.4. \square

The following lemma is classical, see, e.g., Theorem 4.2 in Billingsley (1968).

Lemma C.4. *For $n \geq 1$ and $\varepsilon > 0$, let X_n^ε , X_n , X^ε , X be real random variables (X_n and X_n^ε being defined on the same probability space) and such that*

- (a) $\forall \varepsilon > 0$, $X_n^\varepsilon \rightsquigarrow X^\varepsilon$ as $n \rightarrow \infty$;
- (b) $X^\varepsilon \rightsquigarrow X$ as $\varepsilon \rightarrow 0$;
- (c) $\limsup_{n \rightarrow \infty} \{\mathbb{E}|X_n^\varepsilon - X_n|\} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Then $X_n \rightsquigarrow X$.

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