



**HAL**  
open science

# Singular hypersurfaces characterizing the Lefschetz properties

Roberta Di Gennaro, Giovanna Ilardi, Jean Vallès

► **To cite this version:**

Roberta Di Gennaro, Giovanna Ilardi, Jean Vallès. Singular hypersurfaces characterizing the Lefschetz properties. 2012. hal-00739676v2

**HAL Id: hal-00739676**

**<https://hal.science/hal-00739676v2>**

Preprint submitted on 11 Dec 2012 (v2), last revised 20 May 2014 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CONES RELATED TO THE LEFSCHETZ PROPERTIES

ROBERTA DI GENNARO, GIOVANNA ILARDI AND JEAN VALLÈS

ABSTRACT. In [16] the authors highlight the link between rational varieties satisfying a Laplace equation and artinian ideals that fail the Weak Lefschetz property. Continuing their work we extend this link to the more general situation of artinian ideals failing the Strong Lefschetz Property. We characterize the failure of SLP (that includes WLP) by the existence of special singular hypersurfaces (cones for WLP). This characterization allows us to solve three problems posed in [17] and to give new examples of ideals failing the SLP. Finally, line arrangements are related to artinian ideals and the unstability of the associated derivation bundle is linked with the failure of SLP. Moreover we reformulate the so-called Terao's conjecture for free line arrangements in terms of artinian ideals failing the SLP.

December 11, 2012

## 1. INTRODUCTION

The tangent space  $T_a X$  to an integral projective variety  $X$  of dimension  $n$  in a smooth point has always dimension  $n$ .

It is no longer true for the osculating spaces. For instance, as it was pointed out by Togliatti in [25], the osculating space  $T_a^2 X$ , in a general point  $a$ , of the rational surface  $X$  defined by

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^5, (x, y, z) \mapsto (xz^2, yz^2, x^2z, y^2z, xy^2, x^2y),$$

has projective dimension 4 instead of 5. Indeed there is a non trivial linear relation between the partial derivatives of order 2 of  $\phi$  in  $a$  that define  $T_a^2 X$ . This relation is usually called a *Laplace equation* of order 2. More generally, we will say that  $X$  satisfies a Laplace equation of order  $k$  when its  $k$ -th osculating space  $T_a^k X$  in a general point  $a \in X$  has dimension less than expected.

The study of the surfaces satisfying a Laplace equation was developed in the last century by Togliatti [25] and Terracini [24]. Togliatti [25] gave a complete classification of the rational surfaces embedded with linear systems of plane cubics and satisfying a Laplace equation of order two.

In the dense paper [20], Perkinson gives a complete classification of smooth toric surfaces (thm. 3.2) and threefolds (thm. 3.5) embedded with a monomial linear system and satisfying a Laplace equation of any order.

Very recently Miro-Roig, Mezzetti and Ottaviani [16] establish a nice link between rational varieties (i.e. projections of Veronese varieties) satisfying a Laplace equation and artinian graded rings  $A = \bigoplus_{0 \leq i \leq s} A_i$  such that the multiplication by a general linear form has not maximal rank in one degree  $i$ . On the contrary, when the rank of the multiplication map is maximal in any degree, the ring is said to have the *Weak*

---

2010 *Mathematics Subject Classification.* 13E10, 14L35, 14N20, 53A20.

*Key words and phrases.* Weak and Strong Lefschetz Property, Laplace equations, Line arrangements.

Third author partially supported by ANR-09-JCJC-0097-0 INTERLOW, ANR GEOLMI, the project F.A.R.O. 2010: Algebre di Hopf, differenziali e di vertice in geometria, topologia e teorie di campo classiche e quantistiche and by the program "breve mobilità de l'Universita degli Studi di Napoli Federico II".

*Lefschetz property* (briefly WLP). This name comes from a Stanley's preliminary work. The Hard Lefschetz theorem is used to prove that the ring  $\frac{\mathbb{C}[x_0, \dots, x_n]}{(x_0^{d_0}, \dots, x_n^{d_n})}$  has the WLP [ [23], thm. 2.4]. From this example one can ask if the artinian complete intersection rings have the WLP. Actually  $\frac{\mathbb{C}[x, y, z]}{(F_0, F_1, F_2)}$  has the WLP (first proved in [11] and then also in [3]) but it is still not known for more than three variables. Many other questions derive from this first example.

The same type of problems occurs when we consider the multiplication by powers  $L^k$  ( $k \geq 1$ ) of a general linear form  $L$ . Indeed, if the rank of the multiplication map by  $L^k$  is maximal for any  $k$  and any degree, the ring is said to have the *Strong Lefschetz property* (briefly SLP). For more details about known results and some open problems we refer to [17].

As in [11], many papers about the Lefschetz properties involve the *syzygy bundle*  $K$ . It is associated to an artinian ideal  $I = (F_1, \dots, F_r)$  in the following way:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r \xrightarrow{(F_1, \dots, F_r)} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0.$$

Indeed, in [ [3], prop. 2.1] Brenner and Kaid proved that the graded piece of degree  $d + i$  of the artinian ring  $A = \frac{\mathbb{C}[x_0, \dots, x_n]}{(F_0, \dots, F_r)}$  is  $H^1(K(i))$ . In [ [16], thm. 3.2] the authors characterize the failure of WLP (in degree  $d - 1$ , i.e. for the map  $A_{d-1} \rightarrow A_d$ ) when  $r \leq h^0(\mathcal{O}_L(d))$  by the non injectivity of the restricted map

$$H^0(\mathcal{O}_L)^r \xrightarrow{(F_1, \dots, F_r)} H^0(\mathcal{O}_L(d)),$$

on a general hyperplane  $L$ .

Let us say, in few words, what we are doing in this paper and how is it organized. First of all we recall some definitions, basic facts and we propose a conjecture (section 3). In the section 4 we extend to the SLP the previous characterization of failure of WLP. Then we translate the failure of WLP and SLP in terms of existence of special singular hypersurfaces (section 5). It allows us to give an answer to three unsolved questions in [17]. In the section 6 we construct examples of artinian rings failing the WLP and the SLP by producing the appropriate singular hypersurfaces. In the last section we relate the problem of SLP at the range 2 to the topic of line arrangements (section 7).

Let us now give more details about the different sections of this paper. In the section 4, more precisely in the theorem 4.1, we characterize the failure of SLP by the non maximality of the induced map on section

$$H^0(\mathcal{O}_{L^k}(i))^r \xrightarrow{(F_1, \dots, F_r)} H^0(\mathcal{O}_{L^k}(i + d)).$$

The geometric consequences of this link are explained in the section 5 (see theorem 5.1). The non injectivity is translated in terms of the number of Laplace equations and the non surjectivity is related, via apolarity, to the existence of special singular hypersurfaces. Then we give the propositions 5.3, 5.4 and 5.5 that solve three problems posed in [ [17], problem 5.4 and conjecture 5.13].

In section 6 we produce many ideals (monomial and non monomial) that fail the SLP. The failure of WLP is studied for monomial ideals generated in degree 4 on  $\mathbb{P}^2$  in the theorem 6.1, monomial ideals generated in degree 5 on  $\mathbb{P}^2$  in the proposition 6.2, monomial ideals generated in degree 4 on  $\mathbb{P}^3$  in the proposition 6.5. The failure of SLP is studied for monomial ideal generated in degree 4 in proposition 6.6. In the proposition 6.7 we propose a method to produce non monomial ideals that fail the SLP at any range.

In the last section Lefschetz property and line arrangements are linked. The theory of line arrangements, more generally of hyperplane arrangements, is an old and deep

subject that concerns combinatorics, topology and algebraic geometry. One can say that it began with Jakob Steiner (in the first volume of Crelles's journal, 1826) that determined in how many regions a real plane is divided by a finite number of lines. It is relevant also with the amazing Sylvester-Gallai's problem. Hyperplane arrangements come back in a modern presentation in the Arnold's fundamental work [1] on the cohomology ring of  $\mathbb{P}^n \setminus D$  (where  $D$  is the union of the hyperplanes of the arrangement). For a large part of mathematicians, working on arrangements, it culminates today with the Terao's conjecture (see the last section of this paper or directly [19]). This conjecture concerns more particularly the derivation sheaf (also called logarithmic sheaf) associated to the arrangement. In this paper we recall the conjecture. In the proposition 7.2 we prove that the failure of the SLP at the range 2 of some ideals is equivalent to the unstability of the associated derivation sheaves. Thanks to the important literature on arrangements, we find artinian ideals that fail the SLP. For instance the Coxeter arrangement, called B3, gives an original ideal that fails the SLP at the range 2 in a non trivial way (see proposition 7.3).

We finish by a reformulation of Terao's conjecture in terms of SLP.

## 2. NOTATIONS

The ground field is  $\mathbb{C}$ .

The dual  $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  of a vector space  $V$  is denoted by  $V^*$ .

The dimension of the vector space  $H^0(\mathcal{O}_{\mathbb{P}^n}(t))$  is denoted by  $r_t$  where  $n$  is clearly known in the context.

The vector space generated by the set  $E \subset \mathbb{C}^t$  is  $\langle E \rangle$ .

The join variety of  $s$  projective varieties  $X_i \subset \mathbb{P}^n$  is denoted by  $\text{Join}(X_1, \dots, X_s)$  (see [10] for the definition of join variety).

The fundamental points  $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 0, 1)$  in  $\mathbb{P}^n$  are denoted by  $P_0, P_1, \dots, P_n$ .

We often write in the same way a projective hyperplane and the linear form defining it; we use in general the notation  $L_i$  on  $\mathbb{P}^n$  and the notation  $l_i$  on  $\mathbb{P}^2$  for hyperplanes. The ideal sheaf of a point  $P$  is  $\mathcal{I}_P$

## 3. LEFSCHETZ PROPERTY

Let  $R = \mathbb{C}[x_0, \dots, x_n]$  be the graded polynomial ring in  $n + 1$  variables over  $\mathbb{C}$ . The dimension of the vector space  $R_t$  is  $r_t$ .

$$\text{Let } A = R/I = \sum_{i=0}^m A_i$$

be a graded artinian algebra. Note that  $A$  is finite dimensional over  $\mathbb{C}$ .

**Definition 3.1.** Let  $L$  be a general linear form. The artinian algebra  $A$  (or the artinian ideal  $I$ ) has the Weak Lefschetz Property (WLP) if the homomorphism induced by the multiplication by  $L$ ,

$$\times L : A_i \rightarrow A_{i+1},$$

has maximal rank (i.e. is injective or surjective) for all  $i$ . The artinian algebra  $A$  (or the artinian ideal  $I$ ) has the Strong Lefschetz Property (SLP) if,

$$\times L^k : A_i \rightarrow A_{i+k},$$

has maximal rank (i.e. is injective or surjective) for all  $i$  and  $k$ .

**Remarks.** • It is clear that the SLP for  $k = 1$  corresponds to the WLP.

- We will often be interested in artinian rings  $A$  that fail the SLP (or WLP), i.e. when for a general linear form  $L$  there exist  $i$  and  $k$  such that the multiplication map

$$\times L^k : A_i \rightarrow A_{i+k},$$

has not maximal rank. In that case we will say that  $A$  (or  $I$ ) fails the SLP at range  $k$  and degree  $i$ . When  $k = 1$  we will say simply that  $A$  fails the WLP in degree  $i$ .

One of the main examples comes from the Togliatti's result (see for instance [3], Example 3.1): the ideal  $I = (x^3, y^3, z^3, xyz)$  fails the WLP in degree 2. There are many ways to prove it. One of them comes from the polarity on the rational normal cubic curve. It leads to a generalisation that gives one of the few known non toric examples.

**Proposition 3.2** ([26], Thm. 3.1). *Let  $n \geq 1$  be an integer and  $l_1, \dots, l_{2n+1}$  be non concurrent linear forms on  $\mathbb{P}^2$ . Then the ideal*

$$(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i)$$

*fails the WLP in degree  $2n$ .*

Indeed on the general line  $L$  the  $2n + 2$  forms of degree  $2n + 1$  become dependent thanks to the polarity on the rational normal curve of degree  $2n + 1$ . We propose the following conjecture. For  $n = 1$  it is again the Togliatti's result.

**Conjecture.** Let  $l_i$  be non concurrent linear forms on  $\mathbb{P}^2$  and  $f$  be a form of degree  $2n + 1$  on  $\mathbb{P}^2$ . Then,  $(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, f)$  fails the WLP in degree  $2n$  if and only if modulo linear combination

$$f \in (l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i).$$

#### 4. LEFSCHETZ PROPERTY AND THE SYZYGY BUNDLE

In [16], prop. 2.3] the failure of WLP in degree  $d - 1$  is related to the restriction of the syzygy bundle on a general hyperplane. We extend here this relationship to the SLP situation at any range and in many degrees.

**Theorem 4.1.** *Let  $I = (F_1, \dots, F_r) \subset R$  be an artinian ideal generated by homogeneous forms of degree  $d$  and  $K$  the vector bundle defined by the exact sequence*

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r \xrightarrow{\Phi_I} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0,$$

*where  $\Phi_I(a_1, \dots, a_r) = a_1 F_1 + \dots + a_r F_r$ . Let  $i$  be an integer such that  $h^0(K(i)) = 0$  and  $k$  be an integer such that  $k \geq 1$ . Then  $I$  fails the SLP at the range  $k$  in degree  $d - k + i$  if and only if the induced homomorphism on sections (denoted by  $H^0(\Phi_{I, L^k})$ )*

$$H^0(\mathcal{O}_{L^k}(i))^r \xrightarrow{H^0(\Phi_{I, L^k})} H^0(\mathcal{O}_{L^k}(i + d))$$

*has not maximal rank for a general linear form  $L$ .*

**Remark.** The theorem is not true if  $h^0(K(i)) \neq 0$  i.e. if there exists a syzygy of degree  $i$  between  $F_1, \dots, F_r$ . In [16] the authors consider the injectivity of the map  $H^0(\Phi_{I, L^k})$  for  $i = 0$  and for  $r \leq h^0(\mathcal{O}_L(d))$ . In that case, since the forms  $F_j$  are the generators of  $I$ , we have of course  $h^0(K) = 0$ .

*Proof.* In [prop. 2.1, [3]] the authors proved that  $A_{d+i} = H^1(K(i))$  for any  $i \in \mathbb{Z}$ . Let us consider the canonical exact sequence

$$0 \longrightarrow K(i-k) \xrightarrow{L^k} K(i) \longrightarrow K \otimes \mathcal{O}_{L^k}(i) \longrightarrow 0.$$

We obtain a long exact sequence of cohomology

$$0 \rightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d-k+i} \xrightarrow{\times L^k} A_{d+i} \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^2(K(i-k)) \rightarrow 0.$$

Let us assume first that  $n > 2$ . Then we have always  $h^2(K(i-k)) = 0$  and it gives a shorter exact sequence:

$$0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \longrightarrow A_{d-k+i} \xrightarrow{\times L^k} A_{d+i} \longrightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \longrightarrow 0.$$

Moreover, since  $n > 2$ , we have also  $h^1(\mathcal{O}_{L^k}(i)) = 0$ . Then tensoring the exact sequence defining the syzygy bundle  $K$  by  $\mathcal{O}_{L^k}(i)$  and taking the long cohomology exact sequence, we find:

$$0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \longrightarrow H^0(\mathcal{O}_{L^k}(i))^r \xrightarrow{H^0(\Phi_{I,L^k})} H^0(\mathcal{O}_{L^k}(i+d)) \longrightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow 0.$$

Since the kernel and cokernel of both maps,  $H^0(\Phi_{I,L^k})$  and  $\times L^k$  are the same, the theorem is proved for  $n > 2$ .

If  $n = 2$ , let us introduce the number  $t = h^2(K(i-k))$ . This number is equal to  $t = rr_{k-i-3} - r_{k-i-d-3}$  and we have a long exact sequence:

$$0 \rightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow \mathbb{C}^t \rightarrow 0.$$

Let us consider now the long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(K \otimes \mathcal{O}_{L^k}(i)) & \longrightarrow & H^0(\mathcal{O}_{L^k}(i))^r & \xrightarrow{H^0(\Phi_{I,L^k})} & H^0(\mathcal{O}_{L^k}(i+d)) \\ & & & & & & \downarrow \\ .0 & \longleftarrow & H^1(\mathcal{O}_{L^k}(i+d)) & \longleftarrow & H^1(\mathcal{O}_{L^k}(i))^r & \longleftarrow & H^1(K \otimes \mathcal{O}_{L^k}(i)) \end{array}$$

Since  $h^1(\mathcal{O}_{L^k}(i)) = h^2(\mathcal{O}_{\mathbb{P}^2}(i-k)) = r_{k-i-3}$  (and  $h^1(\mathcal{O}_{L^k}(i+d)) = h^2(\mathcal{O}_{\mathbb{P}^2}(i+d-k)) = r_{k-i-d-3}$ ), it remains a shorter exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(K \otimes \mathcal{O}_{L^k}(i)) & \longrightarrow & H^0(\mathcal{O}_{L^k}(i))^r & \xrightarrow{H^0(\Phi_{I,L^k})} & H^0(\mathcal{O}_{L^k}(i+d)) \\ & & & & & & \downarrow \\ .0 & & \longleftarrow & \mathbb{C}^t & \longleftarrow & & H^1(K \otimes \mathcal{O}_{L^k}(i)) \end{array}$$

As before, since the kernel and cokernel of both maps are the same, the theorem is proved.  $\square$

Let us introduce the numbers  $N(r, i, k, d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k})$ ,

$$N^+ = \sup(0, N(r, i, k, d)) \text{ and } N^- = \sup(0, -N(r, i, k, d)).$$

The following corollary is a didactic reformulation of the theorem above.

**Corollary 4.2.** *Assume that there is no syzygy of degree  $i$  between the  $F_j$ 's. Then  $I$  fails the SLP at the range  $k$  in degree  $d+i-k$  if and only if one of the two following equivalent conditions occurs:*

- $h^0(K \otimes \mathcal{O}_{L^k}(i)) = \ker(H^0(\Phi_{I,L^k})) > N^+$ ,
- $\text{coker}(H^0(\Phi_{I,L^k})) > N^-$ .

In the next section we translate this corollary in geometric terms.

## 5. SYZYGY BUNDLE AND VERONESE VARIETY

We recall that the  $k$ -th osculating space to a  $n$ -dimensional complex projective variety  $X \subset \mathbb{P}^N$  at  $P$  is the subspace of  $\mathbb{P}^N$  spanned by  $P$  and by all the derivative points of degree less or equal to  $k$  of a local parametrization of  $X$ , evaluated at  $P$ . A  $n$ -dimensional variety  $X \subset \mathbb{P}^N$  whose  $k$ -th osculating space at a general point has dimension  $\inf(\binom{n+k}{n} - 1, N) - \delta$  is said to satisfy  $\delta$  independent Laplace equations of order  $k$ . We will say, for shortness, that the *number* of Laplace equations is  $\delta$ .

Let us briefly explain now the link with projections of  $v_d(\mathbb{P}^n)$ .

Let  $R_1$  be a complex vector space of linear form of dimension  $n + 1$  such that  $H^0 \mathcal{O}_{\mathbb{P}^n}(1) = R_1$ . We consider the Veronese embedding

$$v_d : \mathbb{P}(R_1^*) \hookrightarrow \mathbb{P}(R_d^*).$$

The image  $v_d(\mathbb{P}^n)$  is called Veronese  $n$ -fold of order  $d$ . At the point  $[L^d] \in v_d(\mathbb{P}^n)$ , where  $[L] \in \mathbb{P}(R_1^*)$ , the  $s$ -th osculating space,  $1 \leq s \leq d-1$ , denoted by  $T_{[L^d]}^s v_d(\mathbb{P}^n)$  and defined by the partial derivatives of order  $s$ , is the space of degree  $d$  forms possessing a factorization  $L^{d-s}G$  where  $G$  is a form of degree  $s$ . It is identified with  $\mathbb{P}(R_s^*)$ .

Let us think about the projective duality in terms of derivations (it is in fact the so-called apolarity, see [5]). A canonical basis of  $R_d^*$  is given by the  $r_d$  derivations:

$$\frac{\partial^d}{\partial x_0^{i_1} \dots \partial x_n^{i_n}} \text{ with } i_1 + \dots + i_n = d.$$

Let  $I = (F_1, \dots, F_r) \subset R$  an ideal generated by  $r$  forms of degree  $d$ . Note that  $F_i$  are points in  $\mathbb{P}(R_d^*)$ . We denote by  $I_d$  the vector subspace of  $R_d$  generated by the  $F_i^{\vee}$ . Let us introduce the orthogonal vector space to  $I_d$

$$I_d^\perp = \{\delta \in R_d^* \mid \delta(F) = 0, \forall F \in I_d\}.$$

It gives an exact sequence of vector spaces

$$0 \longrightarrow I_d^\perp \longrightarrow R_d^* \longrightarrow I_d^* \longrightarrow 0$$

and the corresponding projection map  $\mathbb{P}(R_d^*) \setminus \mathbb{P}(I_d^*) \xrightarrow{\pi_I} \mathbb{P}(I_d^\perp)$ . Of course one can identify  $R_d/I_d \simeq (I_d^\perp)^*$  and write the decomposition  $R_d = I_d \oplus (I_d^\perp)^*$ .

**Remark.** In the following two situations, the orthogonal space  $I_d^\perp$  is easy to describe:

- (1) When  $I_d$  is generated by  $r$  monomials of degree  $d$ ,  $(I_d^\perp)^*$  is generated by the remainings  $r_d - r$  monomials.
- (2) When  $I_d = (L_1^d, \dots, L_r^d)$  where  $[L_i] \in \mathbb{P}(R_1^*)$ ,  $(I_d^\perp)^*$  is generated by degree  $d$  polynomials that vanish at the points  $[L_i^\vee] \in \mathbb{P}(R_1)$ .

It is well known that the tangent spaces to the Veronese varieties can be interpreted as singular hypersurfaces. More precisely a hyperplane containing the tangent space  $T_{[L^d]} v_d(\mathbb{P}^n)$  corresponds in the dual space  $\mathbb{P}^{n^\vee}$  to a hypersurface of degree  $d$  that is singular at the point  $[L^\vee]$ . More generally a hyperplane containing the  $s$ -th ( $s \leq 1$ ) osculating space  $T_{[L^d]}^s v_d(\mathbb{P}^n)$  corresponds to a hypersurface of degree  $d$  and multiplicity  $(s + 1)$  at the point  $[L^\vee]$  (see for instance [10]).

Thus the dual variety of  $v_d(\mathbb{P}^n)$  is the discriminant variety that parametrizes the singular hypersurfaces of degree  $d$  when the  $s$ -th osculating variety of  $v_d(\mathbb{P}^n)$  parametrizes the hypersurfaces of degree  $d$  with a point of multiplicity  $s + 1$ .

We propose now an extended version of the “main” theorem of [ [16]] (to be precise thm. 3.2).

**Theorem 5.1.** *Let  $I = (F_1, \dots, F_r) \subset R$  be an artinian ideal generated by  $r$  homogeneous polynomials of degree  $d$ . Let  $i, k, \delta$  be integers such that  $i \geq 0, k \geq 1$ . Assume that there is no syzygy of degree  $i$  between the  $F_j$ 's. The following conditions are equivalent:*

- (1) *The ideal  $I$  fails the SLP at the range  $k$  in degree  $d - k + i$ .*
- (2) *There exist  $N^+ + \delta$ , with  $\delta \geq 1$ , independent vectors  $(G_{1j}, \dots, G_{rj})_{j=1, \dots, N^+ + \delta} \in R_i^{\oplus r}$  and  $N^+ + \delta$  forms  $G_j \in R_{d-k+i}$  such that  $G_{1j}F_1 + \dots + G_{rj}F_r = L^k G_j$  for a general linear form  $L$  of  $\mathbb{P}^n$ .*
- (3) *The  $n$ -dimensional variety  $\pi_I(v_{d+i}(\mathbb{P}^n))$  satisfies  $\delta \geq 1$  Laplace equations of order  $d + i - k$ .*
- (4) *For any  $L \in R_1$ ,  $\dim_{\mathbb{C}}((I_{d+i}^{\perp})^* \cap H^0(\mathcal{I}_{L^{\vee}}^{d+i-k+1}(d+i))) \geq N^+ + \delta$ , with  $\delta \geq 1$ .*

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) is proved in the theorem 4.1.

Since  $I$  is generated in degree  $d$ , the map  $R_i \times I_d \rightarrow I_{d+i}$  is surjective and the relation  $G_1F_1 + \dots + G_rF_r = L^kG$  is equivalent to  $\mathbb{P}(I_{d+i}^*) \cap T_{[L^{\vee}]}^{d+i-k} v_{d+i}(\mathbb{P}^n) \neq \emptyset$ . More generally the number of independent relations  $G_{1j}F_1 + \dots + G_{rj}F_r = L^kG_j$  is the dimension of the kernel of the map  $H^0(\Phi_{I, L^k})$  i.e. the dimension of  $H^0(K \otimes \mathcal{O}_{L^k})$ ; this number of independent relations, written in a geometric way, is

$$\delta = \dim[\mathbb{P}(I_{d+i}^*) \cap T_{[L^{\vee}]}^{d+i-k} v_{d+i}(\mathbb{P}^n)] + 1,$$

where the projective dimension is  $-1$  if the intersection is empty. This number is the number of Laplace equations. Indeed, the dimension of the  $(d + i - k)$ -th osculating space to  $\pi_I(v_{d+i}(\mathbb{P}^n))$  is  $r_{d+i-k} - \delta$  since the  $(d + i - k)$ -th osculating space to  $v_{d+i}(\mathbb{P}^n)$  meets the center of projection along a  $\mathbb{P}^{\delta-1}$ . In other words, the  $n$ -dimensional variety  $\pi_I(v_{d+i}(\mathbb{P}^n))$  satisfies  $\delta$  Laplace equations and (3) is equivalent to (2).

The image by  $\pi_I$  of the  $(d + i - k)$ -th osculating space to the Veronese  $v_{d+i}(\mathbb{P}^n)$  in a general point has codimension  $h^0(K \otimes \mathcal{O}_{L^k}) - N^+$  in  $\mathbb{P}(I_{d+i}^{\perp})$ . The codimension corresponds to the number of hyperplanes in  $\mathbb{P}(I_{d+i}^{\perp})$  containing the osculating space to  $\pi_I(v_{d+i}(\mathbb{P}^n))$ . These hyperplanes are images by  $\pi_I$  of hyperplanes in  $\mathbb{P}(R_{d+i}^*)$  containing  $\mathbb{P}(I_{d+i}^*)$  and the  $(d + i - k)$ -th osculating plane to  $v_{d+i}(\mathbb{P}^n)$  at the point  $[L^{\vee}]$ . In the dual setting it means that these hyperplanes are forms of degree  $d + i$  in  $(I_{d+i}^{\perp})^*$  with multiplicity  $(d + i - k + 1)$  at  $[L^{\vee}]$ . It proves that (3) is equivalent to (4).

To summarize, the number of Laplace equation is  $h^0(K \otimes \mathcal{O}_{L^k}) - N^+$  and  $\text{coker}(H^0(\Phi_{I, L^k})) \simeq (I_{d+i}^{\perp})^* \cap H^0(\mathcal{I}_{L^{\vee}}^{d+i-k+1}(d+i))$ .  $\square$

**Remarks.**  $\bullet$  Let  $I = (L_1^d, \dots, L_r^d)$  where  $L_1, \dots, L_r$  be general linear forms. The vector space  $(I_{d+i}^{\perp})^*$ , where  $I_{d+i} = L_1^d R_i + \dots + L_r^d R_i$ , is the vector space of forms of degree  $d + i$  vanishing in  $r$  points  $[L_j^{\vee}]$  with multiplicity  $(i + 1)$ . In other words  $f \in \cap_{j=1}^r H^0(\mathcal{I}_{L_j^{\vee}}^{i+1}(d+i))$  (see [ [7], cor. 3] ). Geometrically it can be described as  $\mathbb{P}(I_{d+i}^*) = \text{Join}(T_{[L_1^{\vee}]}^i v_{d+i}(\mathbb{P}^n), \dots, T_{[L_r^{\vee}]}^i v_{d+i}(\mathbb{P}^n))$ .

- $\bullet$  By the theorem above, when  $N(r, i, k, d) \geq 0$ , the ideal  $I$  fails the SLP at the range  $k$  in degree  $d - k + i$  if and only if there exists a hypersurface  $\{f = 0\}$  with  $f \in I_{d+i}^{\perp}$  passing through the point  $[L^{\vee}]$  with multiplicity  $d + i - k + 1$ , i.e. when the following intersection is not empty:

$$\text{Join}(T_{[L_1^{\vee}]}^i v_{d+i}(\mathbb{P}^n), \dots, T_{[L_r^{\vee}]}^i v_{d+i}(\mathbb{P}^n)) \cap T_{[L^{\vee}]}^{d+i-k} v_{d+i}(\mathbb{P}^n).$$

- $\bullet$  On  $\mathbb{P}^2$  a hypersurface of degree  $d + i$  with a point of multiplicity  $d + i$  is an union of lines. On  $\mathbb{P}^n$ , with  $n > 2$ , a hypersurface of degree  $d + i$  with a point of multiplicity  $d + i$  is more generally a cone over a hypersurface in the hyperplane at infinity.



Thanks to this remark we can answer in the three following propositions to a list of questions posed by Migliore and Nagel ([17], problem 5.4 and conjecture 5.13). When  $I = (L_1^d, \dots, L_r^d)$  we verify now that the hypothesis in the theorem 5.1 on the sygygy is not restrictive.

**Lemma 5.2.** *Let  $I$  be the ideal  $(L_1^d, \dots, L_r^d)$  where the  $L_j$  are linear forms and  $r < r_d$ . Let  $K$  be its syzygy bundle. Then*

$$h^0(K(i)) = 0 \Leftrightarrow rr_i \leq r_{d+i}.$$

*Proof.* One direction is obvious. Let us assume that  $rr_i \leq r_{d+i}$  and that there exists a relation

$$G_1 L_1^d + \dots + G_r L_r^d = 0,$$

with  $G_1, \dots, G_r$  forms of  $R_i$ . Both hypotheses imply that the projective space  $\text{Join}(T_{[L_1^{d+i}]}^i v_{d+i}(\mathbb{P}^n), \dots, T_{[L_r^{d+i}]}^i v_{d+i}(\mathbb{P}^n))$  has dimension strictly less than expected. Since the linear forms are general, it implies that the algebraic closure of  $\cup_{L \in R_1^{d+i}} T_{[L^{d+i}]}^i v_{d+i}(\mathbb{P}^n)$  has not the expected dimension. It contradicts the lemma 3.3 in [2].  $\square$

This following proposition is a generalization of the example 1.1 in [12].

**Proposition 5.3.** *Let  $N$  be an integer such that  $N \geq 3$ . Then the ideal  $(x_0^N, x_1^N, x_2^N, x_3^N, (x_0 + x_1 + x_2 + x_3)^N)$  fails the WLP in degree  $N - 2$ .*

**Remark.** Of course it is equivalent to say that  $(L_1^N, \dots, L_5^N)$  fails the WLP in degree  $N - 2$  for  $L_1, \dots, L_5$  general linear forms.

*Proof.* Let us consider the syzygy bundle associated to the linear system

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^3}^5 \xrightarrow{(x_0^N, x_1^N, x_2^N, x_3^N, (x_0 + x_1 + x_2 + x_3)^N)} \mathcal{O}_{\mathbb{P}^3}(N) \longrightarrow 0.$$

Since  $5r_{N-2} < r_{2N-2}$  the lemma 5.2 implies  $h^0(K(N-2)) = 0$ . Let  $L$  be a linear form. When  $N \geq 3$  we have  $5h^0(\mathcal{O}_L(N-2)) \geq h^0(\mathcal{O}_L(2N-2))$ . According to theorem 5.1 the failure of WLP in degree  $N - 2$  is equivalent to the existence of a surface with multiplicity  $N - 1$  in the points  $P_0, P_1, P_2, P_3$  and  $P(1, 1, 1, 1)$  and multiplicity  $2N - 2$  at the moving point  $M$ . The five concurrent lines in  $M$  passing through  $P_0, P_1, P_2, P_3, P$  belong to a quadric cone with equation  $\{F = 0\}$  (the cone over the conic at infinity through the five points). Since  $F^{N-1} \in H^0(\mathcal{I}_M^{2N-2}(2N-2))$  the hypersurface  $\{F^{N-1} = 0\}$  has the wanted properties.  $\square$

The same idea gives a hint to answer to the third question in the same list of problem (problem 5.4). Indeed, in  $\mathbb{P}^n$  there is always a quadric through  $\frac{n(n+3)}{2}$  points in general position. Then by any general point  $M \in \mathbb{P}^{n+1}$  there is a quadratic cone with a vertex at  $M$  and passing through  $\frac{n(n+3)}{2}$  fixed points in general position. Then we prove,

**Proposition 5.4.** *In the following cases the ideal  $(L_1^N, \dots, L_{\frac{n(n+3)}{2}}^N)$  fails the WLP in degree  $N - 2$ :*

- $N = 3$  and  $n \geq 2$ ,
- $N = 4$  and  $2 \leq n \leq 4$ ,
- $N > 4$  and  $2 \leq n \leq 3$ .

*Proof.* Let us consider the syzygy bundle associated to the linear system

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{\frac{n(n+3)}{2}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(N) \longrightarrow 0.$$

Let  $L$  a linear form. Then the inequality  $\frac{n(n+3)}{2}h^0(\mathcal{O}_L(N-2)) \geq h^0(\mathcal{O}_L(2N-2))$  is true if and only if  $N$  and  $n$  are one of the possibilities stated in the theorem. In all these cases we have  $\frac{n(n+3)}{2}r_{N-2} \leq r_{2N-2}$ , and by the lemma 5.2,  $h^0(K(N-2)) = 0$ .

According to theorem 5.1 the failure of WLP is equivalent to the existence of a hypersurface with multiplicity  $N-1$  in the points  $[L_i^\vee]$  and multiplicity  $2N-2$  at the moving point  $M$ . The lines through  $M$  and  $[L_i^\vee]$  belong to a quadratic cone with equation  $\{F=0\}$  (the cone over the quadric at infinity through the points). Since  $F^{N-1} \in H^0(\mathcal{I}_M^{2N-2}(2N-2))$  the hypersurface  $\{F^{N-1}=0\}$  has the wanted properties.  $\square$

The same idea gives the proof of the following result that was conjectured in [15].

**Proposition 5.5.** *The ideal  $I = (L_1^3, \dots, L_8^3)$  fails the WLP in degree 3 where  $L_1, \dots, L_8$  be general linear forms on  $\mathbb{P}^6$ .*

*Proof.* Since  $8r_1 < r_4$  the lemma 5.2 implies  $h^0(K(1)) = 0$ . We have to prove that, on a general hyperplane  $L$ , the cokernel of  $H^0(\mathcal{O}_L(1))^8 \longrightarrow H^0(\mathcal{O}_L(4))$  has dimension strictly greater than  $h^0(\mathcal{O}_L(4)) - h^0(\mathcal{O}_L(1))^8 = 78$ . The dimension of this cokernel is the dimension of quartics with a quadruple point  $[L^\vee]$  and 8 double points. We consider on the hyperplane at infinity the vector space  $V$  of quadrics through the images of the 8 points  $[L_1^\vee], \dots, [L_8^\vee]$ . It has dimension 13. Let  $Q_1, \dots, Q_{13}$  be a basis of this space of quadrics. Then the vector space  $\text{Sym}^2(V)$  of quartics generated by the products  $Q_i Q_j$  has dimension 91 and all these quartics are singular in the 8 points. In  $\mathbb{P}^6$  the independent quartic cones over these quartics belong to the cokernel.  $\square$

The characterization of SLP in theorem 5.1 allows us to give, in the next section, many examples of ideals failing the WLP or the SLP by producing ad-hoc singular hypersurfaces.

## 6. CLASSES OF IDEALS FAILING THE WLP AND SLP

**6.1. Monomial ideals coming from singular hypersurfaces.** In their nice paper about osculating spaces of Veronese surfaces, Lanteri and Mallavibarena remark that the equation of the curve given by three concurrent lines depends only on six monomials instead of seven. More precisely let us consider a cubic with a triple point at  $(a, b, c)$  passing through  $P_0, P_1$  and  $P_2$ . Its equation is  $(bz - cy)(az - cx)(ay - bx) = 0$  and it depends only on the monomials  $x^2y, xy^2, x^2z, xz^2, y^2z, yz^2$ . So there is a non zero form in

$$(I_3^\perp)^* = \langle x^2y, xy^2, x^2z, xz^2, y^2z, yz^2 \rangle \simeq \frac{R_3}{\langle x^3, y^3, z^3, xyz \rangle}$$

that is triple at a general point. By this way they explain the Togliatti surprising phenomena ([14], Thm. 4.1, [13] and [8]).

We apply this idea in our context. Recall that in the monomial case being artinian to the ideal  $I$  means that it contains the forms  $x_0^d, \dots, x_n^d$ . Let us consider the  $(n+1)$  fundamental point  $P_0, P_1, \dots, P_n$  and let us assume that the number  $r$  of monomials generating  $I$  is chosen such  $N(r, i, k, d) = 0$  for  $i \geq 0, k \geq 1$  fixed integers. Then, as it is proved in theorem 5.1, the ideal  $I$  fails the SLP at the range  $k$  in degree  $d+i-k$  if and only if there exists at any point  $M(a_0, \dots, a_n)$  a hypersurface of degree  $d+i$  with multiplicity  $d+i-k+1$  at  $M$  given by a form in  $(I_{d+i}^\perp)^* \simeq R_{d+i}/I_{d+i}$ . One has to write this equation with a number of monomials as small as possible. Then the orthogonal space becomes bigger and we will cover all the possible choices.

First of all we describe exhaustively the monomial ideals  $(x^4, y^4, z^4, f, g) \subset \mathbb{C}[x, y, z]$  of degree 4 that do not verify the WLP.

**Theorem 6.1.** *Up to permutation of variables the monomial ideals generated by five quartic forms in  $\mathbb{C}[x, y, z]$  that fail the WLP in degree 3 are the following*

- $I_1 = (x^4, y^4, z^4, x^3z, x^3y)$ ,
- $I_2 = (x^4, y^4, z^4, x^2y^2, xyz^2)$ .

**Remark.** Geometrically it is evident that the first ideal  $(x^4, y^4, z^4, x^3z, x^3y)$  fails the WLP. Indeed modulo a linear form  $L$  the restricted monomials  $\bar{x}^i\bar{y}^j$  can be interpreted as points of the projective space  $\mathbb{P}^4$  defined by the rational normal curve of degree four image of  $L$  by the Veronese map. Then the tangent line to the rational quartic curve at the point  $[\bar{x}^4]$  contains the two points  $[\bar{x}^3\bar{y}]$  and  $[\bar{x}^3\bar{z}]$ . This line meets the plane  $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4 \rangle)$  in one point; it implies that

$$\dim_{\mathbb{C}} \langle \bar{x}^4, \bar{y}^4, \bar{z}^4, \bar{x}^3\bar{y}, \bar{x}^3\bar{z} \rangle \leq 4.$$

For the second ideal, it is not evident to see that the line  $\mathbb{P}(\langle \bar{x}^2\bar{y}^2, \bar{x}\bar{y}\bar{z}^2 \rangle)$  always (for any restriction) meets the plane  $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4 \rangle)$ .

*Proof.* Let us consider the points  $P_0$ ,  $P_1$  and  $P_2$  and the degree 4 curves with a quadruple point in  $(a, b, c)$  passing through these three points. These curves are product of four lines:

$$f(x, y, z) = (ay - bx)(cx - az)(cy - bz)(\alpha(ay - bx) + \beta(cx - az)) = 0.$$

Developing  $f$  explicitly in the coordinates  $(x, y, z)$ , we see that the forms  $x^4, y^4, z^4$  are

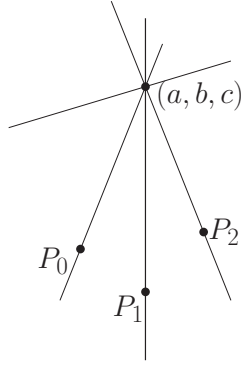


FIGURE 1. Quartic with a quadruple point

missing and that twelve monomials appear to write its equation. Since we want only ten monomials, we have to remove two. The following possibilities occur:

- $\alpha = 0$  (or equivalently by permutation of variables  $[\beta = 0]$  or  $[\alpha \neq 0, \beta \neq 0$  and  $b\alpha = c\beta]$ ) then the remaining linear system is  $(x^4, y^4, z^4, y^3z, xy^3)$ . It corresponds to the first case i.e. to the ideal  $I_1$ .
- $\alpha \neq 0$  and  $\beta \neq 0$  but  $c\beta + b\alpha = 0$  (or equivalently by permutation of variables  $[2b\alpha - c\beta = 0]$  or  $[b\alpha - 2c\beta = 0]$ ) then the remaining linear system is  $(x^4, y^4, z^4, x^2yz, y^2z^2)$ . It corresponds to the second case i.e. to the ideal  $I_2$ .

□

**Remark.** The quartic curve with multiplicity four in  $(a, b, c)$  consists, in the first case, in two simple lines and a double line that are concurrent. In the second case, it consists in four concurrent simple lines in harmonic division.

We cannot apply the same technique to describe exhaustively the monomial ideals  $(x^5, y^5, z^5, f, g, h) \subset \mathbb{C}[x, y, z]$  of degree 5 that do not verify the WLP because the computations become too tricky. But we can give some cases by geometric arguments.

**Proposition 6.2.** *The following monomial ideals*

- $(x^5, y^5, z^5, x^3y^2, x^3z^2, x^3yz)$ ,
- $(x^5, y^5, z^5, x^4z, x^4y, m)$ , where  $m$  is any monomial,
- $(x^5, y^5, z^5, x^3y^2, x^2y^3, x^2y^2z)$ ,

fail the WLP in degree 4.

*Proof.* Modulo a linear form  $L$  the restricted monomials  $\bar{x}^i\bar{y}^j$  can be interpreted as points of the projective  $\mathbb{P}^5$  defined by the rational normal curve of degree 5 image of  $L$  by the Veronese map. Then the tangent line to the rational quintic curve at the point  $[\bar{x}^5]$  contains the two points  $[\bar{x}^4\bar{y}]$  and  $[\bar{x}^4\bar{z}]$ . This line meets the plane  $\mathbb{P}(\langle \bar{x}^5, \bar{y}^5, \bar{z}^5 \rangle)$  in one point; it implies that

$$\dim_{\mathbb{C}} \langle \bar{x}^5, \bar{y}^5, \bar{z}^5, \bar{x}^4\bar{y}, \bar{x}^4\bar{z}, \bar{m} \rangle \leq 5.$$

In the same way the osculating plane at  $[\bar{x}^5]$  i.e.  $\mathbb{P}(\langle \bar{x}^3\bar{y}^2, \bar{x}^3\bar{z}^2, \bar{x}^3\bar{y}\bar{z} \rangle)$  meets the plane  $\mathbb{P}(\langle \bar{x}^5, \bar{y}^5, \bar{z}^5 \rangle)$  in one point.

In the last case, the geometric argument is not so evident. Let us pose  $X = bz - cy$  and  $Y = cx - az$ . Then the equation of the product of the five concurrent lines is  $f(X(x, y, z), Y(x, y, z)) = XY(aX + bY)(\alpha X + \beta Y)(\gamma X + \delta Y) = a\alpha\gamma X^4Y + (a\alpha\gamma + b\alpha\gamma + a\alpha\delta)X^3Y^2 + (b\beta\gamma + b\alpha\delta + a\beta\delta)X^2Y^3 + b\beta\delta XY^4 = 0$ .

For any point  $M(a, b, c, d)$  we choose  $(\alpha, \beta, \gamma, \delta)$  such that  $a\alpha\gamma + b\alpha\gamma + a\alpha\delta = 0$  and  $b\beta\gamma + b\alpha\delta + a\beta\delta = 0$ . Then the equation depends only on 15 monomials and the remaining monomials are  $(x^5, y^5, z^5, x^3y^2, x^2y^3, x^2y^2z)$ .  $\square$

We describe now some monomial ideals in  $\mathbb{C}[x, y, z, t]$ , generated in degree 3, that do not verify the WLP.

**Proposition 6.3.** *The monomial ideals  $I = (x^3, y^3, z^3, t^3, f_1, f_2, f_3, f_4, f_5, f_6)$  where the forms  $f_i$  are chosen among one of the following sets of monomials:*

- $\{x^2y, xy^2, x^2z, x^2t, y^2z, y^2t, z^2t, zt^2, xyz, xyt\}$ , (Case (A1))
- $\{x^2y, xy^2, xz^2, y^2z, yz^2, y^2t, zt^2, z^2t\}$ , (Case (A2))
- $\{x^2y, xy^2, z^2t, zt^2, xyz, xyt, xzt, yzt\}$ , (Case (A3))
- $\{xz^2, yz^2, xyz, xyt, x^2y, xy^2, z^2t, zt^2\}$ , (Case (A4))
- $\{x^2y, xy^2, x^2z, xz^2, x^2t, xt^2, xyz, xzt, xyt, yzt\}$ , (Case (B1))

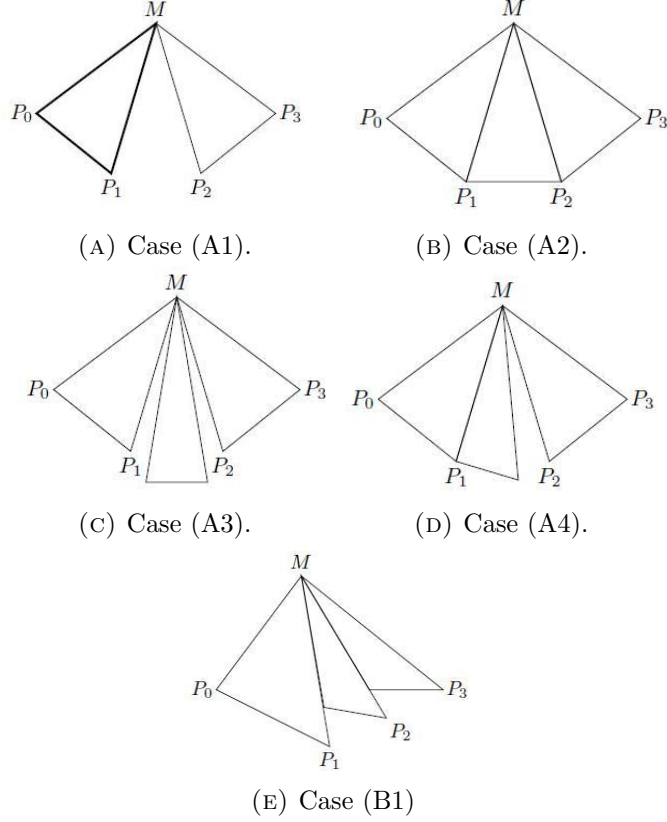
fail the WLP in degree 2.

**Remark.** We do not know if under permutation of variables the description above is exhaustive or not. The singular cubic that we are considering here are union of concurrent planes and not all the cubic cones.

*Proof.* We look for a surface of degree 3 with multiplicity 3 at a general point  $M(a, b, c, d)$  that passes through the points  $P_0, P_1, P_2, P_3$  such that its equation depends only on the remaining monomials in  $R_3/I_3$ . Such a cubic surface is a cone over a cubic curve. Here, instead of a general cubic cone we consider only three concurrent planes. Since these 3 planes have to pass through  $P_0, P_1, P_2$  and  $P_3$  it remains only, after a simple verification, the following cases:

- (A1) The equation of the cubic is  $(bx - ay)(dz - ct)^2 = 0$ .
- (A2) The equation of the cubic is  $(bx - ay)(dz - ct)(cx - az) = 0$ .
- (A3) The equation of the cubic is  $(bx - ay)(dz - ct)(bx + ay + udz + uct) = 0$  where at any point  $(a, b, c, d)$  the function  $u(a, b, c, d)$  verifies  $ab + u(a, b, c, d)cd = 0$ .
- (A4) The equation of the cubic is  $(bx - ay)(dz - ct)(bdx + ady - 2abt) = 0$ .
- (B1) The equation of the cubic is  $(cy - bz)(dz - ct)(dy - bt) = 0$ .

$\square$



If we want  $I_3$  to be of dimension  $r < 10$  (for instance  $r = 8$ ) we need  $10 - r + 1$  independent cubics with a triple point. So we need  $10 - r + 1$  independent cubics with a triple point to describe the failure of WLP. Let us recover with our method two linear systems of eight cubic forms (the complete classification is already done in [ [16], thm. 4.11]) that fail the WLP in degree 2.

**Proposition 6.4.** *The following monomial ideals*

- $I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t)$ ,
- $J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt)$

*fail the WLP in degree 2.*

**Remark.** The ideals  $I$  and  $J$  correspond respectively to the cases (3) and (1) in [ [16], Thm. 4.11].

*Proof.* Let us consider the following three forms defining singular cubics passing through the fundamental points and a general point  $(a, b, c, d)$ :

$$(ct - dz)(at - dx)(ay - bx) = 0, (ct - dz)^2(ay - bx) = 0, (ct - dz)(ay - bx)^2 = 0.$$

They are particular cases of type (A) in the proof of the proposition 6.3. They are linearly independent and can be written with twelve monomials. Then it remains only 8 forms for  $I_3$ :

$$I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t).$$

Let us consider the following three forms defining singular cubics passing through the basis points and the general point  $(a, b, c, d)$ :

$$(bz - cy)(az - cx)(ay - bx) = 0, (bx - ay)(at - dx)(dy - bt) = 0, (az - cx)(dx - at)(dz - ct) = 0.$$

They are cases of type (B1) in the proof of proposition 6.3. They are linearly independent and can be written with twelve monomials:

$$(x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, t^2y, t^2z, ty^2, tz^2, t^2x, x^2t).$$

It remains only

$$J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt).$$

□

Of course the same argument (concurrent planes or hyperplanes) can be used in degree or dimension bigger than 3. For instance let us give a set of monomial ideals in  $\mathbb{C}[x, y, z, t]$ , generated in degree 4 that fail the WLP.

**Proposition 6.5.** *Let  $f_1, \dots, f_{11}$  be eleven monomials chosen among*

$$x^3y, x^3z, x^3t, xy^3, xz^3, xt^3, y^3z, y^3t, yz^3, yt^3, z^3t, zt^3, x^2y^2, z^2t^2, y^2z^2, x^2t^2.$$

*Then the ideal  $I = (x^4, y^4, z^4, t^4, f_1, \dots, f_{11})$  fails the WLP in degree 3.*

*Proof.* At any point  $p = (a, b, c, d)$  an equation of a surface of degree 4 with multiplicity 4 at  $p$  that passes through the points  $P_0, P_1, P_2, P_3$  is given by

$$f(x, y, z, t) = (ct - dz)(at - dx)(ay - bx)(bz - cy) = 0.$$

□

We conclude this section with an example that fails the SLP at the range 2.

**Proposition 6.6.** *The ideal  $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3) \subset \mathbb{C}[x, y, z]$  fails the SLP at the range 2 in degree 2.*

*Proof.* Let  $P_0, P_1, P_2$  and  $M(a, b, c)$  be four points. We consider the quartic curve consisting in the union of the four lines  $(MP_0), (MP_1), (MP_2)$  and  $(P_0P_1)$ . It is a quartic passing through  $P_0, P_1, P_2$  and triple at  $M(a, b, c)$ . It depends on the six monomials  $x^3y, x^3z, x^2y^2, xy^2z, x^2z^2, xyz^2$ . Then it remains  $9 = 15 - 6$  monomials

$$I_4 = \langle x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3 \rangle.$$

The associated syzygy bundle  $K$  verifies  $h^0(K \otimes \mathcal{O}_{L^2}) \neq 0$  for a general linear form  $L$ . It proves that  $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3)$  fails the SLP at the range 2 in degree 2. □

**6.2. Non monomial examples coming from singular hypersurfaces.** Let us study now the interesting case  $I_d^\perp = H^0(\mathcal{I}_Z(d))^*$  where  $Z$  is a finite set of distinct points in  $\mathbb{P}^{2^\vee}$  of length  $|Z|$  and  $\mathcal{I}_Z$  its ideal sheaf. The set  $Z$  corresponds by projective duality to a set of  $|Z|$  distinct lines in  $\mathbb{P}^2$  defined by linear forms  $l_1, \dots, l_{|Z|}$ . We will now consider the ideal  $I \subset R$  generated by  $(l_1^d, \dots, l_{|Z|}^d)$ . We have  $|Z| = \dim_{\mathbb{C}} I_d$ .

**Proposition 6.7.** *Let  $k \geq 1$ ,  $r = r_d - r_{d-k}$  and  $Z = \{l_1^\vee, \dots, l_r^\vee\}$  a finite set of  $r$  distinct points in  $\mathbb{P}^{2^\vee}$  where  $l_i$  are linear forms on  $\mathbb{P}^2$ . Assume that there exists a subset  $Z_1 \subset Z$ , of length  $r - d + k - 1$ , contained in a curve  $\Gamma_1$  of degree  $k - 1$ . Then the ideal  $I = (l_1^d, \dots, l_r^d)$  fails the SLP at the range  $k$  in degree  $d - k$ .*

*Proof.* The union of  $\Gamma_1$  and  $(d - k + 1)$  concurrent lines at a point  $P$  passing through the remaining points  $Z \setminus Z_1$ , is a non zero section of  $H^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{d-k+1}(d))$ . By the theorem 5.1 it proves that  $I$  fails the SLP at the range  $k$  in degree  $d - k$ . □

With this method it is always possible to find systems of any degree that fail the SLP by exhibiting a curve of degree  $d$  with multiplicity  $d - k + 1$  at a general point  $P$ . But one can find some set of points for which these special curves do not split as product of lines (see proposition 7.3 in the next section).

7. SLP AT THE RANGE 2 AND LINE ARRANGEMENTS ON  $\mathbb{P}^2$ 

A line arrangement is a collection of distinct lines in the projective plane. Arrangement of lines or more generally of hyperplanes is a famous and classical object related to combinatorics, commutative algebra, topology and geometry. It has been studied by many different authors for a very long time (see [4] or [19] for a good introduction to this topic).

Let us denote by  $\{f = 0\}$  the equation of the union of lines of the considered arrangement. Another classical object associated to the arrangement is the vector bundle  $\mathcal{D}_0$  defined as the kernel of the jacobian map:

$$0 \longrightarrow \mathcal{D}_0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\partial f)} \mathcal{O}_{\mathbb{P}^2}(d-1).$$

The bundle  $\mathcal{D}_0$  is called *derivation bundle* (sometimes logarithmic bundle) of the line arrangement (see [21] and [22] for an introduction to derivation bundles).

One can think about the lines of the arrangement in  $\mathbb{P}^2$  as a set of distinct points  $Z$  in  $\mathbb{P}^{2V}$ . Then we will denote by  $\mathcal{D}_0(Z)$  the associated derivation bundle.

The arrangement of lines is said *free with exponents*  $(a, b)$  when its derivation bundle splits on  $\mathbb{P}^2$  as a sum of two line bundles, more precisely when

$$\mathcal{D}_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b).$$

The splitting of  $\mathcal{D}_0(Z)$  over a line  $l \subset \mathbb{P}^2$  is related to the existence of curves (with a given degree  $a+1$ ) passing through  $Z$  that are multiple (with multiplicity  $a$ ) at  $l^V \in \mathbb{P}^{2V}$ . More precisely,

**Lemma 7.1** ([27], prop. 2.1). *Let  $Z \subset \mathbb{P}^{2V}$  be a set of  $a+b+1$  distinct points with  $1 \leq a \leq b$  and  $l$  be a general line in  $\mathbb{P}^2$ . Then the following conditions are equivalent:*

- (1)  $\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b)$ .
- (2)  $h^0((\mathcal{J}_Z \otimes \mathcal{J}_V^a)(a+1)) \neq 0$  and  $h^0((\mathcal{J}_Z \otimes \mathcal{J}_V^{a-1})(a)) = 0$ .

In our context it implies the following characterization of instability. We recall that a rank two vector bundle  $E$  on  $\mathbb{P}^n$ ,  $n \geq 2$  is *unstable* if and only if its splitting  $E_l = \mathcal{O}_l(a) \oplus \mathcal{O}_l(b)$  on a general line  $l$  verifies  $|a-b| \geq 2$ . This characterization is a consequence of the Grauert-Müllich theorem, see [18].

**Proposition 7.2.** *Let  $I \subset R = \mathbb{C}[x, y, z]$  be an artinian ideal generated by  $2d+1$  polynomials  $l_1^d, \dots, l_{2d+1}^d$  where  $l_i$  are distinct linear forms in  $\mathbb{P}^2$ . Let  $Z = \{l_1^V, \dots, l_{2d+1}^V\}$  be the corresponding set of points in  $\mathbb{P}^{2V}$ . Then the following conditions are equivalent:*

- (1) *The ideal  $I$  fails the SLP at the range 2 in degree  $d-2$ .*
- (2) *The derivation bundle  $\mathcal{D}_0(Z)$  is unstable.*

*Proof.* The failure of SLP at the range 2 in degree  $d-2$  is equivalent to the existence at a general point  $l^V$  of a curve of degree  $d$  with multiplicity  $d-1$  at  $l^V$  belonging to  $I_d^\perp = H^0(\mathcal{I}_Z(d))$ . By the lemma 7.1 it is equivalent to the following splitting

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(d-s) \oplus \mathcal{O}_l(d+s) \text{ with } s > 0,$$

on a general line  $l$ . In other words the failure of SLP is equivalent to have a non balanced decomposition and according to Grauert-Müllich theorem it is equivalent to instability.  $\square$

Let us give now an ideal generated by non monomials quartic forms that fails the SLP at the range 2. It comes from a line arrangement, called B3 arrangement (see [19], pages 13, 25 and 287), such that the associated derivation bundle is unstable (in fact even decomposed). The existence of a quartic curve with a general triple point is the key argument. But contrary to the previous examples, this quartic is irreducible and consequently not obtainable by the proposition 6.7.

**Proposition 7.3.** *The ideal*

$$I = (x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (x - z)^4, (y + z)^4, (y - z)^4) \subset \mathbb{C}[x, y, z]$$

*fails the SLP at the range 2 and degree 2.*

*Proof.* Consider the set  $Z$  of the nine dual points of the linear forms  $x, y, z, x + y, x - y, x + z, x - z, y + z, y - z$ . Let  $I$  be the artinian ideal  $(x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (x - z)^4, (y + z)^4, (y - z)^4)$  and  $K$  its syzygy bundle. The derivation bundle of the arrangement is  $\mathcal{D}_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)$  (it is free with exponents  $(3, 5)$ ; see [19] for a proof). Then, according to the lemma 7.1 there is at any point  $P$  a degree 4 curve with multiplicity 3 at  $P$  passing through  $Z$ . In other words, by the theorem 5.1,  $I$  fails the SLP at the range 2 and degree 2.  $\square$

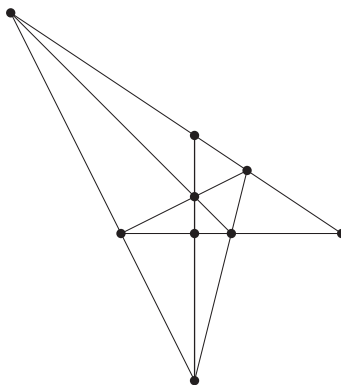


FIGURE 2. Dual set of points of the  $B_3$  arrangement

More generally non balanced free arrangements leads to ideals that fail the SLP.

**Proposition 7.4.** *Let  $\mathcal{A} = \{l_1, \dots, l_{a+b+1}\}$  a line arrangement that is free with exponents  $(a, b)$  such that  $a \leq b$ ,  $b - a \geq 2$  and  $a + b$  even. The ideal  $I = (l_1^{\frac{a+b}{2}}, \dots, l_{a+b+1}^{\frac{a+b}{2}})$  fails the SLP at the range 2 and degree  $\frac{a+b}{2} - 1$ .*

**Remark.** If  $a + b$  is odd we can add to  $Z$  one point  $P$  in general position with  $Z$  and we can prove in the same way that  $I = (l_1^{\frac{a+b+1}{2}}, \dots, l_{a+b+1}^{\frac{a+b+1}{2}}, (P^\vee)^{\frac{a+b+1}{2}})$  fails the SLP at the range 2 and degree  $\frac{a+b}{2}$ .

*Proof.* Let us denote by  $Z = \{l_1^\vee, \dots, l_{a+b+1}^\vee\}$  the dual set of points of  $\mathcal{A}$ . Since there exists at any general point  $l^\vee$  a curve of degree  $a + 1$  passing through  $Z$ , the lemma 7.1 implies that  $\mathcal{D}_0(Z)$  is unstable and the proposition 7.2 implies that  $I$  fails the SLP at the range 2 and degree  $\frac{a+b}{2} - 1$ .  $\square$

**7.1. SLP at the range 2 and Terao's conjecture.** One of the main conjecture about hyperplane arrangements (still open also for line arrangements) is the Terao's conjecture. It concerns the free arrangements. The conjecture says that freeness depends only on the combinatorics of the arrangement. Let us recall that the combinatorics of the arrangement  $\mathcal{A} = \{l_1, \dots, l_n\}$  is determined by an incidence graph. Its vertices are the lines  $l_k$  and the points  $P_{i,j} = l_i \cap l_j$ . Its edges are joining  $l_k$  to  $P_{i,j}$  when  $P_{i,j} \in l_k$ . We refer again to [19] for a good introduction to the subject. The Terao's



conjecture is valid not only for line arrangement but more generally for hyperplane arrangements.

**Conjecture** (Terao). The freeness of a hyperplane arrangement depends only on its combinatorics.

In other words if an arrangement has the same combinatorics than a free arrangement it is also free.

Let us consider a free arrangement  $\mathcal{A}_0 = \{h_1, \dots, h_n\}$  with exponents  $(a, b)$  ( $a \leq b$ ) and let us denote by  $Z_0$  its dual set of points. We assume the Terao's conjecture is not true i.e, that there exists a non free arrangement  $\mathcal{A} = \{l_1, \dots, l_n\}$  with the same combinatorics than  $\mathcal{A}_0$ . Let us add  $b - a$  points  $\{M_1, \dots, M_{b-a}\}$  in general position to  $Z_0$  in order to form  $\Gamma_0$  and to the dual set  $Z$  of  $\mathcal{A}$  to form  $\Gamma$ . Then the length of both sets of points is  $2b + 1$ . On the general line  $l$  we have

$$\mathcal{D}_0(Z_0) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b),$$

when, since  $Z$  is not free, we have a less balanced decomposition for  $\mathcal{D}_0(Z)$  (this affirmation is proved in [6]):

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(s - a) \oplus \mathcal{O}_l(-s - b), \quad s \geq 1.$$

It implies that

$$h^0(\mathcal{I}_Z \otimes \mathcal{I}_V^{a-1}(a)) \neq 0, h^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_V^{a-1}(a)) = 0 \text{ and } h^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_V^a(a+1)) \neq 0.$$

Then adding  $b - a$  lines passing through  $l^\vee$  and the  $b - a$  added points we obtain  $h^0(\mathcal{I}_\Gamma \otimes \mathcal{I}_V^{b-1}(b)) \neq 0$ ,  $h^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_V^{b-1}(b)) = 0$  and  $h^0(\mathcal{I}_{Z_0} \otimes \mathcal{I}_V^b(b+1)) \neq 0$ . The bundle  $\mathcal{D}_0(\Gamma_0)$  is balanced with splitting  $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(-b)$  and

$$\mathcal{D}_0(\Gamma) \otimes \mathcal{O}_l = \mathcal{O}_l(1 - b) \oplus \mathcal{O}_l(-1 - b).$$

Then  $\mathcal{D}_0(\Gamma_0)$  is semistable and  $\mathcal{D}_0(\Gamma)$  is unstable. In other words the ideal

$$(l_1^b, \dots, l_{a+b+1}^b, (M_1^\vee)^b, \dots, (M_{b-a}^\vee)^b)$$

fails the SLP at the range 2 and degree  $b - 2$  when

$$(d_1^b, \dots, d_{a+b+1}^b, (M_1^\vee)^b, \dots, (M_{b-a}^\vee)^b)$$

has the SLP at the range 2 and degree  $b - 2$ .

The following conjecture written in terms of SLP is equivalent to the Terao's conjecture on  $\mathbb{P}^2$ .

**Conjecture.** Let  $Z_0 = \{d_1^\vee, \dots, d_{2b+1}^\vee\}$  a set of points of length  $2b + 1$  in  $\mathbb{P}^{2\vee}$  such that the ideal  $I = (d_1^b, \dots, d_{2b+1}^b)$  has the SLP at the range 2 and degree  $b - 2$ . Assume that  $Z = \{l_1^\vee, \dots, l_{2b+1}^\vee\}$  has the same combinatorics than  $Z_0$ . Then  $J = (l_1^b, \dots, l_{2b+1}^b)$  has the SLP at the range 2 and degree  $b - 2$ .

## REFERENCES

- [1] Valdimir I. Arnold. The cohomology ring of colored braid group. *Mat. Zametki*, 5(2): 227–231, 1969.
- [2] Alessandra Bernardi, Maria Virginia Catalisano, Alessandro Gimigliano and Monica Idá. Osculating varieties of Veronese varieties and their higher secant varieties. *Canad. J. Math.* 59 (2007), no. 3, 488–502.
- [3] Holger Brenner and Almar Kaid. Syzygy bundles on  $\mathbb{P}^2$  and the Weak Lefschetz Property. *Illinois J. Math.*, 51:1299–1308, 2007.
- [4] Pierre Cartier. Les arrangements d'hyperplans: un chapitre de géométrie combinatoire. Bourbaki Seminar, Vol. 1980/81, p. 1–22, Lecture Notes in Math., 901, Springer, Berlin-New York, 1981.
- [5] Igor Dolgachev. *Classical Algebraic Geometry: a modern view*. To be published by Cambridge Univ. press.

- [6] Georges Elencwajg and Otto Forster. Bounding cohomology groups of vector bundles on  $\mathbb{P}_n$ , *Math. Ann.*, 246(3):251–270, 1979/80.
- [7] Jacques Emsalem and Anthony Iarrobino. Inverse System of a Symbolic Power, I, *J. of Algebra* 174:1080–1090, 1995.
- [8] Davide Franco and Giovanna Ilardi. On a theorem of Togliatti. *Int. Math. J.* 2(4):379–397, 2002.
- [9] Daniele Faenzi, Daniel Matei and Jean Vallès. Hyperplane arrangements of Torelli type. *ArXiv e-prints*, November 2010. To appear in *Compositio Math.*
- [10] Joe Harris. *Algebraic geometry, a first course*. volume 133 of *Graduate Texts in Math.* Springer Verlag, 1992.
- [11] Tadahito Harima, Juan C. Migliore, Uwe Nagel and Junzo Watanabe. The weak and Strong Lefschetz Properties for artinian K-Algebras. *J. Algebra*, 262:99–126, 2003.
- [12] Brian Harbourne, Henry K. Schenk and Alexandra Seleanu. Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property. *Journal of the London Mathematical Society* 84:712–730, 2011.
- [13] Giovanna Ilardi. Togliatti systems. *Osaka J. Math.*, 43(1):1–12, 2006.
- [14] Antonio Lanteri and Raquel Mallavibarena. Osculatory behavior and second dual varieties of Del Pezzo surfaces. *Adv. in Geom.*, 2(4):345–363, 2002.
- [15] Juan C. Migliore, Rosa Miró-Roig and Uwe Nagel. On the weak lefschetz property for powers of linear forms. *Algebra and Number theory*, 4:487–526, 2012.
- [16] Emilia Mezzetti, Rosa Miró-Roig and Giorgio Ottaviani. Laplace Equations and the Weak Lefschetz Property, *ArXiv e-prints*, October 2011. To appear in *Canadian Journal of Mathematics*.
- [17] Juan C. Migliore and Uwe Nagel. A tour of the weak and strong Lefschetz properties. *ArXiv e-prints*, September 2011.
- [18] Christian Okonek, Michael Schneider and Heinz Spindler. *Vector bundles on complex projective spaces*. Progress in Mathematics, vol. 3, Birkhäuser, Boston, Mass., 1980.
- [19] Peter Orlik and Hiroaki Terao. *Arrangement of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
- [20] David Perkinson. Inflections of toric varieties. *Mich. Math. J.*, 48:483–516, 2000.
- [21] Kyoji Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.
- [22] Henry K. Schenk. Elementary modifications and line configurations in  $\mathbb{P}^2$ . *Comment. Math. Helv.* 78(3):447–462, 2003.
- [23] Richard P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *Siam J. Alg. Disc. Meth.* 1(2):168–184, 1980.
- [24] Alessandro Terracini. Sulle  $V_k$  che rappresentano più di  $\frac{1}{2}k(k-1)$  equazioni di Laplace linearmente indipendenti. *Rend. Circ. Mat. Palermo*, 33:176–186, 1912.
- [25] Eugenio Togliatti. Alcune osservazioni sulle superficie razionali che rappresentano equazioni di Laplace. *Ann. Mat. Pura Appl.* 25(4):325–339, 1946.
- [26] Jean Vallès. Variétés de type Togliatti. *C.R.A.S.* 343(6):411–414, 2006.
- [27] Jean Vallès. Fibrés logarithmiques sur le plan projectif. *Ann. Fac. Sci. Toulouse Math.* 16(2):385–395, 2007.