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LAPLACE EQUATIONS AND THE STRONG LEFSCHETZ PROPERTY

ROBERTA DI GENNARO, GIOVANNA ILARDI AND JEAN VALLÈS

ABSTRACT. As in [9] we explore the link between artinian ideals that fail the Weak Lefschetz Property (WLP) in degree $d - 1$ and projections of the Veronese varieties satisfying a Laplace equation of order $d - 1$. We extend this link to the more general situation of artinian ideals failing the Strong Lefschetz Property (SLP) at the range $k \geq 1$ in degree $d - k$. This generalization is not artificial; indeed for $k = 2$ it is related to the so-called Terao's conjecture about free arrangements. We reformulate the Terao's conjecture for line arrangements in terms of artinian ideals failing the SLP at the range 2. Using this new link we propose non toric examples of ideals failing the SLP at the range 2.

Moreover we add a new characterization of WLP or SLP in terms of singular hypersurfaces (see thm. 3.1). Thanks to this characterization we produce many toric examples that fail the WLP and also the SLP at the range 2.

October 8, 2012

1. INTRODUCTION

The study of the surfaces satisfying Laplace equations was developed in the last century by Togliatti [16] and Terracini [15]. Togliatti [16] gave a complete classification of the rational surfaces embedded with linear systems of plane cubics and satisfying a Laplace equation of order two. For a linear system of cubic surfaces that satisfy a Laplace equation of degree two, a preliminary work was done by Ilardi in [7], and a complete classification of such smooth threefolds is achieved by Miro-Roig, Mezzetti and Ottaviani ([9], Thm. 4.11). In order to do it they establish and study the link between artinian ideals that fail the weak Lefschetz property (WLP) in degree $d - 1$ and the projections of Veronese varieties satisfying at least one Laplace equation of order $d - 1 \geq 2$.

In this paper we extend this link to the more general situation of artinian ideals failing the Strong Lefschetz Property (SLP) at the range $k \geq 1$ in degree $d - k$.

To explain more precisely what we will do in this paper, let us interrupt briefly this introduction to fix some notations and say, at least, what mean WLP and SLP.

Let $R = \mathbf{k}[x_0, \dots, x_n]$ be the graded polynomial ring in $n + 1$ variables over a field of characteristic zero \mathbf{k} . Let us denote by r_t the dimension (that is $\binom{n+t}{n}$) of the vector space R_t .

$$\text{Let } A = R/I = \sum_{i=0}^m A_i$$

be a graded artinian algebra. Note that A is finite dimensional over \mathbf{k} .

Definition 1.1. Let l be a general linear form. The artinian algebra A (or the artinian ideal I) has the Weak Lefschetz Property (WLP) if the homomorphism induced by the

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multiplication by l ,

$$\times l : A_i \rightarrow A_{i+1},$$

has maximal rank (i.e. is injective or surjective) for all i . The artinian algebra A (or the artinian ideal I) has the Strong Lefschetz Property (SLP) if,

$$\times l^k : A_i \rightarrow A_{i+k},$$

has maximal rank (i.e. is injective or surjective) for all i and k .

The most interesting artinian ideals are certainly those that fail the WLP in some degree or the SLP at some range and degree. To be precise,

Definition 1.2. The artinian algebra A (or the artinian ideal I) fails the WLP in degree i if for a general linear form l the multiplication map

$$\times l : A_i \rightarrow A_{i+1},$$

has not maximal rank. The artinian algebra A (or the artinian ideal I) fails the SLP in degree i at the range k if for a general linear form l the multiplication map

$$\times l^k : A_i \rightarrow A_{i+k},$$

has not maximal rank.

Remark. It is clear that the SLP at the rang $k = 1$ corresponds to the WLP.

So, as we wrote above, we study the link between artinian ideals $I = (F_1, \dots, F_r) \subset \mathbf{k}[x_0, \dots, x_n]$ generated by r ($r \leq r_d - r_{d-k}$) homogeneous forms of degree d that fail SLP in degree $d - k$ at the range $k \geq 1$ and projections (from the subspace $\mathbb{P}(\langle F_1, \dots, F_r \rangle)$) of the Veronese variety $v_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-r-1}$ satisfying at least one Laplace equation of order $d - k$ (i.e. such that the general tangent space of order $d - k$, generated by partial derivatives of order $d - k$, has projective dimension $< r_{d-k}$).

This generalization is meaningful in the topic of hyperplane arrangements and more precisely for line arrangements. Indeed, one associates canonically to a non degenerated line arrangement $\mathcal{A} = \{l_1, \dots, l_r\}$ in \mathbb{P}^2 an artinian ideal $I = (l_1^d, \dots, l_r^d) \subset \mathbf{k}[x, y, z]$. The Terao's conjecture which affirms that the freeness of any arrangement depends only on its combinatorics (see for instance [12] for a good definition of combinatorics of an arrangement) can be reformulated in terms of failure of SLP at the range 2. This link allows us to give new examples of such artinian ideals.

The main idea to relate the notions of WLP or SLP (at the range k) and the projections of Veronese (from a subspace $\mathbb{P}(\langle F_1, \dots, F_r \rangle)$) satisfying a Laplace equation is to consider the following exact sequence:

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r \xrightarrow{\langle F_1, \dots, F_r \rangle} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0,$$

to tensor it by \mathcal{O}_{L^k} for a general linear form L and to compute the dimension of the space $H^0(K \otimes \mathcal{O}_{L^k})$. In the particular case $H^0(\mathcal{O}_{L^k}(d)) = r$ the failure of SLP at the range k corresponds to the existence of special singular hypersurfaces. Thanks to this link and by exploiting an idea given in [8] to explain why Togliatti's surface verifies a Laplace equation, we produce many examples of artinian ideals that fail the WLP (i.e. the SLP at the range 1) or the SLP at the range 2.

2. LINK WLP, SLP AND VERONESE VARIETIES

First of all we extend the key lemma 2.3 in [9] to SLP.

Proposition 2.1. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by $r \leq r_d - r_{d-k}$ (where k is a fixed integer greater than 1) homogeneous forms of degree d . Let L be a general linear form, let $B = R/(L^k)$ and let $J = (f_1, \dots, f_r)$ where the f_i is the image of F_i in B . Then the homomorphism of multiplication by L^k*

$$\phi_{d-k} : (R/I)_{d-k} \longrightarrow (R/I)_d$$

has not maximal rank if and only if f_1, \dots, f_r are linearly dependent.

Remark. We point out that when $r = r_d - r_{d-k}$ the map is injective if and only if it is surjective. Then in that case we will show later that failing SLP (in degree $d-k$ at the range k) is equivalent to the existence of a hypersurface of degree d with multiplicity $d-k+1$ at a general point in the orthogonal vector space of the vector subspace $\langle F_1, \dots, F_r \rangle \subset R_d$ (see Thm. 3.1).

Proof. We note that $(R/I)_{d-i} \simeq R_{d-i}$ when $i < d$. Consider the exact sequence

$$0 \longrightarrow \frac{[I:L^k]}{I} \longrightarrow \frac{R}{I} \xrightarrow{L^k} \frac{R}{I}(k) \longrightarrow \frac{R}{(I,L^k)}(k) \longrightarrow 0.$$

The cokernel of ϕ_{d-k} is $(\frac{R}{(I,L^k)})_d$. According to the upper bound of r we have $\dim(R/I)_{d-k} \leq \dim(R/I)_d$. Hence ϕ_{d-k} is not of maximal rank if and only if ϕ_{d-k} is not injective, if and only if $\text{rk}(\phi_{d-k}) < \binom{n+d-k}{n}$, if and only if $\dim(R/(I,L^k))_d = \dim B_d - \dim J_d = \binom{n-1+d}{n-1} - \dim J_d > \dim(R/I)_d - \binom{n+d-k}{n} = \binom{n+d}{n} - r - \binom{n+d-k}{n}$. Therefore ϕ_{d-k} is not injective if and only if

$$r - \dim J_d > \binom{n+d}{n} - \binom{n-1+d}{n-1} - \binom{n+d-k}{n}.$$

Since the right-hand side is equal to $\binom{n-1+d}{d-1} - \binom{n+d-k}{d-k}$ which is ≥ 0 when $k \geq 1$ the map ϕ_{d-k} is injective if and only if $\dim_{\mathbb{C}} \langle f_1, \dots, f_r \rangle < r$. \square

According to this proposition we can reformulate the WLP and the SLP in the following form.

Proposition 2.2. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by $r \leq r_d - r_{d-k}$ (where $k \geq 1$ is a fixed integer) homogeneous forms of degree d and K the vector bundle defined by the exact sequence*

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r \xrightarrow{(F_1, \dots, F_r)} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0.$$

Then I fails the SLP in degree $d-k$ at the range k if for a general linear form L we have $H^0(K \otimes \mathcal{O}_{L^k}) \neq 0$.

Let us briefly explain now the link with projections of $v_d(\mathbb{P}^n)$ (a more complete description is done in the next section). The ideal $I = (F_1, \dots, F_r)$ with $\deg(F_i) = d$ fails the SLP at the range k in degree $d-k$ when for a general linear form L it exists a_1, \dots, a_r complex numbers and a form G of degree $d-k$ such that

$$a_1 F_1 + \dots + a_r F_r = L^k G.$$

In other terms it means that the projective $d-k$ tangent space to $v_d(\mathbb{P}^n)$ at the point $[L^d]$ intersects $\mathbb{P}(\langle F_1, \dots, F_r \rangle)$ (since this space corresponds to degree d forms divisible by L^k). Let us consider the projection map

$$\mathbb{P}(R_d) \setminus \mathbb{P}(\langle F_1, \dots, F_r \rangle) \xrightarrow{\pi} \mathbb{P}(R_d / \langle F_1, \dots, F_r \rangle).$$

Let us denote by $X_{n,(I-1)_d}$ the image of $v_d(\mathbb{P}^n)$ by π , like in [9]. The dimension of the $d-k$ tangent space to $X_{n,(I-1)_d}$ is strictly less than the dimension of the $d-k$ tangent spaces to $v_d(\mathbb{P}^n)$ since this last one meet the center of projection. In that case,

$X_{n,(I^{-1})_d}$ satisfies a Laplace equation, i.e. there is a linear relation between the partial derivatives of order $d - k$.

We propose an extended version of Tea theorem proved in [9].

Theorem 2.3. *Let $I = (F_1, \dots, F_r) \subset R$ be an artinian ideal generated by r homogeneous polynomials of degree d . Let $1 \leq k < d$ a positive integer. If $r \leq r_d - r_{d-k}$ then the following conditions are equivalent:*

- (1) *The ideal I fails the SLP at the range k in degree $d - k$.*
- (2) *The homogeneous forms F_1, \dots, F_r become linearly dependent on a general multiple hyperplane L^k of \mathbb{P}^n (i.e. $\mathbb{P}(\langle F_1, \dots, F_r \rangle) \cap T_{[L^d]}^{d-k} v_d \neq \emptyset$).*
- (3) *The n -dimensional variety $X_{n,(I^{-1})_d}$ satisfies a Laplace equation of order $d - k$.*

Proof. We have already seen in the above proposition 2.1 that (1) is equivalent to (2). As explained before, the fact that the space $\mathbb{P}(\langle F_1, \dots, F_r \rangle)$ meets the general $d - k$ tangent space of the Veronese $v_d(\mathbb{P}^n)$ is equivalent to say that there is a relation

$$a_1 F_1 + \dots + a_r F_r = L^k G$$

for L general, in other words (3) is equivalent to (2). \square

The key example is of course the one coming from the Togliatti's result (see for instance [1], Example 3.1): the ideal $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2. There are many ways to prove it. One of them comes from the polarity on the rational normal cubic curve. It leads to a generalisation that gives one of the few non toric examples.

Proposition 2.4 ([17], Thm. 3.1). *Let $n \geq 1$ be an integer and l_1, \dots, l_{2n+1} be non concurrent linear forms on \mathbb{P}^2 . Then the ideal*

$$(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1, \dots, 2n+1} l_i)$$

fails the WLP in degree $2n$.

Indeed on the general line L the $2n + 2$ forms of degree $2n + 1$ become dependent thanks to the polarity on the rational normal curve of degree $2n + 1$. We propose the following conjecture. For $n = 1$ it is again the Togliatti's result.

Conjecture. Let l_i be non concurrent linear forms on \mathbb{P}^2 and f be a form of degree $2n + 1$ on \mathbb{P}^2 . Then, $(l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, f)$ fails the WLP in degree $2n$ if and only if modulo linear combination

$$f \in (l_1^{2n+1}, \dots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i).$$

3. SINGULAR HYPERSURFACES, WLP AND SLP

It is well known that according to the duality between a linear form L and its kernel $\{L = 0\}$, that is a hyperplane, the tangent spaces to the Veronese varieties can be interpreted as singular hypersurfaces. More precisely a hyperplane containing the tangent space $T_{L^d} v_d(\mathbb{P}^n)$ corresponds in the dual space \mathbb{P}^{n^\vee} to a hypersurface with degree d that is singular at the point $\{L^\vee\}$. More generally a hyperplane containing $T_{L^d}^{d-k} v_d(\mathbb{P}^n)$ corresponds to a hypersurface with degree d and multiplicity $(d - k + 1)$ at the point $\{L^\vee\}$ (see for instance [5]).

In order to precise this correspondence let us introduce some formalism. Let R_1 be a complex vector space of linear form of dimension $n + 1$ such that $H^0 \mathcal{O}_{\mathbb{P}^n}(1) = R_1$. We consider the Veronese embedding $v_d : \mathbb{P}(R_1) = \mathbb{P}^n \hookrightarrow \mathbb{P}(R_d) = \mathbb{P}(H^0 \mathcal{O}_{\mathbb{P}^n}(d))$. The

image $v_d(\mathbb{P}^n)$ is called Veronese n -fold of order d . At the point $[L^d] \in v_d(\mathbb{P}^n)$, where L is an hyperplane in $\mathbb{P}(R_1)$, the $(k+1)$ -osculating space, $1 \leq k \leq d-1$ (defined by the partial derivatives of order k) is identified to $\mathbb{P}(R_k)$, i.e. to degree d forms possessing a factorization $L^{d-k}.G$ where G is a degree k form. The set of hyperplanes containing the $(k+1)$ tangent space at the point $[L^d]$ is the projective space $\mathbb{P}((H^0(\mathfrak{m}_{L^\vee}^{k+1}(d)))^\vee)$ of degree d hypersurfaces with multiplicity $(k+1)$ at the point L^\vee . Then we have

$$v_d(\mathbb{P}^n)^{(k+1)^\vee} = \overline{\cup_{L^d \in v_d(\mathbb{P}^n)} \mathbb{P}((H^0(\mathfrak{m}_{L^\vee}^{k+1}(d)))^\vee)} =: \mathfrak{X}_{d,k+1},$$

where \mathfrak{m}_{L^\vee} is the ideal sheaf of the point L^\vee . It is the set of degree d hypersurfaces that are $(k+1)$ -singular at one point.

For instance the dual variety of $v_d(\mathbb{P}^2)$ is the discriminant variety that parametrizes the singular degree d curves.

The smooth model of this variety $\mathfrak{X}_{d,k+1}$ is a projective bundle usually called bundle of principal parts. Indeed the derivation map, dual of the multiplication map

$$\partial^k : R_d \longrightarrow R_k \otimes R_{d-k}, \quad f \mapsto \sum_{a_0 + \dots + a_r = k} X_0^{a_0} \dots X_n^{a_n} \frac{\partial^k f}{\partial X_0^{a_0} \dots \partial X_n^{a_n}},$$

induces the following homomorphism of vector bundles on $\mathbb{P}(R_1)$:

$$0 \longrightarrow R_k^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-d+k) \xrightarrow{\partial^k} R_d^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow P_{d,k+1} \longrightarrow 0.$$

The first map can be represented by the matrix ∂^k of partial derivatives of order k of a R_d^\vee -basis. The projective bundle $\mathbb{P}(P_{d,k+1}) \subset \mathbb{P}^n \times \mathbb{P}(R_d^\vee)$ is the incidence variety $\{(x, f) \mid \partial^k f(x) = 0\}$ and the fiber over a point $x \in \mathbb{P}^n$ is identified with the projective space $\mathbb{P}(H^0(\mathfrak{m}_x^{k+1}(d)))^\vee$ (of hypersurfaces of degree d passing through x with multiplicity $(k+1)$) where \mathfrak{m}_x is the ideal sheaf of the point x . Its image in $\mathbb{P}(R_d^\vee)$ by the second projection is $\mathfrak{X}_{d,k+1}$. It is an irreducible variety birational to $\mathbb{P}(P_{n,k+1})$.

3.1. Projections of Veronese and sections of Principal bundles. Let $\wedge \subset R_d$ be a vector subspace of dimension less or equal to $r_d - r_{d-k}$. We consider the projection map:

$$\mathbb{P}(R_d) \setminus \mathbb{P}(\wedge) \longrightarrow \mathbb{P}\left(\frac{R_d}{\wedge}\right).$$

Let K be the kernel of the induced map:

$$0 \longrightarrow K \longrightarrow \wedge \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0.$$

We introduce the multiple incidence variety [point/hyperplane]:

$$\begin{array}{ccc} \mathbb{F}^k = \{(\sum X_i X_i^\vee)^k = 0\} & \xrightarrow{q_k} & \mathbb{P}^{\vee n} \\ & & \downarrow p_k \\ & & \mathbb{P}^n \end{array}$$

Applying the functor $p_{k*} q_k^*$ to the exact sequence defining K , we obtain

$$0 \longrightarrow q_{k*} p_k^* K \longrightarrow \wedge \otimes \mathcal{O}_{\mathbb{P}^{\vee n}} \longrightarrow P_{d,k+1} \longrightarrow R^1 q_{k*} p_k^* K \longrightarrow 0.$$

The surjective homomorphism $\mathrm{Hom}(R_d, \mathbf{k}) \rightarrow \mathrm{Hom}(\wedge, \mathbf{k})$ gives an embedding $\mathbb{P}(\mathrm{Hom}(\wedge, \mathbf{k})) \subset \mathbb{P}(\mathrm{Hom}(R_d, \mathbf{k}))$. Let us consider the diagram

$$\begin{array}{ccccccc}
& & \wedge \otimes \mathcal{O}_{\mathbb{P}^n} & \xlongequal{\quad} & \wedge \otimes \mathcal{O}_{\mathbb{P}^n} & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_{d-k} \otimes \mathcal{O}_{\mathbb{P}^n}(-k) & \longrightarrow & R_d \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & P_{d,k+1} \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & R_{d-k} \otimes \mathcal{O}_{\mathbb{P}^n}(-k) & \longrightarrow & \frac{R_d}{\wedge} \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & R^1 q_{k*} p_k^* K \longrightarrow 0.
\end{array}$$

It is important to understand that $\mathrm{supp}(R^1 q_{k*} p_k^* K) = \{x \mid \wedge \cap H^0(\mathfrak{m}_x^{k+1}(d)) \neq 0\}$. We remark that \wedge induces an artinian ideal since the map given by it is surjective. Then we have the following result when $\dim(\wedge) = r_d - r_{d-k}$.

Theorem 3.1. *Let $I \subset R$ be an artinian ideal generated by r homogeneous polynomials F_1, \dots, F_r of degree d . Let $k \leq d$ a positive integer. If $r = r_d - r_{d-k}$ then the following conditions are equivalent:*

- (1) *The ideal I fails the SLP at the range k in degree $d - k$.*
- (2) *The homogeneous forms F_1, \dots, F_r become linearly dependent on a general multiple hyperplane L^k of \mathbb{P}^n .*
- (3) *The n -dimensional variety $X_{n, (I^{-1})_d}$ satisfies at least one Laplace equation of order $d - k$.*
- (4) *It exists $f \in \frac{R_d}{\langle F_1, \dots, F_r \rangle}$ such that $f \in H^0(\mathfrak{m}_{L^\vee}^{d-k+1})$ for a general linear form $L \in R_1$.*

Proof. In the case of equality $h^0(K \otimes \mathcal{O}_{L^k}) = h^1(K \otimes \mathcal{O}_{L^k})$ for any linear form L . Then $H^0(K \otimes \mathcal{O}_{L^k}) \neq 0$ (see proposition 2.2) implies $H^1(K \otimes \mathcal{O}_{L^k}) \neq 0$ and the non vanishing of this last vector space means that L^\vee belongs to the support of the sheaf $R^1 q_{k*} p_k^* K$. Since $\mathrm{supp}(R^1 q_{k*} p_k^* K) = \{x \mid \langle F_1, \dots, F_r \rangle \cap H^0(\mathfrak{m}_x^{k+1}(d)) \neq 0\}$, we have done. \square

In the next section we will give many examples of ideals failing the WLP or the SLP by producing ad-hoc singular hypersurfaces. Moreover we will give also an exhaustive list of monomial ideals in degree 3 in four variables and degree 4 in three variables that fail the WLP.

3.2. Monomial ideals coming from singular hypersurfaces. In their nice paper about osculating spaces of Veronese surfaces, Lanteri and Mallavibarena remark that the equation of the curve given by three concurrent lines depends only on six monomials instead of seven. More precisely let us consider a cubic with a triple point at (a, b, c) passing through $(1, 0, 0)$, $(0, 1, 0)$ and $((0, 0, 1)$. Its equation is

$$(bz - cy)(az - cx)(ay - bx) = 0$$

and it depends on the monomials $x^2y, xy^2, x^2z, xz^2, y^2z, yz^2$. So there is a non zero form in $\frac{R_3}{\langle x^3, y^3, z^3, xyz \rangle}$ that is triple at a general point. By this way they explain the Togliatti surprising phenomena ([8], Thm. 4.1, [6] and [3]).

We apply this idea in our context. In the monomial case being artinian for the ideal I means that it contains the forms x_0^d, \dots, x_n^d . Let us consider the $(n+1)$ basis point $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 0, 1)$ and let us assume that the number of monomials in I is equal to $r_d - r_{d-k}$ for $k \geq 1$ a fixed integer. Let us denote by \wedge_I the vector subspace of R_d generated by the monomials generating I . Then, as we said above, the ideal I fails the SLP at the range k in degree $d - k$ if and only if there exists at any point $x = (a_0, \dots, a_n)$ a hypersurface of degree d with multiplicity $d - k + 1$ at x given by a form in R_d / \wedge_I .

Let us consider first the case $k = 1$, i.e. the WLP case. Then the wanted singular hypersurface at x is given by d hyperplanes meeting at x and passing through the $n + 1$ basis points. If this equation depends only on the monomials of R_d/\wedge_I then the image of $v_d(\mathbb{P}^n)$ by $\pi : \mathbb{P}(R_d) \setminus \mathbb{P}(\wedge_I) \rightarrow \mathbb{P}(R_d/\wedge_I)$ verifies a Laplace equation. Let us give some examples.

Case $n = 2, d = 4, k = 1$.

We describe exhaustively the monomial ideals $(x^4, y^4, z^4, f, g) \subset \mathbf{k}[x, y, z]$ of degree 4 that do not verify the WLP.

Theorem 3.2. *Up to permutation of variables the monomial ideals generated by 5 forms that fail the WLP in degree 3 are one of the following $(x^4, y^4, z^4, x^3z, x^3y)$ and $(x^4, y^4, z^4, x^2y^2, xyz^2)$.*

Remark. Geometrically it is evident that the first ideal $(x^4, y^4, z^4, x^3z, x^3y)$ fails the WLP. Indeed modulo a linear form L the restricted monomials $\bar{x}^i\bar{y}^j$ can be interpreted as points of the projective \mathbb{P}^4 defined by the rational normal curve of degree four image of L by the Veronese map. Then the tangent line to the rational quartic curve (isomorphic to L) at the point $[\bar{x}^4]$ contains the two points $[\bar{x}^3\bar{y}]$ and $[\bar{x}^3\bar{z}]$. This line meets the plane $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4 \rangle)$ in one point; it implies that

$$\dim_{\mathbb{C}} \langle \bar{x}^4, \bar{y}^4, \bar{z}^4, \bar{x}^3\bar{y}, \bar{x}^3\bar{z} \rangle \leq 4.$$

It is not so evident for the second ideal to see that the line $\mathbb{P}(\langle \bar{x}^2\bar{y}^2, \bar{x}\bar{y}\bar{z}^2 \rangle)$ always (when we restrict at any line) meet the plane $\mathbb{P}(\langle \bar{x}^4, \bar{y}^4, \bar{z}^4 \rangle)$.

Proof. Let us consider the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and the degree 4 curves with a quartuple point in (a, b, c) passing through these three points. These curves are product of four lines:

$$f(x, y, z) = (ay - bx)(az - cx)(cy - bz)(\alpha(ay - bx) + \beta(az - cx)).$$

We develop f . We find the following coefficients for monomials.

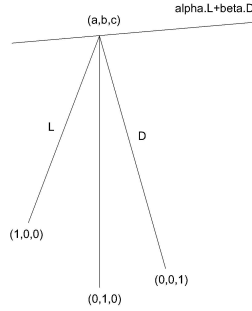


FIGURE 1. quartic with a quartuple point.

$$f(x, y, z) = bc(\beta c^2 - \alpha ab)x^2yz + ac(-\alpha ab - \beta(ac + c^2))xy^2z + ac^2(2\alpha b + \beta c)x^2y^2 + bc^2(-\alpha b - \beta c)x^3y + a^2b(2\alpha b + \beta c)xyz^2 + a^2b(2\alpha b + \beta c)x^2z^2 + b^2c(\alpha b + \beta c)x^3z + \alpha a^3cy^3z - \alpha a^2c^2xy^3 + a^2(-\alpha ab + \beta c^2)y^2z^2 - \beta a^2bcyz^3 + \beta ab^2c.xz^3.$$

Then, twelve monomials appear to write the wanted quartic with a quartuple point. The forms x^4, y^4, z^4 are missing. We want only ten forms. The coefficients of two forms have to vanish. The following possibilities appear:

- $\alpha = 0$,
the remaining linear system is $(x^4, y^4, z^4, y^3z, xy^3)$.
- $\beta = 0$
the remaining linear system is $(x^4, y^4, z^4, yz^3, xz^3)$.
- $\alpha \neq 0$ and $\beta \neq 0$ but $\alpha b + \beta c = 0$,
the remaining linear system is $(x^4, y^4, z^4, x^3z, x^3y)$.
- $\alpha \neq 0$ and $\beta \neq 0$ but $\beta c^2 = \alpha ab$
the remaining linear system is $(x^4, y^4, z^4, x^2yz, y^2z^2)$.
- $\alpha \neq 0$ and $\beta \neq 0$ but $2\alpha b + \beta c = 0$,
the remaining linear system is $(x^4, y^4, z^4, x^2y^2, xyz^2)$.
- $\alpha \neq 0$ and $\beta \neq 0$ but $\alpha ab + \beta(ac + c^2) = 0$,
the remaining linear system is $(x^4, y^4, z^4, x^2z^2, xy^2z)$.

□

Case $n = 2, d = 5, k = 1$.

We cannot apply the same technique to describe exhaustively the monomial ideals $(x^5, y^5, z^5, f, g, h) \subset \mathbf{k}[x, y, z]$ of degree 5 that do not verify the WLP. Indeed the computations become too tricky. We can give some cases by geometric arguments.

Proposition 3.3. *The following monomial ideals $(x^5, y^5, z^5, x^3y^2, x^3z^2, x^3yz)$ and $(x^5, y^5, z^5, x^4z, x^4y, m)$, where m is any monomial, fail the WLP in degree 4.*

Proof. Modulo a linear form L the restricted monomials $\bar{x}^i \bar{y}^j$ can be interpreted as points of the projective \mathbb{P}^5 defined by the rational normal curve of degree 5 image of L by the Veronese map. Then the tangent line to the rational quintic curve (isomorphic to L) at the point $[\bar{x}^5]$ contains the two points $[\bar{x}^4 \bar{y}]$ and $[\bar{x}^4 \bar{z}]$. This line meet the plane $\mathbb{P}(\langle \bar{x}^5, \bar{y}^5, \bar{z}^5 \rangle)$ in one point; it implies that

$$\dim_{\mathbb{C}} \langle \bar{x}^5, \bar{y}^5, \bar{z}^5, \bar{x}^4 \bar{y}, \bar{x}^4 \bar{z}, \bar{m} \rangle \leq 5.$$

In the same way the osculating plane at $[\bar{x}^5]$ i.e. $\mathbb{P}(\langle \bar{x}^3 \bar{y}^2, \bar{x}^3 \bar{z}^2, \bar{x}^3 \bar{y} \bar{z} \rangle)$ meets the plane $\mathbb{P}(\langle \bar{x}^5, \bar{y}^5, \bar{z}^5 \rangle)$ in one point. □

Case $n = 3, d = 3, k = 1$.

We describe exhaustively the monomial ideals

$$(x^3, y^3, z^3, t^3, f_1, f_2, f_3, f_4, f_5, f_6) \subset \mathbf{k}[x, y, z, t]$$

of degree 3 that do not verify the WLP.

Theorem 3.4. *Up to permutation of variables the monomial ideals generated by ten forms that fail the WLP in degree 2 are the ideals $I = (x^3, y^3, z^3, t^3, f_1, f_2, f_3, f_4, f_5, f_6)$ where the forms f_i are chosen among one of the following sets of monomials:*

- (1) $\{x^2y, xy^2, x^2z, x^2t, y^2z, y^2t, z^2t, zt^2, xyz, xyt\}$. (Case (A1))
- (2) $\{x^2y, xy^2, xz^2, y^2z, yz^2, y^2t, zt^2, z^2t\}$. (Case (A2))
- (3) $\{x^2y, xy^2, z^2t, zt^2, xyz, xyt, xzt, yzt\}$. (Case (A2))
- (4) $\{x^2y, xy^2, x^2z, xz^2, x^2t, xt^2, xyz, xzt, xyt, yzt\}$. (Case (B1))

Proof. We look for a vector subspace $\wedge \subset R_3$ of dimension 10 that fails the WLP. According to theorem 3.1, we look for a surface of degree 3 with multiplicity 3 at a general point $M = (a, b, c, d)$ (i.e. three concurrent planes) that passes through the points $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ such that its equation f depends only on the monomials in R_3/\wedge .

Remark. One has to write this equation with a number of monomials as small as possible. Then the orthogonal space becomes bigger and we will cover all the possible choices.

First of all, there is at least one plane passing through (a, b, c, d) and two basis points; without loss of generality we assume that its equation is $dz - ct = 0$. Then two cases occur.

- (A) There is another plane through the two remaining points, and one free, i.e. $f = (bx - ay)(dz - ct)(\alpha x + \beta y + \gamma z + \delta t)$, with $\alpha a + \beta b + \gamma c + \delta d = 0$.
- (B) Each remaining plane passes through one basis point, i.e.

$$f = (dz - ct)(u_1x + v_1y + w_1z)(u_2x + v_2y + w_2t)$$

$$\text{with } u_1ba + v_1b + w_1c = u_2a + v_2b + w_2d = 0.$$

Case (A) The equation of f is the following:

$$f(x, y, z, t) = \alpha bdx^2z - \alpha bcx^2t + d(\beta b - \alpha a)xyz + c(\alpha a - \beta b)xyt - \beta ady^2z + \beta acy^2t + \gamma bdxz^2 + b(\delta d - \gamma c)xzt - \gamma adyz^2 + a(\gamma c - \delta d)yzt - \delta bcx^2t + ac\delta yt^2.$$

In order to remove the biggest number of coefficients, different cases occur.

- (A1) The case $\alpha = 0$ and $\beta = 0$ (i.e. one plane through $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ and one double plane through $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. It needs six coefficients to write its equation that are $xz^2, xzt, yz^2, yzt, xt^2, yt^2$. In R_3 it remains 14 monomials that are the four powers x^3, y^3, z^3, t^3 and ten more $x^2y, xy^2, x^2z, x^2t, xyz, xyt, y^2z, y^2t, z^2t, zt^2$. The ideal generated by the four powers and six forms among these ten fails the WLP in degree 2.
- (A2) The case $\beta = 0$ and $\gamma = 0$. The equation of the cubic is

$$f = (bx - ay)(dz - ct)(cx - az).$$

This equation depends on the eight following monomials

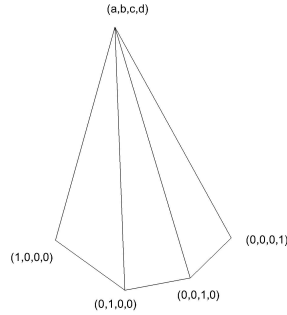


FIGURE 2. cubic with a triple point. Case (A2).

$$yt^2, xyt, yzt, xyz, xt^2, x^2t, xzt, x^2z.$$

In R_3 it remains 12 monomials that are $(x^3, y^3, z^3, t^3, x^2y, xy^2, xz^2, y^2z, yz^2, y^2t, zt^2, z^2t)$. The four powers and eight more. The ideal generated by the four powers and six forms among these eight fails the WLP in degree 2.

- (A3) The case $\beta b = \alpha a$ and $\delta d = \gamma c$. The supplementary plane do not pass through one of the four basis points. There exists at any point (a, b, c, d) a function $u(a, b, c, d)$ such that $ab + u(a, b, c, d)cd = 0$. The equation of the supplementary plane is $bx + ay + udz + uct = 0$. Then the equation of the

cubic depends only on $(x^2z, x^2t, y^2z, y^2t, xz^2, yz^2, xt^2, yt^2)$. In R_3 it remains 12 monomials that are $(x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt, x^2y, xy^2, z^2t, zt^2)$. The four powers and eight more. The ideal generated by the four powers and six forms among these eight fails the WLP in degree 2.

Case (B) The equation of f is $f = (dz - ct)(u_1x + v_1y + w_1z)(u_2x + v_2y + w_2t)$. This equation depends on 14 monomials (i.e. all of them except the four powers and x^2y, xy^2). We have to remove at least the coefficients of four of them. If $w_1 = 0$ or $w_2 = 0$ it is again the first situation. We cannot choose $u_i = v_i = 0$ because the plane has to pass through (a, b, c, d) .

- (B1) If $u_1 = u_2 = 0$ we remove 7 coefficients. Then f depends on 6 monomials that are $(y^2z, yz^2, z^2t, y^2t, yt^2, zt^2)$. In R_3 it remains 14 monomials that are $(x^3, y^3, z^3, t^3, x^2y, xy^2, x^2z, xz^2, x^2t, xt^2, xyz, xzt, xyt, yzt)$. The four powers and ten more. The ideal generated by the four powers and six forms among these ten fails the WLP in degree 2.

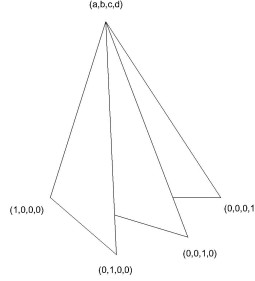


FIGURE 3. cubic with a triple point. Case (B1)

- (B2) If $u_1 = v_2 = 0$ the cubic consists in three plan with equations $dz - ct = 0$, $cy - bz = 0$ and $dx - at = 0$. By permutation it corresponds to the case $\beta = \gamma = 0$ which is done above (see figure 2).
- (B3) If $u_1v_2 + u_2v_1 = du_1w_2 - cu_2w_1 = cv_2w_1 - dv_1w_2 = 0$. Then f depends on ten monomials that are $(x^2z, xz^2, x^2t, xt^2, y^2z, yz^2, y^2t, yt^2, z^2t, zt^2)$. In R_3 it remains 10 monomials that are $(x^3, y^3, z^3, t^3, x^2y, xy^2, xyz, xyt, xzt, yzt)$. The four powers and eight more. The ideal generated by the four powers and six forms among these eight fails the WLP in degree 2. It appears that it is a subcase of A3.

The other cases are obtained by permutation. \square

Remark. If we want \wedge of dimension $r \leq 10$ (for instance 8, like in the examples [9]) we need numerous independent cubics with a triple point. Indeed we have

$$0 \longrightarrow q_{k*}p_k^*K \longrightarrow \mathcal{O}_{\mathbb{P}^3}^r \longrightarrow S^3\Omega_{\mathbb{P}^3}^\vee(-1) \longrightarrow R^1q_{k*}p_k^*K \longrightarrow 0,$$

and the kernel is non zero if and only if the generic rank of the cokernel is $10 - r + 1$ (in other terms $10 - r + 1$ independent cubics with triple points). As an example let us give two linear systems of eight cubic forms that fail the WLP in degree 2.

Proposition 3.5. *The following monomial ideals*

- (1) $I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t)$ and
 (2) $J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt)$

fail the WLP in degree 2.

Remark. It should be possible to give a complete list by this method. Anyway the complete classification is already done, see [9], Thm. 4.11.

Proof. Let us consider the following three forms defining singular cubics passing through the basis points and the general point (a, b, c, d) :

$$(ct - dz)(at - dx)(ay - bx) = 0, (ct - dz)^2(ay - bx) = 0, (ct - dz)(ay - bx)^2 = 0.$$

They are specialisations of the case (1) in the proof of theorem 3.4. They are linearly independent and can be written with twelve monomials. Then it remains only 8 forms for \wedge_I :

$$I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t).$$

Let us consider the following three forms defining singular cubics passing through the basis points and the general point (a, b, c, d) :

$$(bz - cy)(az - cx)(ay - bx) = 0, (bx - ay)(at - dx)(dy - bt) = 0, (az - cx)(dx - at)(dz - ct) = 0.$$

They are specialisations of the case (2) in the proof of theorem 3.4. They are linearly independent and can be written with twelve monomials:

$$(x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, t^2y, t^2z, ty^2, tz^2, t^2x, x^2t).$$

It remains only

$$J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt).$$

The ideals I and J correspond respectively to the cases (3) and (1) of the theorem 4.11 in [9]. \square

Case $n = 4, d = 3, k = 1$.

Again we do not give a complete classification but only one example of monomial ideal in $\subset \mathbf{k}[x, y, z, t, w]$ that fails the WLP.

Proposition 3.6. *The following monomial ideal*

$$I = (x^3, y^3, z^3, t^3, w^3, xyz, xyt, xyw, xzt, xzw, xtw, yzt, yzw, ytw, ztw),$$

fails the WLP in degree 2.

Proof. A cubic solid with a triple point at the general point $x = (a, b, c, d, e)$ is given by three \mathbb{P}^3 through x and through the points

$$(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1).$$

Still copying the Togliatti's case we consider the \mathbb{P}^3 that contain the \mathbb{P}^2 defined by the points $(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$ and the point (a, b, c, d, e) . More precisely we consider three \mathbb{P}^3 obtained by choosing a supplementary point among $(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)$. Like in the following figure. We have to permute in order to have six independent cubics with triple point. Then it remains at the end

$$I = (x^3, y^3, z^3, t^3, w^3, xyz, xyt, xyw, xzt, xzw, xtw, yzt, yzw, ytw, ztw),$$

and I fails the WLP in degree 2. \square

Case $n = 3, d = 4, k = 1$.

Again we do not give a complete classification but only one example of monomial ideal of degree 4 in $\subset \mathbf{k}[x, y, z, t]$ that fails the WLP..

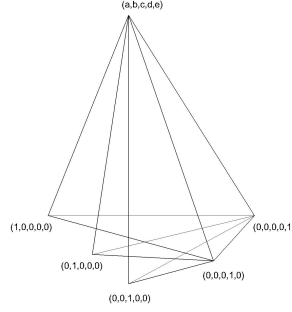


FIGURE 4. cubic with a triple point.

Proposition 3.7. *Let f_1, \dots, f_{11} be eleven monomials chosen among*

$$x^3y, x^3z, x^3t, xy^3, xz^3, xt^3, y^3z, y^3t, yz^3, yt^3, z^3t, zt^3, x^2y^2, z^2t^2, y^2z^2, x^2t^2.$$

Then the ideal $I = (x^4, y^4, z^4, t^4, f_1, \dots, f_{11})$ fails the WLP in degree 3.

Proof. We want at any point $x = (a, b, c, d)$ a surface of degree 4 with multiplicity 4 at x and that passes through the basis points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$. It means four concurrent planes. Let us see an example.

$$f = (ct - dz)(at - dx)(ay - bx)(bz - cy) = 0.$$

Then we need the following monomials to write this f :

$$yzt^2, xyzt, yz^2t, xyz^2, xzt^2, x^2zt, xz^2t, x^2z^2, y^2t^2, xy^2t, y^2zt, xy^2z, xyt^2, x^2yt, x^2yz.$$

Twenty monomials are missing. More precisely:

$$x^4, y^4, z^4, t^4, x^3y, x^3z, x^3t, xy^3, xz^3, xt^3, y^3z, y^3t, yz^3, yt^3, z^3t, zt^3, x^2y^2, z^2t^2, y^2z^2, x^2t^2.$$

Then if we choose 11 monomials f_1, \dots, f_{11} among

$$x^3y, x^3z, x^3t, xy^3, xz^3, xt^3, y^3z, y^3t, yz^3, yt^3, z^3t, zt^3, x^2y^2, z^2t^2, y^2z^2, x^2t^2,$$

the ideal $I = (x^4, y^4, z^4, t^4, f_1, \dots, f_{11})$ fails the WLP in degree 3. \square

Let us give now an example when $k = 2$. We look for singular hypersurfaces of degree d with multiplicity $d - 1$ at a point p that consist of $d - 1$ hyperplanes meeting at p and an extra hyperplane such that their union passes through the $n + 1$ basis points. As before, such a hypersurface has an equation in the canonical monomial basis of degree d forms; if this equation depends only on the monomials of R_d/\wedge_I then the image of $v_d(\mathbb{P}^n)$ by $\pi : \mathbb{P}(R_d) \setminus \mathbb{P}(\wedge_I) \rightarrow \mathbb{P}(R_d/\wedge_I)$ verifies a Laplace equation. Let us give examples.

Case $n = 2, d = 4, k = 2$.

Proposition 3.8. *The ideal $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3) \subset \mathbf{k}[x, y, z]$ fails the SLP at the range 2 in degree 2.*

Proof. We consider the curve of degree 3 through $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$ and $p_3 = (0, 0, 1)$ with multiplicity 3 in (a, b, c) and the line through p_3 and p_2 i.e. $x = 0$. Then the union of both curves is a quartic passing through p_1, p_2, p_3 and triple

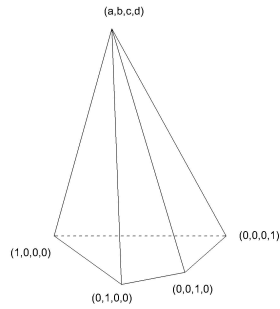


FIGURE 5. quartic with a quartuple point.

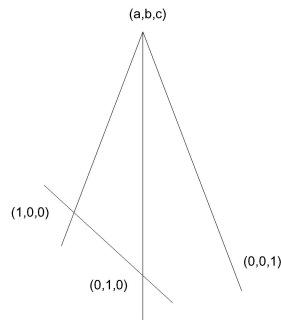


FIGURE 6. quartic with a triple point.

at (a, b, c) . It depends on the six monomials $x^3y, x^3z, x^2y^2, xy^2z, x^2z^2, xyz^2$. Then it remains $9 = 15 - 6$ monomials

$$\wedge_I = \langle x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3 \rangle$$

such that the kernel K of the following map

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^2}^9 \xrightarrow{\wedge_I} \mathcal{O}_{\mathbb{P}^2}(4) \longrightarrow 0,$$

verifies $h^0(K \otimes \mathcal{O}_{L^2}) \neq 0$ for a general linear form L . According to theorem 3.1 it proves that $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3)$ fails the SLP at the range 2 in degree 2. \square

4. SLP AT THE RANGE 2 AND DERIVATION BUNDLE ON \mathbb{P}^2

Let us point out the interesting case $\wedge = H^0(I_Z(n))^\vee$ where Z is a finite set of points in \mathbb{P}^2 . This case leads to non toric examples. But mostly, it leads to derivation bundles associated to line arrangements. More precisely, let us denote by $f = 0$ the equation of

the union of dual lines of the d points of Z and by $\mathcal{D}_0(Z)$ the vector bundle appearing as the kernel of the jacobian map:

$$0 \longrightarrow \mathcal{D}_0(Z) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\partial f)} \mathcal{O}_{\mathbb{P}^2}(d-1).$$

This bundle is called *derivation bundle* of the line arrangement and it is studied by many different authors (see for instance [14] and [13]). The arrangement of lines is said *free with exponents* (a, b) when its derivation bundle splits on \mathbb{P}^2 as a sum of two line bundles, more precisely when $\mathcal{D}_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$.

The splitting of $\mathcal{D}_0(Z)$ over a line $l \subset \mathbb{P}^2$ is related to the existence of curves (with a given degree d) passing through Z that are multiple (with multiplicity $d-1$) at $l^\vee \in \mathbb{P}^{2\vee}$. More precisely,

Theorem 4.1. *Let $Z \subset \mathbb{P}^{2\vee}$ be a set of $2n+r+1$ distinct points with $n \geq 1$, $r \geq 0$ and l be a general line in \mathbb{P}^2 . Then the following conditions are equivalent:*

- (1) $\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(-n) \oplus \mathcal{O}_l(-n-r)$.
- (2) $H^0((\mathcal{J}_Z \otimes \mathcal{J}_l^n)(n+1)) \neq 0$ and $H^0((\mathcal{J}_Z \otimes \mathcal{J}_l^{n-1})(n)) = 0$.

Proof. Let us introduce the variety $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ which is the incidence variety *point-line* in \mathbb{P}^2 , and the projections p and q on \mathbb{P}^2 and $\mathbb{P}^{2\vee}$.

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{q} & \mathbb{P}^{2\vee} \\ p \downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

We recall first that the derivation bundle is obtained by looking at $I_Z(1)$ on \mathbb{P}^2 . More precisely, we have $\mathcal{D}_0(Z) = p_*q^*I_Z(1)$ (see [4], prop.1.3).

Let us denote by $\widehat{\mathbb{P}}$ the blowing up of $\mathbb{P}^{2\vee}$ along the point $x = l^\vee$. We recall that $\widehat{\mathbb{P}} \simeq p^{-1}(x^\vee) \subset \mathbb{F}$ and we consider the induced incidence diagram:

$$\begin{array}{ccc} \widehat{\mathbb{P}} & \xrightarrow{\widehat{q}} & \mathbb{P}^{2\vee} \\ \widehat{p} \downarrow & & \\ l = x^\vee & & \end{array}$$

Moreover we have the following resolution of $\widehat{\mathbb{P}}$ in \mathbb{F} :

$$0 \longrightarrow p^*\mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{F}} \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}} \longrightarrow 0.$$

Then tensoring the exact sequence above by $q^*\mathcal{J}_Z(1)$ and taking the direct image by p we obtain:

$$\begin{aligned} 0 &\longrightarrow \mathcal{D}_0(Z)(-1) \xrightarrow{x^\vee} \mathcal{D}_0(Z) \longrightarrow \widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) \longrightarrow \\ &\longrightarrow [R^1p_*q^*\mathcal{J}_Z(1)](-1) \xrightarrow{x^\vee} [R^1p_*q^*\mathcal{J}_Z(1)] \longrightarrow R^1\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) \longrightarrow 0. \end{aligned}$$

Since x is general, any line through x is at most 1-secant to Z . Then the support of the sheaf $R^1\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1)$, which is the locus of 3-secant lines to Z through x (by base change theorem, see [11] for instance), is empty. So it proves that we have in fact a short exact sequence:

$$0 \longrightarrow \mathcal{D}_0(Z)(-1) \xrightarrow{x^\vee} \mathcal{D}_0(Z) \longrightarrow \widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) \longrightarrow 0,$$

and consequently $\widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1) = T_Z \otimes \mathcal{O}_{x^\vee}$.

Then the decomposition $\mathcal{D}_0(Z) \otimes \mathcal{O}_{x^\vee} = \mathcal{O}_{x^\vee}(-n) \oplus \mathcal{O}_{x^\vee}(-n-r)$ gives an injective homomorphism: $\mathcal{O}_{x^\vee}(-n) \hookrightarrow \widehat{p}_*\widehat{q}^*\mathcal{J}_Z(1)$.

This means that we have a non zero map $\widehat{p}^*\mathcal{O}_{x^\vee}(-n) \hookrightarrow \widehat{q}^*\mathcal{J}_Z(1)$ on $\widehat{\mathbb{P}}$, that we can write also as: $\mathcal{O}_{\widehat{\mathbb{P}}} \hookrightarrow \widehat{q}^*\mathcal{J}_Z(1) \otimes \widehat{p}^*\mathcal{O}_{x^\vee}(n)$.

By the projection formula this last map is equivalent to a non zero map on \mathbb{P}^{2V} :

$$\mathcal{O}_{\mathbb{P}^{2V}} \hookrightarrow \mathcal{J}_Z(1) \otimes \mathcal{J}_x^n(n) = (\mathcal{J}_Z \otimes \mathcal{J}_x^n)(n+1).$$

Let us now prove that $H^0(\mathcal{J}_Z \otimes \mathcal{J}_x^{n-1})(n) = 0$. Assume that $H^0(\mathcal{J}_Z \otimes \mathcal{J}_x^{n-1})(n) \neq 0$ and consider a non zero section: $\mathcal{O}_{\mathbb{P}^{2V}} \hookrightarrow \mathcal{J}_Z \otimes \mathcal{J}_x^{n-1}(n)$. The corresponding section on $\widehat{\mathbb{P}} \mathcal{O}_{x^\vee} \hookrightarrow \widehat{q}^* \mathcal{J}_Z(n)$ vanishes in codimension 1 along $(n-1)$ -times the exceptionnal divisor $q^{-1}(x)$. After simplification by its equation ($f \in H^0(\widehat{p}^* \mathcal{O}_{x^\vee}(n-1) \otimes \widehat{q}^* \mathcal{O}_{\mathbb{P}^{2V}}(1-n))$) we obtain:

$$\widehat{p}^* \mathcal{O}_{x^\vee}(1-n) \hookrightarrow \widehat{q}^* \mathcal{J}_Z(1).$$

It induces on x^\vee a non zero map: $\mathcal{O}_{x^\vee}(1-n) \hookrightarrow \widehat{p}_* \widehat{q}^* \mathcal{J}_Z(1)$, contradicting the decomposition of $\mathcal{D}_0(Z)$ along x^\vee . \square

In our context it implies the following characterization of unstability. We recall that a rank two vector bundle E on \mathbb{P}^n , $n \geq 2$ is *unstable* if and only if its splitting $E_l = \mathcal{O}_l(a) \oplus \mathcal{O}_l(b)$ on a general line l verifies $|a-b| \geq 2$. This characterization is a consequence of the Grauert-Mülich theorem, see [11].

Theorem 4.2. *Let $I \subset R = \mathbb{C}[x, y, z]$ be an artinian ideal generated by $2d+1$ polynomials l_1^d, \dots, l_{2d+1}^d where l_i are distinct linear forms in \mathbb{P}^2 . Let $Z = \{l_1^\vee, \dots, l_{2d+1}^\vee\}$ be the corresponding set of points in \mathbb{P}^{2V} . Then the following conditions are equivalent:*

- (1) *The ideal I fails the SLP at the range 2 in degree $d-2$.*
- (2) *The derivation bundle $\mathcal{D}_0(Z)$ is unstable.*

Proof. The failure of SLP at the range 2 in degree $d-2$ is equivalent to the existence at a general point l^\vee of a curve of degree d with multiplicity $d-1$ at l^\vee belonging to $\wedge = H^0(I_Z(d))$. But, the description of $\mathcal{D}_0(Z)$ as the image of $I_Z(1)$ implies that such a curve imposes the splitting of $\mathcal{D}_0(Z)$ over the line l . More precisely we have the following equivalence:

$$\mathcal{O}_l(1-d) \hookrightarrow \mathcal{D}_0(Z) \otimes \mathcal{O}_l \Leftrightarrow H^0((I_Z \otimes \mathfrak{m}_{l^\vee}^{d-1})(d)) \neq 0.$$

Then on the general line

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(d-s) \oplus \mathcal{O}_l(d+s) \text{ with } s > 0.$$

In other words it gives a non balanced decomposition and according to Grauert-Mülich theorem it is equivalent to unstability, see [11]. \square

Let us give now an ideal (not generated by monomials) that fails the SLP at the range 2. It comes from a line arrangement such that the derivation bundle associated is unstable (in fact even decomposed).

Example 4.3. Consider the set of nine points

$$Z = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, -1, 0), (1, 0, 1), (1, 0, -1), (0, 1, 1), (0, 1, -1)\}.$$

The dual linear forms are $x, y, z, x+y, x-y, x+z, x-z, y+z, y-z$. We consider the map

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^2}^9 \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^2}(4) \longrightarrow 0,$$

where $\phi(a_1, \dots, a_9) =$

$$a_1 x^4 + a_2 y^4 + a_3 z^4 + a_4 (x+y)^4 + a_5 (x-y)^4 + a_6 (x+z)^4 + a_7 (x-z)^4 + a_8 (y+z)^4 + a_9 (y-z)^4.$$

The associated derivation bundle is $\mathcal{D}_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)$ (it is free with exponents $(3, 5)$; see [12] for a proof). Then, according to theorem 4.1 there is at any point p a degree 4 curve with multiplicity 3 at p passing through Z . In other words we have $H^0(K \otimes \mathcal{O}_{L^2}) \neq 0$ for a general linear form L and we have proved that:

Proposition 4.4. *The ideal*

$$I = (x^4, y^4, z^4, (x+y)^4, (x-y)^4, (x+z)^4, (x-z)^4, (y+z)^4, (y-z)^4) \mathbf{k}[x, y, z]$$

fails the SLP at the range 2 and degree 2.

More generally we can produce examples that come from free arrangements.

Proposition 4.5. *Let $\mathcal{A} = \{l_1, \dots, l_{a+b+1}\}$ a line arrangement that is free with exponents (a, b) such that $a \leq b$, $b - a \geq 2$ and $a + b$ even. The ideal $I = (l_1^{\frac{a+b}{2}}, \dots, l_{a+b+1}^{\frac{a+b}{2}})$ fails the SLP at the range 2 and degree $\frac{a+b-1}{2}$.*

Remark. If $a + b$ is odd we can add to Z one point p in general position with Z and we can prove in the same way that $I = (l_1^{\frac{a+b+1}{2}}, \dots, l_{a+b+1}^{\frac{a+b+1}{2}}, (p^\vee)^{\frac{a+b+1}{2}})$ fails the SLP at the range 2 and degree $\frac{a+b}{2}$.

Proof. Let us denote by $Z = \{l_1^\vee, \dots, l_{a+b+1}^\vee\}$ the dual set of points of \mathcal{A} . Since there exists at any general point l^\vee a curve of degree $a + 1$ passing through Z theorem 4.1 implies that $\mathcal{D}_0(Z)$ is unstable and theorem 4.2 implies that I fails the SLP at the range 2 and degree $\frac{a+b-1}{2}$. \square

4.1. SLP at the range 2 and Terao's conjecture. One of the main conjecture about hyperplane arrangements (still open also for line arrangements) is the Terao's conjecture. It concerns the arrangements that are called free. The conjecture says that freeness depends only on the combinatorics of the arrangement (the combinatorics of the arrangement $\mathcal{A} = \{l_1, \dots, l_n\}$ is encoded by a incidence graph with vertices given by the lines l_i and the points $l_i \cap l_j$ and edges joining point and line when the point belongs to the line). We refer to [12] for a good introduction to the subject.

Conjecture (Terao). The freeness depends only on combinatorics.

In other words if an arrangement has the same combinatorics than a free arrangement it is also free.

Let us consider an arrangement of lines and let us denote by Z_0 its dual set of points. We assume that Z_0 is free with exponents (a, b) with $a \leq b$. We assume that there exists a non free arrangement Z with the same combinatorics than Z_0 , i.e. that the Terao's conjecture is not true. Let us add $b - a$ points in general position to Z_0 in order to form Γ_0 and to Z to form Γ . Then the length of both sets of points is $2b + 1$. On the general line l we have

$$\mathcal{D}_0(Z_0) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b),$$

when, since Z is not free we have a less balanced decomposition for $\mathcal{D}_0(Z)$ (this affirmation is not so easy; it is proved in [2]), more precisely

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(s - a) \oplus \mathcal{O}_l(-s - b), \quad s \geq 1.$$

It implies that

$$H^0(I_Z \otimes \mathfrak{m}_{l^\vee}^{a-1}(a)) \neq 0, H^0(I_{Z_0} \otimes \mathfrak{m}_{l^\vee}^{a-1}(a)) = 0 \text{ and } H^0(I_{Z_0} \otimes \mathfrak{m}_{l^\vee}^a(a+1)) \neq 0.$$

Then adding $b - a$ lines passing through l^\vee and the $b - a$ added points we obtain $H^0(I_\Gamma \otimes \mathfrak{m}_{l^\vee}^{b-1}(b)) \neq 0$, $H^0(I_{Z_0} \otimes \mathfrak{m}_{l^\vee}^{b-1}(b)) = 0$ and $H^0(I_{Z_0} \otimes \mathfrak{m}_{l^\vee}^b(b+1)) \neq 0$, in other words $\mathcal{D}_0(\Gamma_0)$ is balanced with splitting $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(-b)$ when

$$\mathcal{D}_0(\Gamma) \otimes \mathcal{O}_l = \mathcal{O}_l(1 - b) \oplus \mathcal{O}_l(-1 - b).$$

Then $\mathcal{D}_0(\Gamma)$ is unstable. In other words if $Z_0 = \{d_1^\vee, \dots, d_{a+b+1}^\vee\}$, $\Gamma_0 = \{d_1^\vee, \dots, d_{a+b+1}^\vee, p_1, \dots, p_{b-a}\}$, $Z = \{l_1^\vee, \dots, l_{a+b+1}^\vee\}$ and $\Gamma = \{l_1^\vee, \dots, l_{a+b+1}^\vee, p_1, \dots, p_{b-a}\}$ then the ideal

$$(l_1^b, \dots, l_{a+b+1}^b, (p_1^\vee)^b, \dots, (p_{b-a}^\vee)^b)$$

fails the SLP at the range 2 and degree $b - 2$ when

$$(d_1^b, \dots, d_{a+b+1}^b, (p_1^\vee)^b, \dots, (p_{b-a}^\vee)^b)$$

has SLP at the range 2 and degree $b - 2$.

The following conjecture written in terms of SLP implies the Terao's conjecture on \mathbb{P}^2 .

Conjecture. Let $Z_0 = \{d_1^\vee, \dots, d_{2b+1}^\vee\}$ a set of points of length $2b+1$ in $\mathbb{P}^{2\vee}$ such that the ideal $I = (d_1^b, \dots, d_{2b+1}^b)$ has the SLP at the range 2 and degree $b - 2$. Assume that $Z = \{l_1^\vee, \dots, l_{2b+1}^\vee\}$ has the same combinatorics than Z_0 . Then $J = (l_1^b, \dots, l_{2b+1}^b)$ has the SLP at the range 2 and degree $b - 2$.

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