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SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN METRIC LIE GROUPS

PIERRE BAYARD, JULIEN ROTH AND BERENICE ZAVALA JIMÉNEZ

ABSTRACT. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Lie groups equipped with left invariant metrics.

Keywords: Spin geometry, metric Lie groups, isometric immersions, Weierstrass representation.

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1. INTRODUCTION

2. PRELIMINARIES

sec preliminaries

2.1. Notation. Let G be a Lie group, endowed with a left invariant metric $\langle \cdot, \cdot \rangle$, and \mathcal{G} its Lie algebra: \mathcal{G} is the space of the left invariant vector fields on G , equipped with the Lie bracket $[\cdot, \cdot]$ and is identified to the linear space tangent to G at the identity. The left multiplication induces a bundle isomorphism

$$(1) \quad TG \simeq G \times \mathcal{G}$$

TG trivial

which preserves the fibre metrics. We note that a vector field $X \in \Gamma(TG)$ is left invariant if, by (1), $X : G \rightarrow \mathcal{G}$ is a constant map. We consider ∇^G the Levi-Civita connection of $(G, \langle \cdot, \cdot \rangle)$ and the linear map

$$\begin{aligned} \Gamma : \quad \mathcal{G} &\rightarrow \Lambda^2 \mathcal{G} \\ X &\mapsto \Gamma(X) \end{aligned}$$

such that, for all $X, Y \in \mathcal{G}$

$$(2) \quad \nabla_X^G Y = \Gamma(X)(Y).$$

def Gamma

By the Koszul formula, Γ is determined by the metric as follows: for all $X, Y, Z \in \mathcal{G}$,

$$(3) \quad \langle \Gamma(X)(Y), Z \rangle = \frac{1}{2} \langle [X, Y], Z \rangle + \frac{1}{2} \langle [Z, X], Y \rangle - \frac{1}{2} \langle [Y, Z], X \rangle.$$

koszul formula

Since ∇^G is without torsion, we have, for all $X, Y \in \mathcal{G}$,

$$(4) \quad \Gamma(X)(Y) - \Gamma(Y)(X) = [X, Y].$$

nablaG without torsion

We note that the curvature of ∇^G is given by

$$(5) \quad R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X, Y]) \in \Lambda^2 \mathcal{G}$$

curvature nablaG

for all $X, Y \in \mathcal{G}$. In the formula the first brackets stand for the commutator of the endomorphisms.

2.2. The spinor bundle of G . Let us denote by $Cl(\mathcal{G})$ the Clifford algebra of \mathcal{G} with its scalar product, and let us consider the representation

$$\begin{aligned} \rho : \quad Spin(\mathcal{G}) &\rightarrow GL(Cl(\mathcal{G})) \\ a &\mapsto \xi \mapsto a\xi. \end{aligned}$$

This representation is not irreducible in general: it is a sum of irreducible representations [11]. By (1) the principal bundle Q_G of the positively oriented and orthonormal frames of G is also trivial

$$Q_G \simeq G \times SO(\mathcal{G}),$$

and we may consider the trivial spin structure

$$\tilde{Q}_G := G \times Spin(\mathcal{G})$$

and the corresponding spinor bundle

$$\Sigma := \tilde{Q}_G \times_{\rho} Cl(\mathcal{G}) \simeq G \times Cl(\mathcal{G}).$$

A spinor field $\varphi \in \Gamma(\Sigma)$ is said to be left invariant if it is constant as a map $G \rightarrow Cl(\mathcal{G})$. The covariant derivative of a left invariant spinor field is

nabla spinor invariant

$$(6) \quad \nabla_X^{\mathcal{G}} \varphi = \frac{1}{2} \Gamma(X) \cdot \varphi$$

where $\Gamma(X) \in \Lambda^2 \mathcal{G} \subset Cl(\mathcal{G})$ and the dot “ \cdot ” stands for the Clifford product.

2.3. The spin representation of $Spin(p) \times Spin(q)$. Let us assume that $p+q = n$, and fix an orthonormal basis $e_1^o, e_2^o, \dots, e_n^o$ of \mathcal{G} ; this gives a splitting $\mathcal{G} = \mathbb{R}^p \oplus \mathbb{R}^q$ (the first factor corresponds to the first p vectors, and the second factor to the last q vectors of the basis) and a natural map

$$Spin(p) \times Spin(q) \rightarrow Spin(\mathcal{G})$$

associated to the isomorphism

$$Cl(\mathcal{G}) = Cl_p \hat{\otimes} Cl_q.$$

We thus also have a representation, still denoted by ρ ,

rep spin p spin q

$$(7) \quad \begin{aligned} \rho : \quad Spin(p) \times Spin(q) &\rightarrow GL(Cl(\mathcal{G})) \\ a &\mapsto \xi \mapsto a\xi. \end{aligned}$$

on twisted spinor bundle

2.4. The twisted spinor bundle. We consider M a p -dimensional Riemannian manifold, $E \rightarrow M$ a bundle of rank q , with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures

$$\tilde{Q}_M \xrightarrow{2:1} Q_M \quad \text{and} \quad \tilde{Q}_E \xrightarrow{2:1} Q_E$$

where Q_M and Q_E are the bundles of positively oriented orthonormal frames of TM and E , and we set

$$\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E;$$

this is a $Spin(p) \times Spin(q)$ principal bundle. We define

$$\Sigma := \tilde{Q} \times_{\rho} Cl(\mathcal{G})$$

and

$$U\Sigma := \tilde{Q} \times_{\rho} Spin(\mathcal{G}) \quad \subset \Sigma$$

where ρ is the representation (7). The vector bundle Σ is equipped with the covariant derivative ∇ naturally associated to the spinorial connections on \tilde{Q}_M and \tilde{Q}_E . Let us denote by $\tau : Cl(\mathcal{G}) \rightarrow Cl(\mathcal{G})$ the anti-automorphism of $Cl(\mathcal{G})$ such that

$$\tau(x_1 \cdot x_2 \cdots x_k) = x_k \cdots x_2 \cdot x_1$$

for all $x_1, x_2, \dots, x_k \in \mathcal{G}$, and set

$$\begin{aligned} \text{def brackets 1} \quad (8) \quad \langle\langle \cdot, \cdot \rangle\rangle : \quad Cl(\mathcal{G}) \times Cl(\mathcal{G}) &\rightarrow Cl(\mathcal{G}) \\ (\xi, \xi') &\mapsto \tau(\xi')\xi. \end{aligned}$$

This map is $Spin(\mathcal{G})$ -invariant: for all $\xi, \xi' \in Cl(\mathcal{G})$ and $g \in Spin(\mathcal{G})$ we have

$$\langle\langle g\xi, g\xi' \rangle\rangle = \tau(g\xi')g\xi = \tau(\xi')\tau(g)g\xi = \tau(\xi')\xi = \langle\langle \xi, \xi' \rangle\rangle,$$

since $Spin(\mathcal{G}) \subset \{g \in Cl^0(\mathcal{G}) : \tau(g)g = 1\}$; this map thus induces a $Cl(\mathcal{G})$ -valued map

$$\begin{aligned} \text{def brackets 2} \quad (9) \quad \langle\langle \cdot, \cdot \rangle\rangle : \quad \Sigma \times \Sigma &\rightarrow Cl(\mathcal{G}) \\ (\varphi, \varphi') &\mapsto \langle\langle [\varphi], [\varphi'] \rangle\rangle \end{aligned}$$

where $[\varphi]$ and $[\varphi'] \in Cl(\mathcal{G})$ represent φ and φ' in some spinorial frame $\tilde{s} \in \tilde{Q}$.

Lemma 2.1. *The map $\langle\langle \cdot, \cdot \rangle\rangle : \Sigma \times \Sigma \rightarrow Cl(\mathcal{G})$ satisfies the following properties: for all $\varphi, \psi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$,*

$$\text{scalar product property1} \quad (10) \quad \langle\langle \varphi, \psi \rangle\rangle = \tau\langle\langle \psi, \varphi \rangle\rangle$$

and

$$\text{scalar product property2} \quad (11) \quad \langle\langle X \cdot \varphi, \psi \rangle\rangle = \langle\langle \varphi, X \cdot \psi \rangle\rangle.$$

Proof. We have

$$\langle\langle \varphi, \psi \rangle\rangle = \tau[\psi] [\varphi] = \tau(\tau[\varphi] [\psi]) = \tau\langle\langle \psi, \varphi \rangle\rangle$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \tau[\psi] [X][\varphi] = \tau([X][\psi])[\varphi] = \langle\langle \varphi, X \cdot \psi \rangle\rangle$$

where $[\varphi]$, $[\psi]$ and $[X] \in Cl(\mathcal{G})$ represent φ , ψ and X in some given frame $\tilde{s} \in \tilde{Q}$. \square

Lemma 2.2. *The connection ∇ is compatible with the product $\langle\langle \cdot, \cdot \rangle\rangle$:*

$$\partial_X \langle\langle \varphi, \varphi' \rangle\rangle = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle$$

for all $\varphi, \varphi' \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$.

Proof. If $\varphi = [\tilde{s}, [\varphi]]$ is a section of $\Sigma = \tilde{Q} \times_{\rho} Cl(\mathcal{G})$, we have

$$\nabla_X \varphi = [\tilde{s}, \partial_X [\varphi] + \rho_*(\tilde{s}^* \alpha(X))([\varphi])], \quad \forall X \in TM,$$

where ρ is the representation (7) and α is the connection form on \tilde{Q} ; the term $\rho_*(\tilde{s}^* \alpha(X))$ is an endomorphism of $Cl(\mathcal{G})$ given by the multiplication on the left by an element belonging to $\Lambda^2 \mathcal{G} \subset Cl(\mathcal{G})$, still denoted by $\rho_*(\tilde{s}^* \alpha(X))$. Such an element satisfies

$$\tau(\rho_*(\tilde{s}^* \alpha(X))) = -\rho_*(\tilde{s}^* \alpha(X)),$$

and we have

$$\begin{aligned} \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle &= \tau\{[\varphi']\} (\partial_X [\varphi] + \rho_*(\tilde{s}^* \alpha(X))[\varphi]) \\ &\quad + \tau\{\partial_X [\varphi'] + \rho_*(\tilde{s}^* \alpha(X))[\varphi']\} [\varphi] \\ &= \tau\{[\varphi']\} \partial_X [\varphi] + \tau\{\partial_X [\varphi']\} [\varphi] \\ &= \partial_X \langle\langle \varphi, \varphi' \rangle\rangle. \end{aligned}$$

□

We finally note that there is a natural action of $Spin(\mathcal{G})$ on $U\Sigma$, by right multiplication: for $\varphi = [\tilde{s}, [\varphi]] \in U\Sigma = \tilde{Q} \times_{\rho} Spin(\mathcal{G})$ and $a \in Spin(\mathcal{G})$ we set

$$\boxed{\text{def right action}} \quad (12) \quad \varphi \cdot a := [\tilde{s}, [\varphi] \cdot a] \in U\Sigma.$$

2.5. The spin geometry of a submanifold of G . We keep the notation of the previous section, assuming moreover here that M is a submanifold of a Lie group G and that $E \rightarrow M$ is its normal bundle. If we consider spin structures on TM and on E whose sum is the trivial spin structure of $TM \oplus E$ [13], we have

$$\Sigma = \tilde{Q} \times_{\rho} Cl(\mathcal{G}) \simeq M \times Cl(\mathcal{G}),$$

where the last bundle is the spinor bundle of G restricted to M . Two connections are thus defined on Σ , the connection ∇ and the connection ∇^G ; they satisfy the following Gauss formula:

$$\boxed{\text{gauss formula}} \quad (13) \quad \nabla_X^G \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi$$

for all $\varphi \in \Gamma(\Sigma)$ and all $X \in \Gamma(TM)$, where $B : TM \times TM \rightarrow E$ is the second fundamental form of M into G . We refer to [1] for the proof (in a slightly different context). Since the covariant derivative of a left invariant spinor field is given by (6), the restriction to M of such a spinor field satisfies

$$\boxed{\text{gauss formula Gamma}} \quad (14) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $X \in TM$.

section main result

3. MAIN RESULT

We consider M a p -dimensional Riemannian manifold, $E \rightarrow M$ a bundle of rank q , with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures. We suppose that a bilinear and symmetric map $B : TM \times TM \rightarrow E$ is given, and we moreover do the following two assumptions:

- (1) There exists a bundle isomorphism

$$\boxed{\text{def bundle iso f}} \quad (15) \quad f : TM \oplus E \rightarrow M \times \mathcal{G}$$

which preserves the metrics; this mapping permits to define a bundle map

$$(16) \quad \Gamma : TM \oplus E \rightarrow \Lambda^2(TM \oplus E)$$

such that, for all $X, Y \in \Gamma(TM \oplus E)$,

$$\boxed{\text{def Gamma TM+E}} \quad (17) \quad f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$$

(on the right hand side Γ is the map defined on \mathcal{G} by (2)), together with the following notion: a section $Z \in \Gamma(TM \oplus E)$ will be said to be left invariant if $f(Z) : M \rightarrow \mathcal{G}$ is a constant map.

- (2) The covariant derivative of a left invariant section $Z \in \Gamma(TM \oplus E)$ is given by

$$\boxed{\text{nabla field invariant}} \quad (18) \quad \nabla_X Z = \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N)$$

where $Z = Z^T + Z^N$ in $TM \oplus E$ and $B^* : TM \times E \rightarrow TM$ is the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X, Y), N \rangle = \langle Y, B^*(X, N) \rangle.$$

Remark 1. *These two assumptions are equivalent to the assumptions made in [10, 15]: they are necessary to write down the equations of Gauss, Codazzi and Ricci in a general metric Lie group, and to obtain a fundamental theorem for immersions in that context; see Section 4.*

$\boxed{\text{rmk cond frame}}$

Remark 2. *Sometimes it is convenient to write these assumptions in some local frames. For sake of simplicity, we assume that E is a trivial line bundle, oriented by a unit section ν . Let $(e_1^o, e_2^o, \dots, e_n^o)$ be an orthonormal basis of \mathcal{G} and $\Gamma_{ij}^k \in \mathbb{R}$, $1 \leq i, j, k \leq n$, be such that*

$$\Gamma(e_i^o)(e_j^o) = \sum_{k=1}^n \Gamma_{ij}^k e_k^o.$$

We set, for $i = 1, \dots, n$, $\underline{e}_i \in \Gamma(TM \oplus E)$ such that $f(\underline{e}_i) = e_i^o$, and $f_i \in C^\infty(M)$, $T_i \in \Gamma(TM)$ such that $\underline{e}_i = T_i + f_i \nu$. Since f preserves the metrics, the vectors $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$ are orthonormal, and we have

$$\boxed{\text{cond e}_i \text{ orth}} \quad (19) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_{ij}$$

for all $i, j = 1, \dots, n$. The assumption (18) then reads as follows: for all $X \in TM$, $j = 1, \dots, n$,

$$\boxed{\text{eqn 1 trad}} \quad (20) \quad \nabla_X T_j = \sum_{i,k} \Gamma_{ij}^k \langle X, T_i \rangle T_k + f_j S(X),$$

$$\boxed{\text{eqn 2 trad}} \quad (21) \quad df_j(X) = \sum_{i,k} \Gamma_{ij}^k f_k \langle X, T_i \rangle - h(X, T_j)$$

where $S(X) = B^*(X, \nu)$ and $h(X, Y) = \langle B(X, Y), \nu \rangle$. Conversely, if vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$, $1 \leq i \leq n$, are given such that (19), (20) and (21) hold, we may define a bundle isomorphism $f : TM \oplus E \rightarrow M \times \mathcal{G}$ preserving the metrics and such that (18) holds: setting $\underline{e}_i = T_i + f_i \nu$, we define f such that $f(\underline{e}_i) = e_i^o$, $i = 1, \dots, n$.

We keep the notation of Section 2 and state the main result of the paper:

$\boxed{\text{thm main result}}$

Theorem 1. *We moreover assume that M is simply connected. The following statements are equivalent:*

- (1) *There exists a section $\varphi \in \Gamma(U\Sigma)$ such that*

$\boxed{\text{killing equation}}$

$$(22) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $X \in TM$.

- (2) *There exists an isometric immersion $F : M \rightarrow G$ with normal bundle E and second fundamental form B .*

More precisely, if φ is a solution of (22), replacing φ by $\varphi \cdot a$ for some $a \in \text{Spin}(\mathcal{G})$ if necessary, the formula $F = \int \xi$ where ξ is the \mathcal{G} -valued 1-form defined by

$$\boxed{\text{def xi}} \quad (23) \quad \xi(X) := \langle X \cdot \varphi, \varphi \rangle$$

for all $X \in TM$, defines an isometric immersion with normal bundle E and second fundamental form B . Here \int stands for the Darboux integral, i.e. $F = \int \xi : M \rightarrow G$ is such that $F^*\omega_G = \xi$, where $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G . Reciprocally, an isometric immersion $M \rightarrow G$ with normal bundle E and second fundamental form B may be written in that form.

The formula $F = \int \xi$ where ξ is defined by (23) may be regarded as a generalized Weierstrass representation formula.

This theorem generalizes the main result of [4] to a Lie group equipped with a left invariant metric.

Remark 3. If φ is a solution of (22) and a belongs to $\text{Spin}(\mathcal{G})$, $\varphi' := \varphi \cdot a$ is also a solution of (22) (see (12) for the definition of $\varphi \cdot a$). Moreover the associated 1-forms ξ_φ and $\xi_{\varphi'}$ are linked by

$$\boxed{\text{xi phi g}} \quad (24) \quad \xi_{\varphi'} = \tau(a) \xi_\varphi \quad a = \text{Ad}(a^{-1}) \circ \xi_\varphi.$$

Let us recall that a 1-form $\xi \in \Omega^1(M, \mathcal{G})$ is Darboux integrable if and only if it satisfies the structure equation $d\xi + [\xi, \xi] = 0$ (M is simply connected). The theorem thus says that if φ is a solution of (22), it is possible to find an other solution φ' of this equation such that $\xi_{\varphi'}$ is Darboux integrable and $F = \int \xi_{\varphi'}$ is an immersion with normal bundle E and second fundamental form B . The proof of (1) \Rightarrow (2) in the theorem will in fact follow these lines. See also Remark 5 below.

Remark 4. We note that (22) implies the Dirac equation

$$D\varphi = (\vec{H} + \gamma) \cdot \varphi$$

where the Dirac operator D is defined by

$$D\varphi = \sum_{j=1}^p e_j \cdot \nabla_{e_j} \varphi$$

and

$$\vec{H} = \frac{1}{2} \sum_{j=1}^p B(e_j, e_j) \in E \quad \text{and} \quad \gamma = \frac{1}{2} \sum_{j=1}^p e_j \cdot \Gamma(e_j) \in \text{Cl}(TM \oplus E).$$

We now prove the theorem: (1) \Rightarrow (2) will be a consequence of Propositions 3.1 and 3.3 below, and (2) \Rightarrow (1) will be proved at the end of the section.

$\boxed{\text{lem xi closed}}$

Proposition 3.1. Assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) and define ξ by (23). Then

- (1) ξ takes its values in $\mathcal{G} \subset \text{Cl}(\mathcal{G})$;
- (2) there exists $T \in \text{SO}(\mathcal{G})$ such that $\xi = T \circ f$;
- (3) replacing φ by $\varphi \cdot a$ where $a \in \text{Spin}(\mathcal{G})$ is such that $\text{Ad}(a) = T$, we have $\xi = f$, and ξ satisfies the structure equation

$\boxed{\text{xi structure equation}}$

$$(25) \quad d\xi + [\xi, \xi] = 0.$$

Proof. 1- By the very definition of ξ , we have

$$\xi(X) = \tau[\varphi][X][\varphi]$$

for all $X \in TM$, where $[X]$ and $[\varphi]$ represent X and φ in a given frame \tilde{s} of \tilde{Q} . Since $[X]$ belongs to $\mathcal{G} \subset Cl(\mathcal{G})$ and $[\varphi]$ is an element of $Spin(\mathcal{G})$, $\xi(X)$ belongs to \mathcal{G} .

2- This amounts to show that for every left invariant section $Z \in \Gamma(TM \oplus E)$, the map $\xi(Z) : M \rightarrow \mathcal{G}$ is constant. Indeed, assuming that this property holds, if (e_1^o, \dots, e_n^o) is a fixed orthonormal basis of \mathcal{G} and denoting by $\underline{e}_1, \dots, \underline{e}_n$ the left invariant sections of $TM \oplus E$ such that $f(\underline{e}_i) = e_i^o$, $i = 1, \dots, n$, we have, for all section $Z = \sum_i Z_i \underline{e}_i \in \Gamma(TM \oplus E)$,

$$\xi(Z) = \sum_{i=1}^n Z_i \xi(\underline{e}_i)$$

where $(\xi(\underline{e}_1), \dots, \xi(\underline{e}_n))$ is a constant orthonormal basis of \mathcal{G} . Considering the orthogonal transformation $T : \mathcal{G} \rightarrow \mathcal{G}$ such that $T(e_i^o) = \xi(\underline{e}_i)$, $i = 1, \dots, n$, we get

$$\xi(Z) = \sum_{i=1}^n Z_i T(e_i^o) = T\left(\sum_{i=1}^n Z_i e_i^o\right) = T(f(Z)),$$

i.e. $\xi = T \circ f$. We thus assume that $Z \in \Gamma(TM \oplus E)$ is left invariant, and compute, for $X \in TM$,

$$\begin{aligned} \partial_X \xi(Z) &= \langle \langle \nabla_X Z \cdot \varphi, \varphi \rangle \rangle + (id + \tau) \langle \langle Z \cdot \varphi, \nabla_X \varphi \rangle \rangle \\ &= \langle \langle \{ \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N) \} \cdot \varphi, \varphi \rangle \rangle \\ &\quad + \frac{1}{2} (id + \tau) \langle \langle Z \cdot \varphi, (-\sum_{j=1}^p e_j \cdot B(X, e_j) + \Gamma(X)) \cdot \varphi \rangle \rangle. \end{aligned}$$

This expression is zero, as a consequence of the following formulas:

lemma proof xi=f

Lemma 3.2. For all $X \in TM$ and $Z \in TM \oplus E$,

proof xi=f id 1

$$(26) \quad \langle \langle \Gamma(X)(Z) \cdot \varphi, \varphi \rangle \rangle = -\frac{1}{2} (id + \tau) \langle \langle Z \cdot \varphi, \Gamma(X) \cdot \varphi \rangle \rangle$$

and

proof xi=f id 2

$$(27) \quad \langle \langle \{ B(X, Z^T) - B^*(X, Z^N) \} \cdot \varphi, \varphi \rangle \rangle = -\frac{1}{2} (id + \tau) \langle \langle Z \cdot \varphi, \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi \rangle \rangle.$$

Proof. We first prove (26): we have

$$\begin{aligned} \frac{1}{2} (id + \tau) \langle \langle Z \cdot \varphi, \Gamma(X) \cdot \varphi \rangle \rangle &= -\langle \langle \varphi, [\Gamma(X), Z] \cdot \varphi \rangle \rangle \\ &= -\langle \langle \Gamma(X)(Z) \cdot \varphi, \varphi \rangle \rangle, \end{aligned}$$

since $\tau_{\mathcal{G}} = id$, $\tau_{\Lambda^2 \mathcal{G}} = -id$ and by Lemma A.1. The proof of (27) is similar:

$$\begin{aligned} \frac{1}{2} (id + \tau) \langle \langle Z \cdot \varphi, \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi \rangle \rangle &= -\langle \langle \varphi, \left[\sum_{j=1}^p e_j \cdot B(X, e_j), Z \right] \cdot \varphi \rangle \rangle \\ &= -\langle \langle \{ B(X, Z^T) - B^*(X, Z^N) \} \cdot \varphi, \varphi \rangle \rangle \end{aligned}$$

by Lemma A.3. \square

3- For all $a \in Spin(\mathcal{G})$ and $X \in TM$, we have

$$\begin{aligned} \langle \langle X \cdot (\varphi \cdot a), \varphi \cdot a \rangle \rangle &= \tau([\varphi]a)[X][\varphi]a \\ &= \tau(a) \langle \langle X \cdot \varphi, \varphi \rangle \rangle a \\ &= Ad(a^{-1})(\xi(X)) \\ &= Ad(a^{-1})(T \circ f(X)); \end{aligned}$$

thus, replacing φ by $\varphi \cdot a$ where $a \in Spin(\mathcal{G})$ is such that $Ad(a) = T$ we get $\xi = f$. We compute, for $X, Y \in \Gamma(TM)$ such that $\nabla X = \nabla Y = 0$ at x_0 ,

$$\begin{aligned} \partial_X \xi(Y) &= \langle \langle Y \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle Y \cdot \varphi, \nabla_X \varphi \rangle \rangle \\ &= (id + \tau) \langle \langle Y \cdot \varphi, \nabla_X \varphi \rangle \rangle \\ &= (id + \tau) \langle \langle \varphi, -\frac{1}{2} \sum_{j=1}^p Y \cdot e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} Y \cdot \Gamma(X) \cdot \varphi \rangle \rangle \end{aligned}$$

and

$$\begin{aligned} d\xi(X, Y) &= \partial_X \xi(Y) - \partial_Y \xi(X) \\ (28) \quad &= (id + \tau) \langle \langle \varphi, \mathcal{C} \cdot \varphi \rangle \rangle + \frac{1}{2} (id + \tau) \langle \langle \varphi, \{Y \cdot \Gamma(X) - X \cdot \Gamma(Y)\} \cdot \varphi \rangle \rangle \end{aligned}$$

dxi interm

with

$$\mathcal{C} := -\frac{1}{2} \sum_{j=1}^p \{Y \cdot e_j \cdot B(X, e_j) - X \cdot e_j \cdot B(Y, e_j)\}.$$

Now, for $X = \sum_{1 \leq k \leq p} x_k e_k$ and $Y = \sum_{1 \leq k \leq p} y_k e_k$,

$$\sum_{j=1}^p X \cdot e_j \cdot B(Y, e_j) = -B(Y, X) + \sum_{j=1}^p \sum_{k \neq j} x_k e_k \cdot e_j \cdot B(Y, e_j)$$

and

$$\sum_{j=1}^p Y \cdot e_j \cdot B(X, e_j) = -B(X, Y) + \sum_{j=1}^p \sum_{k \neq j} y_k e_k \cdot e_j \cdot B(X, e_j),$$

which yields the formula

$$\mathcal{C} = -\frac{1}{2} \sum_{j=1}^p \sum_{k \neq j} e_k \cdot e_j \cdot (y_k B(X, e_j) - x_k B(Y, e_j)).$$

Since a Clifford product of three pairwise orthogonal vectors is changed to its opposite by τ , we deduce that $\tau[\mathcal{C}] = -[\mathcal{C}]$; this implies

$$\tau \langle \langle \varphi, \mathcal{C} \cdot \varphi \rangle \rangle = \tau(\tau[\varphi]\tau[\mathcal{C}][\varphi]) = -\tau[\varphi]\tau[\mathcal{C}][\varphi] = -\langle \langle \varphi, \mathcal{C} \cdot \varphi \rangle \rangle.$$

Thus the first term in (28) is zero and

$$\begin{aligned} d\xi(X, Y) &= \frac{1}{2} (id + \tau) \langle \langle \varphi, \{Y \cdot \Gamma(X) - X \cdot \Gamma(Y)\} \cdot \varphi \rangle \rangle \\ &= \frac{1}{2} \langle \langle \varphi, \{Y \cdot \Gamma(X) - \Gamma(X) \cdot Y + X \cdot \Gamma(Y) - \Gamma(Y) \cdot X\} \cdot \varphi \rangle \rangle \end{aligned}$$

since $\tau_{\mathcal{G}} = id$ and $\tau_{\Lambda^2 \mathcal{G}} = -id$. We finally notice that

$$\frac{1}{2} \{Y \cdot \Gamma(X) - \Gamma(X) \cdot Y + X \cdot \Gamma(Y) - \Gamma(Y) \cdot X\} = -\Gamma(X)(Y) + \Gamma(Y)(X)$$

(Lemma A.1), which yields

$$\begin{aligned} d\xi(X, Y) &= -\xi(\Gamma(X)(Y) - \Gamma(Y)(X)) \\ &= -[\xi(X), \xi(Y)], \end{aligned}$$

since $\xi = f$, Γ satisfies (17), and by (4). \square

We keep the notation of Proposition 3.1, and moreover assume that M is simply connected; we consider

$$F : M \rightarrow G$$

such that $F^*\omega_G = \xi$ (assuming that φ is chosen in such a way that ξ satisfies the structure equation (25)). The next proposition follows from the properties of the Clifford product:

lem F isometry

Proposition 3.3. 1. The map $F : M \rightarrow G$ is an isometry.
2. The map

$$\begin{aligned} \Phi_E : E &\rightarrow M \times \mathcal{G} \\ X \in E_m &\mapsto (F(m), \xi(X)) \end{aligned}$$

is an isometry between E and the normal bundle of $F(M)$ into G , preserving connections and second fundamental forms. Here, for $X \in E$, $\xi(X)$ still stands for the quantity $\langle X \cdot \varphi, \varphi \rangle$.

Proof. For $X, Y \in \Gamma(TM \oplus E)$, we have

$$\begin{aligned} \langle \xi(X), \xi(Y) \rangle &= -\frac{1}{2} (\xi(X)\xi(Y) + \xi(Y)\xi(X)) \\ &= -\frac{1}{2} (\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi] + \tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]) \\ &= -\frac{1}{2} \tau[\varphi] ([X][Y] + [Y][X]) [\varphi] \\ &= \langle X, Y \rangle, \end{aligned}$$

since $[X][Y] + [Y][X] = -2\langle [X], [Y] \rangle = -2\langle X, Y \rangle$. This implies that F is an isometry, and that Φ_E is a bundle map between E and the normal bundle of $F(M)$ into G which preserves the metrics of the fibres. Let us denote by B_F and ∇'^F the second fundamental form and the normal connection of the immersion F ; the aim is now to prove that

$$(29) \quad \xi(B(X, Y)) = B_F(\xi(X), \xi(Y)) \quad \text{and} \quad \xi(\nabla'_X N) = \nabla'_{\xi(X)} \xi(N)$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$. First,

$$B_F(\xi(X), \xi(Y)) = (\nabla_{\xi(X)}^G \xi(Y))^N = \{\partial_X \xi(Y) + \Gamma(\xi(X))(\xi(Y))\}^N$$

where the superscript N means that we consider the component of the vector which is normal to the immersion. We fix a point $x_0 \in M$, assume that $\nabla Y = 0$ at x_0 , and compute:

$$\begin{aligned} \partial_X \xi(Y) &= \langle Y \cdot \nabla_X \varphi, \varphi \rangle + \langle Y \cdot \varphi, \nabla_X \varphi \rangle \\ &= (id + \tau) \langle \varphi, Y \cdot \nabla_X \varphi \rangle \\ &= \frac{1}{2} (id + \tau) \langle \varphi, Y \cdot \left\{ -\sum_{j=1}^p e_j \cdot B(X, e_j) + \Gamma(X) \right\} \cdot \varphi \rangle. \end{aligned}$$

preserves ff connection

We showed in the proof of Proposition 3.1 that

$$Y \cdot \sum_{j=1}^p e_j \cdot B(X, e_j) = -B(X, Y) + \mathcal{D}$$

where \mathcal{D} is a term which satisfies $\tau\mathcal{D} = -\mathcal{D}$. Since moreover $\tau[B(X, Y)] = [B(X, Y)]$ and $\tau[Y \cdot \Gamma(X)] = -[\Gamma(X) \cdot Y]$, we get

$$\begin{aligned} \partial_X \xi(Y) &= \xi(B(X, Y))^N + \langle \langle \varphi, \frac{1}{2}(Y \cdot \Gamma(X) - \Gamma(X) \cdot Y) \cdot \varphi \rangle \rangle^N \\ &= \xi(B(X, Y)) - \langle \langle \varphi, \Gamma(X)(Y) \cdot \varphi \rangle \rangle^N, \end{aligned}$$

since $\xi(B(X, Y))$ is normal to the immersion and by Lemma A.1. This implies that

$$\begin{aligned} B_F(\xi(X), \xi(Y)) &= \xi(B(X, Y)) - \langle \langle \varphi, \Gamma(X)(Y) \cdot \varphi \rangle \rangle^N + \Gamma(\xi(X))(\xi(Y))^N \\ &= \xi(B(X, Y)) \end{aligned}$$

since

$$\begin{aligned} \langle \langle \varphi, \Gamma(X)(Y) \cdot \varphi \rangle \rangle &= \langle \langle \Gamma(X)(Y) \cdot \varphi, \varphi \rangle \rangle \quad (\text{since } \tau[\Gamma(X)(Y)] = [\Gamma(X)(Y)] \in \mathcal{G}) \\ &= \xi(\Gamma(X)(Y)) \\ &= f(\Gamma(X)(Y)) \\ &= \Gamma(f(X))(f(Y)) \quad (\text{by definition of } \Gamma \text{ on } TM \oplus E) \\ &= \Gamma(\xi(X))(\xi(Y)). \end{aligned}$$

We finally show the second identity in (29): we have

$$\begin{aligned} \nabla'_{\xi(X)} \xi(N) &= (\nabla'_{\xi(X)} \xi(N))^N \\ &= (\partial_X \xi(N) + \Gamma(\xi(X))(\xi(N)))^N \\ &= \langle \langle \nabla'_X N \cdot \varphi, \varphi \rangle \rangle^N + \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle^N + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle^N \\ &\quad + \Gamma(\xi(X))(\xi(N))^N. \end{aligned}$$

The first term in the right hand side is $\xi(\nabla'_X N)$, and we only need to show that

$$(30) \quad \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle^N + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle^N + \Gamma(\xi(X))(\xi(N))^N = 0.$$

We have

$$\begin{aligned} \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle + \langle \langle N \cdot \varphi, \nabla_X \varphi \rangle \rangle &= (id + \tau) \langle \langle N \cdot \nabla_X \varphi, \varphi \rangle \rangle \\ &= \frac{1}{2} (id + \tau) \langle \langle \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) + N \cdot \Gamma(X) \rangle \cdot \varphi, \varphi \rangle \\ &= \frac{1}{2} (id + \tau) \langle \langle \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) \cdot \varphi, \varphi \rangle \rangle \\ &\quad - \langle \langle \Gamma(X)(N) \cdot \varphi, \varphi \rangle \rangle \end{aligned}$$

since $\tau[N \cdot \Gamma(X)] = -[\Gamma(X) \cdot N]$ and by Lemma A.1. Taking into account that

$$\langle \langle \Gamma(X)(N) \cdot \varphi, \varphi \rangle \rangle = \Gamma(\xi(X))(\xi(N))$$

(see the first part of the proof above), the identity (30) will be proved if we show that the vector

$$\frac{1}{2} (id + \tau) \langle \langle \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) \cdot \varphi, \varphi \rangle \rangle$$

is tangent to the immersion. We have

$$\begin{aligned}
 \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) &= -\sum_{j=1}^p e_j \cdot B(X, e_j) \cdot N - 2\sum_{j=1}^p e_j \langle B(X, e_j), N \rangle \\
 &= -\sum_{j=1}^p B(X, e_j) \cdot N \cdot e_j - 2B^*(X, N) \\
 &= -\tau \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) - 2B^*(X, N);
 \end{aligned}$$

thus

$$\frac{1}{2}(id + \tau) \langle \langle \sum_{j=1}^p e_j \cdot N \cdot B(X, e_j) \cdot \varphi, \varphi \rangle \rangle = -\langle \langle B^*(X, N) \cdot \varphi, \varphi \rangle \rangle,$$

which is a vector tangent to the immersion since $B^*(X, N)$ belongs to TM ; (30) follows, which finishes the proof. \square

We finally show the converse statement (2) \Rightarrow (1) : we suppose that $F : M \rightarrow G$ is an isometric immersion with normal bundle E and second fundamental form B , we consider the orthonormal frame $s_o = 1_{SO(\mathcal{G})}$ of \mathcal{G} , and the spinor frame $\tilde{s}_o = 1_{Spin(\mathcal{G})}$ (recall that $Q_G = G \times SO(\mathcal{G})$ and $\tilde{Q}_G = G \times Spin(\mathcal{G})$; see Section 2). The spinor field $\varphi = [\tilde{s}_o, 1_{Cl(\mathcal{G})}]$ satisfies (22) as a consequence of the Gauss formulas (13)-(14); moreover, its associated 1-form is, for all $X \in TM$,

$$\xi(X) = \langle \langle F_*X \cdot \varphi, \varphi \rangle \rangle = \tau[\varphi] [F_*X] [\varphi] = [F_*X],$$

where $[F_*X] \in \mathcal{G}$ represents F_*X in s_o , that is $[F_*X] = \omega_G(F_*X)$ ($\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G). Thus $\xi = F^*\omega_G$, that is $F = \int \xi$.

rmk congruence

Remark 5. We proved that if $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) such that ξ_φ satisfies the structure equation (25) then $F = \int \xi_\varphi$ is an immersion with normal bundle E and second fundamental form B . By (24) it is clear that if $a \in Spin(\mathcal{G})$ is such that $Ad(a^{-1}) : \mathcal{G} \rightarrow \mathcal{G} \in SO(\mathcal{G})$ is an automorphism of Lie algebra, then $\xi_{\varphi \cdot a}$ satisfies the structure equation too; in fact, the corresponding immersions $F_\varphi = \int \xi_\varphi$ and $F_{\varphi \cdot a} = \int \xi_{\varphi \cdot a}$ are linked by the following formula: if $\Phi_a : G \rightarrow G$ is the automorphism of G such that $d(\Phi_a)_e = Ad(a^{-1})$, then Φ_a is also an isometry for the left invariant metric, and

expr F phi g

$$(31) \quad F_{\varphi \cdot a} = L_b \circ \Phi_a \circ F_\varphi$$

for some b belonging to G . This relies on the following formula: if $\Phi : G \rightarrow G$ is an automorphism, $\omega_G \in \Omega^1(G, \mathcal{G})$ is the Maurer-Cartan form of G and $F : M \rightarrow G$ is a smooth map, then

$$(\Phi \circ F)^*\omega_G = d(\Phi)_e \circ (F^*\omega_G).$$

This formula applied to $\Phi = \Phi_a$ and $F = F_\varphi$ shows that $\Phi_a \circ F_\varphi$ is a solution of the Darboux equation associated to the form $\xi_{\varphi \cdot a}$; thus, by uniqueness of a solution of the Darboux equation, (31) holds for some b belonging to G .

Remark 6. uniqueness of immersions + correspondence between immersions and solutions belonging to a sub-bundle.

4. AN APPLICATION: THE FUNDAMENTAL THEOREM FOR IMMERSIONS IN A METRIC LIE GROUP

tion fundamental theorem

We now show that the equations of Gauss, Ricci and Codazzi on B are exactly the integrability conditions of (22). We recall these equations for immersions in the metric Lie group G : if R^G denotes the curvature tensor of $(G, \langle \cdot, \cdot \rangle)$, and if R^T and R^N stand for the curvature tensors of the connections on TM and on E (M is a submanifold of G and E is its normal bundle), then we have, for all $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(E)$,

(1) the Gauss equation

Gauss equation G

$$(32) \quad (R^G(X, Y)Z)^T = R^T(X, Y)Z - B^*(X, B(Y, Z)) + B^*(Y, B(X, Z)),$$

(2) the Ricci equation

Ricci equation G

$$(33) \quad (R^G(X, Y)N)^N = R^N(X, Y)N - B(X, B^*(Y, N)) + B(Y, B^*(X, N)),$$

(3) the Codazzi equation

Codazzi equation G

$$(34) \quad (R^G(X, Y)Z)^N = \tilde{\nabla}_X B(Y, Z) - \tilde{\nabla}_Y B(X, Z);$$

in the last equation, $\tilde{\nabla}$ denotes the natural connection on $T^*M \otimes T^*M \otimes E$.

These equations make sense if M is an abstract manifold and $E \rightarrow M$ is an abstract bundle as in Section 3, if we assume the existence of the bundle map f in (15), since f permits to define Γ on $TM \oplus E$ by (17), and R^G may be written in terms of Γ only (see (4)-(5)). We prove the following:

Proposition 4.1. *We assume that M is simply connected. There exists $\varphi \in \Gamma(U\Sigma)$ solution of (22) if and only if $B : TM \times TM \rightarrow E$ satisfies the Gauss, Ricci and Codazzi equations.*

Proof. We first prove that the Gauss, Ricci and Codazzi equations are necessary if we have a non-trivial solution of (22). We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) and compute the curvature

$$R(X, Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X, Y]}\varphi.$$

We fix a point $x_0 \in M$, and assume that $\nabla X = \nabla Y = 0$ at x_0 . We have

$$\begin{aligned} \nabla_X \nabla_Y \varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) \cdot \varphi + B(Y, e_j) \cdot \nabla_X \varphi \right) \\ &\quad + \frac{1}{2} (\nabla_X \Gamma(Y) \cdot \varphi + \Gamma(Y) \cdot \nabla_X \varphi) \\ &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \tilde{\nabla}_X B(Y, e_j) \cdot \varphi - \frac{1}{4} \sum_{j, k=1}^p e_j \cdot e_k \cdot B(Y, e_j) \cdot B(X, e_k) \\ &\quad - \frac{1}{4} \sum_{j=1}^p e_j \cdot B(Y, e_j) \cdot \Gamma(X) \cdot \varphi + \frac{1}{2} \nabla_X \Gamma(Y) \cdot \varphi - \frac{1}{4} \Gamma(Y) \cdot \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi \\ &\quad + \frac{1}{4} \Gamma(Y) \cdot \Gamma(X) \cdot \varphi. \end{aligned}$$

Thus

$$\begin{aligned}
 R(X, Y)\varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j) \right) \cdot \varphi \\
 &+ \frac{1}{4} \sum_{j \neq k} e_j \cdot e_k \cdot (B(X, e_j) \cdot B(Y, e_k) - B(Y, e_j) \cdot B(X, e_k)) \cdot \varphi \\
 &- \frac{1}{4} \sum_{j=1}^p (B(X, e_j) \cdot B(Y, e_j) - B(Y, e_j) \cdot B(X, e_j)) \cdot \varphi \\
 &+ \frac{1}{2} \underbrace{\left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right]}_{c_1} \cdot \varphi + \frac{1}{2} \underbrace{\left[\sum_{j=1}^p e_j \cdot B(Y, e_j), \Gamma(X) \right]}_{c_2} \cdot \varphi \\
 &+ \frac{1}{2} \underbrace{(\nabla_X \Gamma(Y) - \nabla_Y \Gamma(X))}_{c_3} \cdot \varphi + \frac{1}{2} \underbrace{[\Gamma(X), \Gamma(Y)]}_{c_4} \cdot \varphi.
 \end{aligned}
 \tag{35}$$

We computed the second and the third terms in [4]; we only recall the result here:

computation AB

Lemma 4.2. [4] *Let us set*

$$\mathcal{A} := \frac{1}{4} \sum_{j \neq k} e_j \cdot e_k \cdot (B(X, e_j) \cdot B(Y, e_k) - B(Y, e_j) \cdot B(X, e_k))$$

and

$$\mathcal{B} := -\frac{1}{4} \sum_j (B(X, e_j) \cdot B(Y, e_j) - B(Y, e_j) \cdot B(X, e_j)).$$

We have

$$\mathcal{A} = \frac{1}{2} \sum_{j < k} \{ \langle B^*(X, B(Y, e_j)), e_k \rangle - \langle B^*(Y, B(X, e_j)), e_k \rangle \} e_j \cdot e_k$$

and

$$\mathcal{B} = \frac{1}{2} \sum_{k < l} \langle B(X, B^*(Y, n_k)) - B(Y, B^*(X, n_k)), n_l \rangle n_k \cdot n_l.$$

We now compute the other terms in (35). We first compute the covariant derivative of Γ , considering Γ as a map

$$\Gamma : TM \oplus E \rightarrow \text{End}(TM \oplus E).$$

lemma formula nabla gamma

Lemma 4.3. *If $X, Y \in TM$ and $Z \in TM \oplus E$,*

$$\begin{aligned}
 (\nabla_X \Gamma)(Y)Z &= [\Gamma(X), \Gamma(Y)]Z - B(X, (\Gamma(Y)Z)^T) + B^*(X, (\Gamma(Y)Z)^N) \\
 &+ \Gamma(Y)(B(X, Z^T)) - \Gamma(Y)(B^*(X, Z^N)) - \Gamma(\Gamma(X)Y)(Z) \\
 &+ \Gamma(B(X, Y))(Z).
 \end{aligned}$$

Here the brackets stand for the commutator of the endomorphisms.

Proof. Since the expression is tensorial, we may assume that $X, Y, Z \in \Gamma(TM \oplus E)$ are left invariant vector fields. By definition,

edo1_nabla

$$(\nabla_X \Gamma)(Y)Z = \nabla_X(\Gamma(Y)Z) - \Gamma(\nabla_X Y)Z - \Gamma(Y)(\nabla_X Z).
 \tag{36}$$

Since X, Y and Z are left invariant vector fields, so are $\Gamma(Y)Z$, $\nabla_X Y$ and $\nabla_X Z$, and, by (18),

$$\nabla_X(\Gamma(Y)Z) = \Gamma(X)(\Gamma(Y)Z) - B(X, (\Gamma(Y)Z)^T) + B^*(X, (\Gamma(Y)Z)^N),$$

$$\Gamma(Y)(\nabla_X Z) = \Gamma(Y)(\Gamma(X)Z) - \Gamma(Y)B(X, Z^T) + \Gamma(Y)B^*(X, Z^N)$$

and

$$\Gamma(\nabla_X Y)(Z) = \Gamma(\Gamma(X)Y)Z - \Gamma(B(X, Y^T))Z + \Gamma(B^*(X, Y^N))Z.$$

Plugging these formulas in (36), we get the result. \square

We now regard Γ as a map

$$\Gamma : TM \oplus E \rightarrow \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E),$$

and compute the term \mathcal{C}_3 in (35).

Lemma 4.4. *If $X, Y \in TM$,*

$$\begin{aligned} \frac{1}{2}((\nabla_X \Gamma)(Y) - (\nabla_Y \Gamma)(X)) &= [\Gamma(X), \Gamma(Y)] - \frac{1}{2}\Gamma([\Gamma(X), Y] - [\Gamma(Y), X]) \\ &\quad - \frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right] + \frac{1}{2} \left[\sum_{j=1}^p e_j \cdot B(Y, e_j), \Gamma(X) \right]. \end{aligned}$$

Here the brackets stand for the commutator in $Cl(TM \oplus E)$.

Proof. We first note that the linear maps $Z \mapsto [\Gamma(X), \Gamma(Y)]Z$, $Z \mapsto \Gamma(\Gamma(X)Y)Z$ and $Z \mapsto \Gamma(B(X, Y))Z$ appearing in Lemma 4.3 are respectively represented by the bivectors $[\Gamma(X), \Gamma(Y)]$, $\Gamma([\Gamma(X), Y])$ and $\Gamma(B(X, Y))$ (Lemmas A.1 and A.2 in the appendix). Moreover, by Lemma A.4 applied to the linear maps $B(X, \cdot) : TM \rightarrow E$ and $\Gamma(Y) : TM \oplus E \rightarrow TM \oplus E$, the map

$$Z \mapsto -B^*(X, (\Gamma(Y)Z)^N) + \Gamma(Y)(B^*(X, Z^N)) + B(X, (\Gamma(Y)Z)^T) - \Gamma(Y)(B(X, Z^T))$$

is represented by the bivector

$$\left[\sum_{j=1}^p e_j \cdot B(X, e_j), \Gamma(Y) \right] \in Cl(TM \oplus E).$$

The result follows. \square

We now deduce the sum of the last four terms in (35):

Lemma 4.5. *Let us set, for $X, Y \in TM$,*

$$R^G(X, Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([\Gamma(X), Y] - [\Gamma(Y), X]) \in \Lambda^2(TM \oplus E)$$

(note that R^G is the curvature tensor of G , pulled-back to $TM \oplus E$ by the bundle isomorphism f introduced in (15)). Then

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \frac{1}{2}R^G(X, Y).$$

computation other terms

We thus get the formula

$$\begin{aligned}
 \boxed{\text{R function B 2}} \quad (37) \quad R(X, Y)\varphi &= -\frac{1}{2} \sum_{j=1}^p e_j \cdot \left(\tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j) \right) \cdot \varphi \\
 &\quad + \mathcal{A} \cdot \varphi + \mathcal{B} \cdot \varphi + \frac{1}{2} R^G(X, Y) \cdot \varphi
 \end{aligned}$$

where \mathcal{A} and \mathcal{B} are computed in Lemma 4.2 and R^G may be conveniently written in the form

$$\begin{aligned}
 R^G(X, Y) &= \sum_{1 \leq j < k \leq p} \langle R^G(X, Y)(e_j), e_k \rangle e_j \cdot e_k \\
 &\quad + \sum_{j=1}^p \sum_{r=1}^q \langle R^G(X, Y)(e_j), n_r \rangle e_j \cdot n_r \\
 &\quad + \sum_{1 \leq l < r \leq q} \langle R^G(X, Y)(n_r), n_l \rangle n_l \cdot n_r.
 \end{aligned}$$

On the other hand, the curvature of the spinorial connection is given by

$$\begin{aligned}
 \boxed{\text{R function RT RN}} \quad (38) \quad R(X, Y)\varphi &= \frac{1}{2} \left(\sum_{1 \leq j < k \leq p} \langle R^T(X, Y)(e_j), e_k \rangle e_j \cdot e_k \right. \\
 &\quad \left. + \sum_{1 \leq k < l \leq q} \langle R^N(X, Y)(n_k), n_l \rangle n_k \cdot n_l \right) \cdot \varphi.
 \end{aligned}$$

We now compare the expressions (37) and (38): since in a given frame \bar{s} belonging to \tilde{Q} , φ is represented by an element which is invertible in $Cl(\mathcal{G})$ (it is in fact represented by an element belonging to $Spin(\mathcal{G})$), we may identify the coefficients and get

$$\langle R^T(X, Y)(e_j), e_k \rangle = \langle B^*(X, B(Y, e_j)), e_k \rangle - \langle B^*(Y, B(X, e_j)), e_k \rangle + \langle R^G(X, Y)(e_j), e_k \rangle,$$

$$\langle R^N(X, Y)(n_k), n_l \rangle = \langle B(X, B^*(Y, n_k)), n_l \rangle - \langle B(Y, B^*(X, n_k)), n_l \rangle + \langle R^G(X, Y)(n_k), n_l \rangle$$

and

$$\langle \tilde{\nabla}_X B(Y, e_j) - \tilde{\nabla}_Y B(X, e_j), n_r \rangle = \langle R^G(X, Y)(e_j), n_r \rangle$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We now prove that the equations of Gauss, Ricci and Codazzi are also sufficient to get a solution of (22). The calculations above show that the connection on Σ defined by

$$\boxed{\text{def nabla prime}} \quad (39) \quad \nabla'_X \varphi := \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} \Gamma(X) \cdot \varphi$$

for all $\varphi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$ is flat if and only if the equations of Gauss, Ricci and Codazzi hold. But if this connection is flat there exists a solution $\varphi \in \Gamma(U\Sigma)$ of (22); this is because ∇' may be also interpreted as a connection on $U\Sigma$ regarded as a principal bundle (of group $Spin(\mathcal{G})$, acting on the right): indeed, ∇ defines such

a connection (since it comes from a connection on \tilde{Q}), and the right hand side term in (39) defines a linear map

$$\begin{aligned} TM &\rightarrow \chi_V^{inv}(U\Sigma) \\ X &\mapsto \varphi \mapsto \frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi - \frac{1}{2} \Gamma(X) \cdot \varphi \end{aligned}$$

from TM to the vector fields on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are of the form $\varphi \mapsto \eta \cdot \varphi$, $\eta \in \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E)$). Assuming that the equations of Gauss, Codazzi and Ricci hold, we thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (22). \square

The considerations above give a spinorial proof of the fundamental theorem of submanifold theory in the metric Lie group G :

Corollary 1. *We keep the hypotheses and notation of Section 2, and moreover assume that M is simply connected and that $B : TM \times TM \rightarrow E$ is bilinear, symmetric and satisfies the equations of Gauss, Codazzi and Ricci. Then there is an isometric immersion of M into G with normal bundle E and second fundamental form B . The immersion is unique up to a rigid motion in G , that is up to a transformation of the form*

rigid motion

$$(40) \quad \begin{aligned} L_b \circ \Phi_a : G &\rightarrow G \\ g &\mapsto b\Phi_a(g) \end{aligned}$$

where $a \in Spin(\mathcal{G})$ is such that $Ad(a) : \mathcal{G} \rightarrow \mathcal{G}$ is an automorphism of Lie algebra, $\Phi_a : G \rightarrow G$ is the group automorphism such that $d(\Phi_a)_e = Ad(a)$, and b belongs to G .

Proof. The equations of Gauss, Codazzi and Ricci are the integrability conditions of (22). We thus get a solution $\varphi \in \Gamma(U\Sigma)$ of (22); with such a spinor field at hand, $F = \int \xi$ where ξ is defined in (23) is the immersion. Finally, a solution of (22) is unique up to the right action of an element of $Spin(\mathcal{G})$; the right multiplication of φ by $a \in Spin(\mathcal{G})$ and the left multiplication by $b \in G$ in the last integration give also an immersion, if $Ad(a) : \mathcal{G} \rightarrow \mathcal{G}$ is moreover an automorphism of Lie algebra. This immersion is obtained from the immersion defined by φ by a rigid motion, as described in (40). \square

Remark 7. *In \mathbb{R}^n , a rigid motion as in (40) is a transformation of the form*

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto ax + b, \end{aligned}$$

with $a \in SO(n)$ and $b \in \mathbb{R}^n$.

5. SPECIAL CASES

5.1. Submanifolds in \mathbb{R}^n . If the metric Lie group is \mathbb{R}^n with its natural metric, we recover the main result of [4]. We suppose that M is a p -dimensional Riemannian manifold, $E \rightarrow M$ a bundle of rank q , with a fibre metric and a compatible connection. We assume that TM and E are oriented and spin with given spin structures, and that $B : TM \times TM \rightarrow E$ is bilinear and symmetric.

theorem section Rn

Theorem 2. [4] *We moreover assume that M is simply connected. The following statements are equivalent:*

(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that

$$\text{kill equation Rn} \quad (41) \quad \nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi$$

for all $X \in TM$.

(2) There exists an isometric immersion $F : M \rightarrow \mathbb{R}^n$ with normal bundle E and second fundamental form B .

Moreover, $F = \int \xi$ where ξ is the \mathbb{R}^n -valued 1-form defined by

$$\text{def xi Rn} \quad (42) \quad \xi(X) := \langle\langle X \cdot \varphi, \varphi \rangle\rangle$$

for all $X \in TM$.

Proof. We only prove (1) \Rightarrow (2). This will be a consequence of Theorem 1 if we may define a bundle map f as in (15) such that (18) holds. We assume that φ is a solution of (41), and set

$$f : \quad TM \oplus E \quad \rightarrow \quad M \times \mathbb{R}^n \\ Z \quad \mapsto \quad \langle\langle Z \cdot \varphi, \varphi \rangle\rangle.$$

The map Γ defined by (17) is $\Gamma = 0$. We now show that (18) is satisfied for every $Z \in \Gamma(TM \oplus E)$ such that $f(Z) : M \rightarrow \mathbb{R}^n$ is a constant map: for all $X \in TM$, we have $\partial_X \{f(Z)\} = 0$, which reads

$$\langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle = 0.$$

But, by (41) and (11),

$$\begin{aligned} \langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle + \langle\langle Z \cdot \varphi, \nabla_X \varphi \rangle\rangle &= \langle\langle \left[\sum_{j=1}^p e_j \cdot B(X, e_j), Z \right] \cdot \varphi, \varphi \rangle\rangle \\ &= \langle\langle \{B(X, Z^T) - B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle\rangle, \end{aligned}$$

where we use Lemma A.3 in the last step. Thus

$$\langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle = \langle\langle \{-B(X, Z^T) + B^*(X, Z^N)\} \cdot \varphi, \varphi \rangle\rangle$$

and

$$\nabla_X Z = -B(X, Z^T) + B^*(X, Z^N),$$

which is (18) with $\Gamma = 0$. \square

5.2. Submanifolds in \mathbb{H}^n . Spinor representations of submanifolds in \mathbb{H}^n with its natural metric were already given in [14, 3, 4]. We give here an other representation using the group structure of \mathbb{H}^n , with an arbitrary left invariant metric. Let us set

$$\mathbb{H}^n = \{a = (a', a_n) \in \mathbb{R}^n : a_n > 0\},$$

and, for $a \in \mathbb{H}^n$, the similarity of \mathbb{R}^{n-1} (by a similarity we mean an homothety composed by a translation)

$$\varphi_a : \quad \mathbb{R}^{n-1} \quad \rightarrow \quad \mathbb{R}^{n-1} \\ x \quad \mapsto \quad a_n x + a'.$$

The similarities of \mathbb{R}^{n-1} naturally form a group under composition, and the bijection

$$\begin{aligned} \varphi : \quad \mathbb{H}^n &\rightarrow \{ \text{similarities } \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \} \\ a &\mapsto \varphi_a \end{aligned}$$

induces a group structure on \mathbb{H}^n : it is such that

$$\boxed{\text{prod Hn}} \quad (43) \quad ab = (a_n b' + a', a_n b_n)$$

for all $a, b \in \mathbb{H}^n$; the neutral element is $e = (0, 1) \in \mathbb{H}^n$. Let us denote by $(e_1^o, e_2^o, \dots, e_n^o)$ the canonical basis of $T_e \mathbb{H}^n = \mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on \mathbb{H}^n . The Lie bracket may be easily seen to be given by

$$[e_i^o, e_j^o] = 0 \quad \text{and} \quad [e_n^o, e_i^o] = e_i^o$$

for $i, j = 1, \dots, n-1$. This may also be written in the form

$$\boxed{\text{def bracket Rn}} \quad (44) \quad [X, Y] = l(X)Y - l(Y)X$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \rightarrow \mathbb{R}$ is the linear form such that $l(e_i^o) = 0$ if $i \leq n-1$ and $l(e_n^o) = 1$. This property implies that every left invariant metric on \mathbb{H}^n has constant negative curvature $-|l|^2$ [13, 12].

We suppose that a left invariant metric $\langle \cdot, \cdot \rangle$ is given on \mathbb{H}^n , and consider the vector $U_o \in T_e \mathbb{H}^n$ such that $l(X) = \langle U_o, X \rangle$ for all $X \in T_e \mathbb{H}^n$. We have $|U_o| = |l|$, and, by the Koszul formula (3),

$$\boxed{\text{expr Gamma Hn}} \quad (45) \quad \Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o$$

for all $X, Y \in T_e \mathbb{H}^n$.

We keep the hypotheses made at the beginning of Section 5.1. We suppose moreover that $U \in \Gamma(TM \oplus E)$ is given such that $|U| = |l|$ and, for all $X \in TM$,

$$\boxed{\text{nabla U Hn}} \quad (46) \quad \nabla_X U = -|U|^2 X + \langle X, U \rangle U - B(X, U^T) + B^*(X, U^N).$$

We set, for $X \in TM$ and $Y \in TM \oplus E$,

$$\boxed{\text{def Gamma Hn}} \quad (47) \quad \Gamma(X)(Y) = -\langle Y, U \rangle X + \langle X, Y \rangle U.$$

rem U solution eqn Z

Remark 8. Equation (46) implies the following:

- (1) U is a solution of (18), with the definition (47) of Γ .
- (2) The norm of U is constant, since, by a straightforward computation,

$$d|U|^2(X) = 2\langle \nabla_X U, U \rangle = 0$$

for all $X \in TM$. The additional hypothesis $|U| = |l|$ is thus not very restrictive.

We note that it is not necessary to assume the existence of U solution of (46) to get a spinor representation of a submanifold in \mathbb{H}^n if \mathbb{H}^n is regarded as the set of unit vectors in Minkowski space $\mathbb{R}^{n,1}$ [14, 3, 4]. Nevertheless, this hypothesis seems necessary if we consider \mathbb{H}^n as a group, since the group structure introduces an anisotropy: the vector $e_n \in T_e \mathbb{H}^n$ is indeed a special direction for the group structure.

Let us construct the spinor bundles Σ and $U\Sigma$ on M as in Section 2.4 with here $\mathcal{G} = T_e \mathbb{H}^n$.

Theorem 3. We assume that M is simply connected. The following statements are equivalent:

- (1) There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (22) where Γ is defined by (47).

- (2) *There exists an isometric immersion $M \rightarrow \mathbb{H}^n$ with normal bundle E and second fundamental form B .*

Proof. We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) where Γ is defined by (47), and define $f : TM \oplus E \rightarrow M \times T_e\mathbb{H}^n$ by

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle$$

for all $Z \in TM \oplus E$. Let us first observe that if Z is a vector field solution of (18), then $f(Z)$ is constant: we have, for all $X \in TM$,

$$\partial_X f(Z) = \langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + (id + \tau)\langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle;$$

this is 0, by (18), (22) and formulas (26)-(27) in Lemma 3.2. Since U is a solution of (18) (see Remark 8), we deduce that $f(U) \in T_e\mathbb{H}^n$ is a constant, and, since $|f(U)| = |U| = |U_o|$, replacing φ by $\varphi \cdot a$ for some $a \in Spin(T_e\mathbb{H}^n)$ if necessary, we may suppose that $f(U) = U_o$. Since Γ is defined on $T_e\mathbb{H}^n$ by (45) and on $TM \oplus E$ by (47), and since f preserves the metrics, it is straightforward to see that $f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$ for all $X, Y \in TM \oplus E$. Finally, (18) holds for all $Z \in \Gamma(TM \oplus E)$ such that $f(Z)$ is constant: this is the same argument as in the proof of Theorem 2 in Section 5.1, just adding the term Γ . The result then follows from Theorem 1. \square

5.3. Hypersurfaces in a metric Lie group. We assume that G is a simply connected n -dimensional metric Lie group, M is a p -dimensional Riemannian manifold, $n = p+1$, and E is the trivial line bundle on M , oriented by a unit section $\nu \in \Gamma(E)$. We moreover suppose that M is simply connected and that $h : TM \times TM \rightarrow \mathbb{R}$ is a given symmetric bilinear form, and that the hypotheses (1) and (2) of Section 3 with $B = h\nu$ hold. According to Theorem 1, an isometric immersion of M into G with second fundamental form h is equivalent to a section φ of $\Gamma(U\Sigma)$ solution of the Killing equation (22). Note that $Q_E \simeq M$ and the double covering

$$\tilde{Q}_E \rightarrow Q_E$$

is trivial, since M is assumed to be simply connected. Fixing a section \tilde{s}_E of \tilde{Q}_E we get an injective map

$$\begin{aligned} \tilde{Q}_M &\rightarrow \tilde{Q}_M \times_M \tilde{Q}_E =: \tilde{Q} \\ \tilde{s}_M &\mapsto (\tilde{s}_M, \tilde{s}_E). \end{aligned}$$

Using

$$Cl_p \simeq Cl_{p+1}^0 \subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p \rightarrow Cl_{p+1}$, $X \mapsto X \cdot e_{p+1}$), we deduce a bundle isomorphism

identif spineurs

$$(48) \quad \begin{aligned} \tilde{Q}_M \times_\rho Cl_p &\rightarrow \tilde{Q} \times_\rho Cl_{p+1}^0 \subset \Sigma \\ \psi &\mapsto \psi^*. \end{aligned}$$

It satisfies the following properties: for all $X \in TM$ and $\psi \in \tilde{Q}_M \times_\rho Cl_p$,

properties spinors M G

$$(49) \quad (X \cdot_M \psi)^* = X \cdot \nu \cdot \psi^* \quad \text{and} \quad \nabla_X(\psi^*) = (\nabla_X \psi)^*.$$

To write down the Killing equation (22) in the bundle $\tilde{Q}_M \times_\rho Cl_p$, we need to decompose the Clifford action of $\Gamma(X)$ into its tangent and its normal parts:

Lemma 5.1. *Recall the notation introduced in Remark 2. Then, for all $X \in TM$,*

$$(50) \quad \Gamma(X) = \sum_i \langle X, T_i \rangle \sum_{j < k} \Gamma_{ij}^k \left(\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu \right).$$

Proof. We have

$$\begin{aligned} X &= \sum_{i=1}^n \langle X, e_i \rangle e_i = \sum_{i=1}^n \langle X, T_i \rangle e_i, \\ \Gamma(X)(e_j) &= \sum_{i=1}^n \langle X, T_i \rangle \Gamma(e_i)(e_j) \\ &= \sum_{i=1}^n \langle X, T_i \rangle \sum_{k=1}^n \Gamma_{ij}^k e_k \\ &= \sum_{1 \leq i, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_k + f_k \nu), \end{aligned}$$

and thus

$$\begin{aligned} \Gamma(X) &= \frac{1}{2} \sum_{j=1}^n e_j \cdot \Gamma(X)(e_j) \\ &= \frac{1}{2} \sum_{j=1}^n (T_j + f_j \nu) \cdot \sum_{1 \leq i, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_k + f_k \nu) \\ &= \frac{1}{2} \sum_{1 \leq i, j, k \leq n} \Gamma_{ij}^k \langle X, T_i \rangle (T_j + f_j \nu) \cdot (T_k + f_k \nu). \end{aligned}$$

Now

$$(T_j + f_j \nu) \cdot (T_k + f_k \nu) = T_j \cdot T_k + f_k T_j \cdot \nu - f_j T_k \cdot \nu - f_j f_k,$$

and the result follows since $\Gamma_{ij}^k = -\Gamma_{ik}^j$. \square

The section $\varphi \in \Gamma(U\Sigma)$ solution of (22) thus identifies to a section ψ of $\tilde{Q}_M \times_\rho Cl_p$ solution of

$$\begin{aligned} \nabla_X \psi &= -\frac{1}{2} \sum_{j=1}^p h(X, e_j) e_j \cdot_M \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot_M \psi \\ &= -\frac{1}{2} S(X) \cdot_M \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot_M \psi \end{aligned}$$

for all $X \in TM$, where

def Gamma tilde

$$(51) \quad \tilde{\Gamma}(X) = \sum_i \langle X, T_i \rangle \sum_{j < k} \Gamma_{ij}^k \left(\frac{1}{2} (T_j \cdot_M T_k - T_k \cdot_M T_j) + (f_k T_j - f_j T_k) \right).$$

and $S : TM \rightarrow TM$ is the symmetric operator associated to h . We deduce the following result:

thm hypersurfaces

Theorem 4. *Let $S : TM \rightarrow TM$ be a symmetric operator. The following two statements are equivalent:*

- (1) *there exists an isometric immersion of M into G with shape operator S ;*

(2) there exists a normalized spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ solution of

$$\text{equation psi} \quad (52) \quad \nabla_X \psi = -\frac{1}{2}S(X) \cdot_M \psi + \frac{1}{2}\tilde{\Gamma}(X) \cdot_M \psi$$

for all $X \in TM$, where $\tilde{\Gamma}$ is defined in (51).

Here, a spinor field $\psi \in \Gamma(\tilde{Q}_M \times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s} \in \tilde{Q}_M$ by an element $[\psi] \in Cl_p \simeq Cl_{p+1}^0$ belonging to $Spin(p+1)$.

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

5.4. Surfaces in a 3-dimensional metric Lie group. Since $Cl_2 \simeq \Sigma_2$ we have

$$\tilde{Q}_M \times_\rho Cl_2 \simeq \Sigma M,$$

and φ is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (52) and such that $|\psi| = 1$. Moreover, the explicit representation formula $F = \int \xi$ may be written in terms of ψ : it may be proved by a computation that

$$\text{cit representation dim 3} \quad (53) \quad \langle\langle X \cdot \varphi, \varphi \rangle\rangle = i2\mathcal{R}e\langle X \cdot \psi^+, \psi^- \rangle + j(\langle X \cdot \psi^+, \alpha(\psi^+) \rangle - \langle X \cdot \psi^-, \alpha(\psi^-) \rangle)$$

where the brackets $\langle \cdot, \cdot \rangle$ stand here for the natural hermitian product on Σ_2 and $\alpha : \Sigma_2 \rightarrow \Sigma_2$ is the natural quaternionic structure. If $G = \mathbb{R}^3$, this is the explicit representation formula given in [6] (see also [3]).

We also note that the expression (51) of $\tilde{\Gamma}$ simplifies if the Lie group is 3-dimensional:

lem Gamma dim 3

Lemma 5.2. *If (j, k, l) is a permutation of $\{1, 2, 3\}$ and $\epsilon_{jkl} = \pm 1$ denotes its sign, then, for all $\psi \in \Gamma(\Sigma M)$,*

$$\left(\frac{1}{2} (T_j \cdot_M T_k - T_k \cdot_M T_j) + (f_k T_j - f_j T_k) \right) \cdot \psi = \epsilon_{jkl} (f_l - T_l) \cdot \omega \cdot \psi.$$

Proof. Keeping the notation introduced above, we note that

$$\underline{e}_j \cdot \underline{e}_k \cdot \underline{e}_l = \epsilon_{jkl} \omega \cdot \nu,$$

which yields

$$\underline{e}_j \cdot \underline{e}_k = -\epsilon_{jkl} \omega \cdot \nu \cdot \underline{e}_l.$$

Thus

$$\begin{aligned} T_j \cdot T_k + (f_k T_j - f_j T_k) \cdot \nu - f_j f_k &= -\epsilon_{jkl} \omega \cdot \nu \cdot (T_l + f_l \nu) \\ &= \epsilon_{jkl} (f_l - T_l \cdot \nu) \cdot \omega \end{aligned}$$

since $T_l \cdot \nu = -\nu \cdot T_l$, $T_l \cdot \omega = -\omega \cdot T_l$ and $\omega \cdot \nu = \nu \cdot \omega$. Switching the indices j and k we also get

$$T_k \cdot T_j + (f_j T_k - f_k T_j) \cdot \nu - f_k f_j = \epsilon_{kjl} (f_l - T_l \cdot \nu) \cdot \omega = -\epsilon_{jkl} (f_l - T_l \cdot \nu) \cdot \omega$$

and deduce that

$$\frac{1}{2} (T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu = \epsilon_{jkl} (f_l - T_l \cdot \nu) \cdot \omega.$$

The result is then a consequence of the first property in (49). \square

5.4.1. *The metric Lie group S^3 .* A spinor representation of a surface immersed in S^3 was already given in [14] (see also [3, 4]). We give here a spinor representation relying on the group structure; it appears that it coincides with the result in [14].

We regard the sphere S^3 as the set of the unit quaternions, with its natural group structure. The Lie algebra of S^3 identifies to \mathbb{R}^3 , with the bracket $[X, Y] = 2X \times Y$ for all $X, Y \in \mathbb{R}^3$ (\times is the usual cross product). By the Koszul formula (3), for all $X, Y \in \mathbb{R}^3$,

$$\Gamma(X)(Y) = X \times Y.$$

As a bivector, for all $X = X_1e_1^o + X_2e_2^o + X_3e_3^o \in \mathbb{R}^3$,

$$\begin{aligned} \Gamma(X) &= \frac{1}{2}(e_1^o \cdot \Gamma(X)(e_1^o) + e_2^o \cdot \Gamma(X)(e_2^o) + e_3^o \cdot \Gamma(X)(e_3^o)) \\ &= X_1e_2^o \cdot e_3^o + X_2e_3^o \cdot e_1^o + X_3e_1^o \cdot e_2^o \\ &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o). \end{aligned}$$

Thus, if $\varphi \in \tilde{Q} \times_{\rho} Cl_3^0$ represents an immersion of an oriented surface M in S^3 and if $\psi \in \Gamma(\Sigma M)$ is such that $\varphi = \psi^*$, then, for all $X \in TM$,

$$\begin{aligned} \Gamma(X) \cdot \varphi &= -X \cdot (e_1^o \cdot e_2^o \cdot e_3^o) \cdot \varphi \\ &= -X \cdot \omega \cdot \nu \cdot \varphi \\ &= (X \cdot \nu) \cdot \omega \cdot \varphi \\ &= (X \cdot \omega \cdot \psi)^* \end{aligned}$$

where ω is the area form of M , and ν is the vector normal to M in S^3 . Since $\varphi \in \Gamma(U\Sigma)$ is a solution of (22), $\psi \in \Gamma(\Sigma M)$ is a solution of

$$\nabla_X \psi = -\frac{1}{2}S(X) \cdot \psi + \frac{1}{2}X \cdot \omega \cdot \psi$$

and satisfies $|\psi| = 1$. Taking the trace, we get

$$\begin{aligned} D\psi &= e_1 \cdot \nabla_{e_1} \psi + e_2 \cdot \nabla_{e_2} \psi \\ &= H\psi - \omega \cdot \psi \end{aligned}$$

where (e_1, e_2) is a positively oriented and orthonormal basis of TM . Now, setting $\bar{\psi} = \psi^+ - \psi^-$ and since $\omega \cdot \psi = -i\bar{\psi}$ (recall that $i\omega$ acts as the identity on Σ^+M and as -identity on Σ^-M), we get

$$D\psi = H\psi - i\bar{\psi},$$

which is also the spinor characterization given by Morel in [14].

Ce n'est sans doute pas une coïncidence, mais je ne me l'explique pas complètement.

5.4.2. *Surfaces in the 3-dimensional metric Lie groups $E(\kappa, \tau)$, $\tau \neq 0$.* We recover here a spinor characterization of immersions in the 3-dimensional homogeneous spaces $E(\kappa, \tau)$; this result was obtained by one of the authors in [16], using a characterization of immersions in these spaces by Daniel [5]. We give here an independent proof, and rather obtain the result of Daniel as a corollary.

The metric Lie group $E(\kappa, \tau)$, $\tau \neq 0$, is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the vectors e_1^o, e_2^o, e_3^o of the canonical basis by

$$[e_1^o, e_2^o] = 2\tau e_3^o, \quad [e_2^o, e_3^o] = \sigma e_1^o, \quad [e_3^o, e_1^o] = \sigma e_2^o$$

where $\sigma = \frac{\kappa}{2\tau}$. The metric on \mathcal{G} is the canonical metric, ie the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. The Levi-Civita connection is then given by

$$\boxed{\text{Gamma E k t}} \quad (54) \quad \Gamma(X)(Y) = \{\tau(X - \langle X, e_3^o \rangle e_3^o) + (\sigma - \tau)\langle X, e_3^o \rangle e_3^o\} \times Y$$

for $X, Y \in \mathcal{G}$; see e.g. [5].

Let $S : TM \rightarrow TM$ be a symmetric operator. We assume that a vector field $T \in \Gamma(TM)$ and a function $f \in C^\infty(M, \mathbb{R})$ are given such that

$$\boxed{\text{eqn T f E k t}} \quad (55) \quad |T|^2 + f^2 = 1,$$

$$\boxed{\text{eqn T E k t}} \quad (56) \quad \nabla_X T = f(S(X) - \tau JX)$$

and

$$\boxed{\text{eqn f E k t}} \quad (57) \quad df(X) = -\langle S(X) - \tau JX, T \rangle$$

for all $X \in TM$.

Theorem 5. [16] *If M is simply connected, the following two statements are equivalent:*

(1) *There exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*

$$\boxed{\text{eqn spinor E k t}} \quad (58) \quad \nabla_X \psi = -\frac{1}{2}S(X) \cdot \psi + \frac{1}{2}\{(2\tau - \sigma)\langle X, T \rangle(T - f) - \tau X\} \cdot \omega \cdot \psi$$

for all $X \in TM$.

(2) *There exists an isometric immersion of M into $E(\kappa, \tau)$, with shape operator S .*

Proof. We consider the trivial line bundle $E = \mathbb{R}\nu$, where ν is a unit section. The bundle $TM \oplus E$ is of rank 3, and is assumed to be oriented by the orientation of TM and by ν . We suppose that it is endowed with the natural product metric. Let us denote by \times the natural cross product in the fibers. We set

$$e_3 = T + f\nu,$$

and, for all $X, Y \in TM \oplus E$,

$$\boxed{\text{def Gamma TM+E E k t}} \quad (59) \quad \Gamma(X)(Y) = \{\tau(X - \langle X, e_3 \rangle e_3) + (\sigma - \tau)\langle X, e_3 \rangle e_3\} \times Y.$$

Defining $B : TM \times TM \rightarrow E$ and its adjoint $B^* : TM \times E \rightarrow TM$ by

$$\boxed{\text{def B E k t}} \quad (60) \quad B(X, Y) = \langle S(X), Y \rangle \nu \quad \text{and} \quad B^*(X, \nu) = S(X)$$

for all $X, Y \in TM$, the equations (56) and (57) are equivalent to the single equation

$$\boxed{\text{eqn e3 invariant}} \quad (61) \quad \nabla_X e_3 = \Gamma(X)(e_3) - B(X, e_3^T) + B^*(X, e_3^N)$$

for all $X \in TM$, where ∇ is the sum of the Levi-Civita connection on TM and the trivial connection on E . This is (18) for $Z = e_3$. We will need the following expression for Γ :

lemma Gamma E k t **Lemma 5.3.** *For all $X \in TM$, the linear map $\Gamma(X) : TM \oplus E \rightarrow TM \oplus E$ is represented by the bivector*

$$\Gamma(X) = \{(2\tau - \sigma)\langle X, T \rangle(T \cdot \nu - f) - \tau X \cdot \nu\} \cdot \omega.$$

Proof. The linear map $\Gamma(X)$ defined by (59) is represented by the bivector

$$\Gamma(X) = \frac{1}{2}(e_1 \cdot \Gamma(X)(e_1) + e_2 \cdot \Gamma(X)(e_2) + e_3 \cdot \Gamma(X)(e_3))$$

where e_1, e_2 are such that e_1, e_2, e_3 is a positively oriented and orthonormal basis of $TM \oplus E$ (see Lemma A.1); thus, a straightforward computation shows that $\Gamma(X)$ is represented by the bivector

lem Gamma E k t

$$(62) \quad \Gamma(X) = -\tau(X \times e_3) \cdot e_3 + (\sigma - \tau)\langle X, e_3 \rangle e_1 \cdot e_2.$$

The following formula may be checked by a direct computation: for $X, Y \in TM \oplus E$,

$$X \times Y = -(X \cdot Y + \langle X, Y \rangle) e_1 \cdot e_2 \cdot e_3;$$

this gives

$$\begin{aligned} (X \times e_3) \cdot e_3 &= -(X \cdot e_3 + \langle X, e_3 \rangle) e_1 \cdot e_2 \cdot e_3 \cdot e_3 \\ &= (X - \langle X, e_3 \rangle e_3) e_1 \cdot e_2 \cdot e_3 \\ &= (X - \langle X, T \rangle (T + f\nu)) \cdot \omega \cdot \nu \\ &= (X \cdot \nu - \langle X, T \rangle (T \cdot \nu - f)) \cdot \omega. \end{aligned}$$

Moreover,

$$\begin{aligned} \langle X, e_3 \rangle e_1 \cdot e_2 &= \langle X, T \rangle (-e_1 \cdot e_2 \cdot e_3 \cdot e_3) \\ &= \langle X, T \rangle (-\omega \cdot \nu \cdot (T + f\nu)) \\ &= -\langle X, T \rangle (T \cdot \nu - f) \cdot \omega. \end{aligned}$$

Plugging these two formulas in (62) we get the result. \square

We deduce the following key lemma:

prop phi equiv psi

Lemma 5.4. *A spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (22) is equivalent to a spinor field $\psi \in \Gamma(\Sigma M)$ solution of (58).*

Proof. We use the identification $\psi \in \Gamma(\Sigma M) \mapsto \psi^* \in \Gamma(\Sigma)$ described at the beginning of the section; we recall that, for all $X \in TM$,

properties ident psi phi

$$(63) \quad (\nabla_X \psi)^* = \nabla_X(\psi^*) \quad \text{and} \quad (X \cdot \psi)^* = X \cdot \nu \cdot (\psi^*).$$

Thus, if $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) and if $\psi \in \Gamma(\Sigma M)$ is such that $\psi^* = \varphi$, using (63), the formula

$$\sum_{j=1}^p e_j \cdot B(X, e_j) = \sum_{j=1}^p e_j \cdot \langle S(X), e_j \rangle \nu = S(X) \cdot \nu$$

and Lemma 5.3, we get:

$$\begin{aligned} (\nabla_X \psi)^* &= \nabla_X \varphi \\ &= -\frac{1}{2} S(X) \cdot \nu \cdot \varphi + \frac{1}{2} \{(2\tau - \sigma) \langle X, T \rangle (T \cdot \nu - f) - \tau X \cdot \nu\} \cdot \omega \cdot \varphi \\ &= \left(-\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \{(2\tau - \sigma) \langle X, T \rangle (T - f) - \tau X\} \cdot \omega \cdot \psi \right)^*. \end{aligned}$$

This gives (58). Reciprocally, if ψ is a solution of (58), the spinor field $\varphi = \psi^*$ solves (22). This proves the lemma. \square

Instead of $\psi \in \Gamma(\Sigma M)$ solution of (58) we may thus consider $\varphi \in \Gamma(U\Sigma)$ solution of (22). The theorem will thus be a consequence of Theorem 1 if we can define a bundle isomorphism $f : TM \oplus E \rightarrow M \times \mathcal{G}$ such that (17) and (18) hold. Let us set

$$f(Z) = \langle \langle Z \cdot \varphi, \varphi \rangle \rangle.$$

We first observe that $f(e_3)$ is constant: indeed, for all $X \in TM$,

$$\partial_X(f(e_3)) = \langle \langle \nabla_X e_3 \cdot \varphi, \varphi \rangle \rangle + (id + \tau) \langle \langle e_3 \cdot \nabla_X \varphi, \varphi \rangle \rangle = 0$$

in view of (61), (22) and identities (26)-(27) in Lemma 3.2. Moreover, since f preserves the norm of the vectors, $f(e_3)$ is a unit vector. Replacing φ by $\varphi \cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, we may thus assume that $f(e_3) = e_3^o$. We now check (17): since the map f is an orientation preserving isometry and using $f(e_3) = e_3^o$, we have, for all $X, Y \in TM$,

$$\begin{aligned} f(\Gamma(X)(Y)) &= f(\{\tau(X - \langle X, e_3 \rangle e_3) + (\sigma - \tau)\langle X, e_3 \rangle e_3\} \times Y) \\ &= \{\tau(f(X) - \langle f(X), f(e_3) \rangle f(e_3)) + (\sigma - \tau)\langle f(X), f(e_3) \rangle f(e_3)\} \times f(Y) \\ &= \{\tau(f(X) - \langle f(X), e_3^o \rangle e_3^o) + (\sigma - \tau)\langle f(X), e_3^o \rangle e_3^o\} \times f(Y) \\ &= \Gamma(f(X))(f(Y)). \end{aligned}$$

Finally, the proof of (18) is very similar to the proof of this identity made in Section 5.1 for $G = \mathbb{R}^n$: we only have to add the term involving Γ which appears in the expression (22) of the covariant derivative of φ ; we leave the details to the reader. \square

Remark 9. *We also get an explicit representation formula: the immersion is given by the Darboux integral of $\xi : X \mapsto \langle \langle X \cdot \varphi, \varphi \rangle \rangle$, which may be written in terms of ψ by the formula (53).*

We deduce the following result, first obtained by Daniel in [5] using the moving frame method:

Corollary 2. *If S, T, f, κ y τ satisfy (55)-(57), the Gauss equation*

$$(64) \quad K = \det S + \tau^2 + (\kappa - 4\tau^2) f^2$$

and the Codazzi equation

$$(65) \quad \nabla_X(SY) - \nabla_Y(SX) - S([X, Y]) = (\kappa - 4\tau^2)f(\langle Y, T \rangle X - \langle X, T \rangle Y),$$

then there exists an isometric immersion of M into $E(\kappa, \tau)$ with shape operator S . Moreover the immersion is unique up to a global isometry of $E(\kappa, \tau)$ preserving the orientations.

Proof. The equations (64) and (65) are equivalent to the Gauss and Codazzi equations (32) and (34) where B is defined by (60). \square

Peut-on en déduire la transformation de Lawson ?

5.4.3. *The last 3-dimensional riemannian homogeneous space: the metric Lie group Sol_3 .* We describe here the special case of a surface in Sol_3 : this achieves the spinor representation of immersions of surfaces into 3-dimensional riemannian homogeneous spaces [16].

Gauss equation E k t

Codazzi equation E k t

Let us recall that Sol_3 is the only metric Lie group whose isometry group is 3-dimensional. It is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the canonical basis (e_1^o, e_2^o, e_3^o) by

$$[e_1^o, e_2^o] = 0, \quad [e_2^o, e_3^o] = -e_2^o, \quad [e_3^o, e_1^o] = -e_1^o.$$

The metric on \mathcal{G} is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. By the Koszul formula, the Levi-Civita connection is then such that

$$\boxed{\text{Gamma ijk sol3}} \quad (66) \quad \Gamma_{11}^3 = -\Gamma_{13}^1 = -1, \quad \Gamma_{22}^3 = -\Gamma_{23}^2 = 1$$

and $\Gamma_{ij}^k = 0$ for the other indices.

Let us consider an oriented riemannian surface M , and a symmetric operator $S : TM \rightarrow TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leq i \leq 3$ satisfying

$$\boxed{\text{sol1}} \quad (67) \quad \langle T_i, T_j \rangle + f_i f_j = \delta_i^j$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

$$\boxed{\text{sol2}} \quad (68) \quad \begin{aligned} \nabla_X T_i &= (-1)^i \langle X, T_i \rangle T_3 + f_i S(X), \\ df_i(X) &= (-1)^i \langle X, T_i \rangle f_3 - \langle SX, T_i \rangle \end{aligned}$$

for $1 \leq i \leq 2$,

$$\boxed{\text{sol3}} \quad (69) \quad \begin{aligned} \nabla_X T_3 &= \sum_{j=1}^2 (-1)^{j+1} \langle X, T_j \rangle T_j + f_3 S(X), \\ df_3(X) &= \sum_{j=1}^2 (-1)^{j+1} \langle X, T_j \rangle f_j - \langle S(X), T_3 \rangle. \end{aligned}$$

The equations (68) and (69) are the equations (20) and (21) in Remark 2, with the coefficients Γ_{ij}^k given by (66). According to (51) together with the simplification in Lemma 5.2, we set

$$(70) \quad \tilde{\Gamma}(X) = -\{\langle X, T_1 \rangle (T_2 - f_2) + \langle X, T_2 \rangle (T_1 - f_1)\} \cdot \omega$$

for all $X \in TM$. Theorem 4 then yields the following result:

Theorem 6. *If M is simply connected, the following two statements are equivalent:*

- (1) *there exists an isometric immersion of M into Sol_3 with shape operator S ;*
- (2) *there exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and*

$$\boxed{\text{killing equation sol3}} \quad (71) \quad \nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi$$

for all $X \in TM$.

Remark 10. *This theorem implies the result of Lodovici [10] concerning isometric immersions in Sol_3 : by the considerations in Section 4, the equation (71) is solvable if and only if the equations of Gauss and Codazzi hold. Compléter ?*

APPENDIX A. SKEW-SYMMETRIC OPERATORS AND BIVECTORS

We consider \mathbb{R}^n endowed with its canonical scalar product. A skew-symmetric operator $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ naturally identifies to a bivector $\underline{u} \in \Lambda^2\mathbb{R}^n$, which may in turn be regarded as belonging to the Clifford algebra $Cl_n(\mathbb{R})$. We precise here the relations between the Clifford product in $Cl_n(\mathbb{R})$ and the composition of endomorphisms. If a and b belong to the Clifford algebra $Cl_n(\mathbb{R})$, we set

$$[a, b] = \frac{1}{2} (a \cdot b - b \cdot a),$$

where the dot \cdot is the Clifford product. We denote by (e_1, \dots, e_n) the canonical basis of \mathbb{R}^n .

lem1 ap1

Lemma A.1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a skew-symmetric operator. Then the bivector*

biv rep u

$$(72) \quad \underline{u} = \frac{1}{2} \sum_{j=1}^n e_j \cdot u(e_j) \in \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents u , and, for all $\xi \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi).$$

In the paper, and for sake of simplicity, we will use the same letter u to denote \underline{u} .

Proof. For $i < j$, we consider the linear map

$$u : \quad e_i \mapsto e_j, \quad e_j \mapsto -e_i, \quad e_k \mapsto 0 \quad \text{if } k \neq i, j;$$

it is skew-symmetric and corresponds to the bivector $e_i \wedge e_j \in \Lambda^2\mathbb{R}^n$; it is thus naturally represented by $\underline{u} = e_i \cdot e_j = \frac{1}{2} (e_i \cdot e_j - e_j \cdot e_i)$, which is (72). We then compute, for $k = 1, \dots, n$,

$$[\underline{u}, e_k] = \frac{1}{2} (e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j)$$

and easily get

$$[\underline{u}, e_k] = e_j \quad \text{if } k = i, \quad -e_i \quad \text{if } k = j, \quad 0 \quad \text{if } k \neq i, j.$$

The result follows by linearity. \square

lem2 ap1

Lemma A.2. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be two skew-symmetric operators, represented in $Cl_n(\mathbb{R})$ by*

$$u = \frac{1}{2} \sum_{j=1}^n e_j \cdot u(e_j) \quad \text{and} \quad v = \frac{1}{2} \sum_{j=1}^n e_j \cdot v(e_j)$$

respectively. Then $[u, v] \in \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R})$ represents $u \circ v - v \circ u$.

Proof. For $\xi \in \mathbb{R}^n$, the Jacobi equation yields

$$[[u, v], \xi] = [u, [v, \xi]] - [v, [u, \xi]].$$

Thus, using Lemma A.1 repeatedly, $[u, v]$ represents the map

$$\begin{aligned} \xi \mapsto [[u, v], \xi] &= [u, [v, \xi]] - [v, [u, \xi]] \\ &= [u, v(\xi)] - [v, u(\xi)] \\ &= (u \circ v - v \circ u)(\xi), \end{aligned}$$

and the result follows. \square

We now assume that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = n$.

lem3 ap1

Lemma A.3. *Let us consider a linear map $u : \mathbb{R}^p \rightarrow \mathbb{R}^q$ and its adjoint $u^* : \mathbb{R}^q \rightarrow \mathbb{R}^p$. Then the bivector*

$$\underline{u} = \sum_{j=1}^p e_j \cdot u(e_j) \in \Lambda^2 \mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} : \mathbb{R}^p \oplus \mathbb{R}^q \rightarrow \mathbb{R}^p \oplus \mathbb{R}^q,$$

we have

biv u u*

$$(73) \quad \underline{u} = \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) + \sum_{j=p+1}^n e_j \cdot (-u^*(e_j)) \right)$$

and, for all $\xi = \xi_p + \xi_q \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi_p) - u^*(\xi_q).$$

As above, we will simply denote \underline{u} by u .

Proof. In view of Lemma A.1, \underline{u} represents the linear map $\xi \mapsto [\underline{u}, \xi]$. We compute, for $\xi \in \mathbb{R}^p$,

$$\begin{aligned} [\underline{u}, \xi] &= \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^p e_j \cdot u(e_j) \right) \\ &= -\frac{1}{2} \sum_{j=1}^p (e_j \cdot \xi + \xi \cdot e_j) \cdot u(e_j) \\ &= \sum_{j=1}^p \langle \xi, e_j \rangle u(e_j) \\ &= u(\xi), \end{aligned}$$

and, for $\xi \in \mathbb{R}^q$,

$$\begin{aligned} [\underline{u}, \xi] &= \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^p e_j \cdot u(e_j) \right) \\ &= \frac{1}{2} \sum_{j=1}^p e_j \cdot (u(e_j) \cdot \xi + \xi \cdot u(e_j)) \\ &= -\sum_{j=1}^p e_j \langle u(e_j), \xi \rangle \\ &= -\sum_{j=1}^p e_j \langle e_j, u^*(\xi) \rangle \\ &= -u^*(\xi). \end{aligned}$$

Finally,

$$\begin{aligned}\underline{u} &= \sum_{j=1}^p e_j \cdot u(e_j) \\ &= \frac{1}{2} \left(\sum_{j=1}^p e_j \cdot u(e_j) + \sum_{j=1}^p -u(e_j) \cdot e_j \right)\end{aligned}$$

with

$$\begin{aligned}\sum_{j=1}^p -u(e_j) \cdot e_j &= - \sum_{i=p+1}^{p+q} \sum_{j=1}^p \langle u(e_j), e_i \rangle e_i \cdot e_j \\ &= \sum_{i=p+1}^{p+q} e_i \cdot \left(- \sum_{j=1}^p \langle e_j, u^*(e_i) \rangle e_j \right) \\ &= \sum_{i=p+1}^{p+q} e_i \cdot (-u^*(e_i)),\end{aligned}$$

which gives (73). \square

lem4 ap1

Lemma A.4. *Let us consider two linear maps $u : \mathbb{R}^p \rightarrow \mathbb{R}^q$ and $v : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with v skew-symmetric, and the associated bivectors*

$$u = \sum_{j=1}^p e_j \cdot u(e_j), \quad v = \frac{1}{2} \sum_{j=1}^n e_j \cdot v(e_j).$$

Then $[u, v] \in \Lambda^2 \mathbb{R}^n$ represents the map

$$\xi = \xi_p + \xi_q \mapsto -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p)),$$

where the sub-indices p and q mean that we take the components of the vectors in \mathbb{R}^p and \mathbb{R}^q respectively. In view of Lemma A.1, this may also be written in the form

$$[[u, v], \xi] = -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p))$$

for all $\xi \in \mathbb{R}^n$.

Proof. From Lemmas A.2 and A.3, the bivector $[u, v] \in \Lambda^2 \mathbb{R}^n$ represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \circ v - v \circ \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix},$$

that is the map

$$\begin{aligned}\xi &\mapsto \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} v(\xi)_p \\ v(\xi)_q \end{pmatrix} - v \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} \\ &= \begin{pmatrix} -u^*(v(\xi)_q) + v(u^*(\xi_q)) \\ u(v(\xi)_p) - v(u(\xi_p)) \end{pmatrix},\end{aligned}$$

which gives the result. \square

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