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THE FIRST TERMS IN THE EXPANSION OF THE BERGMAN KERNEL IN HIGHER DEGREES

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Abstract. We establish the cancellation of the first $2j$ terms in the diagonal asymptotic expansion of the restriction to the $(0,2j)$-forms of the Bergman kernel associated to the spin^c Dirac operator on high tensor powers of a positive line bundle twisted by a (non necessarily holomorphic) complex vector bundle, over a compact Kähler manifold. Moreover, we give a local formula for the first and the second (non-zero) leading coefficients.

0. Introduction

The Bergman kernel of a Kähler manifold endowed with a positive line bundle L is the smooth kernel of the Kodaira Laplacian $\square^L = \bar\partial^L \bar\partial^{L,*} + \bar\partial^{L,*}\bar\partial^L$. The existence of a diagonal asymptotic expansion of the Bergman kernel associated with the p^{th} tensor power of L when $p \to +\infty$ and the form of the leading term were proved in [Tia90], [Cat99] and [Zel98]. Moreover, the coefficients in this expansion encode geometric informations about the underlying manifold, and therefor they have been studied closely: the second and third terms were computed by Lu [Lu00], X. Wang [Wan05], L. Wang [Wan03] and recently by Ma-Marinescu [MM12] in different degrees of generality. This asymptotic plays an important role in various problems of Kähler geometry, see for instance [Don01] or [Fin]. We refer the reader to the book [MM07] for a comprehensive study of the Bergman kernel and its applications. See also the survey [Ma10].

In fact, Dai-Liu-Ma established the asymptotic of the Bergman kernel in the symplectic case in [DLM06], using the heat kernel (cf. also Ma-Marinescu [MM06]). Recently, this asymptotic in the symplectic case found an application in the study of variation of Hodge structures of vector bundles by Charbonneau and Stern in [CS11]. In their setting, the Bergman kernel is the kernel of a Kodaira-like Laplacian on a twisted bundle $L\otimes E$, where E is a (not necessarily holomorphic) complex vector bundle. Because of that, the Bergman kernel no longer concentrates in degree 0 (unlike it did in the Kähler case), and the asymptotic of its restriction to the $(0,2j)$-forms is related to the degree of 'non-holomorphicity'of E.

In this paper, we will show that the leading term in the asymptotic of the restriction to the $(0,2j)$-forms of the Bergman kernel is of order $p^{\dim X-2j}$ and we will compute it. That will lead to a local version of [CS11, (1.3)], which is the main technical result of their paper, see Remark 0.6. After that, we will also compute the second term in this asymptotic.

We now give more detail about our results. Let (X,ω,J) be a compact Kähler manifold of complex dimension n. Let (L,h^L) be a holomorphic Hermitian line bundle on X, and (E,h^E) a Hermitian complex vector bundle. We endow (L,h^L) with its Chern (i.e. holomorphic and Hermitian) connection ∇^L, and (E,h^E) with a Hermitian connection ∇^E, whose curvatures are respectively $R^L=(\nabla^L)^2$ and $R^E=(\nabla^E)^2$.

Except in the beginning of section 1.1, we will always assume that (L,h^L,∇^L) satisfies the *pre-quantization condition*:

Date: October 4, 2012.

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.1}$$

Let $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω and J. It induces a metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X) := \Lambda^\bullet(T^{*(0,1)}X)$, see Section 1.1.

Let $L^p = L^{\otimes p}$ be the p^{th} tensor power of L. Let $\Omega^{0,\bullet}(X, L^p \otimes E) = \mathscr{C}^\infty(X, \Lambda^{0,\bullet}(T^*X) \otimes L^p \otimes E)$ and $\bar\partial^{L^p\otimes E} : \Omega^{0,\bullet}(X, L^p \otimes E) \to \Omega^{0,\bullet+1}(X, L^p \otimes E)$ be the Dolbeault operator induced by the $(0,1)$-part of ∇^E (cf. (1.3)). Let $\bar\partial^{L^p\otimes E,*}$ be its dual with respect to the L^2-product. We set (see (1.6)):

$$D_p = \sqrt{2}\Big(\bar\partial^{L^p\otimes E} + \bar\partial^{L^p\otimes E,*}\Big), \tag{0.2}$$

which exchanges odd and even forms.

Definition 0.1. Let

$$P_p \colon \Omega^{0,\bullet}(X, L^p \otimes E) \to \ker(D_p) \tag{0.3}$$

be the orthogonal projection onto the kernel $\ker(D_p)$ of D_p. The operator P_p is called the *Bergman projection*. It has a smooth kernel with respect to $dv_X(y)$, denoted by $P_p(x,y)$, which is called the *Bergman kernel*.

Remark 0.2. If E is holomorphic, then by Hodge theory and the Kodaira vanishing theorem (see respectively [MM07, Theorem 1.4.1] and [MM07, Theorem 1.5.6]), we know that, for p large enough, P_p is the orthogonal projection $\mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$. Here, by [MM02, Theorem 1.1], we just know that $\ker(D_p|_{\Omega^{0,\mathrm{odd}}(X,L^p\otimes E)}) = 0$ for p large, so that $P_p \colon \Omega^{0,\mathrm{even}}(X, L^p \otimes E) \to \ker(D_p)$. In particular, $P_p(x,x) \in \mathscr{C}^\infty\big(X, \mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E)\big)$.

By Theorem 1.3, D_p is a Dirac operator, which enables us to apply the following result:

Theorem 0.3 (Dai-Liu-Ma, cf. [DLM06, Thm. 1.1])**.** *There exist smooth sections $\boldsymbol{b}_r$ of* $\mathrm{End}(\Lambda^{0,\mathrm{even}}(T^*X) \otimes E)$ *such that for any $k \in \mathbb{N}$ and for $p \to +\infty$:*

$$p^{-n} P_p(x,x) = \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} + O(p^{-k-1}), \tag{0.4}$$

that is for every $k, l \in \mathbb{N}$, there exists a constant $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$,

$$\left| p^{-n} P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{-r} \right|_{\mathscr{C}^l(X)} \leq C_{k,l}\, p^{-k-1}. \tag{0.5}$$

Here $|\cdot|_{\mathscr{C}^l(X)}$ is the $\mathscr{C}^l$-norm for the variable $x \in X$.

Let $\mathscr{R} = (R^E)^{0,2} \in \Omega^{0,2}(X, \mathrm{End}(E))$ be the $(0,2)$-part of R^E (which is zero if E is holomorphic). For $j \in [\![1, n]\!]$, let

$$I_j \colon \Lambda^{0,\bullet}(T^*X) \otimes E \to \Lambda^{0,j}(T^*X) \otimes E \tag{0.6}$$

be the natural orthogonal projection. The first main result in this paper is

Theorem 0.4. *For any $k \in \mathbb{N}$, $k \geq 2j$, we have when $p \to +\infty$:*

$$p^{-n} I_{2j} P_p(x,x) I_{2j} = \sum_{r=2j}^{k} I_{2j} \boldsymbol{b}_r(x) I_{2j}\, p^{-r} + O(p^{-k-1}), \tag{0.7}$$

and moreover,

$$I_{2j}\boldsymbol{b}_{2j}(x)I_{2j} = \frac{1}{(4\pi)^{2j}}\frac{1}{2^{2j}(j!)^2} I_{2j}\left(\mathscr{R}_x^j\right)\left(\mathscr{R}_x^j\right)^* I_{2j}, \tag{0.8}$$

where $\left(\mathscr{R}_x^j\right)^*$ *is the dual of* $\mathscr{R}_x^j$ *acting on* $(\Lambda^{0,\bullet}(T^*X)\otimes E)_x$.

Theorem 0.4 leads immediately to

Corollary 0.5. *Uniformly in* $x \in X$, *when* $p \to +\infty$, *we have*

$$\mathrm{Tr}\left((I_{2j}P_pI_{2j})(x,x)\right) = \frac{1}{(4\pi)^{2j}}\frac{1}{2^{2j}(j!)^2}\left\|\mathscr{R}_x^j\right\|^2 p^{n-2j} + O(p^{n-2j-1}). \tag{0.9}$$

Remark 0.6. By integrating (0.9) over X, we get

$$\mathrm{Tr}\left(I_{2j}P_pI_{2j}\right) = \frac{1}{(4\pi)^{2j}}\frac{1}{2^{2j}(j!)^2}\left\|\mathscr{R}^j\right\|^2_{L^2} p^{n-2j} + O(p^{n-2j-1}), \tag{0.10}$$

which is the main technical result of Charbonneau and Stern [CS11, (1.3)], thus Corollary 0.5 can be viewed as a local version of [CS11, (1.3)]. The constant in (0.10) differ from the one in [CS11] because our conventions are not the same as theirs (e.g. they choose $\omega = \sqrt{-1}R^L$, etc...).

Let $R^E_\Lambda := -\sqrt{-1}\sum_i R^E(w_i,\overline{w}_i)$ for $(\overline{w}_1,\dots,\overline{w}_n)$ an orthonormal frame of $T^{(0,1)}X$. Let R^{TX} be the curvature of the Levi-Civita connection ∇^{TX} of (X,g^{TX}), and for $(e_1,\dots,e_{2n})$ an orthonormal frame of TX, let $r^X = -\sum_{i,j}\langle R^{TX}(e_i,e_j)e_i,e_j\rangle$ be the scalar curvature of X.

For $j,k\in\mathbb{N}$ and $j\geq k$, we also define $\mathrm{C}_j(k)$ by

$$\mathrm{C}_j(k) := \frac{1}{(4\pi)^j}\frac{1}{2^k k!}\frac{1}{\prod_{s=k+1}^j(2s+1)}, \tag{0.11}$$

with the convention that $\prod_{s\in\emptyset} = 1$.

Let $\nabla^{\Lambda^{0,\bullet}}$ be the connection on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX}. Let $\nabla^{\Lambda^{0,\bullet}\otimes E}$ be the connection on $\Lambda^{0,\bullet}(T^*X)\otimes E$ induced by ∇^E and $\nabla^{\Lambda^{0,\bullet}}$, and let $\Delta^{\Lambda^{0,\bullet}\otimes E}$ be the associated Laplacian. For the precise definitions, see Section 1.1.

The second goal of this paper is to compute the second term in the expansion (0.7). As above, we will prove:

Theorem 0.7. *We can decompose* $I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j}$ *as the sum of 6 terms:*

$$I_{2j}\boldsymbol{b}_{2j+1}(x)I_{2j} = T_{\mathrm{I}_a} + (T_{\mathrm{I}_a})^* + T_{\mathrm{I}_b} + T_{\mathrm{II}_a} + (T_{\mathrm{II}_a})^* + T_{\mathrm{III}}, \tag{0.12}$$

where T_{I_a}, T_{I_b}, *etc. come from the terms* I_a, I_b, *etc. calculated in Section 3. Moreover, we have the following explicit formulas:*

if $j=0,1$, *then* $T_{\mathrm{I}_a}=0$, *and if* $j\geq 2$,

(0.13)

$$\begin{aligned} T_{\mathrm{I}_a} = \frac{\mathrm{C}_j(j)}{2\pi} I_{2j}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\Big\{&\Big(\mathrm{C}_j(j)-\mathrm{C}_j(q+1)\Big)\mathscr{R}_x^{j-(q+2)}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}_\cdot)(x)\mathscr{R}_x^{q-m}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i}\mathscr{R}_\cdot)(x)\mathscr{R}_x^m \\ &+ \mathrm{C}_j(m)\Big[\prod_{s=q+2}^{j}(1+\frac{1}{2s})-1\Big]\mathscr{R}_x^{j-(q+2)}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i}\mathscr{R}_\cdot)(x)\mathscr{R}_x^{q-m}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}_\cdot)(x)\mathscr{R}_x^m\Big\}\left(\mathscr{R}_x^j\right)^* I_{2j}, \end{aligned}$$

if $j=0$, then $T_{\mathrm{I}_b}=0$, and if $j\geq 1$,

$$T_{\mathrm{I}_b}=\frac{1}{2\pi}I_{2j}\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_x^{j-k-1}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\mathscr{R}_x^k\right] \times\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_x^{j-k-1}(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\mathscr{R}_x^k\right]^* I_{2j}, \tag{0.14}$$

if $j=0$, then $T_{\mathrm{II}_a}=0$, and if $j\geq 1$,

$$T_{\mathrm{II}_a}=\frac{\mathrm{C}_j(j)}{4\pi}I_{2j}\sum_{k=0}^{j-1}\left\{\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}_x^{j-(k+1)}(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}.)(x)\mathscr{R}_x^k\right\}\left(\mathscr{R}_x^j\right)^* I_{2j}, \tag{0.15}$$

(0.16)
$$T_{\mathrm{III}}=I_{2j}\mathrm{C}_j(j)\mathscr{R}_x^j\sum_{k=0}^{j}\left[\frac{1}{3}\left(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r^X_x-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x}\right]\left(\mathscr{R}_x^j\right)^* I_{2j}.$$

In particular, for $j=1$, we have:

$$\begin{aligned}128\pi^3 I_2\boldsymbol{b}_3(x)I_2=&\frac{1}{9}I_2(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\Big((\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i}\mathscr{R}.)(x)\Big)^* I_2\\&+\frac{1}{6}I_2\Big((\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}.)(x)(\mathscr{R}_x)^*+(\mathscr{R}_x)^*(\Delta^{\Lambda^{0,\bullet}\otimes E}\mathscr{R}.)(x)\Big)I_2\\&-I_2\mathscr{R}_x\left[\frac{r^X_x}{4}+R^E_\Lambda\right](\mathscr{R}_x)^*I_2.\end{aligned}\tag{0.17}$$

This paper is organized as follows: in Section 1 we compute the square of D_p and use a local trivialization to rescale it, and then give the Taylor expansion of the rescaled operator. In Section 2, we use this expansion to give a formula for the coefficients $\boldsymbol{b}_r$ appearing in (0.4), which will lead to a proof of Theorem 0.4. Finally, in Section 3, we prove Theorem 0.7 using again the formula for $\boldsymbol{b}_r$. In this whole paper, when an index variables appears twice in a single term, it means that we are summing over all its possible values.

1. Rescaling D_p^2 and Taylor expansion

In this section, we follow the method of [MM07, Chapter 4], that enables them to prove the existence of $\boldsymbol{b}_r$ in (0.4) in the case of a holomorphic vector bundle E, and that still applies here (as pointed out in [MM07, Section 8.1.1]). Then, in section 3 and 4, we will use this approach to understand $I_{2j}\boldsymbol{b}_rI_{2j}$ and prove Theorems 0.4 and 0.7.

In Section 1.1, we will first prove Theorem 1.3, and then give a formula for the square of D_p, which will be the starting point of our approach.

In Section 1.2, we will rescale the operator D_p^2 to get an operator $\mathscr{L}_t$, and then give the Taylor expansion of the rescaled operator.

In Section 1.3, we will study more precisely the limit operator $\mathscr{L}_0$.

1.1. **The square of D_p.**

For further details on the material of this subsection, the lector can read [MM07]. First of all let us give some notations.

The Riemannian volume form of (X,g^{TX}) is given by $dv_X=\omega^n/n!$. We will denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX\otimes\mathbb{C}$ induced by g^{TX}.

For the rest of this subsection 1.1, we will fix $(w_1,\dots,w_n)$ a local orthonormal frame of $T^{(1,0)}X$ with dual frame $(w^1,\dots,w^n)$. Then $(\overline{w}_1,\dots,\overline{w}_n)$ is a local orthonormal frame of $T^{(0,1)}X$ whose dual frame is denoted by $(\overline{w}^1,\dots,\overline{w}^n)$, and the vectors

$$e_{2j-1}=\frac{1}{\sqrt{2}}\,(w_j+\overline{w}_j)\ \text{ and } e_{2j+1}=\frac{\sqrt{-1}}{\sqrt{2}}\,(w_j-\overline{w}_j) \tag{1.1}$$

form a local orthonormal frame of TX.

We choose the Hermitian metric $h^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X):=\Lambda^\bullet(T^{*(0,1)}X)$ such that $\{\overline{w}^{j_1}\wedge\cdots\wedge\overline{w}^{j_k}/1\le j_1<\cdots<j_k\le n\}$ is an orthonormal frame of $\Lambda^{0,\bullet}(T^*X)$.

For any Hermitian bundle (F,h^F) over X, let $\mathscr{C}^\infty(X,F)$ be the space of smooth sections of F. It is endowed with the L^2-Hermitian metric:

$$\langle s_1,s_2\rangle=\int_X\langle s_1(x),s_2(x)\rangle_{h^F}\,dv_x(x). \tag{1.2}$$

The corresponding norm will be denoted by $\|\cdot\|_{L^2}$, and the completion of $\mathscr{C}^\infty(X,F)$ with respect to this norm by $L^2(X,F)$.

Let $\bar\partial^E$ be the Dolbeault operator of E: it is the $(0,1)$-part of the connection ∇^E

$$\bar\partial^E:=\left(\nabla^E\right)^{0,1}:\mathscr{C}^\infty(X,E)\to\mathscr{C}^\infty(X,T^{*(0,1)}X\otimes E). \tag{1.3}$$

We extend it to get an operator

$$\bar\partial^E\colon\Omega^{0,\bullet}(X,E)\to\Omega^{0,\bullet+1}(X,E) \tag{1.4}$$

by the Leibniz formula: for $s\in\mathscr{C}^\infty(X,E)$ and $\alpha\in\mathscr{C}^\infty(X,\Lambda^{0,\bullet}(T^*X))$ homogeneous,

$$\bar\partial^E(\alpha\otimes s)=(\bar\partial\alpha)\otimes s+(-1)^{\deg\alpha}\alpha\otimes\bar\partial^E s. \tag{1.5}$$

We can now define the operator

$$D^E=\sqrt{2}\left(\bar\partial^E+\bar\partial^{E,*}\right):\Omega^{0,\bullet}(X,E)\to\Omega^{0,\bullet}(X,E), \tag{1.6}$$

where the dual is taken with respect to the L^2- norm associated to the Hermitian metrics $h^{\Lambda^{0,\bullet}}$ and h^E.

Let $\nabla^{\Lambda(T^*X)}$ be the connection on $\Lambda(T^*X)$ induced by the Levi-Civita connection ∇^{TX} of X. Since X is Kähler, ∇^{TX} preserves $T^{(0,1)}X$ and $T^{(1,0)}X$. Thus it induces a connection $\nabla^{T^{*(0,1)}X}$ on $T^{*(0,1)}X$, and then an Hermitian connexion $\nabla^{\Lambda^{0,\bullet}}$ on $\Lambda^{0,\bullet}(T^*X)$. We then have that for any $\alpha\in\mathscr{C}^\infty(X,\Lambda^{0,\bullet}(T^*X))$,

$$\nabla^{\Lambda^{0,\bullet}}\alpha=\nabla^{\Lambda(T^*X)}\alpha. \tag{1.7}$$

Note the important fact that $\nabla^{\Lambda(T^*X)}$ preserves the bi-grading on $\Lambda^{\bullet,\bullet}(T^*X)$.

Let $\nabla^{\Lambda^{0,\bullet}\otimes E}:=\nabla^{\Lambda^{0,\bullet}}\otimes1+1\otimes\nabla^E$ be the connection on $\Lambda^{0,\bullet}(T^*X)\otimes E$ induced by $\nabla^{\Lambda^{0,\bullet}}$ and ∇^E

Proposition 1.1. *On $\Omega^{0,\bullet}(X,E)$, we have:*

$$\begin{aligned}\bar\partial^E&=\overline{w}^j\wedge\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_j},\\ \bar\partial^{E,*}&=-i_{\overline{w}_j}\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_j}.\end{aligned} \tag{1.8}$$

Proof. We still denote by ∇^E the extension of the connection ∇^E to $\Omega^{\bullet,\bullet}(X,E)$ by the usual formula $\nabla^E(\alpha\otimes s)=d\alpha\otimes s+(-1)^{\deg\alpha}\alpha\wedge\nabla^E s$ for $s\in\mathscr{C}^\infty(X,E)$ and $\alpha\in\mathscr{C}^\infty(X,\Lambda(T^*X))$ homogeneous. We know that $d=\varepsilon\circ\nabla^{\Lambda(T^*X)}$ where ε is the exterior multiplication (see [MM07, (1.2.44)]), so we get that $\nabla^E=\varepsilon\circ\nabla^{\Lambda(T^*X)\otimes E}$. Using (1.7), it follows that

$$\bar\partial^E=(\nabla^E)^{0,1}=\overline{w}^j\wedge\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_j},$$

which is the first part of (1.8).

The second part of our proposition follows classically from the first by exactly the same computation as in [MM07, Lemma 1.4.4]. □

Definition 1.2. Let $v = v^{1,0} + v^{0,1} \in TX = T^{(1,0)}X \oplus T^{(0,1)}X$, and $\bar{v}^{(0,1),*} \in T^{*(0,1)}X$ the dual of $v^{1,0}$ for $\langle \cdot, \cdot \rangle$. We define the *Clifford action of TX on $\Lambda^{0,\bullet}(T^*X)$* by

$$c(v) = \sqrt{2}\left(\bar{v}^{(0,1),*} \wedge -i_{v^{0,1}}\right). \tag{1.9}$$

We verify easily that for $u, v \in TX$,

$$c(u)c(v) + c(v)c(u) = -2\langle u, v\rangle, \tag{1.10}$$

and that for any skew-adjoint endomorphism A of TX,

$$\begin{aligned}\frac{1}{4}\langle Ae_i, e_j\rangle c(e_i)c(e_j) = &-\frac{1}{2}\langle Aw_j, \overline{w}_j\rangle + \langle Aw_\ell, \overline{w}_m\rangle \overline{w}^m \wedge i_{\overline{w}_\ell}\\ &+ \frac{1}{2}\langle Aw_\ell, w_m\rangle i_{\overline{w}_\ell} i_{\overline{w}_m} + \frac{1}{2}\langle A\overline{w}_\ell, \overline{w}_m\rangle \overline{w}^\ell \wedge \overline{w}^m \wedge .\end{aligned} \tag{1.11}$$

Let $\nabla^{\det}$ be the Chern connection of $\det(T^{(1,0)}X) := \Lambda^n(T^{(1,0)}X)$, and ∇^{Cl} the Clifford connexion on $\Lambda^{0,\bullet}(T^*X)$ induced by ∇^{TX} and $\nabla^{\det}$ (see [MM07, (1.3.5)]). We also denote by ∇^{Cl} the connection on $\Lambda^{0,\bullet}(T^*X) \otimes E$ induced by ∇^{Cl} and ∇^E. By [MM07, (1.3.5)], (1.11) and the fact that $\nabla^{\det}$ is holomorphic, we get

$$\nabla^{\mathrm{Cl}} = \nabla^{\Lambda^{0,\bullet}}. \tag{1.12}$$

Let $D^{c,E}$ be the associated spin^c Dirac operator:

$$D^{c,E} = \sum_{j=1}^{2n} c(e_j)\nabla^{\mathrm{Cl}}_{e_j} \colon \Omega^{0,\bullet}(X,E) \to \Omega^{0,\bullet}(X,E). \tag{1.13}$$

By (1.8) and (1.12), we have

Theorem 1.3. *D^E is equal to the spin^c Dirac operator $D^{c,E}$ acting on $\Omega^{0,\bullet}(X,E)$.*

Remark 1.4. Note that all the results proved in the beginning of this subsection hold without assuming the pre-quantization condition (0.1), but from now on, we will use it.

Let (F, h^F) be a Hermitian vector bundle on X and let ∇^F be a Hermitian connection on F. Then the *Bochner Laplacian* Δ^F acting on $\mathscr{C}^\infty(X,F)$ is defined by

$$\Delta^F = -\sum_{j=1}^{2n}\left((\nabla^F_{e_j})^2 - \nabla^F_{\nabla^{TX}_{e_j}e_j}\right). \tag{1.14}$$

On $\Omega^{0,\bullet}(X)$, we define the *number operator* $\mathcal{N}$ by

$$\mathcal{N}|_{\Omega^{0,j}(X)} = j, \tag{1.15}$$

and we also denote by $\mathcal{N}$ the operator $\mathcal{N} \otimes 1$ acting on $\Omega^{0,\bullet}(X,F)$.

The bundle L^p is endowed with the connection ∇^{L^p} induced by ∇^L (which is also its Chern connection). Let $\nabla^{L^p\otimes E} := \nabla^{L^p} \otimes 1 + 1 \otimes \nabla^E$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. We will denote

$$D_p = D^{L^p\otimes E}. \tag{1.16}$$

Theorem 1.5. *The square of D_p is given by*

$$\begin{aligned}D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} &- R^E(w_j, \overline{w}_j) - 2\pi pn + 4\pi p\mathcal{N} + 2\left(R^E + \frac{1}{2}R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell}\\ &+ R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} + R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m.\end{aligned} \tag{1.17}$$

Proof. By Theorem 1.3, we can use [MM07, Theorem 1.3.5]:

$$D_p^2 = \Delta^{\mathrm{Cl}} + \frac{r^X}{4} + \frac{1}{2}\left(R^{L^p\otimes E} + \frac{1}{2}R^{\det}\right)(e_i,e_j)c(e_i)c(e_j), \tag{1.18}$$

where r^X is the scalar curvature of X. Thanks to (1.12), we see that $\Delta^{\mathrm{Cl}} = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E}$. Moreover, $r^X = 2R^{\det}(w_j,\overline{w}_j)$ and $R^{L^p\otimes E} = R^E + pR^L$. Using the equivalent of (1.11) for 2-forms (substituting $A(\cdot,\cdot)$ for $\langle A\cdot,\cdot\rangle$) and the fact that R^L and R^E are $(1,1)$-forms, (1.18) reads

$$\begin{aligned} D_p^2 = \Delta^{\Lambda^{0,\bullet}\otimes L^p\otimes E} &+ \frac{1}{2}R^{\det}(w_j,\overline{w}_j) - \left(R^E(w_j,\overline{w}_j) + pR^L(w_j,\overline{w}_j) + \frac{1}{2}R^{\det}(w_j,\overline{w}_j)\right) \\ &+ 2\left(R^E + pR^L + \frac{1}{2}R^{\det}\right)(w_\ell,\overline{w}_m)\overline{w}^m\wedge i_{\overline{w}_\ell} + R^E(w_\ell,w_m)i_{\overline{w}_\ell}i_{\overline{w}_m} \\ &+ R^E(\overline{w}_\ell,\overline{w}_m)\overline{w}^\ell\wedge\overline{w}^m. \end{aligned}$$

Thanks to (0.1), we have $R^L(w_\ell,\overline{w}_m) = 2\pi\delta_{\ell m}$. Moreover, $\mathcal{N} = \sum_\ell \overline{w}^\ell\wedge i_{\overline{w}_\ell}$, thus we get Theorem 1.5. □

1.2. **Rescaling D_p^2.** In this subsection, we want to rescale D_p^2, but in order to do this, we must define it on a vector space. Therefore, we will use normal coordinates to transfer the problem on the tangent space to X at a fix point. Then we want to give a Taylor expansion of the rescaled operator, but the problem is that each operator acts on a different space, namely

$$\mathrm{E}_p := \Lambda^{0,\bullet}(T^*X)\otimes L^p\otimes E,$$

so we must first handle this issue.

Fix $x_0\in X$. For the rest of this paper, we fix $\{w_j\}$ an orthonormal basis of $T^{(1,0)}_{x_0}X$, with dual basis $\{w^j\}$, and we construct an orthonormal basis $\{e_i\}$ of $T_{x_0}X$ from $\{w_j\}$ as in (1.1).

For $\varepsilon>0$, we denote by $B^X(x_0,\varepsilon)$ and $B^{T_{x_0}X}(0,\varepsilon)$ the open balls in X and $T_{x_0}X$ with center x_0 and 0 and radius ε respectively. If $\exp^X_{x_0}$ is the Riemannian exponential of X, then for ε small enough, $Z\in B^{T_{x_0}X}(0,\varepsilon)\mapsto \exp^X_{x_0}(Z)\in B^X(x_0,\varepsilon)$ is a diffeomorphism, which gives local coordinates by identifying $T_{x_0}X$ with $\mathbb{R}^{2n}$ via the orthonormal basis $\{e_i\}$:

$$(Z_1,\dots,Z_{2n})\in\mathbb{R}^{2n}\mapsto \sum_i Z_ie_i\in T_{x_0}X. \tag{1.19}$$

From now on, we will always identify $B^{T_{x_0}X}(0,\varepsilon)$ and $B^X(x_0,\varepsilon)$. Note that in this identification, the radial vector field $\mathcal{R}=\sum_i Z_ie_i$ becomes $\mathcal{R}=Z$, so Z can be viewed as a point or as a tangent vector.

For $Z\in B^{T_{x_0}X}(0,\varepsilon)$, we identify (L_Z,h^L_Z), (E_Z,h^E_Z) and $(\Lambda^{0,\bullet}_Z(T^*X),h^{\Lambda^{0,\bullet}}_Z)$ with $(L_{x_0},h^L_{x_0})$, $(E_{x_0},h^E_{x_0})$ and $(\Lambda^{0,\bullet}(T^*_{x_0}X),h^{\Lambda^{0,\bullet}}_{x_0})$ by parallel transport with respect to the connection ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$ along the geodesic ray $t\in[0,1]\mapsto tZ$. We denote by Γ^L, Γ^E and $\Gamma^{\Lambda^{0,\bullet}}$ the corresponding connection forms of ∇^L, ∇^E and $\nabla^{\Lambda^{0,\bullet}}$.

Remark 1.6. Note that since $\nabla^{\Lambda^{0,\bullet}}$ preserves the degree, the identification between $\Lambda^{0,\bullet}(T^*X)$ and $\Lambda^{0,\bullet}(T^*_{x_0}X)$ is compatible with the degree. Thus $\Gamma^{\Lambda^{0,\bullet}}_Z\in\bigoplus_j\mathrm{End}(\Lambda^{0,j}(T^*X))$.

Let S_L be a unit vector of L_{x_0}. It gives an isometry $L^p_{x_0}\simeq\mathbb{C}$, which yields to an isometry

$$\mathrm{E}_{p,x_0}\simeq(\Lambda^{0,\bullet}(T^*X)\otimes E)_{x_0} =: \mathbb{E}_{x_0}. \tag{1.20}$$

Thus, in our trivialization, D_p^2 acts on $\mathbb{E}_{x_0}$, but this action may *a priori* depends on the choice of S_L. In fact, since the operator D_p^2 takes values in $\mathrm{End}(\mathrm{E}_{p,x_0})$ which is canonically isomorphic to $\mathrm{End}(\mathbb{E})_{x_0}$ (by the natural identification $\mathrm{End}(L^p) \simeq \mathbb{C}$), all our formulas do not depend on this choice.

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$, and $\kappa(Z)$ be the smooth positive function defined for $|Z| \leq \varepsilon$ by

$$dv_X(Z) = \kappa(Z) dv_{TX}(Z), \tag{1.21}$$

with $\kappa(0) = 1$.

Definition 1.7. We denote by ∇_U the ordinary differentiation operator in the direction U on $T_{x_0}X$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}
&(S_t s)(Z) = s(Z/t),\\
&\nabla_t = t S_t^{-1} \kappa^{1/2} \nabla^{\mathrm{Cl}_0} \kappa^{-1/2} S_t,\\
&\nabla_0 = \nabla + \frac{1}{2} R^L_{x_0}(Z, \cdot),\\
&\mathscr{L}_t = t^2 S_t^{-1} \kappa^{1/2} D_p^2 \kappa^{-1/2} S_t,\\
&\mathscr{L}_0 = -\sum_i (\nabla_{0,e_i})^2 + 4\pi\mathcal{N} - 2\pi n.
\end{aligned} \tag{1.22}$$

Let $||\cdot||_{L^2}$ be the L^2-norm induced by $h^{\mathrm{E}_{x_0}}$ and dv_{TX}. We can now state the key result in our approach of Theorems 0.4 and 0.7:

Theorem 1.8. *There exist second order formally self-adjoint (with respect to $||\cdot||_{L^2}$) differential operators $\mathcal{O}_r$ with polynomial coefficients such that for all $m \in \mathbb{N}$,*

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^m t^r \mathcal{O}_r + O(t^{m+1}). \tag{1.23}$$

Furthermore, each $\mathcal{O}_r$ can be decomposed as

$$\mathcal{O}_r = \mathcal{O}_r^0 + \mathcal{O}_r^{+2} + \mathcal{O}_r^{-2}, \tag{1.24}$$

where $\mathcal{O}_r^k$ changes the degree of the form it acts on by k.

Proof. The first part of the theorem (i.e. equation (1.23)) is contained in [MM08, Theorem 1.4]. We will briefly recall how they obtained this result.

Let Φ_E be the smooth self-adjoint section of $\mathrm{End}(\mathbb{E}_{x_0})$ on $B^{T_{x_0}X}(0,\varepsilon)$:

$$\begin{aligned}
\Phi_E = -R^E(w_j, \overline{w}_j) + 2\left(R^E + \frac{1}{2}R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell}\\
+ R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} + R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m.
\end{aligned} \tag{1.25}$$

We can see that we can decompose $\Phi_E = \Phi_E^0 + \Phi_E^{+2} + \Phi_E^{-2}$, where

$$\begin{aligned}
&\Phi_E^0 = R^E(w_j, \overline{w}_j) + 2\left(R^E + \frac{1}{2}R^{\det}\right)(w_\ell, \overline{w}_m)\overline{w}^m \wedge i_{\overline{w}_\ell} \text{ preserves the degree,}\\
&\Phi_E^{+2} = R^E(\overline{w}_\ell, \overline{w}_m)\overline{w}^\ell \wedge \overline{w}^m \text{ rises the degree by 2,}\\
&\Phi_E^{-2} = R^E(w_\ell, w_m) i_{\overline{w}_\ell} i_{\overline{w}_m} \text{ lowers the degree by 2.}
\end{aligned} \tag{1.26}$$

Using Theorem 1.5, we find that :

$$D_p^2 = \Delta^{\Lambda^{0,\bullet} \otimes L^p \otimes E} + p(-2\pi n + 4\pi\mathcal{N}) + \Phi_E. \tag{1.27}$$

Let $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z)$ and $\left(g^{ij}(Z)\right)_{ij}$ be the inverse of the matrix $(g_{ij}(Z))_{ij}$. Let $\left(\nabla^{TX}_{e_i} e_j\right)(Z) = \Gamma^k_{ij}(Z) e_k$. As in [MM07, (4.1.34)], by (1.22) and (1.27), we get:

$$\begin{aligned} \nabla_{t,\cdot} &= \kappa^{1/2}(tZ)\left(\nabla_{\cdot} + t\Gamma^{\Lambda^{0,\bullet}}_{tZ} + \frac{1}{t}\Gamma^L_{tZ} + t\Gamma^E_{tZ}\right)\kappa^{-1/2}(tZ), \\ \mathscr{L}_t &= -g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}\right) - 2\pi n + 4\pi\mathscr{N} + t^2\Phi_E(tZ). \end{aligned} \tag{1.28}$$

Moreover, $\kappa = (\det(g_{ij}))^{1/2}$, thus we can prove equations (1.23) as in [MM07, Theorem 4.1.7] by taking the Taylor expansion of each term appearing in (1.28) (note that in [MM07], they have to extend every data to $T_{x_0}X$ to make the analysis works, but as we admit the result, we do not have to worry about it and simply restrict ourselves to a neighborhood of x_0).

Now, it is clear that in the formula for $\mathscr{L}_t$ in (1.28), the term

$$\mathscr{L}^0_t := -g^{ij}(tZ)\left(\nabla_{t,e_i}\nabla_{t,e_j} - t\Gamma^k_{ij}(tZ)\nabla_{t,e_k}\right) - 2\pi n + 4\pi\mathscr{N} + t^2\Phi^0_E(tZ) \tag{1.29}$$

preserves the degree, because $\Gamma^{\Lambda^{0,\bullet}}$ does (as explained in Remark 1.6). Thus, using (1.26) and taking Taylor expansion of $\mathscr{L}_t$ in (1.28), we can write

$$\begin{aligned} \mathscr{L}^0_t &= \mathscr{L}_0 + \sum_{r=1}^{\infty} t^r \mathcal{O}^0_r, \\ t^2\Phi^{\pm 2}_E(tZ) &= \sum_{r=2}^{\infty} t^r \mathcal{O}^{\pm 2}_r. \end{aligned} \tag{1.30}$$

From (1.30), we get (1.24).

Finally, due to the presence of the conjugation by $\kappa^{1/2}$ in (1.22), $\mathscr{L}_t$ is a formally self-adjoint operator on $\mathscr{C}^\infty(\mathbb{R}^{2n}, \mathrm{E}_{x_0})$ with respect to $||\cdot||_{L^2}$. Thus $\mathscr{L}_0$ and the $\mathcal{O}_r$'s also are. □

Proposition 1.9. *We have*

$$\mathcal{O}_1 = 0. \tag{1.31}$$

For $\mathcal{O}_2$, we have the formulas:

$$\mathcal{O}^{+2}_2 = \mathscr{R}_{x_0}\,, \quad \mathcal{O}^{-2}_2 = (\mathscr{R}_{x_0})^*\,, \tag{1.32}$$

and

$$\begin{aligned} \mathcal{O}^0_2 = \frac{1}{3}\langle R^{TX}_{x_0}(Z,e_i)Z, e_j\rangle \nabla_{0,e_i}\nabla_{0,e_j} - R^E_{x_0}(w_j, \overline{w}_j) - \frac{r^X_{x_0}}{6} \\ + \left(\left\langle \frac{1}{3}R^{TX}_{x_0}(Z,e_k)e_k + \frac{\pi}{3}R^{TX}_{x_0}(z,\bar{z})Z, e_j\right\rangle - R^E_{x_0}(Z,e_j)\right)\nabla_{0,e_j}. \end{aligned} \tag{1.33}$$

Proof. For $F = L, E$ or $\Lambda^{0,\bullet}(T^*X)$, it is known that (see for instance [MM07, Lemma 1.2.4])

$$\sum_{|\alpha|=r} (\partial^\alpha \Gamma^F)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^F)_{x_0}(Z,e_j)\frac{Z^\alpha}{\alpha!}, \tag{1.34}$$

and in particular,

$$\Gamma^F_Z(e_j) = \frac{1}{2}R^F_{x_0}(Z,e_j) + O(|Z|^2). \tag{1.35}$$

Furthermore, we know that

$$g_{ij}(Z) = \delta_{ij} + O(|Z|^2)\,: \tag{1.36}$$

it is the Gauss lemma (see [MM07, (1.2.19)]). It implies that

$$\kappa(Z) = |\det(g_{ij}(Z))|^{1/2} = 1 + O(|Z|^2). \tag{1.37}$$

Moreover, the second line of [MM07, (4.1.103)] entails

$$\frac{\sqrt{-1}}{2\pi}R^L_Z(Z,e_j)=\langle JZ,e_j\rangle+O(|Z|^3), \tag{1.38}$$

and thus by (1.34) and (1.38)

$$\Gamma^L_Z=\frac{1}{2}R^L_{x_0}(Z,e_j)+O(|Z|^3). \tag{1.39}$$

Using (1.28), (1.35), (1.37) and (1.39), we see that

$$\nabla_t=\nabla_0+O(t^2). \tag{1.40}$$

Finally, using again (1.28), (1.36) and (1.40), we get $\mathcal{O}_1=0$.

Concerning $\mathcal{O}_2^{\pm 2}$, thanks to (1.30), we see that

$$\begin{aligned}\mathcal{O}_2^{+2}&=\Phi_E^{+2}(0)=R^E_{x_0}(\overline{w}_\ell,\overline{w}_m)\overline{w}^\ell\wedge\overline{w}^m=(R^E_{x_0})^{0,2}=\mathscr{R}_{x_0},\\ \mathcal{O}_2^{-2}&=\Phi_E^{-2}(0)=R^E_{x_0}(w_\ell,w_m)i_{\overline{w}_\ell}i_{\overline{w}_m}=\left((R^E_{x_0})^{0,2}\right)^*=(\mathscr{R}_{x_0})^*\,.\end{aligned} \tag{1.41}$$

Finally, by (1.29) and [MM07, (4.1.34)], we see that our $\mathscr{L}^0_t$ corresponds to $\mathscr{L}_t$ in [MM07]. Thus, by (1.30) and [MM07, (4.1.31)], our $\mathcal{O}_2^0$ is equal to their $\mathcal{O}_2$ (this is because in their case, E is holomorphic, so R^E is a $(1,1)$-form and there is no term changing the degree in $(\bar\partial^{L^p\otimes E}+\bar\partial^{L^p\otimes E,*})^2$, but the terms preserving the degree are the same than ours). Hence (1.33) follows from [MM07, Theorem 4.1.25]. □

1.3. **Bergman kernel of the limit operator $\mathscr{L}_0$.** In this subsection we study more precisely the operator $\mathscr{L}_0$.

We introduce the complex coordinates $z=(z_1,\dots,z_n)$ on $\mathbb{C}^n\simeq\mathbb{R}^{2n}$. Thus we get $Z=z+\bar z$, $w_j=\sqrt{2}\frac{\partial}{\partial z_j}$ and $\overline{w}_j=\sqrt{2}\frac{\partial}{\partial \bar z_j}$. We will identify z to $\sum_j z_j\frac{\partial}{\partial z_j}$ and $\bar z$ to $\sum_j \bar z_j\frac{\partial}{\partial \bar z_j}$ when we consider z and $\bar z$ as vector fields.

Set

$$\begin{aligned}b_i&=-2\nabla_{0,\frac{\partial}{\partial z_j}},\quad b_i^+=2\nabla_{0,\frac{\partial}{\partial \bar z_j}},\\ b&=(b_1,\dots,b_n),\quad \mathscr{L}=-\sum_i(\nabla_{0,e_i})^2-2\pi n.\end{aligned} \tag{1.42}$$

By definition, $\nabla_0=\nabla+\frac{1}{2}R^L_{x_0}(Z,\cdot)$ so we get

$$b_i=-2\frac{\partial}{\partial z_i}+\pi\bar z_i,\quad b_i^+=2\frac{\partial}{\partial \bar z_i}+\pi z_i, \tag{1.43}$$

and for any polynomial $g(z,\bar z)$ in z and $\bar z$,

$$\begin{aligned}&[b_i,b_j^+]=-4\pi\delta_{ij}, && [b_i,b_j]=[b_i^+,b_j^+]=0,\\ &[g(z,\bar z),b_j]=2\tfrac{\partial}{\partial z_j}g(z,\bar z), && [g(z,\bar z),b_j^+]=-2\tfrac{\partial}{\partial \bar z_j}g(z,\bar z).\end{aligned} \tag{1.44}$$

Finally, a simple calculation shows:

$$\mathscr{L}=\sum_i b_ib_i^+\text{ and }\mathscr{L}_0=\mathscr{L}+4\pi\mathcal{N}. \tag{1.45}$$

Recall that we denoted by $||\cdot||_{L^2}$ the L^2-norm associated to $h^{\mathbb{E}_{x_0}}$ and dv_{TX}. As for this norm $b_i^+=(b_i)^*$, we see that $\mathscr{L}$ and $\mathscr{L}_0$ are self-adjoint with respect to this norm.

Next theorem is proved in [MM07, Theorem 4.1.20]:

Theorem 1.10. *The spectrum of the restriction of $\mathscr{L}$ to $L^2(\mathbb{R}^{2n})$ is $\mathrm{Sp}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = 4\pi\mathbb{N}$ and an orthogonal basis of the eigenspace for the eigenvalue $4\pi k$ is*

$$b^\alpha\left(z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right)\right), \text{ with } \alpha,\beta \in \mathbb{N}^n \text{ and } \sum_i \alpha_i = k. \tag{1.46}$$

Especially, an orthonormal basis of $\ker(\mathscr{L}|_{L^2(\mathbb{R}^{2n})})$ is

$$\left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta \exp\left(-\frac{\pi}{2}|z|^2\right), \tag{1.47}$$

and thus if $\mathscr{P}(Z,Z')$ is the smooth kernel of $\mathscr{P}$ the orthogonal projection from $(L^2(\mathbb{R}^{2n}), ||\cdot||_0)$ onto $\ker(\mathscr{L})$ (where $||\cdot||_0$ is the L^2-norm associated to $g^{TX}_{x_0}$) with respect to $dv_{TX}(Z')$, we have

$$\mathscr{P}(Z,Z') = \exp\left(-\frac{\pi}{2}(|z|^2+|z'|^2-2z\cdot\bar{z}')\right). \tag{1.48}$$

Now let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}), ||\cdot||_{L^2})$ onto $N := \ker(\mathscr{L}_0)$, and $P^N(Z,Z')$ be its smooth kernel with respect to $dv_{TX}(Z')$. From (1.45), we have:

$$P^N(Z,Z') = \mathscr{P}(Z,Z')I_0. \tag{1.49}$$

2. The first coefficient in the asymptotic expansion

In this Section we prove Theorem 0.4. We will proceed as follows.

In Section 2.1, following [MM07, Section 4.1.7], we will give a formula for $\boldsymbol{b}_r$ involving the $\mathcal{O}_k$'s and $\mathscr{L}_0$.

In Section 2.2, we will see how this formula entails Theorem 0.4.

2.1. A formula for $\boldsymbol{b}_r$.

By Theorem 1.10 and (1.45), we know that for every $\lambda \in \delta$ the unit circle in $\mathbb{C}$, $(\lambda-\mathscr{L}_0)^{-1}$ exists.

Let $f(\lambda,t)$ be a formal power series on t with values in $\mathrm{End}(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0}))$:

$$f(\lambda,t) = \sum_{r=0}^{+\infty} t^r f_r(\lambda) \text{ with } f_r(\lambda) \in \mathrm{End}(L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})). \tag{2.1}$$

Consider the equation of formal power series on t for $\lambda \in \delta$:

$$\left(\lambda - \mathscr{L}_0 - \sum_{r=1}^{+\infty} t^r \mathcal{O}_r\right) f(\lambda,t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})}. \tag{2.2}$$

We then find that

$$\begin{aligned} f_0(\lambda) &= (\lambda-\mathscr{L}_0)^{-1}, \\ f_r(\lambda) &= (\lambda-\mathscr{L}_0)^{-1}\sum_{j=1}^{r} \mathcal{O}_j f_{r-j}(\lambda). \end{aligned} \tag{2.3}$$

Thus by (1.31) and by induction,

$$f_r(\lambda) = \left(\sum_{\substack{r_1+\cdots+r_k=r \\ r_j\geq 2}} (\lambda-\mathscr{L}_0)^{-1}\mathcal{O}_{r_1}\cdots(\lambda-\mathscr{L}_0)^{-1}\mathcal{O}_{r_k}\right)(\lambda-\mathscr{L}_0)^{-1}. \tag{2.4}$$

Definition 2.1. Following [MM07, (4.1.91)], we define $\mathscr{F}_r$ by

$$\mathscr{F}_r = \frac{1}{2\pi\sqrt{-1}}\int_\delta f_r(\lambda)d\lambda, \tag{2.5}$$

and we denote by $\mathscr{F}_r(Z,Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

Theorem 2.2. *The following equation holds:*

$$\boldsymbol{b}_r(x_0) = \mathscr{F}_{2r}(0,0). \tag{2.6}$$

Proof. This formula follows from [MM07, Theorem 8.1.4], as [MM07, (4.1.97)] follows from [MM07, Theorem 4.1.24], remembering that in our situation, the Bergman kernel P_p does not concentrate in degree 0. □

2.2. **Proof of Theorem 0.4.** Let $T_{\mathbf{r}}(\lambda) = (\lambda - \mathscr{L}_0)^{-1}\mathcal{O}_{r_1}\cdots(\lambda-\mathscr{L}_0)^{-1}\mathcal{O}_{r_k}(\lambda-\mathscr{L}_0)^{-1}$ be the term in the sum (2.4) corresponding to $\mathbf{r} = (r_1,\dots,r_k)$. Let $N^\perp$ be the orthogonal of N in $L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$, and $P^{N^\perp}$ be the associated orthogonal projector. In $T_{\mathbf{r}}(\lambda)$, each term $(\lambda-\mathscr{L}_0)^{-1}$ can be decomposed as

$$(\lambda-\mathscr{L}_0)^{-1} = (\lambda-\mathscr{L}_0)^{-1}P^{N^\perp} + \frac{1}{\lambda}P^N. \tag{2.7}$$

Set

$$L^{N^\perp}(\lambda) = (\lambda-\mathscr{L}_0)^{-1}P^{N^\perp}, \qquad L^N(\lambda) = \frac{1}{\lambda}P^N. \tag{2.8}$$

By (1.45), $\mathscr{L}_0$ preserves the degree, and thus so do $(\lambda-\mathscr{L}_0)^{-1}$, $L^{N^\perp}$ and L^N.

For $\eta = (\eta_1,\dots,\eta_{k+1}) \in \{N,N^\perp\}^{k+1}$, let

$$T^\eta_{\mathbf{r}}(\lambda) = L^{\eta_1}(\lambda)\mathcal{O}_{r_1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda). \tag{2.9}$$

We can decompose:

$$T_{\mathbf{r}}(\lambda) = \sum_{\eta=(\eta_1,\dots,\eta_{k+1})} T^\eta_{\mathbf{r}}(\lambda), \tag{2.10}$$

and by (2.4) and (2.5)

$$\mathscr{F}_{2r} = \frac{1}{2\pi\sqrt{-1}}\sum_{\substack{|\mathbf{r}|=2r\\ |\eta|=2r+1}}\int_\delta T^\eta_{\mathbf{r}}(\lambda)d\lambda. \tag{2.11}$$

Note that $L^{N^\perp}(\lambda)$ is an holomorphic function of λ, so

$$\int_\delta L^{N^\perp}(\lambda)\mathcal{O}_{r_1}\cdots L^{N^\perp}(\lambda)\mathcal{O}_{r_k}L^{N^\perp}(\lambda)d\lambda = 0. \tag{2.12}$$

Thus, in (2.11), every non-zero term that appears contains at least one $L^N(\lambda)$:

$$\int_\delta T^\eta_{\mathbf{r}}(\lambda)d\lambda \neq 0 \Rightarrow \text{there exists } i_0 \text{ such that } \eta_{i_0} = N. \tag{2.13}$$

Now fix k and j in $\mathbb{N}$. Let $s \in L^2(\mathbb{R}^{2n},\mathbb{E}_{x_0})$ be a form of degree $2j$, $\mathbf{r} \in (\mathbb{N}\setminus\{0,1\})^k$ such that $\sum_i r_i = 2r$ and $\eta = (\eta_1,\dots,\eta_{k+1}) \in \{N,N^\perp\}^{k+1}$ such that there is a i_0 satisfying $\eta_{i_0} = N$. We want to find a necessary condition for $I_{2j}T^\eta_{\mathbf{r}}(\lambda)I_{2j}s$ to be non-zero.

Suppose then that $I_{2j}T^\eta_{\mathbf{r}}(\lambda)I_{2j}s \neq 0$. Since $L^{\eta_{i_0}} = \frac{1}{\lambda}P^N$, and N is concentrated in degree 0, we must have

$$\deg\left(\mathcal{O}_{r_{i_0}}L^{\eta_{i_0+1}}(\lambda)\mathcal{O}_{r_{i_0+1}}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s\right) = 0,$$

but each $L^{\eta_i}(\lambda)$ preserves the degree, and by Theorem 1.8 each $\mathcal{O}_{r_i}$ lowers the degree at most by 2, so

$$0 = \deg\left(\mathcal{O}_{r_{i_0}}L^{\eta_{i_0+1}}(\lambda)\mathcal{O}_{r_{i_0+1}}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s\right) \geq 2j - 2(k-i_0+1),$$

and thus

$$2j \leq 2(k-i_0+1). \tag{2.14}$$

Similarly, $L^{\eta_1}(\lambda)\mathcal{O}_{r_1}\cdots L^{\eta_k}(\lambda)\mathcal{O}_{r_k}L^{\eta_{k+1}}(\lambda)I_{2j}s$ must have a non-zero component in degree $2j$ and by Theorem 1.8 each $\mathcal{O}_{r_i}$ rises the degree at most by 2, so $2j$ must be less or equal to the number of $\mathcal{O}_{r_i}$'s appearing before $\mathcal{O}_{r_{i_0}}$, that is

$$2j \leq 2(i_0 - 1). \tag{2.15}$$

With (2.14) and (2.15), we find

$$4j \leq 2k. \tag{2.16}$$

Finally, since for every i, $r_i \geq 2$ and $\sum_{i=1}^{k} r_i = 2r$, we have $2k \leq 2r$, and thus

$$4j \leq 2r. \tag{2.17}$$

Consequently, if $r < 2j$ we have $I_{2j}T^{\eta}_{\mathbf{r}}(\lambda)I_{2j} = 0$, and by (2.11), we find $I_{2j}\mathscr{F}_{2r}I_{2j} = 0$. Using Theorem 2.2, we find

$$I_{2j}\boldsymbol{b}_r I_{2j} = 0,$$

which, combined with Theorem 0.3, entails the first part of Theorem 0.4.

For the second part of this theorem, let us assume that we are in the limit case where $r = 2j$. We also suppose that $j \geq 1$, because in the case $j = 0$, [MM07, (8.1.5)] implies that $\boldsymbol{b}_0(x_0) = \mathscr{F}_0(0,0) = I_0\mathscr{P}(0,0) = I_0$, so Theorem 0.4 is true for $j = 0$.

In $I_{2j}\mathscr{F}_{4j}I_{2j}$, there is only one term satisfying equations (2.14), (2.15), (2.16) and (2.17): first we see that (2.16) and (2.17) imply that $r = k = 2j$ and for all i, $r_i = 2$, while (2.14) and (2.15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j. Moreover, since the degree must decrease by $2j$ and then increase by $2j$ with only $k = 2j$ $\mathcal{O}_{r_i}$'s available, only $\mathcal{O}_2^{+2}$ and $\mathcal{O}_2^{-2}$ appear in $I_{2j}\mathscr{F}_{4j}I_{2j}$, and not $\mathcal{O}_2^{0}$. To summarize:

$$\begin{aligned}
I_{2j}\mathscr{F}_{4j}I_{2j} &= \frac{1}{2\pi\sqrt{-1}}\int_\delta I_{2j}\left((\lambda-\mathscr{L}_0)^{-1}P^{N^\perp}\mathcal{O}_2^{+2}\right)^j \frac{1}{\lambda}P^N\left(\mathcal{O}_2^{-2}(\lambda-\mathscr{L}_0)^{-1}P^{N^\perp}\right)^j I_{2j}d\lambda \\
&= I_{2j}\left({\mathscr{L}_0}^{-1}P^{N^\perp}\mathcal{O}_2^{+2}\right)^j P^N\left(\mathcal{O}_2^{-2}{\mathscr{L}_0}^{-1}P^{N^\perp}\right)^j I_{2j} \\
&= I_{2j}\left({\mathscr{L}_0}^{-1}\mathcal{O}_2^{+2}\right)^j P^N\left(\mathcal{O}_2^{-2}{\mathscr{L}_0}^{-1}\right)^j I_{2j},
\end{aligned} \tag{2.18}$$

because by (1.45), $L^2(\mathbb{R}^{2n}, (\Lambda^{0,>0}(T^*X)\otimes E)_{x_0}) \subset N^\perp$, so we can remove the $P^{N^\perp}$'s.

Let $A = I_{2j}\left({\mathscr{L}_0}^{-1}\mathcal{O}_2^{+2}\right)^j P^N$. Since $(\mathcal{O}_2^{+2})^* = \mathcal{O}_2^{-2}$ (see Proposition 1.9) and $\mathscr{L}_0$ is self-adjoint, the adjoint of A is $A^* = P^N\left(\mathcal{O}_2^{-2}{\mathscr{L}_0}^{-1}\right)^j I_{2j}$, and thus

$$I_{2j}\mathscr{F}_{4j}I_{2j} = AA^*. \tag{2.19}$$

Recall that $P^N = \mathscr{P}I_0$ (see (1.49)). Let $s \in L^2(\mathbb{R}^{2n}, E_{x_0})$, since $\mathscr{L}_0 = \mathscr{L} + 4\pi\mathscr{N}$ and $\mathscr{L}\mathscr{P}s = 0$, the term $(\mathscr{P}s)\mathscr{R}_{x_0}$ is an eigenfunction of $\mathscr{L}_0$ for the eigenvalue $2\times 4\pi$. Thus we get

$${\mathscr{L}_0}^{-1}\mathcal{O}_2^{+2}P^N s = {\mathscr{L}_0}^{-1}\mathcal{O}_2^{+2}\mathscr{P}s = {\mathscr{L}_0}^{-1}\left((\mathscr{P}s)\mathscr{R}_{x_0}\right) = \frac{1}{4\pi}\frac{1}{2}\mathscr{R}_{x_0}\mathscr{P}s.$$

Now, an easy induction shows that

$$A = \frac{1}{(4\pi)^j}\frac{1}{2\times 4\times\cdots\times 2j}I_{2j}\mathscr{R}_{x_0}^j\mathscr{P} = \frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}_{x_0}^j\mathscr{P}. \tag{2.20}$$

Let $A(Z,Z')$ and $A^*(Z,Z')$ be the smooth kernels of A and A^* with respect to $dv_{TX}(Z')$. By (2.19), $I_{2j}\mathscr{F}_{4j}I_{2j}(0,0) = \int_{\mathbb{R}^{2n}} A(0,Z)A^*(Z,0)dZ$. Thanks to

$$\int_{\mathbb{R}^{2n}} \mathscr{P}(0,Z)\mathscr{P}(Z,0)dZ = (\mathscr{P}\circ\mathscr{P})(0,0) = \mathscr{P}(0,0) = 1 \tag{2.21}$$

and (2.20), we find (0.8).

3. The second coefficient in the asymptotic expansion

In this section, we prove Theorem 0.7. Using (2.6), we know that

$$I_{2j}\boldsymbol{b}_{2j+1}I_{2j}(0,0) = I_{2j}\mathscr{F}_{4j+2}I_{2j}(0,0). \tag{3.1}$$

In Section 3.1, we decompose this term into 3 terms, and then in Sections 3.2 and 3.3 we handle them separately.

For the rest of the section we fix $j \in [\![0,n]\!]$. For every smoothing operator F acting on $L^2(\mathbb{R}^{2n}, \mathbb{E}_{x_0})$ that appears in this section, we will denote by $F(Z,Z')$ its smooth kernel with respect to $dv_{TX}(Z')$.

3.1. **Decomposition of the problem.** Applying inequalities (2.16) and (2.17) with $r = 2j+1$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$, the non-zero terms $\int_\delta T^\eta_{\mathbf{r}}(\lambda)d\lambda$ appearing in decomposition (2.11) satisfy $k = 2j$ or $k = 2j+1$. Since $\sum_i r_i = 4j+2$ and $r_i \geq 2$, we see that in $I_{2j}\mathscr{F}_{4j+2}I_{2j}$ there are 3 types of terms $T^\eta_{\mathrm{r}}(\lambda)$ with non-zero integral, in which:

- for $k = 2j$:
  - there are $2j-2$ $\mathscr{O}_{r_i}$'s equal to $\mathscr{O}_2$ and 2 equal to $\mathscr{O}_3$: we will denote by I the sum of these terms,
  - there are $2j-1$ $\mathscr{O}_{r_i}$'s equal to $\mathscr{O}_2$ and 1 equal to $\mathscr{O}_4$: we will denote by II the sum of these terms,
- for $k = 2j+1$:
  - all the $\mathscr{O}_{r_i}$'s are equal to $\mathscr{O}_2$: we will denote by III the sum of these terms.

We thus have a decomposition

$$I_{2j}\mathscr{F}_{4j+2}I_{2j} = \mathrm{I} + \mathrm{II} + \mathrm{III}. \tag{3.2}$$

Remark 3.1. Note that for the two sums I and II to be non-zero, we must have $j \geq 1$. Moreover, in the two first cases, as $k = 2j$, by the same reasoning as in Section 2.2, (2.14) and (2.15) imply that the i_0 such that $\eta_{i_0} = N$ is unique and equal to j, and that only $\mathscr{O}_2^{\pm 2}$, $\mathscr{O}_3^{\pm 2}$ and $\mathscr{O}_4^{\pm 2}$ appear in I and II, and not some $\mathscr{O}_{r_i}^0$.

3.2. **The term involving only $\mathscr{O}_2$.**

Lemma 3.2. *In any term $T^\eta_{\mathrm{r}}(\lambda)$ appearing in the term* III *(with non-vanishing integral), the i_0 such that $\eta_{i_0} = N$ is unique and equal to j or $j+1$. If we denote by* III_a *and* III_b *the sum of the terms corresponding to this two cases, we have:*

$$\begin{aligned}
&\mathrm{III}_a = \sum_{k=0}^{j} I_{2j}(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathscr{O}_2^0)(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^k P^N(\mathscr{O}_2^{-2}\mathscr{L}_0{}^{-1})^j I_{2j},\\
&\mathrm{III}_b = (\mathrm{III}_a)^*,\\
&\mathrm{III} = \mathrm{III}_a + \mathrm{III}_b.
\end{aligned} \tag{3.3}$$

Remark 3.3. For the same reason as for (2.18), we have removed the $P^{N^\perp}$'s in (3.3) without getting any trouble with the existence of $\mathscr{L}_0{}^{-1}$.

Proof. Fix a term $T^\eta_{\mathrm{r}}(\lambda)$ appearing in the term III with non-vanishing integral. Using again the same reasoning as in Section 2.2, we see that there exists at most two indices i_0 such that $\eta_{i_0} = N$, and that they are in $\{j, j+1\}$. Indeed, with only $2j+1$ $\mathscr{O}_{r_i}$'s at our disposal, we need j of them before the first P^N, and j after the last one.

Now, the only possible term with $\eta_j = \eta_{j+1} = N$ is:

$$(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^j P^N \mathscr{O}_2^0 P^N (\mathscr{O}_2^{-2}\mathscr{L}_0{}^{-1})^j.$$

To prove that this term is vanishing, we will use [MM12]. By (1.33), [MM12, (3.13),(3.16b)] and [MM12, (4.1a)] we see that $\mathscr{P}\mathcal{O}_2^0\mathscr{P}=0$, and so

$$P^N\mathcal{O}_2^0P^N=\mathscr{P}\mathcal{O}_2^0\mathscr{P}I_0=0,$$

we have proved the first part of the lemma.

The second part follows from the reasoning made at the beginning of this proof, and the facts that i_0 is unique, $\mathcal{O}_2^0$ is self-adjoint and $(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^*=\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1}$. □

Let us compute the term that appears in (3.3):

$$\mathrm{III}_{a,k}:=I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^jI_{2j}. \tag{3.4}$$

With (2.20), we know that

$$P^N(\mathcal{O}_2^{-2}\mathscr{L}_0{}^{-1})^jI_{2j}=\frac{1}{(4\pi)^j}\frac{1}{2^jj!}\mathscr{P}\left(\mathscr{R}_{x_0}^j\right)^*I_{2j}, \tag{3.5}$$

and

$$\begin{aligned}&I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N\\ &\qquad=\frac{1}{(4\pi)^k}\frac{1}{2^kk!}I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}\mathscr{L}_0{}^{-1}\mathscr{R}_{x_0}^k(\mathcal{O}_2^0\mathscr{P})I_0.\end{aligned} \tag{3.6}$$

Let

$$\begin{aligned}R_{k\bar m\ell\bar q}&=\left\langle R^{TX}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial\bar z_m}\right)\frac{\partial}{\partial z_\ell},\frac{\partial}{\partial\bar z_q}\right\rangle_{x_0},\\ R^E_{k\bar\ell}&=R^E_{x_0}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial\bar z_\ell}\right).\end{aligned} \tag{3.7}$$

By [MM12, Lemma 3.1], we know that

$$R_{k\bar m\ell\bar q}=R_{\ell\bar mk\bar q}=R_{k\bar q\ell\bar m}=R_{\ell\bar qk\bar m}\text{ and }r^X_{x_0}=8R_{m\bar mq\bar q}. \tag{3.8}$$

Once again, our $\mathcal{O}_2^0$ correspond to the $\mathcal{O}_2$ of [MM12] (see (1.33) and [MM12, (3.13),(3.16b)]), so we can use [MM12, (4.6)] to get:

$$\mathcal{O}_2^0\mathscr{P}=\left(\frac16b_mb_qR_{k\bar m\ell\bar q}z_kz_\ell+\frac43b_qR_{\ell\bar kk\bar q}z_\ell-\frac{\pi}{3}b_qR_{k\bar m\ell\bar q}z_kz_\ell\bar z_m'+b_qR^E_{\ell\bar q}z_\ell\right)\mathscr{P}, \tag{3.9}$$

Set

$$\begin{aligned}a&=\frac16b_mb_qR_{k\bar m\ell\bar q}z_kz_\ell, &\quad b&=\frac43b_qR_{\ell\bar kk\bar q}z_\ell,\\ c&=-\frac{\pi}{3}b_qR_{k\bar m\ell\bar q}z_kz_\ell\bar z_m', &\quad d&=b_qR^E_{\ell\bar q}z_\ell.\end{aligned} \tag{3.10}$$

Thanks to (1.46) and (3.9), we find

$$\mathscr{L}_0{}^{-1}\mathscr{R}^k_{x_0}\mathcal{O}_2^0\mathscr{P}I_0=\mathscr{R}^k_{x_0}\left(\frac{a}{4\pi(2+2k)}+\frac{b+c+d}{4\pi(1+2k)}\right)\mathscr{P}I_0, \tag{3.11}$$

and by induction, (3.6) becomes

$$\begin{aligned}&I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{0})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}P^N\\ &\qquad=\frac{1}{(4\pi)^{j+1}}\frac{1}{2^kk!}I_{2j}\mathscr{R}^j_{x_0}\left(\frac{a}{(2+2k)\cdots(2+2j)}+\frac{b+c+d}{(1+2k)\cdots(1+2j)}\right)\mathscr{P}I_0.\end{aligned}$$

Lemma 3.4. *We have:*

$$(a\mathscr{P})(0,Z)=\frac{1}{6}r^X_{x_0}\mathscr{P}(0,Z),\qquad (b\mathscr{P})(0,Z)=-\frac{1}{3}r^X_{x_0}\mathscr{P}(0,Z),$$
$$(c\mathscr{P})(0,Z)=0,\qquad (d\mathscr{P})(0,Z)=-2R^E_{q\bar{q}}\mathscr{P}(0,Z). \tag{3.12}$$

Proof. This lemma is a consequence of the relations (1.44) and (3.8). For instance we will compute $(b\mathscr{P})(0,Z)$, the other terms are similar.

$$\begin{aligned}(b\mathscr{P})(0,Z)&=\left(\frac{4}{3}b_qR_{\ell\bar{k}k\bar{q}}z_\ell\mathscr{P}\right)(0,Z)\\&=\frac{4}{3}R_{\ell\bar{k}k\bar{q}}\left((z_\ell b_q-2\delta_{\ell q})\mathscr{P}\right)(0,Z)\\&=-\frac{8}{3}R_{\ell\bar{k}k\bar{\ell}}\mathscr{P}(0,Z)=-\frac{1}{3}r^X_{x_0}\mathscr{P}(0,Z).\end{aligned}$$

□

Using (2.21), (3.4), (3.5) and (3.11), we find

$$\mathrm{III}_{a,k}(0,0)=I_{2j}\mathrm{C}_j(j)\mathscr{R}^j_{x_0}\left[\frac{1}{6}\left(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r^X_{x_0}-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}R^E_{q\bar{q}}\right]\left(\mathscr{R}^j_{x_0}\right)^*I_{2j}. \tag{3.13}$$

Notice that $2R^E_{q\bar{q}}=R^E_{x_0}\left(\sqrt{2}\frac{\partial}{\partial z_q},\sqrt{2}\frac{\partial}{\partial\bar{z}_q}\right)=R^E_{x_0}(w_q,\overline{w}_q)=\sqrt{-1}R^E_{\Lambda,x_0}$ by definition. Consequently, by (3.3) and the fact that $(R^E_{k\bar{q}})^*=R^E_{q\bar{k}}$,

$$\mathrm{III}(0,0)=I_{2j}\mathrm{C}_j(j)\mathscr{R}^j_{x_0}\sum_{k=0}^{j}\left[\frac{1}{3}\left(\mathrm{C}_{j+1}(j+1)-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\right)r^X_{x_0}-\frac{\mathrm{C}_j(k)}{2\pi(2k+1)}\sqrt{-1}R^E_{\Lambda,x_0}\right]\left(\mathscr{R}^j_{x_0}\right)^*I_{2j}. \tag{3.14}$$

3.3. **The two other terms.** In this subsection, we suppose that $j\geq 1$ (cf. remark 3.1). Moreover, the existence of any ${\mathscr{L}_0}^{-1}$ appearing in this section follows from the reasoning done in Remark 3.3, and this operator will be used without further precision.

Due to (1.30), we have

$$\mathcal{O}^{+2}_3=\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^{+2}_{E_0}(tZ)\right)|_{t=0}=z_i\frac{\partial\mathscr{R}.}{\partial z_i}(0)+\bar{z}_i\frac{\partial\mathscr{R}.}{\partial\bar{z}_i}(0)\text{ and} \tag{3.15}$$

$$\mathcal{O}^{+2}_4=\frac{z_iz_j}{2}\frac{\partial^2\mathscr{R}.}{\partial z_i\partial z_j}(0)+z_i\bar{z}_j\frac{\partial^2\mathscr{R}.}{\partial z_i\partial\bar{z}_j}(0)+\frac{\bar{z}_i\bar{z}_j}{2}\frac{\partial^2\mathscr{R}.}{\partial\bar{z}_i\partial\bar{z}_j}(0). \tag{3.16}$$

The sum I can be decomposed in 3 'sub-sums': I_a, I_b and I_c in which the two $\mathcal{O}_3$'s appearing are respectively both at the left of P^N, either side of P^N or both at the right of P^N (see remark 3.1). As usually, we have $\mathrm{I}_c=(\mathrm{I}_a)^*$.

In the same way, we can decompose $\mathrm{II}=\mathrm{II}_a+\mathrm{II}_b$: in II_a the $\mathcal{O}_4$ appears at the left of P^N, and in II_b at the right of P^N. Once again, $\mathrm{II}_b=(\mathrm{II}_a)^*$.

Computation of $\mathrm{I}_b(0,0)$. To compute I_b, we first compute the value at $(0,Z)$ of the kernel of

$$A_k:=I_{2j}({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_2)^{j-k-1}({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_3)({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_2)^k\mathscr{P}I_0.$$

By (2.20) and (3.15),

$$\begin{aligned}A_k&=I_{2j}({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_2)^{j-k-1}({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_3)\frac{1}{(4\pi)^k}\frac{1}{2^kk!}\mathscr{R}^k_{x_0}\mathscr{P}I_0\\&=\frac{1}{(4\pi)^k}\frac{1}{2^kk!}I_{2j}({\mathscr{L}_0}^{-1}\mathcal{O}^{+2}_2)^{j-k-1}{\mathscr{L}_0}^{-1}\left[z_i\frac{\partial\mathscr{R}.}{\partial z_i}(0)+\bar{z}_i\frac{\partial\mathscr{R}.}{\partial\bar{z}_i}(0)\right]\mathscr{R}^k_{x_0}\mathscr{P}I_0.\end{aligned} \tag{3.17}$$

Now by Theorem 1.10, if $s \in N$, then $z_i s \in N$, so by the same calculation as in (2.20),

$$\begin{aligned}
&\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\left[z_i\frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\right]\mathscr{R}^k_{x_0}\mathscr{P}I_0\right)(0,Z)\\
&= \frac{1}{(4\pi)^j}\frac{1}{2^j j!}\left(I_{2j}\left[\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial z_i}(0)\mathscr{R}^k_{x_0}\right]z_i\mathscr{P}I_0\right)(0,Z)=0.
\end{aligned} \tag{3.18}$$

Now by (1.43) and the formula (1.48), we have

$$(b_i^+\mathscr{P})(Z,Z')=0 \text{ and } (b_i\mathscr{P})(Z,Z')=2\pi(\bar z_i-\bar z'_i)\mathscr{P}(Z,Z'). \tag{3.19}$$

Thus,

$$\begin{aligned}
&\frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\left[\bar z_i\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\right]\mathscr{R}^k_{x_0}\mathscr{P}I_0\right)(Z,Z')\\
&= \frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\left[\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\left(\frac{b_i}{2\pi}+\bar z'_i\right)\mathscr{P}I_0\right)(Z,Z')\\
&= \frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\left[\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\right.\\
&\qquad\qquad \left.\times\left(\frac{1}{4\pi(2k+2+1)}\frac{b_i}{2\pi}+\frac{1}{4\pi(2k+2)}\bar z'_i\right)\mathscr{P}I_0\right)(Z,Z')\\
&= \frac{1}{(4\pi)^j}\frac{1}{2^j j!}\left(I_{2j}\left[\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\bar z'_i\mathscr{P}I_0\right)(Z,Z')\\
&+\frac{1}{(4\pi)^j}\frac{1}{2^k k!\prod_{k+1}^j(2\ell+1)}\left(I_{2j}\left[\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\frac{b_i}{2\pi}\mathscr{P}I_0\right)(Z,Z').
\end{aligned} \tag{3.20}$$

For the last two lines, we used that if $s \in N$, then $\mathscr{L}(b_i s)=4\pi b_i s$ (see Theorem 1.10). Thus, by (0.11) and (3.17)–(3.20)

$$\begin{aligned}
A_k(0,Z) &= \frac{1}{(4\pi)^k}\frac{1}{2^k k!}\left(I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-1}\mathscr{L}_0{}^{-1}\left[\bar z_i\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\right]\mathscr{R}^k_{x_0}\mathscr{P}I_0\right)(0,Z)\\
&= \Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)I_{2j}\left[\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\bar z_i\mathscr{P}(0,Z)I_0.
\end{aligned}$$

We know that $(\bar z_i\mathscr{P})^*=z_i\mathscr{P}$, and $\int_{\mathbb{C}^n} z_m\bar z_q e^{-\pi|z|^2}dZ=\frac{1}{\pi}\delta_{mq}$, so

$$\begin{aligned}
(A_{k_1}A_{k_2}{}^*)(0,0) &= \frac{1}{\pi}I_{2j}\left[\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}^{j-k_1-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^{k_1}_{x_0}\right]\\
&\qquad\times\left[\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}^{j-k_2-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^{k_2}_{x_0}\right]^* I_{2j}.
\end{aligned} \tag{3.21}$$

Finally,

$$\begin{aligned}
\mathrm{I}_b(0,0) &= \frac{1}{\pi}I_{2j}\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]\\
&\qquad\times\left[\sum_{k=0}^{j-1}\Big(\mathrm{C}_j(j)-\mathrm{C}_j(k)\Big)\mathscr{R}^{j-k-1}_{x_0}\frac{\partial \mathscr{R}_\cdot}{\partial \bar z_i}(0)\mathscr{R}^k_{x_0}\right]^* I_{2j}.
\end{aligned} \tag{3.22}$$

Computation of $\mathrm{I}_a(0,0)$ *and* $\mathrm{I}_c(0,0)$. First recall that $\mathrm{I}_c(0,0) = (\mathrm{I}_a(0,0))^*$, so we just need to compute $\mathrm{I}_a(0,0)$. By the definition of $\mathrm{I}_a(0,0)$, for it to be non-zero, it is necessary to have $j \geq 2$, which will be supposed in this paragraph. Let

$$A_{k,\ell} := I_{2j}(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{j-k-\ell-2}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{\ell}\mathscr{P}I_0,$$

the sum $\mathrm{I}_a(0,0)$ is then given by

$$\mathrm{I}_a(0,0) = \int_{\mathbb{R}^{2n}} \left(\sum_{k,\ell} A_{k,\ell}(0,Z)\right) \times \left(\frac{1}{(4\pi)^j}\frac{1}{2^j j!} I_{2j}\mathscr{R}_{x_0}^j \mathscr{P} I_0\right)^* (Z,0) dv_{TX}(Z). \tag{3.23}$$

In the following, we will set

$$\tilde{b}_i := \frac{b_i}{2\pi}. \tag{3.24}$$

Using the same method as in (1.44), (3.18), (3.19) and (3.20), we find that there exist constants $C^1_{k,\ell}$, $C^2_{k,\ell}$ given by

$$\begin{aligned} C^1_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}} \frac{1}{2^{k+\ell+1}(k+\ell+1)!}, \\ C^2_{k,\ell} &= \frac{1}{(4\pi)^{k+\ell+1}} \frac{1}{2^\ell \ell! \prod_{\ell+1}^{k+\ell+1}(2s+1)}, \end{aligned} \tag{3.25}$$

such that

(3.26)

$$\begin{aligned} &(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{k}(\mathscr{L}_0{}^{-1}\mathcal{O}_3^{+2})(\mathscr{L}_0{}^{-1}\mathcal{O}_2^{+2})^{\ell}\mathscr{P}I_0 \\ &= \mathscr{L}_0{}^{-1}\Big\{ \frac{\partial \mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_i z_{i'} + \frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_{i'}(\tilde{b}_i + \bar{z}'_i) \\ &\qquad + \frac{\partial \mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0} z_i (C^2_{k,\ell}\tilde{b}_{i'} + C^1_{k,\ell}\bar{z}'_{i'}) \\ &\qquad\qquad + \frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0} (\tilde{b}_i + \bar{z}'_i)(C^2_{k,\ell}\tilde{b}_{i'} + C^1_{k,\ell}\bar{z}'_{i'}) \Big\} \mathscr{P} I_0 \\ &= \mathscr{L}_0{}^{-1}\Big\{ \frac{\partial \mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell} z_i z_{i'} + \frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial z_{i'}}(0)\mathscr{R}^\ell_{x_0} C^1_{k,\ell}\Big(\tilde{b}_i z_{i'} + \frac{\delta_{ii'}}{\pi} + z_{i'}\bar{z}'_i\Big) \\ &\qquad + \frac{\partial \mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0} \Big(C^2_{k,\ell}\Big(\tilde{b}_{i'} z_i + \frac{\delta_{ii'}}{\pi}\Big) + C^1_{k,\ell} z_i \bar{z}'_{i'}\Big) \\ &\qquad + \frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_i}(0)\mathscr{R}^k_{x_0}\frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_{i'}}(0)\mathscr{R}^\ell_{x_0} \Big(C^2_{k,\ell}(\tilde{b}_i\tilde{b}_{i'} + \bar{z}'_i\tilde{b}_{i'}) + C^1_{k,\ell}(\tilde{b}_i\bar{z}'_{i'} + \bar{z}'_i\bar{z}'_{i'})\Big)\Big\} \mathscr{P} I_0, \end{aligned}$$

Using Theorem 1.10, (1.44) and (3.19), we see that there exist constants $C^i_{j,k,\ell}$, $i = 3, \ldots, 10$, such that

$$\begin{aligned} C^3_{j,k,\ell} &= C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s)}, \quad C^4_{j,k,\ell} = C^1_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s+1)}, \\ C^5_{j,k,\ell} &= C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s+1)}, \quad C^6_{j,k,\ell} = C^2_{k,\ell}\frac{1}{(4\pi)^{j-(k+\ell+1)}}\frac{1}{\prod_{k+\ell+2}^{j}(2s)}, \end{aligned} \tag{3.27}$$

and

(3.28)

$$
\begin{aligned}
A_{k,\ell}(0,Z) = I_{2j}\Bigg(&\mathscr{R}_{x_0}^{j-k-\ell-2}\bigg\{\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_{i'}}(0)\Big(C^3_{j,k,\ell}\frac{\delta_{ii'}}{\pi}+C^4_{j,k,\ell}\tilde{b}_i z_{i'}\Big)\\
&+\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_{i'}}(0)\Big(C^5_{j,k,\ell}\tilde{b}_{i'}z_i+\frac{\delta_{ii'}}{\pi}C^6_{j,k,\ell}\Big)\\
&+\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_{i'}}(0)\Big(C^7_{j,k,\ell}\tilde{b}_i\tilde{b}_{i'}+C^8_{j,k,\ell}\bar{z}_i\tilde{b}_{i'}+C^9_{j,k,\ell}\tilde{b}_i\bar{z}_{i'}+C^{10}_{j,k,\ell}\bar{z}_i\bar{z}_{i'}\Big)\bigg\}\mathscr{R}_{x_0}^{\ell}\mathscr{P}I_0\Bigg)(0,Z)\\
=I_{2j}\mathscr{R}_{x_0}^{j-k-\ell-2}&\bigg\{\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\frac{C^3_{j,k,\ell}-C^4_{j,k,\ell}}{\pi}+\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\frac{C^6_{j,k,\ell}-C^5_{j,k,\ell}}{\pi}\\
&+\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_{i'}}(0)\Big(4C^7_{j,k,\ell}-2C^8_{j,k,\ell}-2C^9_{j,k,\ell}+C^{10}_{j,k,\ell}\Big)\bar{z}_i\bar{z}_{i'}\bigg\}\mathscr{R}_{x_0}^{\ell}\mathscr{P}(0,Z)I_0.
\end{aligned}
$$

Now with $\int\bar{z}_i\bar{z}_{i'}\mathscr{P}(0,Z)\mathscr{P}(Z,0)dZ=0$, we can rewrite (3.23):

$$
\begin{aligned}
(3.29)\quad \mathrm{I}_a(0,0)=\frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{k,\ell}\mathscr{R}_{x_0}^{j-k-\ell-2}\bigg\{&(C^3_{j,k,\ell}-C^4_{j,k,\ell})\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\\
&+(C^6_{j,k,\ell}-C^5_{j,k,\ell})\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{k}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\bigg\}\mathscr{R}_{x_0}^{\ell}\left(\mathscr{R}_{x_0}^{j}\right)^*I_{2j}.
\end{aligned}
$$

By (0.11), (3.25) and (3.27),

$$
(3.30)\qquad
\begin{aligned}
C^3_{j,k,\ell}&=\mathrm{C}_j(j), \qquad C^4_{j,k,\ell}=\mathrm{C}_j(k+\ell+1),\\
C^5_{j,k,\ell}&=\mathrm{C}_j(\ell), \qquad C^6_{j,k,\ell}=\mathrm{C}_j(\ell)\prod_{s=k+\ell+2}^{j}\left(1+\frac{1}{2s}\right).
\end{aligned}
$$

We can now write (3.29) more precisely:

$$
\begin{aligned}
\mathrm{I}_a(0,0)=\frac{\mathrm{C}_j(j)}{\pi}I_{2j}\sum_{q=0}^{j-2}\sum_{m=0}^{q}\bigg\{&\Big(\mathrm{C}_j(j)-\mathrm{C}_j(q+1)\Big)\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{m}\\
(3.31)\qquad &+\mathrm{C}_j(m)\Big[\prod_{q+2}^{j}(1+\frac{1}{2s})-1\Big]\mathscr{R}_{x_0}^{j-(q+2)}\frac{\partial\mathscr{R}_{\cdot}}{\partial z_i}(0)\mathscr{R}_{x_0}^{q-m}\frac{\partial\mathscr{R}_{\cdot}}{\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{m}\bigg\}\left(\mathscr{R}_{x_0}^{j}\right)^*I_{2j}.
\end{aligned}
$$

Computation of $\mathrm{II}(0,0)$. Recall that $\mathrm{II}(0,0)=\mathrm{II}_a(0,0)+(\mathrm{II}_a(0,0))^*$. The computation of $\mathrm{II}_a(0,0)$ is very similar to the computation of $\mathrm{I}_a(0,0)$, and is simpler, so we will follow the same method.

Let

$$A_k:=I_{2j}(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^{j-k-1}(\mathscr{L}_0{}^{-1}\mathscr{O}_4^{+2})(\mathscr{L}_0{}^{-1}\mathscr{O}_2^{+2})^{k}\mathscr{P}I_0,$$

the sum $\mathrm{II}_a(0,0)$ is then given by

$$
(3.32)\qquad \mathrm{II}_a(0,0)=\int_{\mathbb{R}^{2n}}\left(\sum_k A_k(0,Z)\right)\times\left(\frac{1}{(4\pi)^j}\frac{1}{2^j j!}I_{2j}\mathscr{R}_{x_0}^{j}\mathscr{P}I_0\right)^*(Z,0)dv_{TX}(Z).
$$

Using (3.16), we can repeat what we have done for (3.26) and (3.28). We find

$$
\begin{aligned}
(3.33)\quad A_k(0,Z)=I_{2j}\bigg\{&\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial z_i\partial\bar{z}_i}(0)\mathscr{R}_{x_0}^{k}\frac{\mathrm{C}_j(j)-\mathrm{C}_j(k)}{\pi}\\
&+\mathscr{R}_{x_0}^{j-(k+1)}\frac{\partial^2\mathscr{R}_{\cdot}}{\partial\bar{z}_i\partial\bar{z}_{i'}}(0)\mathscr{R}_{x_0}^{k}\mathrm{C}_j(j)\frac{\bar{z}_i\bar{z}_{i'}}{2}\bigg\}\mathscr{P}(0,Z)I_0.
\end{aligned}
$$

Thus we get

$$\mathrm{II}_a(0,0) = \frac{\mathrm{C}_j(j)}{\pi} I_{2j} \sum_{k=0}^{j-1} \Big(\mathrm{C}_j(j) - \mathrm{C}_j(k)\Big) \mathscr{R}_{x_0}^{j-(k+1)} \frac{\partial^2 \mathscr{R}_{\cdot}}{\partial z_i \partial \bar{z}_i}(0) \mathscr{R}_{x_0}^{k} \left(\mathscr{R}_{x_0}^{j}\right)^* I_{2j}. \tag{3.34}$$

Conclusion. In order to conclude the proof of Theorem 0.7, we just have to put the pieces together. But before that, as we want to write the formulas in a more intrinsic way, we have to note that since we trivialized $\Lambda^{0,\bullet}(T^*X) \otimes E$ with $\nabla^{\Lambda^{0,\bullet} \otimes E}$, since $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \bar{z}_i}$, and thanks to [MM12, (5.44),(5.45)], we have

$$\frac{\partial \mathscr{R}_{\cdot}}{\partial \bar{z}_i}(0) = \frac{1}{\sqrt{2}} \left(\nabla^{\Lambda^{0,\bullet}\otimes E}_{\overline{w}_i} \mathscr{R}_{\cdot}\right)(x_0), \quad \frac{\partial \mathscr{R}_{\cdot}}{\partial z_i}(0) = \frac{1}{\sqrt{2}} \left(\nabla^{\Lambda^{0,\bullet}\otimes E}_{w_i} \mathscr{R}_{\cdot}\right)(x_0) \text{ and}$$
$$\frac{\partial^2 \mathscr{R}_{\cdot}}{\partial z_i \partial \bar{z}_i}(0) = \frac{1}{4}(\Delta^{\Lambda^{0,\bullet}\otimes E} \mathscr{R}_{\cdot})(x_0).$$

With these remaks and equations (3.14), (3.22), (3.31), (3.34) used in decomposition (3.2), we get Theorem 0.7.

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