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Nonparametric estimation of the density of the alternative hypothesis in a multiple testing setup. Application to local false discovery rate estimation

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Abstract

In a multiple testing context, we consider a semiparametric mixture model with two components where one component is known and corresponds to the distribution of p -values under the null hypothesis and the other component f is nonparametric and stands for the distribution under the alternative hypothesis. Motivated by the issue of local false discovery rate estimation, we focus here on the estimation of the nonparametric unknown component f in the mixture, relying on a preliminary estimator of the unknown proportion θ of true null hypotheses. We propose and study the asymptotic properties of two different estimators for this unknown component. The first estimator is a randomly weighted kernel estimator. We establish an upper bound for its pointwise quadratic risk, exhibiting the classical nonparametric rate of convergence over a class of Hölder densities. To our knowledge, this is the first result establishing convergence as well as corresponding rate for the estimation of the unknown component in this nonparametric mixture. The second estimator is a maximum smoothed likelihood estimator. It is computed through an iterative algorithm, for which we establish a descent property. In addition, these estimators are used in a multiple testing procedure in order to estimate the local false discovery rate. Their respective performances are then compared on synthetic data.

Key words and phrases: False discovery rate; kernel estimation; local false discovery rate; maximum smoothed likelihood; multiple testing; p -values; semiparametric mixture model.

1 Introduction

In the framework of multiple testing problems (microarray analysis, neuro-imaging, etc), a mixture model with two populations is considered

$$\forall x \in \mathbb{R}^d, \quad g(x) = \theta\phi(x) + (1 - \theta)f(x), \quad (1)$$

where θ is the unknown proportion of true null hypotheses, ϕ and f are the densities of the observations generated under the null and alternative hypotheses, respectively. More precisely, assume the test statistics are independent and identically distributed (iid) with a continuous distribution under the corresponding null hypotheses and we observe the p -values X_1, X_2, \dots, X_n associated with n independent tested hypotheses, then the density function ϕ is the uniform distribution on $[0, 1]$ while the density function f is assumed unknown. The parameters of the model are (θ, f) , where θ is a Euclidean parameter while f is an infinite-dimensional one and the model becomes

$$\forall x \in [0, 1], \quad g(x) = \theta + (1 - \theta)f(x). \quad (2)$$

In the following, we focus on model (2). A central problem in the multiple testing setup is the control of type I (*i.e.* false positive) and type II (*i.e.* false negative) errors. The most popular criterion regarding type I errors is the false discovery rate (FDR), proposed by Benjamini and Hochberg (1995). To set up the notation, let H_i be the i -th (null) hypothesis. The outcome of testing n hypotheses simultaneously can be summarized as indicated in Table 1.

Table 1: Possible outcomes from testing n hypotheses H_1, \dots, H_n .

	Accepts H_i	Rejects H_i	Total
H_i is true	TN	FP	n_0
H_i is false	FN	TP	n_1
Total	N	P	n

Benjamini and Hochberg (1995) define FDR as the expected proportion of rejections that are incorrect,

$$\text{FDR} = \mathbb{E} \left[\frac{\text{FP}}{\max(\text{P}, 1)} \right] = \mathbb{E} \left[\frac{\text{FP}}{\text{P}} | \text{P} > 0 \right] \mathbb{P}(\text{P} > 0).$$

They provide a multiple testing procedure that guarantees the bound $\text{FDR} \leq \alpha$, for a desired level α . Storey (2003) proposes to modify FDR so as to obtain a new criterion, the positive FDR (or pFDR), defined by

$$\text{pFDR} = \mathbb{E} \left[\frac{\text{FP}}{\text{P}} | \text{P} > 0 \right],$$

and argues that it is conceptually more sound than FDR. For microarray data for instance, there is a large value of the number of hypotheses n and the difference between pFDR and FDR is generally small as the extra factor $\mathbb{P}(P > 0)$ is very close to 1 (see Liao et al., 2004). In a mixture context, the pFDR is given by

$$\text{pFDR}(x) = \mathbb{P}(H_i \text{ being true} | X \leq x) = \frac{\theta\Phi(x)}{\theta\Phi(x) + (1 - \theta)F(x)},$$

where Φ and F are the cumulative distribution functions (cdfs) for densities ϕ and f , respectively. (It is notationally convenient to consider events of the form $X \leq x$, but we could just as well consider tail areas to the right, two-tailed events, etc).

Efron et al. (2001) define the local false discovery rate (ℓ FDR) to quantify the plausibility of a particular hypothesis being true, given its specific test statistic or p -value. In a mixture framework, the ℓ FDR is the Bayes posterior probability

$$\ell\text{FDR}(x) = \mathbb{P}(H_i \text{ being true} \mid X = x) = 1 - \frac{(1 - \theta)f(x)}{\theta\phi(x) + (1 - \theta)f(x)}. \quad (3)$$

In many multiple testing frameworks, we need information at the individual level about the probability for a given observation to be a false positive (Aubert et al., 2004). This motivates estimating the local false discovery rate ℓ FDR. Moreover, the quantities pFDR and ℓ FDR are analytically related by $\text{pFDR}(x) = \mathbb{E}[\ell\text{FDR}(X) \mid X \leq x]$. As a consequence (and recalling that the difference between pFDR and FDR is generally small), Robin et al. (2007) propose to estimate FDR by

$$\widehat{\text{FDR}}(x_i) = \frac{1}{i} \sum_{j=1}^i \widehat{\ell\text{FDR}}(x_j),$$

where $\widehat{\ell\text{FDR}}$ is an estimator of ℓ FDR and the observations $\{x_i\}$ are increasingly ordered. A natural strategy to estimate ℓ FDR is to start by estimating both the proportion θ and either f or g . Another motivation for estimating the parameters in this mixture model comes from the works of Sun and Cai (2007; 2009), who develop adaptive compound decision rules for false discovery rate control. These rules are based on the estimation of the parameters in model (1) (dealing with z -scores) rather than model (2) (dealing with p -values). However, it appears that in some very specific cases (when the alternative is symmetric about the null), the oracle version of their procedure based on the p -values (and thus relying on estimators of the parameters in model (2)) may outperform the one based on model (1) (see Sun and Cai, 2007, for more details). In the following, we are thus interested in estimating parameters in model (2).

In a previous work (Nguyen and Matias, 2012), we discussed the estimation of the Euclidean part of the parameter θ in model (2). Thus, we will not consider further this point here. We rather focus on the estimation of the unknown density f , relying on a preliminary estimator of θ . We just mention that many estimators of θ have been proposed in the literature. One of the most well-known is the one proposed by Storey and Tibshirani (2003), motivating its use in our simulations. Note that some of these estimators are proved to be consistent (under suitable model assumptions). This is for instance the case for the one proposed by Celisse and Robin (2010). Moreover, this latter estimator has been shown to be \sqrt{n} -consistent in Nguyen and Matias (2012). This will be used later when assessing the rate of convergence of one of our estimators of f .

Now, different modeling assumptions on the marginal density f have been proposed in the literature. For instance, parametric models have been used with Beta distribution for the p -values (see for example Allison et al., 2002; Liao et al., 2004; Pounds and Morris, 2003) or Gaussian distribution of the probit transformation of the p -values (McLachlan et al., 2006). In the framework of nonparametric estimation, Strimmer (2008) proposed a modified Grenander density estimator for f , which has been initially suggested by Langaas et al. (2005). This approach requires monotonicity constraints on the density f . Other nonparametric approaches consist in relying on regularity assumptions on f . This is done for

instance in Neuvial (2010), who is primarily interested in estimating θ under the assumption that it is equal to $g(1)$. Relying on a kernel estimator of g , he derives nonparametric rates of convergence for θ . Another kernel estimator has been proposed by Robin et al. (2007), along with a multiple testing procedure, called `kerfdr`. This iterative algorithm is inspired by an expectation-maximization (`em`) procedure (Dempster et al., 1977). It is proved to be convergent as the number of iterations increases. However, it does not optimize any criterion and contrarily to the original `em` algorithm, it does not increase the observed data likelihood function. Besides, the asymptotic properties (with the number of hypotheses n) of the kernel estimator underlying Robin et al.’s approach have not been studied. Indeed, its iterative form prevents from obtaining any theoretical result on its convergence properties.

The first part of the present work focuses on the properties of a randomly weighted kernel estimator, which in essence, is very similar to the iterative approach proposed by Robin et al. (2007). Thus, this part may be viewed as a theoretical validation of `kerfdr` approach that gives some insights about the convergence properties (as the sample size increases) of this method. In particular, we establish that relying on a preliminary estimator of θ that converges at parametric rate, we obtain an estimator of the unknown density f that converges at the usual minimax nonparametric rate. To our knowledge, this is the first result establishing convergence as well as corresponding rate for the estimation of the unknown component in model (2). In a second part, we are interested in a new iterative algorithm for estimating the unknown density f , that aims at maximizing a smoothed likelihood. We refer to Paragraph 4.1 in Eggermont and LaRiccia (2001) for an interesting presentation of kernel estimators as maximum smoothed likelihood ones. Here, we base our approach on the work of Levine et al. (2011), who study a maximum smoothed likelihood estimator for multivariate mixtures. The main idea consists in introducing a nonlinear smoothing operator on the unknown component f as proposed in Eggermont and LaRiccia (1995). We prove that the resulting algorithm possesses a desirable descent property, just as an `em` algorithm does. We also show that it is competitive with respect to `kerfdr` algorithm when used to estimate ℓ FDR.

The article is organized as follows. We start by recalling in Section 2 the construction of the kernel estimator underlying `kerfdr`’s approach. Then in Section 3, we study a randomly weighted kernel estimator of the unknown density f and establish an upper bound of its pointwise quadratic risk. This estimator is similar in essence to the one underlying `kerfdr` algorithm and whose properties are out of reach because of its iterative form. In a second part (Section 4), we introduce a new iterative algorithm for estimating f by maximizing a smoothed likelihood, and establish that it possesses a descent property. In Section 5, we rely on our different estimators to estimate the local false discovery rate and present some simulations to compare their performances. All the proofs have been postponed to Section 6. Moreover, some of the more technical proofs have been further postponed to Appendix A.

2 Motivating procedure: `kerfdr` algorithm

2.1 Estimator’s basis

We start by explaining a natural construction of a kernel estimator of f . Since f is completely unspecified, it has to be estimated in a nonparametric way, for example by a kernel

estimator. For any hypothesis, we introduce a (latent) random variable Z_i that equals 0 if the null hypothesis H_i is true and 1 otherwise,

$$\forall i = 1, \dots, n \quad Z_i = \begin{cases} 0 & \text{if } H_i \text{ is true,} \\ 1 & \text{otherwise.} \end{cases} \quad (4)$$

Since f corresponds to the density of the observations distributed under the alternative hypothesis, a *weighted* kernel estimator seems to be suited. Intuitively, it would be convenient to introduce a weight for each observation X_i , meant to select this observation only if it comes from f . Equivalently, the weights are used to select the indexes i such that $Z_i = 1$. Thus, a natural kernel estimate of f would be

$$f_1(x) = \frac{1}{h} \sum_{i=1}^n \frac{Z_i}{\sum_{k=1}^n Z_k} K_1\left(\frac{x - X_i}{h}\right), \quad x \in [0, 1],$$

where K_1 denotes a kernel function (namely a real-valued integrable function such that $\int K_1(u)du = 1$) and $h > 0$ is the bandwidth. However, f_1 is not an estimator and cannot be directly used since the random variables Z_i are unknown. A natural approach is to replace them with their conditional expectation given the data $\{X_i\}_{1 \leq i \leq n}$, namely with the posterior probabilities $\tau(X_i) = \mathbb{E}(Z_i|X_i)$ defined by

$$\forall x \in [0, 1], \quad \tau(x) = \mathbb{E}(Z_i|X_i = x) = \frac{(1 - \theta)f(x)}{g(x)} = 1 - \frac{\theta}{g(x)}. \quad (5)$$

This leads to the following definition

$$\forall x \in [0, 1], \quad f_2(x) = \frac{1}{h} \sum_{i=1}^n \frac{\tau(X_i)}{\sum_{k=1}^n \tau(X_k)} K_1\left(\frac{x - X_i}{h}\right). \quad (6)$$

Once again, the weight $\tau_i = \tau(X_i)$ depends on the unknown parameters θ and f and thus f_2 is not an estimator but rather an oracle. Thus it is natural to replace the posterior probabilities τ_i by estimators to obtain a randomly weighted kernel estimator of f . This is what is done, in an iterative way, by `kerfdr` algorithm below and further pursued in our approach developed in Section 3.

2.2 kerfdr procedure

Let us first recall the `kerfdr` algorithm proposed by Robin et al. (2007) as an approximation to the estimator suggested by (6). This algorithm constructs an iterative sequence $\{\hat{f}^{(s)}\}_{s \geq 0}$ of estimates of density f . Let us be given a preliminary estimator $\hat{\theta}$ of proportion θ . We define a weighted sequence $\hat{\tau}$ and a weighted kernel estimator \hat{f} which depend on each other, by

$$\hat{\tau}_i = \frac{(1 - \hat{\theta})\hat{f}(x_i)}{\hat{\theta} + (1 - \hat{\theta})\hat{f}(x_i)} \quad \text{and} \quad \hat{f}(x) = \frac{1}{h} \sum_{i=1}^n \frac{\hat{\tau}_i}{\sum_{k=1}^n \hat{\tau}_k} K\left(\frac{x - X_i}{h}\right),$$

where K is a kernel and $h > 0$ a bandwidth. In the following, we let

$$K_i(\cdot) = \frac{1}{h} K\left(\frac{\cdot - X_i}{h}\right).$$

Algorithm 1: kerfdr algorithm

```

// Initialization;
Set  $\hat{\tau}_i^0 \sim \mathcal{U}([0, 1]), i = 1, 2, \dots, n.$ 

while  $\max_i |\hat{\tau}_i^{(s)} - \hat{\tau}_i^{(s-1)}| / \hat{\tau}_i^{(s-1)} \geq \epsilon$  do
    // Update estimation of  $f$ ;
     $\hat{f}^{(s)}(x_i) = \sum_j \hat{\tau}_j^{(s-1)} K_j(x_i) / \sum_k \hat{\tau}_k^{(s-1)}$ 

    // Update estimation of  $g$ ;
     $\hat{g}^{(s)}(x_i) = \hat{\theta} + (1 - \hat{\theta}) \hat{f}^{(s)}(x_i)$ 

    // Update of weights;
     $\hat{\tau}_i^{(s)} = (1 - \hat{\theta}) \hat{f}^{(s)}(x_i) / \hat{g}^{(s)}(x_i)$ 

     $s \leftarrow s + 1;$ 

// Return;
 $\hat{f}^{(s)}(\cdot) = \sum_i \hat{\tau}_i^{(s-1)} K_i(\cdot) / \sum_k \hat{\tau}_k^{(s-1)}$ 

```

Now, **kerfdr** algorithm is described below as Algorithm 1.

This algorithm has some **em** flavor (Dempster et al., 1977). Actually, updating the weights $\hat{\tau}_i^{(s)}$ is equivalent to **expectation**-step, and $\hat{f}^{(s)}(x)$ can be seen as an average of $\{K_i(x)\}_{1 \leq i \leq n}$ so that updating the estimator \hat{f} may look like a **maximization**-step. However, the algorithm does not optimize any given criterion. Besides, it does not increase the observed data likelihood function.

The relation between $\hat{f}^{(s)}$ and $\hat{\tau}^{(s)}$ implies that the sequence $\{\hat{\tau}^{(s)}\}_{s \geq 0}$ satisfies $\hat{\tau}^{(s)} = \psi(\hat{\tau}^{(s-1)})$, where

$$\psi : [0, 1]^n \setminus \{0\} \rightarrow [0, 1]^n, \quad \psi_i(u) = \frac{\sum_j u_j b_{ij}}{\sum_j u_j b_{ij} + \sum_i u_i}, \quad \text{with } b_{ij} = \frac{1 - \hat{\theta}}{\hat{\theta}} \times \frac{K_i(x_j)}{\phi(x_j)}.$$

Thus, if the sequence $\{\hat{\tau}^{(s)}\}_{s \geq 0}$ is convergent, it has to converge towards a fixed point of ψ . Robin et al. (2007) prove that under some mild conditions, the estimator described in Algorithm 1 is self-convergent, meaning that as the number of iterations s increases, the sequence $\hat{f}^{(s)}$ converges towards the function

$$f_3(x) = \frac{1}{h} \sum_{i=1}^n \frac{\hat{\tau}_i^*}{\sum_k \hat{\tau}_k^*} K\left(\frac{x - X_i}{h}\right),$$

where $\hat{\tau}_i^*$ is the (unique) limit of $\{\hat{\tau}_i^{(s)}\}_{s \geq 0}$. Note that contrarily to f_2 , function f_3 is a randomly weighted kernel estimator of f . However, nothing is known about the convergence of f_3 nor $\hat{f}^{(s)}$ towards the true density f when n tends to infinity (while the bandwidth $h = h_n$ tends to 0). Indeed, the weights $\{\hat{\tau}_i^{(s)}\}_{s \geq 0}$ used by the kernel estimator $\hat{f}^{(s)}$ form an iterative sequence. Thus it is very difficult to study the convergence properties of this

weight sequence or of the corresponding estimator. In the following Section 3, we propose another randomly weighted kernel estimator, whose weights are slightly different from those used in the construction of \hat{f}^s . More precisely, those weights are not defined iteratively but they mimic the sequence of weights $\{\hat{\tau}_i^{(s)}\}_{s \geq 0}$.

3 Weighted kernel estimation of the unknown density f

As already said, Equation (6) suggests to replace the posterior probabilities τ_i by estimators to obtain a randomly weighted kernel estimator of f . Specifically, let $\hat{\theta}$ be a given estimator of the proportion θ and \hat{g}_n be a nonparametric estimator of the density g . We propose here to rely on a (direct) kernel estimator of the density g

$$\hat{g}_n(x) = \frac{1}{nh} \sum_{i=1}^n K_2\left(\frac{x - X_i}{h}\right), \quad (7)$$

where K_2 is a kernel to be chosen later. We then propose an estimator of the posterior probability $\tau(x)$ defined by

$$\forall x \in [0, 1], \hat{\tau}(x) = 1 - \frac{\hat{\theta}}{\hat{g}_n(x)}. \quad (8)$$

By defining the weight

$$\hat{\tau}_i = \hat{\tau}(X_i) = 1 - \frac{\hat{\theta}}{\tilde{g}_n(X_i)}, \text{ where } \tilde{g}_n(X_i) = \frac{1}{(n-1)h} \sum_{j \neq i}^n K_2\left(\frac{X_i - X_j}{h}\right), \quad (9)$$

we get a randomly weighted kernel estimator of the density f

$$\forall x \in [0, 1], \hat{f}_n(x) = \frac{1}{h} \sum_{i=1}^n \frac{\hat{\tau}_i}{\sum_{k=1}^n \hat{\tau}_k} K_1\left(\frac{x - X_i}{h}\right). \quad (10)$$

Note that it is not necessary to use the same kernel in defining \hat{g}_n and \hat{f}_n . However in practise, the choice of the kernel has a negligible influence on the performances of a kernel estimator. We provide below the properties of convergence of the estimator \hat{f}_n . In fact, these naturally depend on the properties of the plug-in estimators $\hat{\theta}$ and \hat{g}_n . We are interested here in controlling the pointwise quadratic risk of estimator \hat{f}_n . This is possible on a class of densities f that are regular enough. In the following, we let \mathbb{P}_g and \mathbb{E}_g respectively denote probability and expectation of iid random variables with density g . In the same way, we denote by $\mathbb{P}_{\theta, f}$ and $\mathbb{E}_{\theta, f}$ the probability and corresponding expectation in the more specific model (2). Moreover, $\lfloor x \rfloor$ denotes the largest integer strictly smaller than x . Now, we recall that the order of a kernel is defined as its first nonzero moment (Tsybakov, 2009) and we recall below the definition of Hölder classes of functions.

Definition 1. Fix $\beta > 0, L > 0$ and denote by $H(\beta, L)$ the set of functions $\psi : [0, 1] \rightarrow \mathbb{R}$ that are l -times continuously differentiable on $[0, 1]$ with $l = \lfloor \beta \rfloor$ and satisfy

$$|\psi^{(l)}(x) - \psi^{(l)}(y)| \leq L |x - y|^{\beta - l}, \quad \forall x, y \in [0, 1].$$

The set $H(\beta, L)$ is called the (β, L) -Hölder class of functions.

We denote by $\Sigma(\beta, L)$ the set

$$\Sigma(\beta, L) = \left\{ \psi : \psi \geq 0, \int_{\mathbb{R}} \psi(x) dx = 1 \quad \text{and} \quad \psi \in H(\beta, L) \right\}.$$

According to the proof of Theorem 1.1 in Tsybakov (2009), we remark that

$$\sup_{\psi \in \Sigma(\beta, L)} \|\psi\|_{\infty} < +\infty.$$

In order to obtain the rate of convergence of the kernel density estimator \hat{f}_n to f , we introduce the following assumptions:

(A1) The kernel K is a right-continuous function.

(A2) K is of bounded variation.

(A3) The kernel K is of order $l = \lfloor \beta \rfloor$ and satisfies

$$\int K(u) du = 1, \quad \int K^2(u) du < \infty, \quad \text{and} \quad \int |u|^{\beta} |K(u)| du < \infty.$$

(B1) f is a uniformly continuous density function.

(C1) The bandwidth h is of order $\alpha n^{-1/(2\beta+1)}$, $\alpha > 0$.

Note that there exist kernels satisfying Assumptions **(A1)**-**(A3)** (see for instance Section 1.2.2 in Tsybakov, 2009). Note also that if $f \in \Sigma(\beta, L)$, it automatically satisfies Assumption **(B1)**.

Remark 1. *i) We first remark that if kernel K_2 satisfies Assumptions **(A1)**, **(A2)** and if Assumptions **(B1)** and **(C1)** hold, then the kernel density estimator \hat{g}_n defined by (7) converges uniformly almost surely to g (Wied and Weißbach, 2012). In other words*

$$\|\hat{g}_n - g\|_{\infty} \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

*ii) If kernel K_2 satisfies Assumption **(A3)** and if Assumption **(C1)** holds, then for all $n \geq 1$*

$$\sup_{x \in [0,1]} \sup_{f \in \Sigma(\beta, L)} \mathbb{E}_{\theta, f} (|\hat{g}_n(x) - g(x)|^2) \leq C n^{\frac{-2\beta}{2\beta+1}},$$

where $C = C(\beta, L, \alpha, K_2)$ (see Theorem 1.1 in Tsybakov, 2009).

In the following theorem, we give the rate of convergence to zero of the pointwise quadratic risk of the estimator \hat{f}_n defined by (10).

Theorem 1. Assume that kernel K_1 satisfies Assumption **(A3)**, $K_1 \in \mathbb{L}_4(\mathbb{R})$ and kernel K_2 satisfies Assumptions **(A1)**-**(A3)**. If $\hat{\theta}$ converges almost surely towards θ and the bandwidth $h = \alpha n^{-1/(2\beta+1)}$ with $\alpha > 0$, then for any $\delta > 0$, the pointwise quadratic risk of \hat{f}_n satisfies

$$\sup_{x \in [0,1]} \sup_{\theta \in [\delta, 1-\delta]} \sup_{f \in \Sigma(\beta, L)} \mathbb{E}_{\theta, f} (|\hat{f}_n(x) - f(x)|^2) \leq C_1 \sup_{\theta \in [\delta, 1-\delta]} \sup_{f \in \Sigma(\beta, L)} \left[\mathbb{E}_{\theta, f} (|\hat{\theta} - \theta|)^4 \right]^{\frac{1}{2}} + C_2 n^{\frac{-2\beta}{2\beta+1}},$$

where C_1, C_2 are two positive constants depending only on β, L, α, δ and K_1, K_2 .

The proof of this theorem is postponed to Section 6.1. It works as follows: we first start by proving that the pointwise quadratic risk of f_2 (which is not an estimator) is of order $n^{-2\beta/(2\beta+1)}$. Then we compare the estimator \hat{f}_n with the function f_2 to conclude the proof. In Section 6.2, we prove the following corollary.

Corollary 1. Under the assumptions of Theorem 1, using the estimator $\hat{\theta}$ proposed in Celisse and Robin (2010), we obtain that for any fixed value (θ, f) , there is some positive constant C such that

$$\sup_{x \in [0,1]} \mathbb{E}_{\theta, f} (|\hat{f}_n(x) - f(x)|^2) \leq C n^{\frac{-2\beta}{2\beta+1}}.$$

Note that the rate $n^{-\beta/(2\beta+1)}$ is the usual nonparametric minimax rate over the class $\Sigma(\beta, L)$ of Hölder densities. However, the corollary states nothing about uniform convergence of $\hat{f}_n(x)$ with respect to the parameter value (θ, f) since the convergence of the estimator $\hat{\theta}$ is not known to be uniform.

4 Maximum smoothed likelihood estimator of the unknown density f

In this section, following the lines of Levine et al. (2011), we construct an iterative estimator sequence of the density f that relies on the maximisation of a smoothed likelihood. Assume that the kernel K is positive and symmetric on \mathbb{R} . We define its rescaled version $K_h(x) = h^{-1}K(h^{-1}x)$. We consider a linear smoothing operator $\mathcal{S} : \mathbb{L}_1([0, 1]) \rightarrow \mathbb{L}_1([0, 1])$ defined as

$$\mathcal{S}f(x) = \int_0^1 \frac{K_h(u-x)f(u)}{\int_0^1 K_h(s-u)ds} du.$$

We remark that if f is a density on $[0, 1]$ then $\mathcal{S}f$ is also a density on $[0, 1]$. Let us consider a submodel of model (2) restricted to

$$\mathcal{F} = \{\text{densities } f \text{ on } [0, 1] \text{ such that } \log f \in \mathbb{L}_1([0, 1])\}.$$

We denote by $\mathcal{S}^* : \mathbb{L}_1([0, 1]) \rightarrow \mathbb{L}_1([0, 1])$ the operator

$$\mathcal{S}^*f(x) = \frac{\int_0^1 K_h(u-x)f(u)du}{\int_0^1 K_h(s-x)ds}.$$

Note the difference between \mathcal{S} and \mathcal{S}^* . The operator \mathcal{S}^* is in fact the adjoint operator of \mathcal{S} . Then for a density $f \in \mathcal{F}$, we approach it by a nonlinear smoothing operator \mathcal{N} defined as

$$\mathcal{N}f(x) = \exp\{(\mathcal{S}^*(\log f))(x)\}, \quad x \in [0, 1].$$

Note that $\mathcal{N}f$ is not necessarily a density. Now, instead of the classical log-likelihood, we consider (the opposite of) a smoothed version of it as our criterion, namely

$$l_n(\theta, f) = \frac{-1}{n} \sum_{i=1}^n \log[\theta + (1 - \theta)\mathcal{N}f(X_i)].$$

In this section, we denote by g_0 the true density of the observations X_i . For any fixed value of θ , up to the additive constant $\int_0^1 g_0(x) \log g_0(x) dx$, the smoothed log-likelihood $l_n(\theta, f)$ converges almost surely towards $l(\theta, f)$ defined as

$$l(\theta, f) := \int_0^1 g_0(x) \log \frac{g_0(x)}{\theta + (1 - \theta)\mathcal{N}f(x)} dx.$$

This quantity may be viewed as a penalized Kullback-Leibler divergence between the true density g_0 and its smoothed approximation for parameters (θ, f) . Indeed, let $D(a | b)$ denote the Kullback-Leibler divergence between (positive) measures a and b , defined as

$$D(a | b) = \int_0^1 \left\{ a(x) \log \frac{a(x)}{b(x)} + b(x) - a(x) \right\} dx.$$

Note that in the above definition, a and b are not necessarily probability measures. Moreover it can be seen that we still have the property $D(a|b) \geq 0$ with equality if and only if $a = b$ (Eggermont, 1999). We now obtain

$$l(\theta, f) = D(g_0 | \theta + (1 - \theta)\mathcal{N}f) + (1 - \theta) \left(1 - \int_0^1 \mathcal{N}f(x) dx \right).$$

The second term in the right-hand side of the above equation acts as a penalization term (Eggermont, 1999; Levine et al., 2011). Our goal is to construct an iterative sequence of estimators of f that possesses a descent property with respect to the criterion $l(\theta, \cdot)$ (for fixed value θ). We start by describing such a procedure, relying on the knowledge of the parameters (thus an oracle procedure) in the next section. Then in Section 4.2, we derive the procedure without this knowledge, obtaining an iterative sequence of estimators.

4.1 An oracle iterative procedure to approximate f

In this section, we fix the value of θ or consider θ as a given estimator. Let us denote by $l_n(f)$ the smoothed log-likelihood $l_n(\theta, f)$ and by $l(f)$ the limit function $l(\theta, f)$. Our goal is to construct an iterative algorithm which ensures that the value of $l(\cdot)$ decreases at each iteration. Specifically, we construct a sequence of densities $\{f^t\}_{t \geq 0}$ such that

$$l(f^t) - l(f^{t+1}) \geq cD(f^{t+1} | f^t) \geq 0,$$

where c is a positive constant depending on θ , the bandwidth h and the kernel K . Let us first introduce a weight function

$$\forall f \in \mathcal{F}, \forall x \in [0, 1], \quad \omega_f(x) = \frac{(1 - \theta)\mathcal{N}f(x)}{\theta + (1 - \theta)\mathcal{N}f(x)},$$

and an operator $G : \mathcal{F} \rightarrow \mathbb{L}_1([0, 1])$

$$G(f)(x) = \alpha_f \int_0^1 \frac{K_h(u - x)\omega_f(u)g_0(u)}{\int_0^1 K_h(s - u)ds} du, \quad x \in [0, 1],$$

where $\alpha_f^{-1} = \int_0^1 \omega_f(u)g_0(u)du$.

Remark 2. We remark that $G(f)$ is automatically a density on $[0, 1]$. Moreover, for every density $f \in \mathcal{F}$, we have $G(f) = \mathcal{S}\varphi$ where

$$\varphi(x) = \frac{\omega_f(x)g_0(x)}{\int_0^1 \omega_f(u)g_0(u)du}, \quad x \in [0, 1].$$

We iteratively define a sequence of densities $\{f^t\}_{t \geq 0}$ in \mathcal{F} as

$$f^{t+1}(x) = G(f^t)(x) = \alpha_t \int_0^1 \frac{K_h(u - x)\omega_t(u)g_0(u)}{\int_0^1 K_h(s - u)ds} du, \quad x \in [0, 1], \quad (11)$$

where

$$\alpha_t = \frac{1}{\int_0^1 \omega_t(u)g_0(u)du} \quad \text{and} \quad \omega_t(x) = \frac{(1 - \theta)\mathcal{N}f^t(x)}{\theta + (1 - \theta)\mathcal{N}f^t(x)}.$$

Note that the sequence $\{f^t\}_{t \geq 0}$ does not define an estimator sequence as it depends on the true density g_0 which is unknown. This is why we call it an oracle. We shall define an estimating sequence $\{\hat{f}^t\}_{t \geq 0}$ as a second step in Section 4.2. Let us now denote by

$$m = \inf_{x \in [-1, 1]} K_h(x) \quad \text{and} \quad M = \sup_{x \in [-1, 1]} K_h(x),$$

then m and M are two positive constants depending on the bandwidth h and the kernel K . We note that for all $x \in [0, 1]$,

$$m \leq \int_0^1 K_h(u - x)du \leq \min(M, 1).$$

We also introduce

$$\mathcal{B} = \{\mathcal{S}\varphi; \varphi \text{ density on } [0, 1]\}.$$

According to Remark 2, every function f^t belongs to \mathcal{B} . For all $f \in \mathcal{B}$, we remark that $m \leq f(\cdot) \leq M/m$. As a consequence, we obtain that the sequence $\{l(f^t)\}_{t \geq 0}$ is lower bounded (by $\int g_0 \log g_0 - m$). We now state the descent property on the sequence $\{f^t\}_{t \geq 0}$. Its proof may be found in Section 6.2.

Proposition 1. Any iterative sequence of densities $\{f^t\}_{t \geq 0}$ defined as $f^{t+1} = G(f^t)$, with $f^0 \in \mathcal{F}$ satisfies the descent property

$$l(f^t) - l(f^{t+1}) \geq cD(f^{t+1} | f^t) \geq 0,$$

where c is a positive constant depending on θ , the bandwidth h and the kernel K .

Note that the sequence $\{l(f^t)\}_{t \geq 0}$ is decreasing and lower bounded, thus it is convergent. Moreover, each sequence $\{f^t\}_{t \geq 0}$ converges (simply) to a local minimum of l .

Remark 3. If f^* is a global minimum of l then f^* is a fixed point of G . Indeed, we have

$$0 \geq l(f^*) - l(G(f^*)) \geq cD(G(f^*) | f^*) \geq 0.$$

It entails that $D(G(f^*) | f^*) = 0$ and thus $f^* = G(f^*)$. Note that the converse is not true: a fixed point of G is not necessarily a minimum of the criterion l .

Now, by using convexity arguments, we can further prove the uniqueness of a minimum of l and thus the (simple) convergence of the sequence $\{f^t\}_{t \geq 0}$ to this minimum.

Corollary 2. The criterion l has a unique minimum f^* on \mathcal{B} . Moreover, any iterative sequence of densities $\{f^t\}_{t \geq 0}$ defined as $f^{t+1} = G(f^t)$, with $f^0 \in \mathcal{F}$ converges simply to f^* .

We conclude the study of the convergence properties of the sequence $\{f^t\}_{t \geq 0}$ by further establishing its uniform convergence below. The proof of this proposition and of the former corollary may be found in Section 6.2.

Proposition 2. If there exists a constant L depending on h such that for all $x, y \in [-1, 1]$

$$|K_h(x) - K_h(y)| \leq L|x - y|,$$

then the sequence of densities $\{f^t\}_{t \geq 0}$ converges uniformly to f^* .

Note that the previous assumption may be satisfied by many different kernels. For instance, if K is the density of the standard normal distribution, then this assumption is satisfied with

$$L = \frac{1}{h^2\sqrt{2\pi}}e^{-1/2}.$$

4.2 Estimation procedure

As previously said, the sequence $\{f^t\}_{t \geq 0}$ is an oracle as it depends on the knowledge of the true density g_0 that is unknown. We now want to construct an estimating sequence $\{\hat{f}^t\}_{t \geq 0}$ of the sequence of functions $\{f^t\}_{t \geq 0}$. Relying on a Monte-Carlo method to approximate the integral involved in the definition (11) of f^t , we propose an iterative algorithm to estimate the density f : given an initial value $\hat{\omega}_0 = (\hat{\omega}_0(1), \dots, \hat{\omega}_0(n)) \in (0, 1)^n$ of the weights, iterate the following steps for $t = 0, 1, 2, \dots$

$$\begin{aligned} \hat{f}^{t+1}(x) &= \frac{1}{\sum_{i=1}^n \hat{\omega}_t(X_i)} \sum_{i=1}^n \frac{K_h(x - X_i) \hat{\omega}_t(X_i)}{\int_0^1 K_h(s - X_i) ds} \\ &= \sum_{i=1}^n \frac{\hat{\omega}_t(X_i)}{\sum_{k=1}^n \hat{\omega}_t(X_k)} \frac{K_h(x - X_i)}{\int_0^1 K_h(s - X_i) ds}, \end{aligned}$$

and

$$\hat{\omega}_{t+1}(X_i) = \frac{(1-\theta)\mathcal{N}\hat{f}^{t+1}(X_i)}{\theta + (1-\theta)\mathcal{N}\hat{f}^{t+1}(X_i)}.$$

The procedure is summarized in Algorithm 2. Its properties are described in Proposition 3 whose proof is very similar to the proof of Proposition 1 and therefore omitted.

Algorithm 2: Smoothed-likelihood kernel algorithm

```

// Initialization;
Set  $\hat{\omega}_0 = (\hat{\omega}_0(1), \dots, \hat{\omega}_0(n)) \in [0, 1]^n$ .

while  $\max_i |\hat{\omega}_t(i) - \hat{\omega}_{t-1}(i)| / \hat{\omega}_{t-1}(i) \geq \epsilon$  do

    // Update estimation of  $f$ ;
     $\hat{f}^t(X_i) = \sum_{j \neq i} \frac{\hat{\omega}_{t-1}(j)}{\sum_{k=1}^n \hat{\omega}_{t-1}(k)} \frac{K_h(X_i - X_j)}{\int_0^1 K_h(s - X_j) ds}$ 

    // Update of weights;
     $\hat{\omega}_t(i) = \frac{(1-\theta)\mathcal{N}\hat{f}^t(X_i)}{\theta + (1-\theta)\mathcal{N}\hat{f}^t(X_i)}$ 

     $t \leftarrow t + 1$ ;

// Return;
 $\hat{f}^t(\cdot) = \sum_{i=1}^n \frac{\hat{\omega}_{t-1}(i)}{\sum_{k=1}^n \hat{\omega}_{t-1}(k)} \frac{K_h(\cdot - X_i)}{\int_0^1 K_h(s - X_i) ds}$ 

```

Proposition 3. For any initial value of the weights $\hat{\omega}_0 \in (0, 1)^n$, the sequence of estimators $\{\hat{f}^t\}_{t \geq 0}$ satisfies

$$l_n(\hat{f}^t) - l_n(\hat{f}^{t+1}) \geq cD(\hat{f}^{t+1} | \hat{f}^t) \geq 0,$$

where c is a positive constant depending on θ , the bandwidth h and the kernel K .

As a consequence and since l_n is lower bounded, the sequence $\{\hat{f}^t\}_{t \geq 0}$ converges to a local minimum of l_n as t increases. Moreover, we recall that as the sample size n increases, the criterion l_n converges (up to a constant) to l . Thus, the outcome of Algorithm 2 is an approximation of the minimizer f^* of l .

5 Estimation of local false discovery rate and simulation study

5.1 Estimation of local false discovery rate

In this section, we study the estimation of local false discovery rate (ℓ FDR) by using the previously introduced estimators of the density f and compare these different approaches on simulated data. Let us recall the definition (3) of the local false discovery rate

$$\ell\text{FDR}(x) = \mathbb{P}(H_i \text{ being true} | X = x) = \frac{\theta}{\theta + (1-\theta)f(x)}, \quad x \in [0, 1].$$

For a given estimator $\hat{\theta}$ of the proportion θ and an estimator \hat{f} of the density f , we obtain a natural estimator of the local false discovery rate for observation x_i

$$\widehat{\ell\text{FDR}}(x_i) = \frac{\hat{\theta}}{\hat{\theta} + (1 - \hat{\theta})\hat{f}(x_i)}. \quad (12)$$

Let us now denote by \hat{f}_{wk} the weighted kernel estimator of f constructed in Section 2.1, by \hat{f}_{kerfdr} the estimator of f presented in Algorithm 1 and by \hat{f}_{msl} the maximum smoothed likelihood estimator of f presented in Algorithm 2. Note that \hat{f}_{kerfdr} is available through the R package `kerfdr`. We also let $\widehat{\ell\text{FDR}}_m, m \in \{\text{wk}, \text{kerfdr}, \text{msl}\}$ be the estimators of ℓFDR induced by a plug-in of the estimators \hat{f}_m in (12). We compute the root mean squared error (RMSE) between the estimates and the true values

$$\text{RMSE}_m = \frac{1}{S} \sum_{s=1}^S \sqrt{\frac{1}{n} \sum_{i=1}^n \{\widehat{\ell\text{FDR}}_m(x_i) - \ell\text{FDR}(x_i)\}^2},$$

for $m \in \{\text{wk}, \text{kerfdr}, \text{msl}\}$ and where $s = 1, \dots, S$ denotes the simulation index (S being the total number of repeats). The quality of the estimates provided by method m is measured by the mean RMSE_m : the smaller the RMSE_m , the better the performance of the method.

5.2 Simulation study

In this section, we give an illustration of the previous results on some simulated experiments. We simulate sets of p -values according to the mixture model (2). We consider four different values for the proportion ($\theta = 0.7, 0.8, 0.9$ and 0.95) and three different cases for the alternative distribution f . In the first case, we simulate p -values under the alternative with distribution

$$f(x) = \rho(1-x)^{\rho-1} \mathbf{1}_{[0,1]}(x),$$

where $\rho = 4$, as proposed in Celisse and Robin (2010). In the second case, the p -value corresponds to the statistic T which has a mixture distribution $\theta\mathcal{N}(0, 1) + (1 - \theta)\mathcal{N}(\mu, 1)$, with $\mu = 2$. In the third case, the p -value corresponds to the statistic T which has a mixture density $\theta(1/2) \exp\{-|t|\} + (1 - \theta)(1/2) \exp\{-|t - \mu|\}$, with $\mu = 1$. For each of the $4 \times 3 = 12$ configurations, we generate $S = 100$ samples of size $n = 1000$. In these experiments, we choose to consider the estimator of θ initially proposed by Schweder and Spjøtvoll (1982), namely

$$\hat{\theta} = \frac{\#\{X_i > \lambda; i = 1, \dots, n\}}{n(1 - \lambda)},$$

with parameter value $\lambda = 0.5$, as recommended by Storey and Tibshirani (2003). Figure 1 shows the RMSEs for the twelve configurations and the three different methods.

First, note that the three methods exhibit small RMSEs and are thus efficient for estimating ℓFDR . We note that the first method (weighted kernel, wk) tends to have lower performances than the other two methods. Remember that we introduced it only as a way of approaching the theoretical performances of `kerfdr` method. Now, the maximum smoothed likelihood (msl) method tends to have identical performances than `kerfdr`, with some cases where it outperforms `kerfdr`. Thus it appears as a competitive method for ℓFDR estimation.

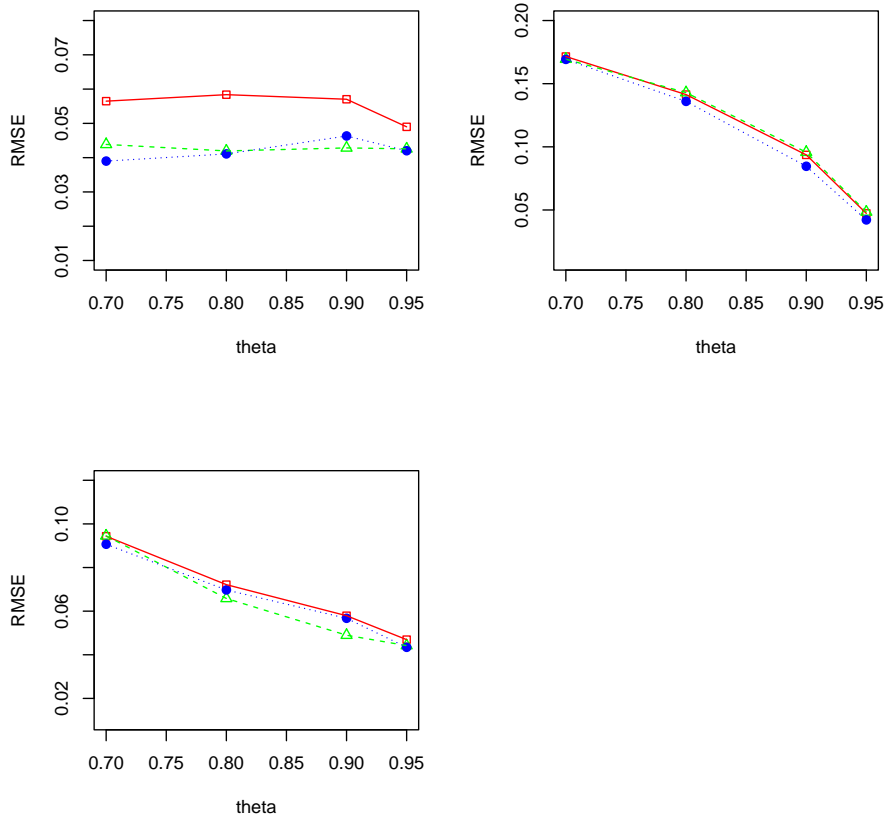


Figure 1: Root mean square error (RMSE) between the true local false discovery rate and the estimates as a function of the proportion θ . Methods: " \square " = wk, " \triangle " = kerfdr, " \bullet " = msl. Top left: first model, top right: second model, bottom left: third model.

6 Proofs

6.1 Proof of Theorem 1

The proof works as follows: we first start by proving that the pointwise quadratic risk of function f_2 defined by (6) is order of $n^{-2\beta/(2\beta+1)}$ in the following proposition. Then we compare the estimator \hat{f}_n with the function f_2 to conclude the proof.

We shall need the following two lemmas. The proof of the first one may be found for instance in Proposition 1.2 in Tsybakov (2009). The second one is known as Bochner's lemma and is a classical result in kernel density estimation. Therefore its proof is omitted.

Lemma 1. (Proposition 1.2 in Tsybakov (2009)). *Let p be a density in $\Sigma(\beta, L)$ and K a kernel function of order $l = \lfloor \beta \rfloor$ such that*

$$\int_{\mathbb{R}} |u|^\beta |K(u)| du < \infty.$$

Then there exists a positive constant C_3 depending only on β, L and K such that for all $x_0 \in \mathbb{R}$,

$$\left| \int_{\mathbb{R}} K(u) [p(x_0 + uh) - p(x_0)] du \right| \leq C_3 h^\beta, \quad \forall h > 0.$$

Lemma 2. (Bochner's lemma). *Let g be a bounded function on \mathbb{R} , continuous in a neighborhood of $x_0 \in \mathbb{R}$ and Q a function which satisfies*

$$\int_{\mathbb{R}} |Q(x)| dx < \infty.$$

Then, we have

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{R}} Q\left(\frac{x - x_0}{h}\right) g(x) dx = g(x_0) \int_{\mathbb{R}} Q(x) dx.$$

Now, we come to the first step in the proof.

Proposition 4. *Assume that the kernel K_1 satisfies Assumption (A3) and the bandwidth $h = \alpha n^{-1/(2\beta+1)}$, with $\alpha > 0$. Then the pointwise quadratic risk of function f_2 , defined by (6) and depending on (θ, f) , satisfies*

$$\sup_{x \in [0,1]} \sup_{\theta \in [\delta, 1-\delta]} \sup_{f \in \Sigma(\beta, L)} \mathbb{E}_{\theta, f} (|f_2(x) - f(x)|^2) \leq C_4 n^{\frac{-2\beta}{2\beta+1}},$$

where C_4 is a positive constant depending only on β, L, α, δ and K_1 .

Proof of Proposition 4. Let us denote by

$$S_n = \sum_{i=1}^n \frac{f(X_i)}{g(X_i)}.$$

The pointwise quadratic risk of f_2 can be written as the sum of a bias term and a variance term

$$\mathbb{E}_{\theta, f} (|f_2(x) - f(x)|^2) = [\mathbb{E}_{\theta, f} (f_2(x)) - f(x)]^2 + \text{Var}_{\theta, f} [f_2(x)].$$

Let us first study the bias term. According to (6) and the definition (5) of the weights, we have

$$\begin{aligned}
\mathbb{E}_{\theta,f}[f_2(x)] &= \frac{n}{h} \mathbb{E}_{\theta,f} \left[\tau_1 K_1 \left(\frac{x - X_1}{h} \right) \left(\sum_{k=1}^n \tau_k \right)^{-1} \right] \\
&= \frac{n}{h} \mathbb{E}_{\theta,f} \left[\frac{f(X_1)}{g(X_1)} K_1 \left(\frac{x - X_1}{h} \right) S_n^{-1} \right] \\
&= \frac{n}{h} \int_0^1 f(t) K_1 \left(\frac{x-t}{h} \right) \mathbb{E}_{\theta,f} \left[\left(\frac{f(t)}{g(t)} + S_{n-1} \right)^{-1} \right] dt \\
&= n \int_{-x/h}^{(1-x)/h} K_1(t) f(x+th) \mathbb{E}_{\theta,f} \left[\left(\frac{f(x+th)}{g(x+th)} + S_{n-1} \right)^{-1} \right] dt. \quad (13)
\end{aligned}$$

Since the functions f and g are related by the equation $g(t) = \theta + (1-\theta)f(t)$ for all $t \in [0, 1]$, the ratio $f(t)/g(t)$ is well defined and satisfies

$$0 \leq \frac{f(t)}{g(t)} \leq \frac{1}{1-\theta} \leq \delta^{-1}, \quad \forall t \in [0, 1], \text{ and } \forall \theta \in [\delta, 1-\delta].$$

Then for all $t \in [-x/h, (1-x)/h]$, we get

$$\frac{1}{S_{n-1} + \delta^{-1}} \leq \left(\frac{f(x+th)}{g(x+th)} + S_{n-1} \right)^{-1} \leq \frac{1}{S_{n-1}},$$

where the bounds are uniform with respect to t .

By combining this inequality with (13), we obtain

$$\begin{aligned}
&n \left(\int_{-x/h}^{(1-x)/h} K_1(t) f(x+th) dt \right) \mathbb{E}_{\theta,f} \left(\frac{1}{S_{n-1} + \delta^{-1}} \right) \leq \mathbb{E}_{\theta,f}[f_2(x)] \\
\text{and } \mathbb{E}_{\theta,f}[f_2(x)] &\leq n \left(\int_{-x/h}^{(1-x)/h} K_1(t) f(x+th) dt \right) \mathbb{E}_{\theta,f} \left(\frac{1}{S_{n-1}} \right).
\end{aligned}$$

Then, we apply the following lemma, whose proof is postponed to Appendix A.1.

Lemma 3. *There exist some positive constants c_1, c_2, c_3, c_4 (depending on δ) such that for n large enough,*

$$\mathbb{E}_{\theta,f} \left(\frac{1}{S_n} \right) \leq \frac{1}{n} + \frac{c_1}{n^2}, \quad (14)$$

$$\mathbb{E}_{\theta,f} \left(\frac{1}{S_n^2} \right) \leq \frac{c_2}{n^2}, \quad (15)$$

$$\mathbb{E}_{\theta,f} \left(\frac{1}{S_n + 2\delta^{-1}} \right) \geq \frac{1}{n} - \frac{c_3}{n^2}, \quad (16)$$

$$\text{and } \mathbb{E}_{\theta,f} \left(\frac{1}{S_n^2} \right) - \mathbb{E}_{\theta,f}^2 \left(\frac{1}{\delta^{-1} + S_n} \right) \leq \frac{c_4}{n^3}. \quad (17)$$

Relying on Inequalities (14) and (16), we have for n large enough

$$\int_{-x/h}^{(1-x)/h} K_1(t) f(x+th) dt - \frac{c_3}{n} \leq \mathbb{E}_{\theta,f}[f_2(x)] \leq \int_{-x/h}^{(1-x)/h} K_1(t) f(x+th) dt + \frac{c_1}{n}.$$

Since $f(x + th) = 0$ for all $t \notin [-x/h, (1-x)/h]$, we may write

$$\int_{-x/h}^{(1-x)/h} K_1(t)f(x + th)dt = \int_{\mathbb{R}} K_1(t)f(x + th)dt.$$

Thus, the bias of $f_2(x)$ satisfies

$$|b(x)| = |\mathbb{E}_{\theta,f}[f_2(x)] - f(x)| \leq \int_{\mathbb{R}} K_1(t)|f(x + th) - f(x)|dt + \frac{C_5}{n}.$$

By using Lemma 1 and the choice of bandwidth h , we obtain that

$$b^2(x) \leq C_5 h^{2\beta},$$

where $C_5 = C_5(\beta, L, K_1)$. Let us study now the variance term of $f_2(x)$. We have

$$\text{Var}_{\theta,f}[f_2(x)] = \frac{1}{h^2} [n\text{Var}_{\theta,f}(Y_1) + n(n-1)\text{Cov}_{\theta,f}(Y_1, Y_2)], \quad (18)$$

where

$$Y_i = \frac{f(X_i)}{g(X_i)} K\left(\frac{x - X_i}{h}\right) S_n^{-1}.$$

The variance of Y_1 is bounded by its second moment and

$$\begin{aligned} \mathbb{E}_{\theta,f}(Y_1^2) &= \mathbb{E}_{\theta,f} \left[\left(\frac{f(X_1)}{g(X_1)} \right)^2 K_1^2\left(\frac{x - X_1}{h}\right) S_n^{-2} \right] \\ &= \int_0^1 \frac{f^2(t)}{g(t)} K_1^2\left(\frac{x - t}{h}\right) \mathbb{E}_{\theta,f} \left[\left(\frac{f(t)}{g(t)} + S_{n-1} \right)^{-2} \right] dt. \end{aligned}$$

Now, recalling that $0 \leq f/g \leq \delta^{-1}$ and using Inequality (15) of Lemma 3, we get

$$\begin{aligned} \mathbb{E}_{\theta,f}(Y_1^2) &\leq h \left(\int_{-x/h}^{(1-x)/h} \frac{f^2(x + th)}{g(x + th)} K_1^2(t) dt \right) \mathbb{E}_{\theta,f} \left(\frac{1}{S_{n-1}^2} \right) \\ &\leq h \delta^{-1} \sup_{f \in \Sigma(\beta, L)} \|f\|_{\infty} \left(\int K_1^2(t) dt \right) \frac{C_2}{n^2} \\ &\leq \frac{C_6 h}{n^2}. \end{aligned} \quad (19)$$

We now study the covariance of Y_1 and Y_2

$$\begin{aligned} \text{Cov}_{\theta,f}(Y_1, Y_2) &= \mathbb{E}_{\theta,f}(Y_1 Y_2) - \mathbb{E}_{\theta,f}^2(Y_1) \\ &= \mathbb{E}_{\theta,f} \left[\frac{f(X_1)f(X_2)}{g(X_1)g(X_2)} K_1\left(\frac{x - X_1}{h}\right) K_1\left(\frac{x - X_2}{h}\right) S_n^{-2} \right] - \mathbb{E}_{\theta,f}^2 \left[\frac{f(X_1)}{g(X_1)} K_1\left(\frac{x - X_1}{h}\right) S_n^{-1} \right] \\ &= \int_{[0,1]^2} f(t)f(u) K_1\left(\frac{x - t}{h}\right) K_1\left(\frac{x - u}{h}\right) \mathbb{E}_{\theta,f} \left[\left(\frac{f(t)}{g(t)} + \frac{f(u)}{g(u)} + S_{n-2} \right)^{-2} \right] dt du \\ &\quad - \left(\int_0^1 f(t) K_1\left(\frac{x - t}{h}\right) \mathbb{E}_{\theta,f} \left[\left(\frac{f(t)}{g(t)} + S_{n-1} \right)^{-1} \right] dt \right)^2 \\ &= \int_{[0,1]^2} f(t)f(u) K_1\left(\frac{x - t}{h}\right) K_1\left(\frac{x - u}{h}\right) A(t, u) dt du, \end{aligned}$$

where

$$\begin{aligned} A(t, u) &= \mathbb{E}_{\theta, f} \left[\left(\frac{f(t)}{g(t)} + \frac{f(u)}{g(u)} + S_{n-2} \right)^{-2} \right] - \mathbb{E}_{\theta, f} \left[\left(\frac{f(t)}{g(t)} + S_{n-1} \right)^{-1} \right] \mathbb{E}_{\theta, f} \left[\left(\frac{f(u)}{g(u)} + S_{n-1} \right)^{-1} \right] \\ &\leq \mathbb{E}_{\theta, f} \left(\frac{1}{S_{n-2}^2} \right) - \mathbb{E}_{\theta, f}^2 \left(\frac{1}{2\delta^{-1} + S_{n-2}} \right). \end{aligned}$$

Hence

$$\begin{aligned} \text{Cov}(Y_1, Y_2) &\leq \int_{[0,1]^2} f(t)f(u)K_1\left(\frac{x-t}{h}\right)K_1\left(\frac{x-u}{h}\right) \left[\mathbb{E}_{\theta, f} \left(\frac{1}{S_{n-2}^2} \right) - \mathbb{E}_{\theta, f}^2 \left(\frac{1}{2\delta^{-1} + S_{n-2}} \right) \right] dtdu \\ &\leq h^2 \left(\int_{\mathbb{R}} f(x+th)K_1(t)dt \right)^2 \left[\mathbb{E}_{\theta, f} \left(\frac{1}{S_{n-2}^2} \right) - \mathbb{E}_{\theta, f}^2 \left(\frac{1}{2\delta^{-1} + S_{n-2}} \right) \right] \\ &\leq C_7 h^2 \left[\mathbb{E}_{\theta, f} \left(\frac{1}{S_{n-2}^2} \right) - \mathbb{E}_{\theta, f}^2 \left(\frac{1}{2\delta^{-1} + S_{n-2}} \right) \right]. \end{aligned}$$

According to Inequality (17) of Lemma 3, we have

$$\mathbb{E}_{\theta, f} \left(\frac{1}{S_{n-2}^2} \right) - \mathbb{E}_{\theta, f}^2 \left(\frac{1}{2\delta^{-1} + S_{n-2}} \right) \leq \frac{c_4}{n^3},$$

hence

$$\text{Cov}_{\theta, f}(Y_1, Y_2) \leq \frac{C_8 h^2}{n^3}. \quad (20)$$

By returning to Equality (18) and combining with (19) and (20), we obtain

$$\text{Var}_{\theta, f}[f_2(x)] \leq \frac{1}{h^2} \left[\frac{C_6 h}{n} + n(n-1)h^2 \frac{C_8 h^2}{n^3} \right] \leq \frac{C_9}{nh}.$$

Thus, as the bandwidth h is of order $n^{-1/(2\beta+1)}$, the pointwise quadratic risk of $f_2(x)$ satisfies

$$\mathbb{E}_{\theta, f}(|f_2(x) - f(x)|^2) \leq C_4 n^{\frac{-2\beta}{2\beta+1}}.$$

□

Proof of Theorem 1. First, the pointwise quadratic risk of $\hat{f}_n(x)$ is bounded in the following way

$$\mathbb{E}_{\theta, f}(|\hat{f}_n(x) - f(x)|^2) \leq 2\mathbb{E}_{\theta, f}(|f_2(x) - f(x)|^2) + 2\mathbb{E}_{\theta, f}(|\hat{f}_n(x) - f_2(x)|^2). \quad (21)$$

According to Proposition 4, we have

$$\mathbb{E}_{\theta, f}(|f_2(x) - f(x)|^2) \leq C_4 n^{\frac{-2\beta}{2\beta+1}}, \quad (22)$$

and it remains to study the second term appearing in the right-hand side of (21). We write

$$\begin{aligned}
\hat{f}_n(x) - f_2(x) &= \frac{1}{h} \sum_{i=1}^n \left(\frac{\hat{\tau}_i}{\sum_k \hat{\tau}_k} - \frac{\tau_i}{\sum_k \tau_k} \right) K_1 \left(\frac{x - X_i}{h} \right) \\
&= \frac{1}{h} \sum_{i=1}^n \frac{\hat{\tau}_i - \tau_i}{\sum_k \hat{\tau}_k} K_1 \left(\frac{x - X_i}{h} \right) + \frac{1}{h} \sum_{i=1}^n \tau_i \left(\frac{1}{\sum_k \hat{\tau}_k} - \frac{1}{\sum_k \tau_k} \right) K_1 \left(\frac{x - X_i}{h} \right) \\
&= \frac{n}{\sum_k \hat{\tau}_k} \times \frac{1}{nh} \sum_{i=1}^n (\hat{\tau}_i - \tau_i) K_1 \left(\frac{x - X_i}{h} \right) \\
&\quad + \frac{n^2}{\sum_k \hat{\tau}_k \sum_k \tau_k} \times \frac{\sum_k (\tau_k - \hat{\tau}_k)}{n} \times \frac{1}{nh} \sum_{i=1}^n \tau_i K_1 \left(\frac{x - X_i}{h} \right).
\end{aligned}$$

Moreover, recalling the definition of the weights (9), we have for all $1 \leq i \leq n$,

$$\hat{\tau}_i - \tau_i = \frac{\hat{\theta}}{\tilde{g}_n(X_i)} - \frac{\theta}{g(X_i)} = \hat{\theta} \left[\frac{1}{\tilde{g}_n(X_i)} - \frac{1}{g(X_i)} \right] + \frac{1}{g(X_i)} (\hat{\theta} - \theta),$$

and thus get

$$\begin{aligned}
\hat{f}_n(x) - f_2(x) &= \frac{n\hat{\theta}}{\sum_k \hat{\tau}_k} \times \frac{1}{nh} \sum_{i=1}^n \left[\frac{1}{\tilde{g}_n(X_i)} - \frac{1}{g(X_i)} \right] K_1 \left(\frac{x - X_i}{h} \right) \\
&\quad + \frac{n(\hat{\theta} - \theta)}{\sum_k \hat{\tau}_k} \times \frac{1}{nh} \sum_{i=1}^n \frac{1}{g(X_i)} K_1 \left(\frac{x - X_i}{h} \right) \\
&\quad + \frac{n^2\hat{\theta}}{\sum_k \hat{\tau}_k \sum_k \tau_k} \times \frac{1}{n} \sum_k \left[\frac{1}{\tilde{g}_n(X_k)} - \frac{1}{g(X_k)} \right] \times \frac{1}{nh} \sum_{i=1}^n \tau_i K_1 \left(\frac{x - X_i}{h} \right) \\
&\quad + \frac{n^2(\hat{\theta} - \theta)}{\sum_k \hat{\tau}_k \sum_k \tau_k} \times \frac{1}{n} \sum_k \frac{1}{g(X_k)} \times \frac{1}{nh} \sum_{i=1}^n \tau_i K_1 \left(\frac{x - X_i}{h} \right). \tag{23}
\end{aligned}$$

Let us control the different terms appearing in this latter equality. We first remark that for all i ,

$$0 \leq \tau_i \leq 1 \text{ and } \frac{1}{g(X_i)} \leq \frac{1}{\theta} \leq \delta^{-1}. \tag{24}$$

Since by assumption $\hat{\theta} \xrightarrow[n \rightarrow \infty]{as} \theta \in [0, 1]$, for n large enough we also get $|\hat{\theta}| < 3/2$, a.s. According to the law of large numbers and $\mathbb{E}_{\theta, f}(\tau_1) = 1 - \theta$, we also obtain that for n large enough

$$\frac{\delta}{2} \leq \frac{1 - \theta}{2} \leq \frac{1}{n} \sum_{i=1}^n \tau_i \leq \frac{3(1 - \theta)}{2} \leq \frac{3(1 - \delta)}{2} \quad \text{a.s.} \tag{25}$$

Moreover, by using a Taylor expansion of the function $u \mapsto 1/u$ with an integral form of the remainder term, we have for all i ,

$$\left| \frac{1}{\tilde{g}_n(X_i)} - \frac{1}{g(X_i)} \right| = \frac{|\tilde{g}_n(X_i) - g(X_i)|}{g^2(X_i)} \int_0^1 \left(1 + s \frac{\tilde{g}_n(X_i) - g(X_i)}{g(X_i)} \right)^{-2} ds.$$

Since convergence of \hat{g}_n to g is valid pointwise and in \mathbb{L}_∞ norm (see Remark 1), and since \tilde{g}_n is a slight modification of \hat{g}_n , we have almost surely, for n large enough and for all $s \in [0, 1]$ and all $x \in [0, 1]$,

$$1 + s \frac{\tilde{g}_n(x) - g(x)}{g(x)} \geq 1 - s \frac{\|\hat{g}_n - g\|_\infty}{\theta} \geq 1 - \frac{s}{2} > 0.$$

Hence, for all $x \in [0, 1]$ and large enough n ,

$$\int_0^1 \left(1 + s \frac{\tilde{g}_n(x) - g(x)}{g(x)}\right)^{-2} ds \leq \int_0^1 \frac{4ds}{(2-s)^2} = 2,$$

and we obtain

$$\left| \frac{1}{\tilde{g}_n(X_i)} - \frac{1}{g(X_i)} \right| \leq 2\delta^{-2} |\tilde{g}_n(X_i) - g(X_i)| \quad \text{a.s.} \quad (26)$$

We also use the following lemma, whose proof is postponed to Appendix A.2.

Lemma 4. *For large enough n , we have*

$$\frac{n}{|\sum_k \hat{\tau}_k|} \leq c_7 \quad \text{a.s.} \quad (27)$$

By returning to Equality (23) and combining with (24), (25), (26) and (27), we obtain

$$\begin{aligned} |\hat{f}_n(x) - f_2(x)|^2 &\leq c_8 \left(\frac{1}{nh} \sum_{i=1}^n |\tilde{g}_n(X_i) - g(X_i)| \times \left| K_1\left(\frac{x - X_i}{h}\right) \right| \right)^2 \\ &\quad + c_9 |\hat{\theta} - \theta|^2 \left(\frac{1}{nh} \sum_{i=1}^n \left| K_1\left(\frac{x - X_i}{h}\right) \right| \right)^2 \\ &\quad + c_{10} \left(\frac{1}{n} \sum_{i=1}^n |\tilde{g}_n(X_i) - g(X_i)| \right)^2 \left(\frac{1}{nh} \sum_{i=1}^n \left| K_1\left(\frac{x - X_i}{h}\right) \right| \right)^2 \quad \text{a.s.} \end{aligned} \quad (28)$$

We now successively control the expectations T_1, T_2 and T_3 of the three terms appearing in this upper-bound. For the first term, we have

$$\begin{aligned} T_1 &= \mathbb{E}_{\theta, f} \left[\left(\frac{1}{nh} \sum_{i=1}^n |\tilde{g}_n(X_i) - g(X_i)| \times \left| K_1\left(\frac{x - X_i}{h}\right) \right| \right)^2 \right] \\ &= \mathbb{E}_{\theta, f} \left[\frac{1}{n^2 h^2} \sum_{i, j=1}^n |\tilde{g}_n(X_i) - g(X_i)| |\tilde{g}_n(X_j) - g(X_j)| \times \left| K_1\left(\frac{x - X_i}{h}\right) K_1\left(\frac{x - X_j}{h}\right) \right| \right] \\ &= \frac{1}{nh} \mathbb{E}_{\theta, f} \left[\frac{1}{h} |\tilde{g}_n(X_1) - g(X_1)|^2 K_1^2\left(\frac{x - X_1}{h}\right) \right] \\ &\quad + \frac{n-1}{n} \mathbb{E}_{\theta, f} \left[\frac{1}{h^2} |\tilde{g}_n(X_1) - g(X_1)| |\tilde{g}_n(X_2) - g(X_2)| \times \left| K_1\left(\frac{x - X_1}{h}\right) K_1\left(\frac{x - X_2}{h}\right) \right| \right]. \end{aligned}$$

Now,

$$\begin{aligned}
T_{11} &= \mathbb{E}_{\theta,f} \left[\frac{1}{h} |\tilde{g}_n(X_1) - g(X_1)|^2 K_1^2 \left(\frac{x - X_1}{h} \right) \right] \\
&= \int_0^1 \mathbb{E}_{\theta,f} (|\hat{g}_{n-1}(t) - g(t)|^2) K_1^2 \left(\frac{x-t}{h} \right) \frac{g(t)}{h} dt \quad (\text{according to definition (9)}) \\
&\leq C_{10} n^{\frac{-2\beta}{2\beta+1}} \int_0^1 K_1^2 \left(\frac{x-t}{h} \right) \frac{g(t)}{h} dt \quad (\text{according to Remark 1}) \\
&\leq C_{11} n^{\frac{-2\beta}{2\beta+1}} \quad (\text{according to Lemma 2}), \tag{29}
\end{aligned}$$

and in the same way

$$\begin{aligned}
T_{12} &= \mathbb{E}_{\theta,f} \left[\frac{1}{h^2} |\tilde{g}_n(X_1) - g(X_1)| |\tilde{g}_n(X_2) - g(X_2)| K_1 \left(\frac{x - X_1}{h} \right) K_1 \left(\frac{x - X_2}{h} \right) \right] \\
&= \int_0^1 \int_0^1 \mathbb{E}_{\theta,f} \left[\left| \frac{n-2}{n-1} \hat{g}_{n-2}(t) - g(t) + \frac{1}{(n-1)h} K_2 \left(\frac{t-s}{h} \right) \right| \right. \\
&\quad \times \left. \left| \frac{n-2}{n-1} \hat{g}_{n-2}(s) - g(s) + \frac{1}{(n-1)h} K_2 \left(\frac{s-t}{h} \right) \right| \right] \left| K_1 \left(\frac{x-t}{h} \right) K_1 \left(\frac{x-s}{h} \right) \right| \frac{g(t)g(s)}{h^2} dt ds.
\end{aligned}$$

This last term is upper-bound by

$$\begin{aligned}
T_{12} &\leq \int_0^1 \int_0^1 \mathbb{E}_{\theta,f} \left[\left(|\hat{g}_{n-2}(t) - g(t)| + \frac{1}{n-1} g(t) + \frac{1}{(n-1)h} \left| K_2 \left(\frac{t-s}{h} \right) \right| \right) \right. \\
&\quad \times \left. \left(|\hat{g}_{n-2}(s) - g(s)| + \frac{1}{n-1} g(s) + \frac{1}{(n-1)h} \left| K_2 \left(\frac{s-t}{h} \right) \right| \right) \right] \\
&\quad \times \left| K_1 \left(\frac{x-t}{h} \right) K_1 \left(\frac{x-s}{h} \right) \right| \frac{g(t)g(s)}{h^2} dt ds \\
&\leq \int_0^1 \int_0^1 \left\{ \mathbb{E}_{\theta,f}^{1/2} [|\hat{g}_{n-2}(t) - g(t)|^2] \mathbb{E}_{\theta,f}^{1/2} [|\hat{g}_{n-2}(s) - g(s)|^2] + o\left(\frac{1}{nh}\right) \right\} \\
&\quad \times \left| K_1 \left(\frac{x-t}{h} \right) K_1 \left(\frac{x-s}{h} \right) \right| \frac{g(t)g(s)}{h^2} dt ds \\
&\leq C_{12} n^{\frac{-2\beta}{2\beta+1}} \left[\int_0^1 \left| K_1 \left(\frac{x-t}{h} \right) \right| \frac{g(t)}{h} dt \right]^2 \quad (\text{according to Remark 1}) \\
&\leq C_{13} n^{\frac{-2\beta}{2\beta+1}} \quad (\text{according to Lemma 2}). \tag{30}
\end{aligned}$$

Thus we get that

$$T_1 = \mathbb{E}_{\theta,f} \left[\left(\frac{1}{nh} \sum_{i=1}^n |\tilde{g}_n(X_i) - g(X_i)| \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^2 \right] \leq C_{14} n^{\frac{-2\beta}{2\beta+1}}. \tag{31}$$

For the second term in the right hand side of (28), we have

$$\begin{aligned}
T_2 &= \mathbb{E}_{\theta,f} \left[|\hat{\theta} - \theta|^2 \left(\frac{1}{nh} \sum_{i=1}^n \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^2 \right] \\
&\leq \mathbb{E}_{\theta,f}^{1/2} [|\hat{\theta} - \theta|^4] \mathbb{E}_{\theta,f}^{1/2} \left[\left(\frac{1}{nh} \sum_{i=1}^n \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^4 \right].
\end{aligned}$$

The proof of the following lemma is postponed to Appendix A.3.

Lemma 5. *There exist some positive constant C_{15} such that*

$$\mathbb{E}_{\theta,f} \left[\left(\frac{1}{nh} \sum_{i=1}^n \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^4 \right] \leq C_{15}. \quad (32)$$

This lemma entails that

$$T_2 \leq C_{15} \left[\mathbb{E}_{\theta,f} \left(|\hat{\theta} - \theta|^4 \right) \right]^{\frac{1}{2}}. \quad (33)$$

Now, we turn to the third term in the right hand side of (28). We have

$$\begin{aligned} T_3 &= \mathbb{E}_{\theta,f} \left[\left(\frac{1}{n} \sum_{i=1}^n |\tilde{g}_n(X_i) - g(X_i)| \right)^2 \left(\frac{1}{nh} \sum_{i=1}^n \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^2 \right] \\ &= \mathbb{E}_{\theta,f} \left[\frac{1}{n^4 h^2} \sum_{i,j,k,l=1}^n |\tilde{g}_n(X_i) - g(X_i)| |\tilde{g}_n(X_j) - g(X_j)| \left| K_1 \left(\frac{x - X_k}{h} \right) K_1 \left(\frac{x - X_l}{h} \right) \right| \right]. \end{aligned}$$

By using the same arguments as for obtaining (29) and (30), we can get that

$$T_3 \leq C_{16} n^{\frac{-2\beta}{2\beta+1}}. \quad (34)$$

According to (31), (33) and (34), we may conclude

$$\mathbb{E}_{\theta,f} (|\hat{f}_n(x) - f_2(x)|^2) \leq C_{15} \left[\mathbb{E}_{\theta,f} \left(|\hat{\theta} - \theta|^4 \right) \right]^{\frac{1}{2}} + C_{17} n^{\frac{-2\beta}{2\beta+1}}. \quad (35)$$

By returning to Inequality (21) and combining it with (22) and (35), we achieve that

$$\mathbb{E}_{\theta,f} (|\hat{f}_n(x) - f(x)|^2) \leq C_1 \left[\mathbb{E}_{\theta,f} \left(|\hat{\theta} - \theta|^4 \right) \right]^{\frac{1}{2}} + C_2 n^{\frac{-2\beta}{2\beta+1}}.$$

□

6.2 Other proofs

Proof of Corollary 1. We now consider the estimator $\hat{\theta}_n$ initially proposed by Celisse and Robin (2010) and further studied in Nguyen and Matias (2012). From the proof of Theorem 3 in Nguyen and Matias (2012), it may be easily seen that for any value $\gamma \in (\beta/(2\beta + 1), 1/2)$,

$$n^{2\gamma} \left[\mathbb{E}_{\theta,f} \left(|\hat{\theta} - \theta| \right)^4 \right]^{\frac{1}{2}} \rightarrow_{n \rightarrow \infty} 0.$$

Thus, there exist some constant $C > 0$ such that

$$\sup_{x \in [0,1]} \mathbb{E}_{\theta,f} (|\hat{f}_n(x) - f(x)|^2) \leq C n^{\frac{-2\beta}{2\beta+1}}.$$

□

Proof of Proposition 1. We control the difference

$$\begin{aligned} l(f^t) - l(f^{t+1}) &= \int_0^1 g_0(x) \log \frac{\theta + (1-\theta)\mathcal{N}f^{t+1}(x)}{\theta + (1-\theta)\mathcal{N}f^t(x)} dx \\ &= \int_0^1 g_0(x) \log \left\{ 1 - \omega_t(x) + \omega_t(x) \frac{\mathcal{N}f^{t+1}(x)}{\mathcal{N}f^t(x)} \right\} dx. \end{aligned}$$

By the concavity of the logarithm function, we get that

$$\begin{aligned} l(f^t) - l(f^{t+1}) &\geq \int_0^1 g_0(x) \omega_t(x) \log \frac{\mathcal{N}f^{t+1}(x)}{\mathcal{N}f^t(x)} dx \\ &\geq \int_0^1 g_0(x) \omega_t(x) \left[\mathcal{S}^*(\log f^{t+1})(x) - \mathcal{S}^*(\log f^t)(x) \right] dx \\ &\geq \int_0^1 g_0(x) \omega_t(x) \left(\int_0^1 K_h(s-x) ds \right)^{-1} \left(\int_0^1 K_h(u-x) \log \frac{f^{t+1}(u)}{f^t(u)} du \right) dx \\ &\geq \int_0^1 \left(\int_0^1 \frac{g_0(x) \omega_t(x) K_h(u-x)}{\int_0^1 K_h(s-x) ds} dx \right) \log \frac{f^{t+1}(u)}{f^t(u)} du \\ &\geq \frac{1}{\alpha_t} \int_0^1 f^{t+1}(u) \log \frac{f^{t+1}(u)}{f^t(u)} du \\ &\geq \frac{1}{\alpha_t} D(f^{t+1} | f^t). \end{aligned}$$

We now establish a lower bound on α_t^{-1} . As already mentioned, for all $t \geq 0$, the function f^t is lower bounded by m . Since the operator \mathcal{N} is increasing, it follows that $\mathcal{N}f^t$ is also lower bounded by m . Now the function

$$x \mapsto \frac{(1-\theta)x}{\theta + (1-\theta)x}$$

is increasing, so that we finally obtain

$$\alpha_t^{-1} = \int_0^1 \omega_t(x) g_0(x) dx = \int_0^1 \frac{(1-\theta)\mathcal{N}f^t(x)}{\theta + (1-\theta)\mathcal{N}f^t(x)} g_0(x) dx \geq \frac{(1-\theta)m}{\theta + (1-\theta)m} = c.$$

This concludes the proof. \square

Proof of Corollary 2. We start by stating a lemma, whose proof is postponed to Appendix A.4.

Lemma 6. *The function $l : \mathcal{B} \rightarrow \mathbb{R}$ is continuous with respect to the topology induced by uniform convergence on the set of functions defined on $[0, 1]$.*

Now, it is easy to see that l is a convex function. Moreover, since the densities in \mathcal{B} are bounded (remember that any $f \in \mathcal{B}$ satisfies $m \leq f(\cdot) \leq M/m$), it comes that $\mathcal{B} \subset \mathcal{F}$ and since \mathcal{S} is linear, the set \mathcal{B} is convex. Existence and uniqueness of the minimum f^* of l in \mathcal{B} thus follows, as well as the simple convergence of the iterative sequence $\{f^t\}_{t \geq 0}$ to this unique minimum. \square

Proof of Proposition 2. For all $x, y \in [0, 1]$ and for all t , we have

$$\begin{aligned} |f^t(x) - f^t(y)| &= \frac{1}{\int_0^1 \omega_t(u)g_0(u)du} \left| \int_0^1 \frac{[K_h(u-x) - K_h(u-y)]\omega_t(u)g_0(u)}{\int_0^1 K_h(s-u)ds} du \right| \\ &\leq \frac{1}{\int_0^1 \omega_t(u)g_0(u)du} \int_0^1 \frac{|K_h(u-x) - K_h(u-y)|\omega_t(u)g_0(u)}{m} du \\ &\leq \frac{L}{m}|x-y|. \end{aligned}$$

Furthermore, for all $x \in [0, 1]$ and for all t

$$m \leq f^t(x) = \frac{1}{\int_0^1 \omega_t(u)g_0(u)du} \int_0^1 \frac{K_h(u-x)\omega_t(u)g_0(u)}{\int_0^1 K_h(s-u)ds} du \leq \frac{M}{m},$$

so that the sequence $\{f^t\}$ is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem, there exists a subsequence $\{f^{t_k}\}$ of $\{f^t\}$ which converges uniformly to some limit. However, this uniform limit must be the simple limit of the sequence, namely the minimum f^* of l . Now, uniqueness of the uniform limit value of the sequence $\{f^t\}_{t \geq 0}$ entails its convergence. □

A Proofs of technical lemmas

A.1 Proof of Lemma 3

Proof. We first show (15). According to the law of large numbers, since $\mathbb{E}_{\theta,f}(f(X_1)/g(X_1)) = 1$, we have

$$\frac{S_n}{n} = \frac{1}{n} \sum_{i=1}^n \frac{f(X_i)}{g(X_i)} \xrightarrow[n \rightarrow \infty]{as} 1. \quad (36)$$

Hence

$$\frac{n^2}{S_n^2} = \left(\frac{S_n}{n}\right)^{-2} \xrightarrow[n \rightarrow \infty]{as} 1.$$

By the dominated convergence theorem, there exists a constant $c_2 > 0$ such that for n large enough

$$\mathbb{E}_{\theta,f}\left[\frac{1}{S_n^2}\right] = \frac{1}{n^2} \mathbb{E}_{\theta,f}\left[\frac{n^2}{S_n^2}\right] \leq \frac{c_2}{n^2},$$

establishing (15). Let us now prove (14). By using a Taylor's expansion, we have

$$\frac{1}{S_n} = \frac{1}{n} \times \frac{1}{1 + (\frac{S_n}{n} - 1)} = \frac{1}{n} \left[2 - \frac{S_n}{n} + \left(\frac{S_n}{n} - 1\right)^2 \frac{1}{(1 + \gamma_n(\frac{S_n}{n} - 1))^3} \right],$$

where $\gamma_n \in]0, 1[$ depends on S_n . Combining this with (36), we obtain

$$\frac{1}{(1 + \gamma_n(\frac{S_n}{n} - 1))^3} \xrightarrow[n \rightarrow \infty]{as} 1.$$

Thus, there exist some positive constants c, c' such that for n large enough,

$$\frac{1}{n} \left[2 - \frac{S_n}{n} + c' \left(\frac{S_n}{n} - 1\right)^2 \right] \leq \frac{1}{S_n} \leq \frac{1}{n} \left[2 - \frac{S_n}{n} + c \left(\frac{S_n}{n} - 1\right)^2 \right] \quad \text{a.s.} \quad (37)$$

This implies in particular that

$$\mathbb{E}_{\theta,f}\left[\frac{1}{S_n}\right] \leq \frac{1}{n} \left[2 - \frac{\mathbb{E}_{\theta,f}[S_n]}{n} + c \mathbb{E}_{\theta,f}\left[\left(\frac{S_n}{n} - 1\right)^2\right] \right] = \frac{1}{n} + \frac{c}{n} \mathbb{E}_{\theta,f}\left[\left(\frac{S_n}{n} - 1\right)^2\right].$$

In addition,

$$\mathbb{E}_{\theta,f}\left[\left(\frac{S_n}{n} - 1\right)^2\right] = \text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n} \text{Var}\left(\frac{f(X_1)}{g(X_1)}\right).$$

Remember that the ratio f/g is bounded (by δ^{-1}) and thus has finite variance. Hence, there exists a positive constant c_1 such that for n large enough

$$\mathbb{E}_{\theta,f}\left[\frac{1}{S_n}\right] \leq \frac{1}{n} + \frac{c_1}{n^2}.$$

We now prove (16). By using again a Taylor expansion, we have

$$\frac{1}{S_n + \delta^{-1}} = \frac{1}{S_n} \times \frac{1}{1 + 1/(\delta S_n)} = \frac{1}{S_n} - \frac{1}{\delta S_n^2} \times \frac{1}{[1 + \beta_n/(\delta S_n)]^2},$$

where $\beta_n \in]0, 1[$ depends on S_n . We also have

$$\frac{1}{[1 + \beta_n/(\delta S_n)]^2} \xrightarrow[n \rightarrow \infty]{as} 1.$$

Thus, there exists a positive constant c'' such that for n large enough

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n + \delta^{-1}} \right] = \mathbb{E}_{\theta,f} \left[\frac{1}{S_n} - \frac{1}{\delta S_n^2} \times \frac{1}{[1 + \beta_n/(\delta S_n)]^2} \right] \geq \mathbb{E}_{\theta,f} \left[\frac{1}{S_n} \right] - \mathbb{E}_{\theta,f} \left[\frac{c''}{S_n^2} \right] \quad \text{a.s.}$$

According to (37), we have

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n} \right] \geq \frac{1}{n} \left[2 - \frac{\mathbb{E}_{\theta,f}[S_n]}{n} + c' \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^2 \right] \right] = \frac{1}{n} + \frac{c'}{n^2} \text{Var} \left(\frac{f(X_1)}{g(X_1)} \right),$$

and it is proved above that

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n^2} \right] \leq \frac{c_2}{n^2}.$$

Thus we obtain Inequality (16), namely

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n + \delta^{-1}} \right] \geq \frac{1}{n} - \frac{c_3}{n^2}.$$

Finally, we show (17). In the same way as we proved (16) above, we have for large enough n ,

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n + 2\delta^{-1}} \right] \geq \frac{1}{n} - \frac{c'_3}{n^2} > 0$$

and thus

$$\mathbb{E}_{\theta,f}^2 \left[\frac{1}{S_n + 2\delta^{-1}} \right] \geq \frac{1}{n^2} \left(1 - \frac{2c'_3}{n} + \frac{c_3'^2}{n^2} \right) \geq \frac{1}{n^2} \left(1 - \frac{2c'_3}{n} \right). \quad (38)$$

According to Inequality (37) (containing only positive terms for n large enough), we have

$$\begin{aligned} \frac{1}{S_n^2} &\leq \frac{1}{n^2} \left[4 + \frac{S_n^2}{n^2} + c^2 \left(\frac{S_n}{n} - 1 \right)^4 - 4 \frac{S_n}{n} + 4c \left(\frac{S_n}{n} - 1 \right)^2 - 2c \frac{S_n}{n} \left(\frac{S_n}{n} - 1 \right)^2 \right] \quad (\text{as}) \\ &\leq \frac{1}{n^2} \left[4 + \frac{S_n^2}{n^2} + c^2 \left(\frac{S_n}{n} - 1 \right)^4 - 4 \frac{S_n}{n} + 4c \left(\frac{S_n}{n} - 1 \right)^2 \right] \quad \text{a.s.} \end{aligned}$$

Since

$$\mathbb{E}_{\theta,f}[S_n] = n, \quad \mathbb{E}_{\theta,f}[S_n^2] = n \text{Var} \left(\frac{f(X_1)}{g(X_1)} \right) + n^2 \quad \text{and} \quad \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^2 \right] = \frac{1}{n} \text{Var} \left(\frac{f(X_1)}{g(X_1)} \right),$$

we have

$$\begin{aligned} \mathbb{E}_{\theta,f} \left[\frac{1}{S_n^2} \right] &\leq \frac{1}{n^2} \left[4 + \frac{\mathbb{E}_{\theta,f}[S_n^2]}{n^2} + c^2 \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right] - 4 \frac{\mathbb{E}_{\theta,f}[S_n]}{n} + 4c \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^2 \right] \right] \\ &\leq \frac{1}{n^2} \left[4 + \frac{1}{n} \text{Var} \left(\frac{f(X_1)}{g(X_1)} \right) + 1 + c^2 \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right] - 4 + \frac{4c}{n} \text{Var} \left(\frac{f(X_1)}{g(X_1)} \right) \right] \\ &\leq \frac{1}{n^2} \left[1 + \frac{C_4}{n} + c^2 \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right] \right]. \quad (39) \end{aligned}$$

Combining (38) and (39), we get that

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n^2} \right] - \mathbb{E}_{\theta,f}^2 \left[\frac{1}{S_n + 2\delta^{-1}} \right] \leq \frac{C}{n^3} + \frac{c^2}{n^2} \mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right]. \quad (40)$$

We now upper-bound the quantity $\mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right]$. Let us denote by

$$U_i = \frac{f(X_i)}{g(X_i)} - 1.$$

We have

$$\begin{aligned} \left(\frac{S_n}{n} - 1 \right)^4 &= \frac{1}{n^4} \left(\sum_{i=1}^n U_i \right)^4 \\ &= \frac{1}{n^4} \sum_{i=1}^n U_i^4 + \frac{1}{n^4} \sum_{i \neq j}^n U_i^3 U_j + \frac{1}{n^4} \sum_{i \neq j}^n U_i^2 U_j^2 + \frac{1}{n^4} \sum_{i \neq j \neq k}^n U_i^2 U_j U_k + \\ &\quad + \frac{1}{n^4} \sum_{i \neq j \neq k \neq l}^n U_i U_j U_k U_l. \end{aligned}$$

Since the random variables U_i are iid with mean zero, we obtain

$$\mathbb{E}_{\theta,f} \left[\left(\frac{S_n}{n} - 1 \right)^4 \right] = \frac{1}{n^4} [n \mathbb{E}_{\theta,f}(U_1^4) + n(n-1) \mathbb{E}_{\theta,f}(U_1^2 U_2^2)] = O\left(\frac{1}{n^2}\right). \quad (41)$$

Finally, according to (40) and (41) we have

$$\mathbb{E}_{\theta,f} \left[\frac{1}{S_n^2} \right] - \mathbb{E}_{\theta,f}^2 \left[\frac{1}{S_n + 2\delta^{-1}} \right] = O\left(\frac{1}{n^3}\right).$$

□

A.2 Proof of Lemma 4

Proof. We write

$$\frac{1}{\sum_k \hat{\tau}_k} = \frac{1}{\sum_k \tau_k + \sum_k (\hat{\tau}_k - \tau_k)} = \frac{1}{\sum_k \tau_k} - \frac{\sum_k (\hat{\tau}_k - \tau_k)}{(\sum_k \tau_k)^2} \times \int_0^1 \left(1 + s \frac{\sum_k (\hat{\tau}_k - \tau_k)}{\sum_k \tau_k} \right)^{-2} ds.$$

Let us establish that $\|\hat{\tau} - \tau\|_{\infty, [0,1]} = \sup_{x \in [0,1]} |\hat{\tau}(x) - \tau(x)|$ converges almost surely to zero. Indeed,

$$\hat{\tau}(x) - \tau(x) = (\theta - \hat{\theta}) \frac{1}{g(x)} + \hat{\theta} \left(\frac{1}{g(x)} - \frac{1}{\tilde{g}_n(x)} \right)$$

and using the same argument as for establishing (26), we get that for n large enough and for all $x \in [0, 1]$,

$$|\hat{\tau}(x) - \tau(x)| \leq \frac{|\hat{\theta} - \theta|}{\theta} + 2|\hat{\theta}| \frac{\|\hat{g}_n - g\|_{\infty}}{\theta^2} \leq \delta^{-1} |\hat{\theta} - \theta| + 2\delta^{-2} \|\hat{g}_n - g\|_{\infty}.$$

By using consistency of $\hat{\theta}_n$ and Remark 1, we obtain that $\|\hat{\tau} - \tau\|_{\infty, [0,1]}$ converges almost surely to zero. Now,

$$\begin{aligned} \forall s \in [0, 1], \quad 1 + s \frac{\sum_k (\hat{\tau}_k - \tau_k)}{\sum_k \tau_k} &\geq 1 - s \frac{n \|\hat{\tau}_k - \tau_k\|_{\infty, [0,1]}}{\sum_k \tau_k} \\ &\geq 1 - s \frac{2 \|\hat{\tau}_k - \tau_k\|_{\infty, [0,1]}}{\theta} \\ &\geq 1 - \frac{s}{2} > 0 \quad \text{a.s.} \end{aligned}$$

We obtain that

$$\begin{aligned} \frac{n}{|\sum_k \hat{\tau}_k|} &\leq \frac{n}{\sum_k \tau_k} + \frac{n \sum_k |\hat{\tau}_k - \tau_k|}{(\sum_k \tau_k)^2} \times \int_0^1 \left(1 + s \frac{\sum_k (\hat{\tau}_k - \tau_k)}{\sum_k \tau_k}\right)^{-2} ds \\ &\leq \frac{n}{\sum_k \tau_k} + \frac{n^2 \|\hat{\tau} - \tau\|_{\infty, [0,1]}}{(\sum_k \tau_k)^2} \times \int_0^1 \left(1 - \frac{s}{2}\right)^{-2} ds \\ &\leq \frac{2}{1 - \theta} + \frac{8 \|\hat{\tau} - \tau\|_{\infty, [0,1]}}{(1 - \theta)^2} \leq c_7 \quad \text{a.s.} \end{aligned}$$

□

A.3 Proof of Lemma 5

Proof. In order to prove (32), let us consider iid random variables U_1, \dots, U_n defined as

$$U_i = \left| K_1 \left(\frac{x - X_i}{h} \right) \right|.$$

For all $1 \leq p \leq 4$, we have

$$\mathbb{E}_{\theta, f}(U_i^p) = \int \left| K_1^p \left(\frac{x - t}{h} \right) \right| g(t) dt = h \int |K_1^p(t)| g(x + th) dt \leq C_{15} h.$$

We then write

$$\left(\frac{1}{nh} \sum_{i=1}^n \left| K_1 \left(\frac{x - X_i}{h} \right) \right| \right)^4 = \frac{1}{n^4 h^4} \left(\sum_i U_i \right)^4, \quad (42)$$

where

$$\left(\sum_i U_i \right)^4 = \sum_i U_i^4 + \sum_{i \neq j} U_i^3 U_j + \sum_{i \neq j} U_i^2 U_j^2 + \sum_{i \neq j \neq k} U_i^2 U_j U_k + \sum_{i \neq j \neq k \neq l} U_i U_j U_k U_l.$$

And for all choice of the bandwidth $h > 0$ such that $nh \rightarrow \infty$,

$$\begin{aligned} &\mathbb{E}_{\theta, f} \left[\left(\sum_i U_i \right)^4 \right] \\ &= n \mathbb{E}_{\theta, f}(U_1^4) + n(n-1) \mathbb{E}_{\theta, f}(U_1^3 U_2) + n(n-1) \mathbb{E}_{\theta, f}(U_1^2 U_2^2) + \\ &\quad + n(n-1)(n-2) \mathbb{E}_{\theta, f}(U_1^2 U_2 U_3) + n(n-1)(n-2)(n-3) \mathbb{E}_{\theta, f}(U_1 U_2 U_3 U_4) \\ &= n \mathbb{E}_{\theta, f}(U_1^4) + n(n-1) \mathbb{E}_{\theta, f}(U_1^3) \mathbb{E}_{\theta, f}(U_1) + n(n-1) \mathbb{E}_{\theta, f}^2(U_1^2) + \\ &\quad + n(n-1)(n-2) \mathbb{E}_{\theta, f}(U_1^2) \mathbb{E}_{\theta, f}^2(U_1) + n(n-1)(n-2)(n-3) \mathbb{E}_{\theta, f}^4(U_1) \\ &\leq C_{15} n^4 h^4. \end{aligned} \quad (43)$$

According to (42) and (43) we obtain the result. □

A.4 Proof of Lemma 6

Proof. Let f be a function in \mathcal{B} and $\{f_n\}$ be a sequence of densities on $[0, 1]$ such that $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$. Let us recall that every $f \in \mathcal{B}$ satisfies the bounds $m \leq f \leq M/m$. We have

$$\begin{aligned} |l(f_n) - l(f)| &= \left| \int_0^1 g_0(x) \log \frac{\theta + (1 - \theta)\mathcal{N}f(x)}{\theta + (1 - \theta)\mathcal{N}f_n(x)} dx \right| \\ &\leq \int_0^1 g_0(x) \left| \log \left\{ 1 + \frac{(1 - \theta)[\mathcal{N}f_n(x) - \mathcal{N}f(x)]}{\theta + (1 - \theta)\mathcal{N}f_n(x)} \right\} \right| dx, \end{aligned}$$

and

$$\begin{aligned} |\mathcal{N}f_n(x) - \mathcal{N}f(x)| &= \mathcal{N}f(x) \left| \exp \frac{\int_0^1 K_h(u - x)[\log f_n(u) - \log f(u)] du}{\int_0^1 K_h(s - x) ds} - 1 \right| \\ &\leq \frac{M}{m} \left| \exp \frac{\int_0^1 K_h(u - x)[\log f_n(u) - \log f(u)] du}{\int_0^1 K_h(s - x) ds} - 1 \right|. \end{aligned}$$

For $|x| < \epsilon$ small enough, we have $|\log(1 + x)| \leq 2|x|$ and $|\exp(x) - 1| \leq 2|x|$. Combining with the fact that f is bounded, we get that

$$\begin{aligned} \left| \int_0^1 K_h(u - x)[\log f_n(u) - \log f(u)] du \right| &\leq \int_0^1 K_h(u - x) \left| \log \left\{ 1 + \frac{f_n(u) - f(u)}{f(u)} \right\} \right| du \\ &\leq 2\|f_n - f\|_\infty \end{aligned}$$

and thus

$$\|\mathcal{N}f_n - \mathcal{N}f\|_\infty \leq \frac{4M}{m^2} \|f_n - f\|_\infty.$$

We finally obtain

$$|l(f_n) - l(f)| \leq C \|f_n - f\|_\infty,$$

where C is a constant depending on h, K and θ . □

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