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KNUTH RELATIONS, TABLEAUX AND MV-CYCLES

STÉPHANE GAUSSENT, PETER LITTELMANN AND AN HOA NGUYEN

Dedicated to C. S. Seshadri on the occasion of his 80th birthday

Abstract: We give a geometric interpretation of the Knuth equivalence relations in terms of the affine Grassmann variety. The Young tableaux are seen as sequences of coweights, called galleries. We show that to any gallery corresponds a Mirković-Vilonen cycle and that two galleries are equivalent if, and only if, their associated MV cycles are equal. Words are naturally identified to some galleries. So, as a corollary, we obtain that two words are Knuth equivalent if, and only if, their associated MV cycles are equal.

1. INTRODUCTION

In the theory of finite dimensional representations of complex reductive algebraic groups, the group $GL_n(\mathbb{C})$ is singled out by the fact that besides the usual language of weight lattices, roots and characters, there exists an additional important combinatorial tool: the plactic monoid of Knuth [9], Lascoux and Schützenberger [12], and the Young tableaux.

The plactic monoid is the monoid of all words in the alphabet $\mathbb{A} = \{1, \dots, n\}$ modulo Knuth equivalence (see (1)). The map associating to a word its class in the plactic monoid has a remarkable section: the semi-standard Young tableaux. In the framework of crystal bases [6], [5], [7], [8], [15] and the path model of representations [14], this monoid got a new interpretation, which made it possible to generalize the classical tableaux character formula to integrable highest weight representations of Kac-Moody algebras.

Mirković and Vilonen [17] gave a geometric interpretation of weight multiplicities for finite dimensional representations of a semi simple algebraic group G . Given a dominant coweight λ^\vee , let $X_{\lambda^\vee} \subset \mathcal{G}$ be the Schubert variety in the associated affine Grassmann variety \mathcal{G} [15, 10]. Let U^- be the maximal unipotent subgroup opposite to a fixed Borel subgroup and denote by \mathcal{O} the ring of formal power series $\mathbb{C}[[t]]$ and by \mathcal{K} its quotient field. The irreducible components of $\overline{G(\mathcal{O}) \cdot \lambda^\vee \cap U^-(\mathcal{K}) \cdot \mu^\vee} \subset X_{\lambda^\vee}$ (Section 9) are called MV-cycles and the number of irreducible components is the weight multiplicity of the weight μ^\vee in the complex irreducible representation $V(\lambda^\vee)$ for the Langlands dual group G^\vee .

This leads naturally to the question: is it possible to give a geometric formulation of the Knuth relations? The aim of this article is to do this for $G = SL_n(\mathbb{C})$. For $1 \leq d \leq n$ denote by $I_{d,n}$ the set $\{\mathbf{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}$, and set $\mathbb{W} = I_{1,n} \cup I_{2,n} \cup \dots \cup I_{n-1,n} \cup I_{n,n}$. We can identify \mathbb{A} with $I_{1,n}$ and hence view \mathbb{A} as a subset of \mathbb{W} . Words in the alphabet \mathbb{W} are called *galleries*, and we introduce on the set of galleries an equivalence relation \sim_K which, restricted to words in the alphabet \mathbb{A} , is

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exactly the Knuth equivalence. To a given gallery γ , we associate in a canonical way a Bott-Samelson type variety Σ , a cell $C_\gamma \subset \Sigma$, a dominant coweight ν^\vee , a map $\pi : \Sigma \rightarrow \mathcal{G}$ and show (see Theorem 2):

Theorem. *a) The closure of the image $\overline{\pi(C_\gamma)}$ is a MV-cycle in the Schubert variety X_{ν^\vee} .
b) Given a second gallery γ' with associated Bott-Samelson type variety Σ' , cell $C_{\gamma'} \subset \Sigma'$ and map $\pi' : \Sigma' \rightarrow \mathcal{G}$, then $\gamma \sim_K \gamma'$ if and only if $\overline{\pi(C_\gamma)} = \overline{\pi'(C_{\gamma'})}$.*

In particular we get:

Corollary. *Two words in the alphabet \mathbb{A} are Knuth equivalent if and only if the closure of the images of the corresponding cells define the same MV-cycle.*

The theorem generalizes for $G = SL_n(\mathbb{C})$ the result in [4], where it was proved that (for semisimple algebraic groups) the image of a cell associated to a Lakshmibai-Seshadri galleries is an MV-cycle.

The construction of the Bott-Samelson type variety and the cell can be sketched out as follows: the elements of the alphabet can be identified with weights occurring in the fundamental representations of the group $SL_n(\mathbb{C})$ (for example $i \leftrightarrow \epsilon_i$), so words and galleries can be viewed as polylines in the apartment associated to the maximal torus T of diagonal matrices in $G = SL_n(\mathbb{C})$. The vertices and edges of a polyline γ give naturally rise to a sequence $P_0 \supset Q_0 \subset P_1 \supset \dots \subset P_r$ of parabolic subgroups of the associated affine Kac-Moody group, and hence we can associate to a gallery the Bott-Samelson type variety $\Sigma = P_0 \times_{Q_0} P_1 \times_{Q_1} \dots \times_{Q_{r-1}} P_r / P_r$. In the language of buildings (which we do not use in this article) the variety Σ can be seen as the variety of all galleries in the building of the same type as γ and starting in 0, making Σ into a canonical object associated to γ . By choosing a generic anti-dominant one parameter subgroup η of T , we get a natural Białyński-Birula cell [1] decomposition of Σ . The η -fixed points in Σ correspond exactly to galleries of the same type as γ (section 3), and $C_\gamma = \{x \in \Sigma \mid \lim_{t \rightarrow 0} \eta(t)x = \gamma\}$. The variety Σ is a desingularization of a Schubert variety in \mathcal{G} and is hence naturally endowed with a morphism $\pi : \Sigma \rightarrow \mathcal{G}$ [4].

Given a gallery γ , we get a natural sequence of vertices $(\mu_0^\vee, \dots, \mu_r^\vee)$ crossed by the gallery. We attach to each vertex a subgroup U_j^- of $U^-(\mathcal{K})$ and show that the image of the cell is $U_0^- \cdots U_r^- \mu^\vee$, where μ^\vee is the endpoint of the associated polyline. Using this description of the image and commuting rules for root subgroups, we show that the image of cells associated to Knuth equivalent galleries have a common dense subset. Since every equivalence class has a unique gallery associated to a semi-standard Young tableaux, the results in [4] show that the closure of such an image is a MV-cycle, and this correspondence semi-standard Young tableaux \leftrightarrow MV-cycles is bijective.

We conjecture that, as in the special case of Lakshmibai-Seshadri galleries in [4], a generalization of the theorem above holds (after an appropriate reformulation and using [13] with Lakshmibai-Seshadri tableaux / galleries as section instead of Young tableaux) for arbitrary complex semi-simple algebraic groups.

2. WORDS AND KEYS

We fix as alphabet the set $\mathbb{A} = \{1, 2, \dots, n\}$ with the usual order $1 < 2 < \dots < n$. By a word we mean a finite sequence (i_1, \dots, i_k) of elements in \mathbb{A} , k is called the length of the word. The set of words \mathcal{W} forms a monoid with the concatenation of words as an operation and with the empty word as a neutral element. We define an equivalence relation on the set of words as follows. We say that two words u, w are related by a *Knuth relation* if we can find a decomposition of u and w such that v_1, v_2 are words, $x, y, z \in \mathbb{A}$ and

$$(1) \quad \begin{cases} u = v_1 x z y v_2 \text{ and } w = v_1 z x y v_2 \text{ where } x < y \leq z \\ \text{or } u = v_1 y z x v_2 \text{ and } w = v_1 y x z v_2 \text{ where } x \leq y < z \end{cases} .$$

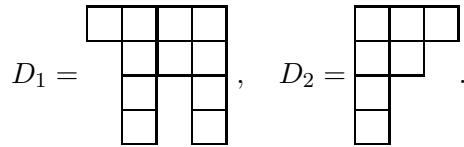
Definition 1. Two words are called *Knuth equivalent*: $u \sim_K w$, if there exists a sequence of words $u = v_0, v_1, \dots, v_r = w$ such that v_{j-1} is related to v_j by a Knuth relation, $j = 1, \dots, r$.

Remark 1. The set of equivalence classes forms a monoid (called the *plactic monoid*, see [12, 18, 13]) with the concatenation of classes of words as an operation and with the class of the empty word as a neutral element.

Remark 2. Though we mostly refer to [3, 11, 12] for results concerning the word algebra and Knuth relations, note that there is a difference in the way the Knuth relations are stated. The reason for the change is the way we translate tableaux into words, see Remark 3.

Definition 2. A *key diagram* of *column shape* (n_1, \dots, n_r) is a sequence of columns of boxes aligned in the top row, having n_1 boxes in the right most column, n_2 boxes in the second right most column etc. The key diagram is called a *Young diagram* if $n_1 \leq \dots \leq n_r$.

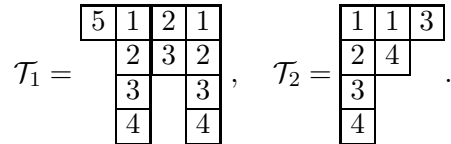
Example 1.



The key diagram D_1 is of column shape $(4, 2, 4, 1)$, the Young diagram D_2 is of column shape $(1, 2, 4)$

Definition 3. A *key tableau* of column shape (n_1, \dots, n_r) is a filling of the corresponding key diagram with elements from the alphabet \mathbb{A} such that the entries are *strictly increasing in the columns* (from top to bottom). A *semistandard Young tableau* of shape (n_1, \dots, n_r) , $n_1 \leq \dots \leq n_r$, is a filling of the corresponding Young diagram with elements from the alphabet \mathbb{A} such that the entries are strictly increasing in the columns (from top to bottom) and weakly increasing in the rows (left to right).

Example 2.



The key tableau \mathcal{T}_1 is of column shape $(4, 2, 4, 1)$, \mathcal{T}_2 is an example for a semistandard Young tableau of column shape $(1, 2, 4)$.

Definition 4. Let \mathcal{T} be a key tableau. The associated word $w_{\mathcal{T}}$ is the sequence (i_1, \dots, i_N) of elements in \mathbb{A} obtained from \mathcal{T} by reading the entries in the key tableau boxwise, from top to bottom in each column, starting with the right most column.

Example 3. For the key tableaux \mathcal{T}_1 and \mathcal{T}_2 in Example 2 we get

$$w_{\mathcal{T}_1} = (1, 2, 3, 4, 2, 3, 1, 2, 3, 4, 5), \quad w_{\mathcal{T}_2} = (3, 1, 4, 1, 2, 3, 4).$$

The map $\mathcal{T} \mapsto w_{\mathcal{T}}$, which associates to a key tableaux the word in the alphabet \mathbb{A} is not bijective, in general there are several key tableaux giving rise to the same word. All the following key tableaux have $(1, 2, 3)$ as the associated word:

$$(2) \quad \boxed{3} \boxed{2} \boxed{1}, \quad \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 3 & 1 \\ \hline & 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

Remark 3. Let $\tilde{w}_{\mathcal{T}}$ be the word obtained from $w_{\mathcal{T}}$ by reading the word backwards, then $\tilde{w}_{\mathcal{T}}$ is the column word (the product of the column words, read from left to right, in each column from bottom to top), which is Knuth equivalent (using the definition of the relation in [12]) to the row word as defined in [12]. The change in the definition of the Knuth relation is due to the fact that the "backwards reading map" $w \mapsto \tilde{w}$ respects the Knuth relations for $x < y < z$ but not for $x = y < z$ respectively $x < y = z$ (see [12]), but it sends equivalence classes for the relations in (1) onto the equivalence classes for the Knuth relations as defined in [12].

Definition 5. Two key tableaux $\mathcal{T}, \mathcal{T}'$ are called *Knuth equivalent*: $\mathcal{T} \sim_K \mathcal{T}'$ if and only if the two associated words $w_{\mathcal{T}}$ and $w_{\mathcal{T}'}$ are Knuth equivalent, i.e. $w_{\mathcal{T}} \sim_K w_{\mathcal{T}'}$.

Example 4. The tableaux in (2) are all Knuth equivalent because they have the same associated word, but also

$$\mathcal{T}_1 = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \\ \hline \end{array} \sim_K \mathcal{T}_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$$

because $w_{\mathcal{T}_1} = (1, 3, 2) \sim_K w_{\mathcal{T}_2} = (3, 1, 2)$.

Recall (see for example [3] or [13]):

Theorem 1. *i) Given a key tableau \mathcal{T} , there exists a unique semistandard Young tableau \mathcal{T}' such that $\mathcal{T} \sim_K \mathcal{T}'$.*
ii) Given a word w in the alphabet \mathbb{A} , there exists a unique semistandard Young tableau \mathcal{T} such that $w \sim_K w_{\mathcal{T}}$.

Remark 4. The unique semistandard Young tableau can be obtained from the given key tableau or word by using the Jeu de Taquin algorithm of Lascoux and Schützenberger or the bumping algorithm, see for example [3].

3. GALLERIES AND KEYS

For $1 \leq d \leq n$ denote by $I_{d,n}$ the set of all strictly increasing sequences of length d :

$$I_{d,n} = \{\mathbf{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < i_2 < \dots < i_d \leq n\}.$$

We fix a second alphabet

$$\mathbb{W} = I_{1,n} \cup I_{2,n} \cup \dots \cup I_{n-1,n} \cup I_{n,n}.$$

We can identify \mathbb{A} with $I_{1,n}$ and hence view \mathbb{A} as a subset of \mathbb{W} .

Definition 6. A *gallery* is a finite sequence $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ of elements in the alphabet \mathbb{W} , r is called the length of the gallery. We say that the gallery is of type (d_1, \dots, d_r) if $\mathbf{i}_1 \in I_{d_1,n}$, $\mathbf{i}_2 \in I_{d_2,n}$, \dots , $\mathbf{i}_r \in I_{d_r,n}$.

By the inclusion of $\mathbb{A} \hookrightarrow \mathbb{W}$ we can identify words in the alphabet \mathbb{A} with galleries of type $(1, \dots, 1)$ (where the number of 1's is equal to the length of the word).

Definition 7. Given a key tableau \mathcal{T} of column shape (n_1, \dots, n_r) , we associate to \mathcal{T} the *gallery* $\gamma_{\mathcal{T}} = (\mathbf{i}_1, \dots, \mathbf{i}_r)$, of type (n_1, \dots, n_r) , where the sequence $\mathbf{i}_\ell = (j_1, \dots, j_\ell) \in I_{d_\ell,n}$, $\ell = 1, \dots, r$, is obtained by reading the entries in the ℓ -th column (counted from right to left) of the key tableau \mathcal{T} from top to bottom.

Example 5. For the key tableaux \mathcal{T}_1 and \mathcal{T}_2 in Example 2 we get the galleries

$$\gamma_{\mathcal{T}_1} = ((1, 2, 3, 4), (2, 3), (1, 2, 3, 4), (5)) , \quad \gamma_{\mathcal{T}_2} = ((3), (1, 4), (1, 2, 3, 4)).$$

Definition 8. Given a gallery $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ of type $\mathbf{d} = (d_1, \dots, d_r)$, let \mathcal{T}_γ be the key tableau of column shape (d_1, \dots, d_r) such that the entries in the j -th column (counted from right to left) are (counted from top to bottom) given by the sequence \mathbf{i}_j .

Example 6. For the gallery $\gamma = ((3, 6, 7), (2), (2, 3, 4), (1, 4))$ we get the key tableau:

$$\mathcal{T}_\gamma = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 4 & 3 & & 6 \\ \hline & 4 & & 7 \\ \hline \end{array}.$$

One obtains a natural bijection between galleries and key tableaux, so in the following we identify often a gallery with its key tableau and vice versa. For example, the word w_γ associated to a gallery is the word associated to the key tableau \mathcal{T}_γ , and we say that *two galleries are Knuth equivalent* if their corresponding key tableaux are Knuth equivalent.

4. THE LOOP GROUP $G(\mathcal{K})$ AND THE KAC-MOODY GROUP $\hat{\mathcal{L}}(G)$

Let $G = SL_n(\mathbb{C})$. For a \mathbb{C} -algebra \mathcal{R} let $G(\mathcal{R})$ be the set of \mathcal{R} -rational points of G , i.e., the set of algebra homomorphisms from the coordinate ring $\mathbb{C}[G] \rightarrow \mathcal{R}$. Then $G(\mathcal{R})$ comes naturally equipped again with a group structure. In our case we can identify $SL_n(\mathcal{R})$ with the set of $n \times n$ -matrices with entries in \mathcal{R} and determinant 1.

Denote $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series in one variable and let $\mathcal{K} = \mathbb{C}((t))$ be its fraction field, the field of formal Laurent series. Denote $v : \mathcal{K}^* \rightarrow \mathbb{Z}$ the standard

valuation on \mathcal{K} such that $\mathcal{O} = \{f \in \mathcal{K} \mid v(f) \geq 0\}$. The *loop group* $G(\mathcal{K})$ is the set of \mathcal{K} -valued points of G , we denote by $G(\mathcal{O})$ its subgroup of \mathcal{O} -valued points. The latter has a decomposition as a semi-direct product $G \ltimes G^1(\mathcal{O})$, where we view $G \subset G(\mathcal{O})$ as the subgroup of constant loops and $G^1(\mathcal{O})$ is the subgroup of elements congruent to the identity modulo t . Note that we can describe $G^1(\mathcal{O})$ also as the image of $\mathfrak{sl}_n \otimes_{\mathbb{C}} t\mathbb{C}[[t]]$ via the exponential map (where $\mathfrak{sl}_n = \text{Lie } G$).

The *rotation operation* $\gamma : \mathbb{C}^* \rightarrow \text{Aut}(\mathcal{K})$, $\gamma(z)(f(t)) = f(zt)$ gives rise to group automorphisms $\gamma_G : \mathbb{C}^* \rightarrow \text{Aut}(G(\mathcal{K}))$, we denote $\mathcal{L}(G(\mathcal{K}))$ the semidirect product $\mathbb{C}^* \ltimes G(\mathcal{K})$. The rotation operation on \mathcal{K} restricts to an operation on \mathcal{O} and hence we have a natural subgroup $\mathcal{L}(G(\mathcal{O})) := \mathbb{C}^* \ltimes G(\mathcal{O})$ (for this and the following see [10], Chapter 13).

Let $\hat{\mathcal{L}}(G)$ be the affine Kac-Moody group associated to the affine Kac-Moody algebra

$$\hat{\mathcal{L}}(\mathfrak{g}) = \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $0 \rightarrow \mathbb{C}c \rightarrow \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \rightarrow \mathfrak{g} \otimes \mathcal{K} \rightarrow 0$ is the universal central extension of the *loop algebra* $\mathfrak{g} \otimes \mathcal{K}$ and d denotes the scaling element. We have corresponding exact sequences also on the level of groups, i.e., $\hat{\mathcal{L}}(G)$ is a central extension of $\mathcal{L}(G(\mathcal{K}))$

$$1 \rightarrow \mathbb{C}^* \rightarrow \hat{\mathcal{L}}(G) \xrightarrow{pr} \mathcal{L}(G(\mathcal{K})) \rightarrow 1$$

(see [10], Chapter 13).

Fix as *maximal torus* $T \subset G$ the diagonal matrices and fix as *Borel subgroup* B the upper triangular matrices in G . Denote by $B^- \subset G$ the Borel subgroup of lower triangular matrices. We denote $\langle \cdot, \cdot \rangle$ the non-degenerate pairing between the *character group* $\text{Mor}(T, \mathbb{C}^*)$ of T and the group $\text{Mor}(\mathbb{C}^*, T)$ of *cocharacters*. We identify $\text{Mor}(\mathbb{C}^*, T)$ with the quotient $T(\mathcal{K})/T(\mathcal{O})$, so we use the same symbol λ^\vee for the cocharacter and the point in $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$.

Let $N = N_G(T)$ be the normalizer in G of the fixed maximal torus $T \subset G$, we denote by W the *Weyl group* N/T of G . Let $N_{\mathcal{K}}$ be the subgroup of $G(\mathcal{K})$ generated by N and $T(\mathcal{K})$, the affine Weyl group is defined as $W^a = N_{\mathcal{K}}/T \simeq W \ltimes \text{Mor}(\mathbb{C}^*, T)$.

5. ROOTS, WEIGHTS AND COWEIGHTS

We use for the root system and the (abstract) weights and coweights the same notation as in [2]: Let \mathbb{R}^n be the real vector space equipped with the canonical basis $\epsilon_1, \dots, \epsilon_n$ and a scalar product (\cdot, \cdot) such that the latter basis is an orthonormal basis. Denote by $\mathbb{V} \subset \mathbb{R}^n$ the subspace orthogonal to $\epsilon_1 + \dots + \epsilon_n$. The root system $\Phi \subset \mathbb{V}$ and the coroot system Φ^\vee can be identified with each other:

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n\} = \{\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi\} = \Phi^\vee.$$

Hence we can also identify the root lattice R and the coroot lattice R^\vee . Similarly, we can identify the abstract weight lattice $X = \{\mu \in \mathbb{V} \mid \forall \alpha^\vee \in \Phi^\vee : (\mu, \alpha^\vee) \in \mathbb{Z}\}$ and the abstract coweight lattice $X^\vee = \{\mu \in \mathbb{V} \mid (\mu, \alpha) \in \mathbb{Z} \forall \alpha \in \Phi\}$. To avoid confusion between weights and coweights, we write $\lambda \in X$ for the (abstract) weights and $\lambda^\vee \in X^\vee$ instead of $\lambda \in X^\vee$ for the (abstract) coweights.

For the group $G = SL_n(\mathbb{C})$ we have $X = \text{Mor}(T, \mathbb{C}^*)$, so the character group is the full abstract weight lattice, and the group of cocharacters is the coroot lattice $R^\vee = \text{Mor}(\mathbb{C}^*, T)$. For the adjoint group $G' = PSL_n(\mathbb{C})$ let $p : SL_n(\mathbb{C}) \rightarrow G'$ be the isogeny and set $T' = p(T)$. We have in this case $R = \text{Mor}(T', \mathbb{C}^*)$, so the character group is the root lattice, and the group of cocharacters is the full lattice of abstract coweights $X^\vee = \text{Mor}(\mathbb{C}^*, T)$.

A basis of the root system is $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \dots, n-1\}$, let Φ^+ be the corresponding set of positive roots. We often identify \mathbb{V} with $\mathbb{R}^n / \mathbb{R}(\epsilon_1 + \dots + \epsilon_n)$, the description for the fundamental weights as $\omega_i = \epsilon_1 + \dots + \epsilon_i$, $i = 1, \dots, n-1$ is to be understood with respect to this identification. The abbreviation $\epsilon_{\mathbf{i}} = \epsilon_{i_1} + \dots + \epsilon_{i_d}$ and $\epsilon_{\mathbf{i}}^\vee = (\epsilon_{i_1} + \dots + \epsilon_{i_d})^\vee$ for $\mathbf{i} \in I_{d,n}$ is meant in the same way and provides a natural bijection between elements of $I_{d,n}$ and Weyl group conjugates of ω_d respectively ω_d^\vee .

Given a root and coroot datum, we have the hyperplane arrangement defined by the set $\{(\alpha, n) \mid \alpha \in \Phi, n \in \mathbb{Z}\}$ of affine roots. The couple (α, n) corresponds to the real affine root $\alpha + n\delta$ with δ being the smallest positive imaginary root of the affine Kac-Moody algebra $\hat{\mathcal{L}}(\mathfrak{g})$. We associate to an affine root (α, n) the affine reflection $s_{\alpha, n} : x^\vee \mapsto x^\vee - ((\alpha, x^\vee) + n)\alpha^\vee$ for $x^\vee \in \mathbb{V}$, and the affine hyperplane $H_{\alpha, n} = \{x^\vee \in \mathbb{V} \mid (\alpha, x^\vee) + n = 0\}$. We write

$$H_{\alpha, n}^+ = \{x^\vee \in \mathbb{V} \mid (\alpha, x^\vee) + n \geq 0\}$$

for the corresponding closed positive half-space and analogously

$$H_{\alpha, n}^- = \{x^\vee \in \mathbb{V} \mid (\alpha, x^\vee) + n \leq 0\}$$

for the negative half space.

Definition 9. Given a gallery $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ of type \mathbf{d} , the associated *polyline of type* (d_1, \dots, d_r) in \mathbb{V} is the sequence of coweights $\varphi(\gamma) = (\mu_0^\vee, \mu_1^\vee, \dots, \mu_r^\vee)$ defined by

$$\mu_0^\vee = 0, \mu_1^\vee = \epsilon_{\mathbf{i}_1}^\vee, \mu_2^\vee = \epsilon_{\mathbf{i}_1}^\vee + \epsilon_{\mathbf{i}_2}^\vee, \mu_3^\vee = \epsilon_{\mathbf{i}_1}^\vee + \epsilon_{\mathbf{i}_2}^\vee + \epsilon_{\mathbf{i}_3}^\vee, \dots, \mu_r^\vee = \epsilon_{\mathbf{i}_1}^\vee + \dots + \epsilon_{\mathbf{i}_r}^\vee.$$

We often identify the sequence with the polyline joining successively the origin 0 with μ_1^\vee , μ_1^\vee with μ_2^\vee , μ_2^\vee with μ_3^\vee etc. The last coweight μ_r^\vee is called the coweight of the gallery.

The Weyl group W is the finite subgroup of $\text{GL}(\mathbb{V})$ generated by the reflections $s_{\alpha, 0}$ for $\alpha \in \Phi$, the affine Weyl group W^a is the group of affine transformations of \mathbb{V} generated by the affine reflections $s_{\alpha, n}$ for $(\alpha, n) \in \Phi \times \mathbb{Z}$.

A fundamental domain for the action of W on \mathbb{V} is given by the dominant Weyl chamber $C^+ = \{x^\vee \in \mathbb{V} \mid (\alpha, x^\vee) \geq 0, \forall \alpha \in \Phi^+\}$. Similarly, the fundamental alcove $\Delta_f = \{x^\vee \in \mathbb{V} \mid 0 \leq (\alpha, x^\vee) \leq 1, \forall \alpha \in \Phi^+\}$ is a fundamental domain for the action of W^a on \mathbb{V} .

6. THE AFFINE GRASSMANN VARIETY

Let $\hat{\mathcal{L}}(G)$ be the affine Kac-Moody group as in section 4. Recall the exact sequences

$$1 \rightarrow \mathbb{C}^* \rightarrow \hat{\mathcal{L}}(G) \xrightarrow{pr} \mathcal{L}(G(\mathcal{K})) \rightarrow 1$$

(see [10], Chapter 13). Denote $P_0 \subset \hat{\mathcal{L}}(G)$ the maximal parabolic subgroup $pr^{-1}(\mathcal{L}(G(\mathcal{O})))$. We have four incarnations of the affine Grassmann variety:

$$(3) \quad \mathcal{G} = G(\mathcal{K})/G(\mathcal{O}) = \mathcal{L}(G(\mathcal{K}))/\mathcal{L}(G(\mathcal{O})) = \hat{\mathcal{L}}(G)/\mathcal{P}_0 = G(\mathbb{C}[t, t^{-1}])/G(\mathbb{C}[t]).$$

Note that $G(\mathcal{K})$ and \mathcal{G} are *ind*-schemes and $G(\mathcal{O})$ is a group scheme (see [10], [16]).

Let now G' be a simple complex algebraic group with the same Lie algebra \mathfrak{sl}_n as G and let $p : G \rightarrow G'$ be an isogeny with G' being simply connected. Then $G'(\mathcal{O}) \simeq G' \times (G')^1(\mathcal{O})$, the natural map $p_{\mathcal{O}} : G(\mathcal{O}) \rightarrow G'(\mathcal{O})$ is surjective and has the same kernel as p . Let T' be a maximal torus such that $p(T) = T'$ and consider the character group $\text{Mor}(T', \mathbb{C}^*)$ the group $\text{Mor}(\mathbb{C}^*, T')$ of cocharacters. The map $p : T \rightarrow T'$ induces an inclusion $\text{Mor}(\mathbb{C}^*, T) \hookrightarrow \text{Mor}(\mathbb{C}^*, T')$.

In particular, if we start with $PSL_n(\mathbb{C})$ (instead of $SL_n(\mathbb{C})$), then $\text{Mor}(T', \mathbb{C}^*)$ is the root lattice R and we can identify $\text{Mor}(\mathbb{C}^*, T')$ with the full coweight lattice X^\vee :

$$X^\vee = \text{Mor}(\mathbb{C}^*, T') = \mathbb{Z}\omega_1^\vee \oplus \dots \oplus \mathbb{Z}\omega_{n-1}^\vee.$$

The quotient $\text{Mor}(\mathbb{C}^*, T')/\text{Mor}(\mathbb{C}^*, T)$ measures the difference between \mathcal{G} and the affine Grassmann variety $\mathcal{G}' = G'(\mathcal{K})/G'(\mathcal{O})$. In fact, \mathcal{G} is connected, and the connected components of \mathcal{G}' are indexed by $\text{Mor}(\mathbb{C}^*, T')/\text{Mor}(\mathbb{C}^*, T)$. The natural maps $p_{\mathcal{K}} : G(\mathcal{K}) \rightarrow G'(\mathcal{K})$ and $p_{\mathcal{O}} : G(\mathcal{O}) \rightarrow G'(\mathcal{O})$ induce a $G(\mathcal{K})$ -equivariant inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$, which is an isomorphism onto the component of \mathcal{G} containing the class of 1. Now $G(\mathcal{K})$ acts via $p_{\mathcal{K}}$ on all of \mathcal{G}' , and each connected component is a homogeneous space for $G(\mathcal{K})$, isomorphic to $G(\mathcal{K})/\mathcal{Q}$ for some parahoric subgroup \mathcal{Q} of $G(\mathcal{K})$ which is conjugate to $G(\mathcal{O})$ by an (outer) automorphism. In the same way the action of $\hat{\mathcal{L}}(G)$ on \mathcal{G} extends to an action on \mathcal{G}' , each connected component being a homogeneous space.

Let $ev : G'(\mathcal{O}) \rightarrow G'$ be the evaluation maps at $t = 0$, and let $\mathcal{B}_{\mathcal{O}} = ev^{-1}(B)$ be the corresponding Iwahori subgroup. Denote by \mathcal{B} the corresponding Borel subgroup of $\hat{\mathcal{L}}(G)$. Recall the orbit decomposition (see for example [10]):

$$\mathcal{G}' = \bigcup_{\mu^\vee \in X^{\vee+}} G(\mathcal{O})\mu^\vee$$

where $X^{\vee+}$ denotes the set of dominant (abstract) coweights.

Throughout the remaining part of the paper, we study for $G = SL_n(\mathbb{C})$ the action of the groups $G(\mathcal{O})$, $G(\mathcal{K})$ (and other relevant subgroups of $G(\mathcal{K})$) on the affine Grassmann variety \mathcal{G}' for $G' = PSL_n$.

For a dominant coweight $\lambda^\vee \in X^{\vee+}$ denote by X_{λ^\vee} the $G(\mathcal{O})$ -stable Schubert variety

$$X_{\lambda^\vee} = \overline{G(\mathcal{O})\lambda^\vee} \subseteq \mathcal{G}'.$$

7. GALLERIES AND SUBGROUPS

Let $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ be a gallery of type \mathbf{d} and denote $\varphi(\gamma) = (\mu_0^\vee, \mu_1^\vee, \dots, \mu_r^\vee)$ the associated polyline. Let $[\mu_{j-1}^\vee, \mu_j^\vee]$ be the convex hull of the two vertices. We associate to the gallery and the vertices on the polyline some subgroups:

Denote $P_{\mu_j^\vee}$ the parabolic subgroup of $\hat{\mathcal{L}}(G)$ containing $\mathbb{C}^* \ltimes T$ and the root subgroups associated to the real roots

$$(4) \quad \hat{\Phi}_j = \{(\alpha, n) \mid \mu_j^\vee \in H_{\alpha, n}^+\}, \quad j = 0, \dots, r.$$

Denote $Q_{\mu_j^\vee}$ the parabolic subgroups of $\hat{\mathcal{L}}(G)$ containing $\mathbb{C}^* \ltimes T$ and the root subgroups associated to the real roots

$$(5) \quad \hat{\Phi}_{j, j+1} = \{(\alpha, n) \mid [\mu_j^\vee, \mu_{j+1}^\vee] \subset H_{\alpha, n}^+\}, \quad j = 0, \dots, r-1,$$

Note that $P_{\mu_j^\vee} \supset Q_{\mu_j^\vee} \subset P_{\mu_{j+1}^\vee}$. Another important subgroup of $P_{\mu_j^\vee}$ is generated by its root subgroups corresponding to roots which are negative in the classical sense:

$$(6) \quad U_{\mu_j^\vee}^- = \langle U_{(\beta, m)} \mid \beta \prec 0, (\beta, m) \in \hat{\Phi}_j \rangle \subset P_{\mu_j^\vee}, \quad j = 0, \dots, r.$$

The following set of roots is relevant later for the description of certain cells. Let

$$\Psi_{\gamma, j} = \{(\alpha, n) \mid \alpha \succ 0, \mu_j^\vee \in H_{\alpha, n}, [\mu_j^\vee, \mu_{j+1}^\vee] \not\subset H_{\alpha, n}^-\}$$

and set

$$\mathbb{U}_{\gamma, j}^- = \prod_{(\alpha, n) \in \Psi_{\gamma, j}} U_{(-\alpha, -n)} \subset U_{\mu_j^\vee}^-.$$

We associate to the gallery the product of root subgroups

$$\mathbb{U}_\gamma^- = \mathbb{U}_{\gamma, 0}^- \times \mathbb{U}_{\gamma, 1}^- \times \dots \times \mathbb{U}_{\gamma, r-1}^-.$$

8. GALLERIES AND BOTT-SAMELSON VARIETIES

For a gallery $\gamma = (\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_r)$ of type $\mathbf{d} = (d_1, \dots, d_r)$ let $\varphi(\gamma) = (\mu_0^\vee, \dots, \mu_r^\vee)$ be the associated polyline. Recall that $P_{\mu_j^\vee} \supset Q_{\mu_j^\vee} \subset P_{\mu_{j+1}^\vee}$, so we can associate to γ the Bott-Samelson variety

$$(7) \quad \Sigma_{\mathbf{d}} = P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} P_{\mu_1^\vee} \times_{Q_{\mu_1^\vee}} \dots \times_{Q_{\mu_{r-1}^\vee}} P_{\mu_r^\vee} / P_{\mu_r^\vee},$$

which is defined as the quotient of $P_{\mu_0^\vee} \times P_{\mu_1^\vee} \times \dots \times P_{\mu_r^\vee}$ by $Q_{\mu_0^\vee} \times Q_{\mu_1^\vee} \times \dots \times Q_{\mu_{r-1}^\vee} \times P_{\mu_r^\vee}$ with respect to the action given by:

$$(q_0, q_1, \dots, q_{r-1}, q_r) \circ (p_0, p_1, \dots, p_r) = (p_0 q_0, q_0^{-1} p_1 q_1, \dots, q_{r-1}^{-1} p_r q_r).$$

The fibred product $\Sigma_{\mathbf{d}}$ is a smooth projective complex variety, its points are denoted by $[p_0, \dots, p_r]$.

Corresponding to the parabolic subgroups $Q_{\mu_j^\vee}, P_{\mu_j^\vee}$ of $\hat{\mathcal{L}}(G)$ one has the parahoric subgroups $\mathcal{P}_{\mu_j^\vee}, \mathcal{Q}_{\mu_j^\vee}$ of $G(\mathcal{K})$ such that $P_{\mu_j^\vee} = pr^{-1}(\mathbb{C}^* \ltimes \mathcal{P}_{\mu_j^\vee})$ and $Q_{\mu_j^\vee} = pr^{-1}(\mathbb{C}^* \ltimes \mathcal{Q}_{\mu_j^\vee})$. We get another description of $\Sigma_{\mathbf{d}}$ as

$$(8) \quad \Sigma_{\mathbf{d}} = \mathcal{P}_{\mu_0^\vee} \times_{\mathcal{Q}_{\mu_0^\vee}} \mathcal{P}_{\mu_1^\vee} \times_{\mathcal{Q}_{\mu_1^\vee}} \dots \times_{\mathcal{Q}_{\mu_{r-1}^\vee}} \mathcal{P}_{\mu_r^\vee} / \mathcal{P}_{\mu_r^\vee}.$$

Set $\mathcal{F}_\gamma = \hat{\mathcal{L}}(G)/P_{\mu_0^\vee} \times \hat{\mathcal{L}}(G)/P_{\mu_1^\vee} \times \dots \times \hat{\mathcal{L}}(G)/P_{\mu_r^\vee}$ and consider the natural morphism $\iota: \Sigma_{\mathbf{d}} \rightarrow \mathcal{F}_\gamma$ given by sending the class $[p_0, p_1, \dots, p_r]$ to the sequence

$$(P_{\mu_0^\vee}, p_0 Q_{\mu_0^\vee} p_0^{-1}, p_0 P_{\mu_1^\vee} p_0^{-1}, p_0 p_1 Q_{\mu_1^\vee} p_1^{-1} p_0^{-1}, \dots, p_0 \dots p_{r-1} P_r p_{r-1}^{-1} \dots p_0^{-1}) \in \mathcal{F}_\gamma.$$

The image of $\Sigma_{\mathbf{d}}$ can be described as the set of sequences of parabolic subgroups (see for example [4]) satisfying the following inclusion relations:

$$(9) \quad \left\{ \begin{array}{l} P_{\mu_0^\vee} \supset Q_0 \subset P_1 \supset \dots \subset P_r \\ (P_{\mu_0^\vee}, Q_0, \dots, P_r) \mid \begin{array}{l} P_j \text{ is conjugate to } P_{\mu_j^\vee}, \forall j = 1, \dots, r \\ Q_j \text{ is conjugate to } Q_{\mu_j^\vee}, \forall j = 0, \dots, r-1. \end{array} \end{array} \right\}$$

The morphism ι is an isomorphism onto the image. If γ' is a gallery of the same type as γ with associated polyline $\varphi(\gamma') = (\mu_0^{\vee'}, \dots, \mu_r^{\vee'})$, then $\mu_0^\vee = \mu_0^{\vee'} = 0$, $\mu_1^{\vee'}$ is conjugate to μ_1^\vee by an element τ_0 in the Weyl group of P_{μ_0} and hence $P_{\mu_1^{\vee'}} = \tau_0 P_{\mu_1} \tau_0^{-1}$, $\mu_2^{\vee'}$ is conjugate to $\tau_0(\mu_2^\vee)$ by an element τ_1 in the Weyl group of $P_{\mu_1} = \tau_0 P_{\mu_1} \tau_0^{-1}$ and hence $P_{\mu_2^{\vee'}} = \tau_1 \tau_0 P_{\mu_2} \tau_0^{-1} \tau_1^{-1}$ etc.

So γ and γ' define in (9) the same variety and hence $\Sigma_{\mathbf{d}}$ depends only on the type \mathbf{d} and not on the gallery. We say that we consider $\Sigma_{\mathbf{d}}$ with center γ if we want to emphasize that the parabolic subgroups in (7) correspond to the coweights lying on the polyline associated to γ .

For a fixed type \mathbf{d} let \mathcal{T}_0 be the unique key tableau of column shape \mathbf{d} having as filling only 1's in the top row, 2's in the second row etc. Denote by γ_0 the associated gallery and let $\varphi(\gamma_0) = (\mu_0^\vee, \dots, \mu_r^\vee)$ be the associated polyline. The variety $\Sigma_{\mathbf{d}}$ is by (8) naturally endowed with a T -action.

Lemma 1. *We consider $\Sigma_{\mathbf{d}}$ with center γ_0 . There exists a natural bijection between T -fixed points in $\Sigma_{\mathbf{d}}$ and galleries of type \mathbf{d} such that the gallery γ_0 corresponds to the class $[1, \dots, 1]$ in $P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} \dots \times_{Q_{\mu_{r-1}^\vee}} P_{\mu_r^\vee} / P_{\mu_r^\vee}$.*

Proof. We have a sequence of projections of the fibered spaces:

$$(10) \quad \Sigma_{\mathbf{d}} \xrightarrow{\phi_{r-1}} P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} \dots \times_{Q_{\mu_{r-3}^\vee}} P_{\mu_{r-2}^\vee} / Q_{\mu_{r-2}^\vee} \xrightarrow{\phi_{r-2}} \dots \xrightarrow{\phi_2} P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} P_{\mu_1^\vee} / Q_{\mu_1^\vee} \xrightarrow{\phi_1} P_{\mu_0^\vee} / Q_{\mu_0^\vee}$$

such that the fibres of ϕ_j are isomorphic to $P_{\mu_j^\vee} / Q_{\mu_j^\vee}$. The fibrations are locally trivial in the Zariski topology and equivariant with respect to the T -action. It is now easy to see that the T -fixed points in $\Sigma_{\mathbf{d}}$ are in a natural one-to one correspondence with T -fixed points in

$$P_{\mu_0^\vee} / Q_{\mu_0^\vee} \times P_{\mu_1^\vee} / Q_{\mu_1^\vee} \times \dots \times P_{\mu_{r-1}^\vee} / Q_{\mu_{r-1}^\vee}.$$

Let $L_{\mu_j^\vee} \subset P_{\mu_j^\vee}$ be the Levi subgroup containing $\mathbb{C}^* \ltimes T$, then the semisimple part $SL_{\mu_j^\vee}$ of $L_{\mu_j^\vee}$ is isomorphic to $SL_n(\mathbb{C})$. With respect to this isomorphism, the intersection $P = SL_{\mu_j^\vee} \cap Q_{\mu_j^\vee}$ is a maximal parabolic subgroup of $SL_{\mu_j^\vee}$ such that

$$P_{\mu_j^\vee} / Q_{\mu_j^\vee} \simeq SL_{\mu_j^\vee} / P \simeq SL_n(\mathbb{C}) / P_{\omega_{d_j}^\vee}.$$

The T -fixed points of this variety are naturally indexed by elements in $I_{d_j, n}$, $j = 1, \dots, r$, proving the claim. \square

9. MV-CYCLES AND GALLERIES

We fix a type $\underline{d} = (d_1, \dots, d_r)$. Since the Bott-Samelson variety $\Sigma_{\underline{d}}$ depends only on the type and not on the choice of gallery, we fix during this section for convenience γ such that the associated polyline is wrapped around the fundamental alcove, i.e. if $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ and $\varphi(\gamma) = (\mu_0^\vee = 0, \mu_1^\vee, \dots, \mu_r^\vee)$, then all the coweights $\mu_0^\vee, \mu_1^\vee, \dots, \mu_r^\vee$ are vertices of the fundamental alcove. Since the type of the gallery is fixed, there exists only one gallery of this type with this property.

Set $\lambda^\vee = \sum_{j=1}^r \omega_{d_j}^\vee$ and consider the Bott-Samelson variety of center γ :

$$\Sigma_{\underline{d}} = P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} P_{\mu_1^\vee} \times_{Q_{\mu_1^\vee}} \times \dots \times_{Q_{\mu_{r-1}^\vee}} P_{\mu_r^\vee} / P_{\mu_r^\vee}.$$

Note that for this choice of γ all the associated parabolic subgroups are standard parabolic subgroups, i.e. $\mathcal{B} \subset P_{\mu_j^\vee}, Q_{\mu_j^\vee}$.

The last parabolic subgroup $P_{\mu_r^\vee}$ in the fibered product corresponds to a maximal standard parabolic subgroup of $\hat{\mathcal{L}}(G)$. In \mathcal{G}' , this group corresponds to the point μ_r^\vee (i.e. $P_{\mu_r^\vee} \subset \hat{\mathcal{L}}(G)$ is the stabilizer of this point with respect to the action of $\hat{\mathcal{L}}(G)$ on \mathcal{G}'), so we get a canonical map

$$\begin{aligned} \pi : \Sigma_{\underline{d}} = P_{\mu_0^\vee} \times_{Q_{\mu_0^\vee}} P_{\mu_1^\vee} \times_{Q_{\mu_1^\vee}} \times \dots \times_{Q_{\mu_{r-1}^\vee}} P_{\mu_r^\vee} / P_{\mu_r^\vee} &\rightarrow \mathcal{G}' \\ [p_0, \dots, p_{r-1}, p_r] &\mapsto p_0 \cdots p_{r-1} \mu_r^\vee. \end{aligned}$$

It is well known that the image $\pi(\Sigma_{\underline{d}})$ is the Schubert variety X_{λ^\vee} , and the map is in fact a desingularization (see for example [4]).

Let $\eta : \mathbb{C}^* \rightarrow T$ be a generic anti-dominant one parameter subgroup. $\Sigma_{\underline{d}}$ is a smooth projective variety endowed with a T -action and hence a η -action. Since η is generic, the T -fixed points are the same as the η -fixed points, so the latter are naturally indexed by galleries of type \underline{d} (see Lemma 1). To each fixed point y_ζ corresponding to a gallery ζ of type \underline{d} , we can associate the Białynicki-Birula cell [1]:

$$C_\zeta = \{x \in \Sigma_{\underline{d}} \mid \lim_{t \rightarrow 0} \eta(t)x = \zeta\}.$$

Recall that $\Sigma_{\underline{d}}$ is the disjoint union $\bigcup C_\zeta$, where ζ is running over the set of all galleries of type \underline{d} .

The Schubert variety X_{λ^\vee} has a similar decomposition, but which is in general not given by cells. Let U^- be the unipotent radical of B^- . Given a coweight μ^\vee , the closure of an irreducible component of

$$(U^-(\mathcal{K}).\mu^\vee) \cap (G(\mathcal{O}).\lambda^\vee) \subset \mathcal{G}'$$

is called a Mirković-Vilonen cycle, or short just MV-cycle [17] of coweight μ^\vee in X_{λ^\vee} . It is known that the intersection is nonempty if and only if in the finite dimensional irreducible representation of $SL_n(\mathbb{C})$ (viewed as Langlands dual group of $PSL_n(\mathbb{C})$) of highest weight λ^\vee the weight space corresponding to μ^\vee is non-zero. Further, the number of irreducible components is equal to the dimension of this weight space. One has a disjoint union

$$X_{\lambda^\vee} = \bigcup_{\mu^\vee \in X^\vee} (X_{\lambda^\vee} \cap U^-(\mathcal{K}).\mu^\vee),$$

and since $\lim_{t \rightarrow 0} \eta(t)u\eta(t)^{-1} = 1$ for all $u \in U^-(\mathcal{K})$, it follows that

$$X_{\lambda^\vee} \cap U^-(\mathcal{K}) \cdot \mu^\vee = \{y \in X_{\lambda^\vee} \mid \lim_{t \rightarrow 0} \eta(t)y = \mu^\vee\}.$$

The desingularization map $\pi : \Sigma_{\underline{\mathbf{d}}} \rightarrow X_{\lambda^\vee}$ is T -equivariant, hence if μ^\vee is the coweight of a gallery γ of type $\underline{\mathbf{d}}$, then the image $\pi(C_\zeta)$ of the cell is contained in $X_{\lambda^\vee} \cap U^-(\mathcal{K}) \cdot \mu^\vee$. Since the image is irreducible, it is contained in some MV-cycle.

10. MAIN THEOREM

Let γ be a gallery of type $\underline{\mathbf{d}}$ with associated polyline $\varphi(\gamma) = (\mu_0^\vee, \dots, \mu_r^\vee)$, set $\lambda^\vee = \sum_{j=1}^r \omega_{d_j}^\vee$ and denote by $\pi_{\underline{\mathbf{d}}} : \Sigma_{\underline{\mathbf{d}}} \rightarrow X_{\lambda^\vee}$ the desingularization map. Let \mathcal{T} be the unique semistandard Young tableau such that the words w_γ and $w_{\mathcal{T}}$ are Knuth equivalent. Let $\underline{\mathbf{c}} = (c_1, \dots, c_s)$ be the shape of \mathcal{T} , set $\nu^\vee = \sum_{j=1}^s \omega_{c_j}^\vee$ and let $X_{\nu^\vee} \subseteq X_{\lambda^\vee}$ be the corresponding Schubert variety.

Theorem 2. *a) The closure $\overline{\pi_{\underline{\mathbf{d}}}(C_\gamma)} \subseteq X_{\lambda^\vee}$ is a MV-cycle for the (possibly smaller) Schubert variety $X_{\nu^\vee} \subseteq X_{\lambda^\vee}$. The weight of the MV-cycle is μ_r^\vee .*
b) Given a second gallery γ' of type $\underline{\mathbf{d}}'$, set $\lambda'^\vee = \sum_{j=1}^r \omega_{d'_j}^\vee$ and let $\pi_{\underline{\mathbf{d}}'} : \Sigma_{\underline{\mathbf{d}}'} \rightarrow X_{\lambda'^\vee}$ be the desingularization map. The following conditions are equivalent:
i) γ and γ' are Knuth equivalent.
ii) $\overline{\pi_{\underline{\mathbf{d}}}(C_\gamma)} = \overline{\pi_{\underline{\mathbf{d}}'}(C_{\gamma'})}$

As a consequence we get a geometric interpretation of the Knuth relations:

Corollary 1. *For two words w_1, w_2 of the same length N in the alphabet \mathbb{A} the following conditions are equivalent:*

- i) w_1 and w_2 are Knuth equivalent*
- ii) $\pi(C_{w_1}) = \pi(C_{w_2})$, where π is the canonical map $\pi : \Sigma_{N\epsilon_1^\vee} \rightarrow X_{N\epsilon_1^\vee}$, and the words w_1, w_2 are viewed as galleries of type $\underbrace{(1, 1, \dots, 1)}_N$.*

The proof will be given in section 13.

11. THE CELL C_γ

We recall a description of the cell and determine its dimension. For a gallery $\gamma = (\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_r)$ of type $\underline{\mathbf{d}} = (d_1, \dots, d_r)$ set $\lambda^\vee = \sum_{j=1}^r \omega_{d_j}^\vee$. Let $\varphi(\gamma) = (\mu_0^\vee, \mu_1^\vee, \dots, \mu_r^\vee)$ be the associated polyline and denote by $\Sigma_{\underline{\mathbf{d}}}$ the associated Bott-Samelson variety of center γ . The coweights μ_j^\vee and μ_{j+1}^\vee are joined by the segment $[\mu_j^\vee, \mu_{j+1}^\vee]$, $j = 0, \dots, r-1$. Let $H_{\alpha, n}$, $\alpha > 0$, be a hyperplane such that $\mu_j^\vee \in H_{\alpha, n}$. We say that γ crosses the wall $H_{\alpha, n}$ in μ_j^\vee in the positive (negative) direction if $[\mu_j^\vee, \mu_{j+1}^\vee] \not\subset H_{\alpha, n}^-$ (respectively $[\mu_j^\vee, \mu_{j+1}^\vee] \not\subset H_{\alpha, n}^+$). Set

$$\begin{aligned} \#^+(\gamma) &= \sum_{j=0}^{r-1} (\# \text{ positive wall crossings in } \mu_j^\vee) = \sum_{j=0}^{r-1} |\Psi_{\gamma, j}|; \\ \#^-(\gamma) &= \sum_{j=0}^{r-1} (\# \text{ negative wall crossings in } \mu_j^\vee); \\ \#^\pm(\gamma) &= \#^+(\gamma) + \#^-(\gamma). \end{aligned}$$

The value of the sum $\sharp^\pm(\gamma)$ depends, by definition, only on the type of the gallery and not on the actual choice of the gallery. It is easy to check that

$$\sharp^\pm(\gamma) = (\lambda^\vee, 2\rho^\vee),$$

where ρ^\vee is the sum of the fundamental coweights.

The following proposition is proved in [4], Proposition 4.19.

Proposition 1. *The map*

$$\begin{aligned} \mathbb{U}_{\gamma,0}^- \times \mathbb{U}_{\gamma,1}^- \times \mathbb{U}_{\gamma,2}^- \times \dots \times \mathbb{U}_{\gamma,r-1}^- &\rightarrow \Sigma_{\mathbf{d}}, \\ (u_0, \dots, u_{r-1}) &\mapsto [u_0, \dots, u_{r-1}, 1] \end{aligned} ,$$

is an isomorphism onto the cell C_γ . In particular, $\sharp^+(\gamma)$ is the dimension of the cell.

The image of the map

$$\begin{aligned} \chi : U_{\mu_0^\vee}^- \times U_{\mu_1^\vee}^- \times \dots \times U_{\mu_{r-1}^\vee}^- &\rightarrow \Sigma_{\mathbf{d}}, \\ (u_0, \dots, u_{r-1}) &\mapsto [u_0, \dots, u_{r-1}, 1] \end{aligned} ,$$

is obviously (by the choice of η) a subset of C_γ . But $\mathbb{U}_{\gamma,j}^-$ is a subset of the group $U_{\mu_j^\vee}^-$, so we get as an immediate consequence:

Corollary 2. *The map χ has as image the cell C_γ .*

The description of the cell in the proposition above has as an immediate consequence:

Corollary 3. $\pi(C_\gamma) = \mathbb{U}_{\gamma,0}^- \mathbb{U}_{\gamma,1}^- \mathbb{U}_{\gamma,2}^- \dots \mathbb{U}_{\gamma,r-1}^- \cdot \mu_r^\vee = U_{\mu_0^\vee}^- U_{\mu_1^\vee}^- \dots U_{\mu_{r-1}^\vee}^- \cdot \mu_r^\vee$.

Lemma 2. *The dimension of the cell C_γ depends only on the coweight of the gallery. More precisely, $\dim C_\gamma = \sharp^+(\gamma) = (\lambda^\vee + \mu_r^\vee, \rho^\vee)$.*

Proof. The proof is by induction on the height of $\lambda^\vee - \mu_r^\vee$. If $\lambda^\vee = \mu_r^\vee$, then there exists only one gallery of this type with coweight λ^\vee , it must be $\gamma = (\omega_{d_1}^\vee, \dots, \omega_{d_r}^\vee)$. The hyperplanes crossed positively are exactly all the affine hyperplanes lying between the origin and λ^\vee , for each positive root there are exactly $(\lambda^\vee, \alpha^\vee)$ such hyperplanes and hence for this gallery we get

$$\sharp^+(\gamma) = \sum_{\alpha \in \Phi^+} (\lambda^\vee, \alpha^\vee) = (\lambda^\vee, 2\rho^\vee) = (2\lambda^\vee, \rho^\vee),$$

proving the claim in this case. Suppose now the claim holds if the height of $\lambda^\vee -$ coweight of the gallery is strictly smaller than s . Suppose $\gamma = (\mathbf{i}_1, \dots, \mathbf{i}_r)$ is a gallery such that the height of $\lambda^\vee - \mu_r^\vee$ is equal to s and $s > 0$. Then there exists a simple root α and $1 \leq t \leq r$ such that $\epsilon_{\mathbf{i}_t}^\vee \neq \omega_{d_t}^\vee$ and $s_\alpha(\epsilon_{\mathbf{i}_t}^\vee) = \epsilon_{\mathbf{i}_t}^\vee + \alpha^\vee$. Let \mathbf{i}'_t be such that $\epsilon_{\mathbf{i}'_t}^\vee = \epsilon_{\mathbf{i}_t}^\vee + \alpha^\vee$ and let γ' be the gallery

$$\gamma' = (\mathbf{i}_1, \dots, \mathbf{i}_{t-1}, \mathbf{i}'_t, \mathbf{i}_{t+1}, \dots, \mathbf{i}_r),$$

and denote by $\mu'_r{}^\vee$ the coweight of γ' . Then the height of $\lambda^\vee - \mu'_r{}^\vee$ is equal to $s - 1$ and hence $\sharp^+(\gamma') = (\lambda^\vee + \mu'_r{}^\vee, \rho^\vee)$. It follows:

$$\begin{aligned} \sharp^+(\gamma) &= \sum_{j=1}^r \sharp\Psi_{\gamma,j} = (\sum_{j=1}^r \sharp\Psi_{\gamma',j}) + 1 &= (\lambda^\vee + \mu'_r{}^\vee, \rho^\vee) + 1 \\ & &= (\lambda^\vee + \mu'_r{}^\vee, \rho^\vee) + (\alpha^\vee, \rho^\vee) \\ & &= (\lambda^\vee + \mu'_r{}^\vee, \rho^\vee) \end{aligned}$$

□

12. TAIL OF A GALLERY

Let γ be a gallery of type $\underline{\mathbf{d}}$. Let \mathcal{T}_γ be the corresponding key tableau. In Section 2, we have associated a to a key tableau a word in the alphabet \mathbb{A} , we just write w_γ for the word associated to \mathcal{T}_γ .

Corresponding to the type, we have the associated Bott-Samelson variety $\Sigma_{\underline{\mathbf{d}}}$, the dominant coweight $\lambda^\vee = \sum \omega_{d_j}^\vee$ and the desingularization map:

$$\pi_1 : \Sigma_{\underline{\mathbf{d}}} \rightarrow X_{\lambda^\vee}.$$

And, associated to the gallery, we have the cell $C_\gamma \subset \Sigma_{\underline{\mathbf{d}}}$. Recall that the map $\gamma \rightarrow w_\gamma$ which associates to a gallery γ the word w_γ in the alphabet \mathbb{A} is far from being injective. Let us first show the following.

Proposition 2. *The image $Z_\gamma = \pi_1(C_\gamma)$ depends only on the word w_γ .*

The proof will be given at the end of this section. Set $Z_\gamma = \pi_1(C_\gamma)$ and note that $pr(U_{\mu_0^\vee}^-) \subseteq U^-(\mathcal{O}) = U^-(\mathcal{K}) \cap SL_n(\mathcal{O})$ by definition. Recall that $\Sigma_{\underline{\mathbf{d}}}$ inherits by (8) naturally an $SL_n(\mathcal{O})$ -action, and the map π_1 is equivariant with respect to this action. The cell C_γ is not $SL_n(\mathcal{O})$ -stable, but the cells are stable under the action of $U^-(\mathcal{O})$: recall that

$$C_\gamma = \{z \in \Sigma_{\underline{\mathbf{d}}} \mid \lim_{t \rightarrow 0} \eta(t).z = \gamma\}$$

for a fixed anti-dominant one-parameter subgroup $\eta : \mathbb{C}^* \rightarrow T$. Since

$$\lim_{t \rightarrow 0} \eta(t)u\eta(t)^{-1} = 1,$$

it follows that $u.z \in C_\gamma$ for all $z \in C_\gamma$ and all $u \in U^-(\mathcal{O})$. By the definition of Ψ_0 we have as an immediate consequence:

Lemma 3. *Z_γ is $U_{\mu_0^\vee}^-$ -stable.*

We want to reformulate Lemma 3 to make it available in a more general situation. Let $\gamma = (\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_r)$ be a gallery of type $\underline{\mathbf{d}}$, denote by $\varphi(\gamma) = (\mu_0^\vee, \dots, \mu_r^\vee)$ the associated polyline. For $k \geq 0$ consider the tail $\gamma^{\geq k} = (\mathbf{i}_k, \dots, \mathbf{i}_r)$ of the gallery γ , denote by $\varphi(\gamma^{\geq k}) = (\mu_0^\vee, \dots, \mu_{r-k}^\vee)$ the associated polyline and by $\varphi^{shift}(\gamma^{\geq k}) = (\mu_k^\vee, \dots, \mu_r^\vee)$ the shifted polyline (starting in μ_k^\vee instead of 0) of the tail gallery. Denote by $Z_{\gamma^{\geq k}}$ the image of the cell $C_{\gamma^{\geq k}} \subset \Sigma_{(d_k, \dots, d_r)}$:

$$Z_{\gamma^{\geq k}} = U_{\gamma^{\geq k}, 0}^- \cdots U_{\gamma^{\geq k}, r-k-1}^- \cdot \mu_{r-k}^\vee = U_{\mu_0^\vee}^- U_{\mu_1^\vee}^- \cdots U_{\mu_{r-k-1}^\vee}^- \cdot \mu_{r-k}^\vee$$

and let $Z_{\gamma}^{\geq k}$ be the *truncated image*:

$$Z_{\gamma}^{\geq k} = \mathbb{U}_{\gamma,k}^- \mathbb{U}_{\gamma,k+1}^- \cdots \mathbb{U}_{\gamma,r-1}^- \cdot \mu_r^{\vee}.$$

Proposition 3. $Z_{\gamma}^{\geq k}$ is $U_{\mu_k^{\vee}}^-$ -stable.

Proof. Let $T' \subset PSL_n(\mathbb{C})$ be as in section 6. Corresponding to a coweight μ_k^{\vee} let $t_{\mu_k^{\vee}}$ be the translation in the extended affine Weyl group. The translations act on the affine roots by shifts: $t_{\mu_k^{\vee}} : (\beta, \ell) \mapsto (\beta, \ell + (\mu_k^{\vee}, \beta))$ and $t_{-\mu_k^{\vee}} : (\beta, k) \mapsto (\beta, \ell - (\mu_k^{\vee}, \beta))$.

Denote by $U_{\beta} : \mathbb{C}((t)) \rightarrow G(\mathcal{K})$ the root subgroup corresponding to the root β . Note that the conjugation by $t_{\mu_k^{\vee}}$ acts on the root subgroups by a shift: given $a \in \mathbb{C}$ and $\ell \in \mathbb{Z}$, then

$$t_{\mu_k^{\vee}} U_{\beta}(at^{\ell})(t_{\mu_k^{\vee}})^{-1} = U_{\beta}(at^{\ell + (\mu_k, \beta)}),$$

so the conjugation on the group and the translation on the coweight lattice correspond to each other. It follows:

$$\begin{aligned} (11) \quad t_{\mu_k^{\vee}} \cdot Z_{\gamma \geq k} &= t_{\mu_k^{\vee}} \cdot (\mathbb{U}_{\gamma \geq k, 0}^- \cdots \mathbb{U}_{\gamma \geq k, r-k-1}^- \cdot \mu_{r-k}^{\vee}) \\ &= \mathbb{U}_{\gamma, k}^- \mathbb{U}_{\gamma, k+1}^- \cdots \mathbb{U}_{\gamma, r-1}^- \cdot \mu_r^{\vee} \\ &= Z_{\gamma}^{\geq k}. \end{aligned}$$

By Lemma 3, $Z_{\gamma \geq k}$ is stabilized by $\mathbb{U}_{\gamma, 0}^-$, so $t_{\mu_k^{\vee}} \mathbb{U}_{\gamma, 0}^- (t_{\mu_k^{\vee}})^{-1} = \mathbb{U}_{\gamma, k}^-$ stabilizes $Z_{\gamma}^{\geq k}$. \square

Proof of Proposition 2. Let w_{γ} be the word (in the alphabet \mathbb{A}) associated to γ . Since $\mathbb{A} \subset \mathbb{W}$, we can view w_{γ} also as a gallery of type $\mathbf{1}_N = (1, 1, \dots, 1)$, where $N = \sum_{j=1}^r d_j$ is the length of the word. For $\gamma = (\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_r)$ let $\mathbf{i}_1 = (j_1, \dots, j_{d_1})$, $\mathbf{i}_2 = (h_1, \dots, h_{d_2})$ etc. and set $\eta_{0,0}^{\vee} = 0$ and

$$\eta_{0,1}^{\vee} = \epsilon_{j_1}^{\vee}, \eta_{0,2}^{\vee} = \eta_{0,1}^{\vee} + \epsilon_{j_2}^{\vee}, \dots, \eta_{0,d_1-1}^{\vee} = \eta_{0,d_1-2}^{\vee} + \epsilon_{j_{d_1-1}}^{\vee}, \eta_{1,0}^{\vee} = \epsilon_{\mathbf{i}_1}, \eta_{1,1}^{\vee} = \epsilon_{\mathbf{i}_1} + \epsilon_{h_1}^{\vee}, \dots$$

Let

$$(\eta_{0,0}^{\vee}, \eta_{0,1}^{\vee}, \eta_{0,2}^{\vee}, \dots, \eta_{0,d_1-1}^{\vee}, \eta_{1,0}^{\vee}, \eta_{1,1}^{\vee}, \dots, \eta_{1,d_2-1}^{\vee}, \dots, \eta_{r-1,d_r-1}^{\vee}, \eta_{r,0}^{\vee})$$

be the polyline associated to w_{γ} (viewed as a gallery of type $\mathbf{1}_N$). The indexing is such that $\eta_{k,0}^{\vee} = \mu_k^{\vee}$. We use now Corollary 3 and compare the product of unipotent groups $\mathbb{U}_{\gamma,k}^-$ and the unipotent group $U_{\mu_k^{\vee}}^-$ associated to the gallery γ and the product of unipotent groups $\mathbb{U}_{w_{\gamma},(k,0)}^- \cdots \mathbb{U}_{w_{\gamma},(k,d_k-1)}^-$ associated to the gallery w_{γ} .

For a positive root $\epsilon_{\ell} - \epsilon_{\ell'}$, $1 \leq \ell < \ell' \leq n$, denote by $p_{\ell, \ell'}^k$ the integer such that $\mu_k^{\vee} \in H_{\epsilon_{\ell} - \epsilon_{\ell'}, -p_{\ell, \ell'}^k}$. Let $\mathbf{i}_{k+1} = (j_1, \dots, j_{d_k})$, then we have for $k = 0, \dots, r-1$:

$$\mathbb{U}_{\gamma,k}^- = \prod_{\substack{j_1 < m \leq n \\ m \neq j_2, \dots, j_{d_k}}} U_{-\epsilon_{j_1} + \epsilon_m, p_{j_1, m}^k} \prod_{\substack{j_2 < m \leq n \\ m \neq j_3, \dots, j_{d_k}}} U_{-\epsilon_{j_2} + \epsilon_m, p_{j_2, m}^k} \cdots \prod_{j_{d_k} < m \leq n} U_{-\epsilon_{j_{d_k}} + \epsilon_m, p_{j_{d_k}, m}^k}.$$

Note that $\mu_k^\vee \in H_{\epsilon_{j_t} - \epsilon_m, -p_{\ell, m}^k}$ if and only if $\mu_k^\vee + \epsilon_{j_1}^\vee + \dots + \epsilon_{j_{t-1}}^\vee \in H_{\epsilon_{j_t} - \epsilon_m, -p_{\ell, m}^k}$, and hence

$$\begin{aligned} & \mathbb{U}_{w_\gamma, (k, 0)}^- \cdots \mathbb{U}_{w_\gamma, (k, d_k - 1)}^- \\ &= \prod_{j_1 < m \leq n} U_{-\epsilon_{j_1} + \epsilon_m, p_{j_1, m}^k} \prod_{j_2 < m \leq n} U_{-\epsilon_{j_2} + \epsilon_m, p_{j_2, m}^k} \cdots \prod_{j_{d_k} < m \leq n} U_{-\epsilon_{j_{d_k}} + \epsilon_m, p_{j_{d_k}, m}^k}. \end{aligned}$$

It follows that $\mathbb{U}_{\gamma, k}^- \subseteq \mathbb{U}_{w_\gamma, (k, 0)}^- \cdots \mathbb{U}_{w_\gamma, (k, d_k - 1)}^- \subseteq U_{\mu_k}^-$ and hence by Corollary 3:

$$\begin{aligned} Z_\gamma = \mathbb{U}_{\gamma, 0}^- \mathbb{U}_{\gamma, 1}^- \cdots \mathbb{U}_{\gamma, r-1}^- \cdot \mu_r^\vee &\subseteq Z_{w_\gamma} = \mathbb{U}_{w_\gamma, (0, 0)}^- \mathbb{U}_{w_\gamma, (0, 1)}^- \cdots \mathbb{U}_{w_\gamma, (r-1, d_r - 1)}^- \cdot \mu_r^\vee \\ &\subseteq Z_\gamma = U_{\mu_0}^- U_{\mu_1}^- \cdots U_{\mu_{r-1}}^- \cdot \mu_r^\vee \end{aligned}$$

This shows that Z_γ depends only on the word w_γ , which proves the proposition. \square

13. PROOF OF THEOREM 2

We have seen that the image $\pi(C_\gamma)$ of a cell depends only on the associated word, to prove part *b*) of Theorem 2 it remains to show that the closure of the image depends only on the equivalence class of the associated word modulo the Knuth relation. We discuss first the simplest case at length. The proof in the general case is very similar, we leave the details to the reader. As in the section before, we view a word as a gallery of type $(1, 1, \dots, 1)$.

Lemma 4. *If $w_1 = yxz$ and $w_2 = yzx$, $x \leq y < z$, then*

$\pi(C_{w_1}) = \mathbb{U}_{w_1, 0}^- \mathbb{U}_{w_1, 1}^- \mathbb{U}_{w_1, 2}^- \cdot (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$ and $\pi(C_{w_2}) = \mathbb{U}_{w_2, 0}^- \mathbb{U}_{w_2, 1}^- \mathbb{U}_{w_2, 2}^- \cdot (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$ have a common dense subset.

Proof. Assume first $x < y < z$. The key associated to w_1 is $\mathcal{T}_1 = \boxed{z \mid x \mid y}$, let \mathcal{T}'_1 be the key $\boxed{x \mid y \mid z}$ having the same associated word. Set $\nu^\vee = (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$, then we have:

$$\begin{aligned} \pi(C_{w_1}) &= \mathbb{U}_{\gamma_{\mathcal{T}_1}}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma_{\mathcal{T}'_1}}^- \cdot \nu^\vee \\ &= \left(\prod_{\ell > y} U_{(-\epsilon_y + \epsilon_\ell, 0)} \right) \left(\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, 0)} \right) U_{(-\epsilon_x + \epsilon_y, -1)} \left(\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, 0)} \right) \cdot \nu^\vee \\ &= \left(\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, 0)} \right) U_{(-\epsilon_y + \epsilon_z, 0)} \left(\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, 0)} \right) U_{(-\epsilon_x + \epsilon_y, -1)} \\ &\quad \left(\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, 0)} \right) \cdot \nu^\vee \\ &= \left(\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, 0)} \right) \left(\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, 0)} \right) U_{(-\epsilon_y + \epsilon_z, 0)} U_{(-\epsilon_x + \epsilon_y, -1)} \\ &\quad \left(\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, 0)} \right) \cdot \nu^\vee \end{aligned}$$

The first equality is implied by Corollary 3, the second follows from Proposition 2, the third equation is just expressing $\mathbb{U}_{\gamma_{\mathcal{T}'_1}}$ as a product of the unipotent factors corresponding to the vertices of the gallery $\gamma_{\mathcal{T}'_1}$, the fourth equation is obtained by separating $U_{(-\epsilon_y + \epsilon_z, 0)}$ from the remaining factors of the product $\left(\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, 0)} \right)$, which is possible since all the

factors of this product commute. The last equality is obtained by switching the commuting factors $U_{(-\epsilon_y+\epsilon_z,0)}$ and $(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)})$.

The terms $\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}$ and $U_{(-\epsilon_x+\epsilon_y,-1)}$ commute, so we have

$$\pi(C_{w_1}) = \left(\prod_{\substack{\ell>y \\ \ell\neq z}} U_{(-\epsilon_y+\epsilon_\ell,0)}\right) \left(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)}\right) U_{(-\epsilon_y+\epsilon_z,0)} \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}\right) U_{(-\epsilon_x+\epsilon_y,-1)} \cdot \nu^\vee.$$

Recall that for $m > z$ and $s, t \in \mathbb{C}$ (see [19] or [20]):

$$U_{(-\epsilon_y+\epsilon_z,0)}(s)U_{(-\epsilon_z+\epsilon_m,0)}(t) = U_{(-\epsilon_y+\epsilon_m,0)}(st)U_{(-\epsilon_z+\epsilon_m,0)}(t)U_{(-\epsilon_y+\epsilon_z,0)}(s)$$

Now $U_{(-\epsilon_y+\epsilon_m,0)}$, $m > z$, commutes with $U_{(-\epsilon_x+\epsilon_k,0)}$, $k \neq y, z$, $U_{(-\epsilon_y+\epsilon_z,0)}$ and $U_{(-\epsilon_z+\epsilon_q,0)}$, $z < q$, so we can join the factor $U_{(-\epsilon_y+\epsilon_m,0)}(st)$ into the first product (where this unipotent subgroup occurs already with a free parameter) and obtain:

$$\pi(C_{w_1}) = \left(\prod_{\substack{\ell>y \\ \ell\neq z}} U_{(-\epsilon_y+\epsilon_\ell,0)}\right) \left(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)}\right) \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}\right) U_{(-\epsilon_y+\epsilon_z,0)} U_{(-\epsilon_x+\epsilon_y,-1)} \cdot \nu^\vee.$$

Now for $s, t \in \mathbb{C}$ we have

$$U_{(-\epsilon_y+\epsilon_z,0)}(s)U_{(-\epsilon_x+\epsilon_y,-1)}(t) = U_{(-\epsilon_x+\epsilon_y,-1)}(t)U_{(-\epsilon_x+\epsilon_z,-1)}(st)U_{(-\epsilon_y+\epsilon_z,0)}(s).$$

Since $U_{(-\epsilon_y+\epsilon_z,0)}(s) \subseteq P_{\nu^\vee}$ for all $s \in \mathbb{C}$, we can omit this term and see that the set

$$\left(\prod_{\substack{\ell>y \\ \ell\neq z}} U_{(-\epsilon_y+\epsilon_\ell,0)}\right) \left(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)}\right) \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}\right) U_{(-\epsilon_x+\epsilon_y,-1)}(*)U_{(-\epsilon_x+\epsilon_z,-1)}(**) \cdot \nu^\vee$$

for parameters $*, ** \neq 0$ forms a dense subset in $\pi(C_{w_1})$. After switching the commuting factors $(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)})$ and $(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)})$ we see that

$$\left(\prod_{\substack{\ell>y \\ \ell\neq z}} U_{(-\epsilon_y+\epsilon_\ell,0)}\right) \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}\right) \left(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)}\right) U_{(-\epsilon_x+\epsilon_y,-1)}(*)U_{(-\epsilon_x+\epsilon_z,-1)}(**) \cdot \nu^\vee$$

is also a dense subset in

$$\begin{aligned} \pi(C_{w_2}) &= \mathbb{U}_{\gamma_{\mathcal{T}_2}}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma_{\mathcal{T}'_2}}^- \cdot \nu^\vee \\ &= \left(\prod_{\substack{\ell>y \\ \ell\neq z}} U_{(-\epsilon_y+\epsilon_\ell,0)}\right) \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)}\right) \left(\prod_{\substack{k>x \\ k\neq y,z}} U_{(-\epsilon_x+\epsilon_k,0)}\right) \\ &\quad U_{(-\epsilon_x+\epsilon_y,-1)} U_{(-\epsilon_x+\epsilon_z,-1)} \cdot \nu^\vee, \end{aligned}$$

where $\mathcal{T}_2 = \begin{bmatrix} x & z & y \end{bmatrix}$ and \mathcal{T}'_2 is the key $\begin{bmatrix} x & y \\ z \end{bmatrix}$ having the same associated word, which finishes the proof in this case.

Assume now $x = y < z$. The key associated to the word $w_1 = xxz$ is a $\mathcal{T}_1 = \begin{bmatrix} z & x & x \\ & & \end{bmatrix}$, let \mathcal{T}'_1 be the key $\begin{bmatrix} x & x \\ z & \end{bmatrix}$. Set $\nu^\vee = 2\epsilon_x^\vee + \epsilon_z^\vee$. We get as image of the associated cell:

$$\pi(C_{w_1}) = \mathbb{U}_{\gamma_{\mathcal{T}_1}}^- \cdot \nu^\vee = \mathbb{U}_{\gamma_{\mathcal{T}'_1}}^- \cdot \nu^\vee = \prod_{k>x} U_{(-\epsilon_x+\epsilon_k,0)} \prod_{\substack{\ell>x \\ \ell \neq z}} U_{(-\epsilon_x+\epsilon_\ell,1)} \prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)} \cdot \nu^\vee.$$

The key associated to the word $w_2 = xzx$ is $\mathcal{T}_2 = \begin{bmatrix} x & z & x \\ & & \end{bmatrix}$ and we get as image of the associated cell

$$\begin{aligned} \pi(C_{w_2}) &= \mathbb{U}_{\mathcal{T}_2}^- \cdot \nu^\vee \\ &= \prod_{k>x} U_{(-\epsilon_x+\epsilon_k,0)} \prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)} \prod_{\ell>x} U_{(-\epsilon_x+\epsilon_\ell,1)} \cdot \nu^\vee \end{aligned}$$

But $U_{(-\epsilon_x+\epsilon_z,1)} \subset P_{\nu^\vee}$ (since $(2\epsilon_x^\vee + \epsilon_z^\vee, \epsilon_x - \epsilon_z) - 1 = 0$) and the subgroups $U_{(-\epsilon_x+\epsilon_\ell,1)}$ and $U_{(-\epsilon_z+\epsilon_m,0)}$ commute for $\ell > x, \ell \neq z$ and $m > z$, so

$$\pi(C_{w_2}) = \mathbb{U}_{\mathcal{T}_2}^- \cdot \nu^\vee = \prod_{k>x} U_{(-\epsilon_x+\epsilon_k,0)} \prod_{\substack{\ell>x \\ \ell \neq z}} U_{(-\epsilon_x+\epsilon_\ell,1)} \prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)} \cdot \nu^\vee = \pi(C_{w_1}),$$

which finishes the proof of the lemma. \square

Lemma 5. *If $w_1 = xzy$ and $w_2 = zxy$, $x < y \leq z$, then*

$\pi(C_{w_1}) = \mathbb{U}_{w_1,0}^- \mathbb{U}_{w_1,1}^- \mathbb{U}_{w_1,2}^- \cdot (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$ and $\pi(C_{w_2}) = \mathbb{U}_{w_2,0}^- \mathbb{U}_{w_2,1}^- \mathbb{U}_{w_2,2}^- \cdot (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$ have a common dense subset.

Proof. We only give a sketch of proof and leave the details to the reader. Assume first $x < y < z$. The key associated to w_1 is $\mathcal{T}_1 = \begin{bmatrix} y & z & x \\ & & \end{bmatrix}$, let \mathcal{T}'_1 be the key $\begin{bmatrix} y & x \\ & z \end{bmatrix}$ having the same associated word. Set $\nu^\vee = (\epsilon_x + \epsilon_y + \epsilon_z)^\vee$, then, as in the proof of Lemma 4, we have, on one side:

$$\begin{aligned} \pi(C_{w_1}) &= \mathbb{U}_{\gamma_{\mathcal{T}_1}}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma_{\mathcal{T}'_1}}^- \cdot \nu^\vee \\ &= \left(\prod_{\substack{\ell>x \\ \ell \neq z}} U_{(-\epsilon_x+\epsilon_\ell,0)} \right) \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)} \right) \left(\prod_{\substack{k>y \\ k \neq z}} U_{(-\epsilon_y+\epsilon_k,0)} \right) U_{(-\epsilon_y+\epsilon_z,-1)} \cdot \nu^\vee \end{aligned}$$

On the other side, the key associated to w_2 is $\mathcal{T}_2 = \begin{bmatrix} y & x & z \\ & & \end{bmatrix}$. Let \mathcal{T}'_2 be the key $\begin{bmatrix} x & z \\ y & \end{bmatrix}$ having the same associated word. Then, we have:

$$\begin{aligned} \pi(C_{w_2}) &= \mathbb{U}_{\gamma_{\mathcal{T}_2}}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma_{\mathcal{T}'_2}}^- \cdot \nu^\vee \\ &= \left(\prod_{m>z} U_{(-\epsilon_z+\epsilon_m,0)} \right) \left(\prod_{\substack{\ell>x \\ \ell \neq z, y}} U_{(-\epsilon_x+\epsilon_\ell,0)} \right) U_{(-\epsilon_x+\epsilon_z,-1)} \\ &\quad \left(\prod_{\substack{k>y \\ k \neq z}} U_{(-\epsilon_y+\epsilon_k,0)} \right) U_{(-\epsilon_y+\epsilon_k,0)} \cdot \nu^\vee \end{aligned}$$

The same kind of computations as in the proof of Lemma 4 shows that those two images share a common dense subset. Now assume that $x < y = z$. Again, using an analogous

calculation with the keys $\mathcal{T}'_1 = \begin{array}{|c|c|} \hline y & x \\ \hline \hline & y \\ \hline \end{array}$ and $\mathcal{T}'_2 = \begin{array}{|c|c|} \hline x & y \\ \hline \hline y & \\ \hline \end{array}$, one can show that $\pi(C_{w_1})$ and $\pi(C_{w_2})$ have a common dense subset. \square

Let now w_1, w_2 be words such that

$$w_1 = (i_1, \dots, i_r, y, x, z, i_{r+4}, \dots, i_N) \quad w_2 = (i_1, \dots, i_r, y, z, x, i_{r+4}, \dots, i_N),$$

where $x \leq y < z$, and let $C_{w_1}, C_{w_2} \subset \Sigma(\underbrace{1, \dots, 1}_N)$ be the associated cells. Denote by

$$\pi : \Sigma(1, \dots, 1) \rightarrow X_{N\epsilon_1}$$

the desingularization map.

Lemma 6. $\overline{\pi(C_{w_1})} = \overline{\pi(C_{w_2})}$.

Proof. Assume first $x < y < z$. Let γ_1 be the gallery of shape $(1, \dots, 1)$ associated to w_1 and with key tableau $\begin{array}{|c|c|c|c|} \hline i_N & \dots & i_{r+4} & i_1 \\ \hline \end{array}$. Let γ'_1 be the gallery corresponding to the key tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline i_N & \dots & i_{r+4} & x & y & i_r & \dots & i_1 \\ \hline & & & z & & & & \\ \hline \end{array}$$

Let $\varphi(\gamma'_1) = (\mu_0^\vee = 0, \mu_1^\vee, \dots, \mu_{N-1}^\vee)$ be the associated polyline. Fix $p_{x,k}^r, p_{y,\ell}^r, p_{z,m}^r \in \mathbb{Z}$ such that $\mu_r^\vee \in H_{(\epsilon_x - \epsilon_k, -p_{x,k}^r)}$, $\mu_r^\vee \in H_{(\epsilon_y - \epsilon_\ell, -p_{y,\ell}^r)}$ and $\mu_r^\vee \in H_{(\epsilon_z - \epsilon_m, -p_{z,m}^r)}$. Recall $\mu_{r+1}^\vee = \mu_r^\vee + \epsilon_y$, so $\mu_{r+1}^\vee \in H_{(\epsilon_x - \epsilon_k, -p_{x,k}^r)}$ for $k > x$, $k \neq y$, $\mu_{r+1}^\vee \in H_{(\epsilon_z - \epsilon_m, -p_{z,m}^r)}$ for $m > z$ and $\mu_{r+1}^\vee \in H_{(\epsilon_x - \epsilon_y, -p_{x,y}^r + 1)}$. Using Corollary 3 and Proposition 3 we get:

$$\begin{aligned} \pi(C_{w_1}) &= \mathbb{U}_{\gamma_1}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma'_1}^- \cdot \nu^\vee \\ &= U_{\mu_0^\vee}^- \cdots U_{\mu_{r-1}^\vee}^- (\prod_{\ell > y} U_{(-\epsilon_y + \epsilon_\ell, p_{y,\ell}^r)}) (\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, p_{x,k}^r)}) U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)} \\ &\quad (\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)}) U_{\mu_{r+2}^\vee}^- \cdots U_{\mu_{N-2}^\vee}^- \nu^\vee \\ &= U_{\mu_0^\vee}^- \cdots U_{\mu_{r-1}^\vee}^- (\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, p_{y,\ell}^r)}) (\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, p_{x,k}^r)}) U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)} \\ &\quad (\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, -p_{z,m}^r)}) U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)} U_{\mu_{r+2}^\vee}^- \cdots U_{\mu_{N-2}^\vee}^- \nu^\vee, \end{aligned}$$

here we use the same arguments (switching commuting subgroups) as in the proof of Lemma 4. Recall that for $m > z$ and $s, t \in \mathbb{C}$ (see [19] or [20]):

$$\begin{aligned} U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)}(s) U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)}(t) \\ = U_{(-\epsilon_y + \epsilon_m, p_{y,m}^r)}(st) U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)}(t) U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)}(s). \end{aligned}$$

As in the proof of Lemma 4, after switching and gathering the root subgroups we get:

$$\begin{aligned} \pi(C_{w_1}) &= U_{\mu_0^\vee}^- \cdots U_{\mu_{r-1}^\vee}^- (\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, p_{y,\ell}^r)}) (\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, p_{x,k}^r)}) \\ &\quad (\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)}) U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)} U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)} U_{\mu_{r+2}^\vee}^- \cdots U_{\mu_{N-2}^\vee}^- \nu^\vee. \end{aligned}$$

Now for $s, t \in \mathbb{C}$ we have

$$U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)}(s) U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)}(t) = U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)}(t) U_{(-\epsilon_x + \epsilon_z, p_{x,z}^r - 1)}(st) U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)}(s).$$

Since $U_{(-\epsilon_y + \epsilon_z, p_{y,z}^r)}(s) \subseteq U_{\mu_{r+2}^\vee}^-$ for all $s \in \mathbb{C}$ (recall that $\mu_{r+2}^\vee = \mu_r^\vee + \epsilon_x^\vee + \epsilon_y^\vee + \epsilon_z^\vee$), we can omit this term and we see (after switching commuting terms as in the proof of Lemma 4) that the set (with parameters $*, ** \neq 0$)

(12)

$$\mathfrak{S} = U_{\mu_0^\vee}^- \cdots U_{\mu_{r-1}^\vee}^- \left(\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, p_{y,\ell}^r)} \right) \left(\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)} \right) \left(\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, p_{x,k}^r)} \right) U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)}(*) U_{(-\epsilon_x + \epsilon_z, p_{x,z}^r - 1)}(**) U_{\mu_{r+2}^\vee}^- \cdots U_{\mu_{N-2}^\vee}^- \nu^\vee$$

is a dense subset in $\pi(C_{w_1})$. Now let γ_2 be the gallery of shape $(1, \dots, 1)$ associated to w_2 and let γ_2' be the gallery corresponding to the key tableau

$$\begin{array}{ccccccccccc} \boxed{i_N} & \cdot & \cdot & \cdot & \boxed{i_{t+4}} & x & y & i_r & \cdot & \cdot & \cdot & \boxed{i_1} \\ & & & & & & & z & & & & \end{array}$$

We get

$$\begin{aligned} \pi(C_{w_2}) &= \mathbb{U}_{\gamma_2}^- \cdot \nu^\vee \\ &= \mathbb{U}_{\gamma_2'}^- \cdot \nu^\vee \\ &= U_{\mu_0^\vee}^- \cdots U_{\mu_{r-1}^\vee}^- \left(\prod_{\substack{\ell > y \\ \ell \neq z}} U_{(-\epsilon_y + \epsilon_\ell, p_{y,\ell}^r)} \right) \left(\prod_{m > z} U_{(-\epsilon_z + \epsilon_m, p_{z,m}^r)} \right) \\ &\quad \left(\prod_{\substack{k > x \\ k \neq y, z}} U_{(-\epsilon_x + \epsilon_k, p_{x,k}^r)} \right) U_{(-\epsilon_x + \epsilon_y, p_{x,y}^r - 1)} U_{(-\epsilon_x + \epsilon_z, p_{x,z}^r - 1)} U_{\mu_{r+2}^\vee}^- \cdots U_{\mu_{N-2}^\vee}^- \nu^\vee. \end{aligned}$$

It follows that the set \mathfrak{S} in (12) is a common dense subset of $\pi(C_{w_1})$ and $\pi(C_{w_2})$, proving the lemma in this case.

The arguments in the case $x = y < z$ used in Lemma 4 are modified in the same way to prove that in this case we have $\pi(C_{w_1}) = \pi(C_{w_2})$. \square

It remains to consider the second Knuth relation. Let now w_1, w_2 be words such that

$$w_1 = (i_1, \dots, i_{s-1}, x, z, y, i_{s+3}, \dots, i_N) \quad w_2 = (i_1, \dots, i_{s-1}, z, x, y, i_{s+3}, \dots, i_N),$$

where $x < y \leq z$, and let $C_{w_1}, C_{w_2} \subset \Sigma_{(1, \dots, 1)}$ be the associated cells. The proof of the corresponding version of Lemma 6 for the Knuth relation $w_1 \sim_K w_2$ is nearly identical to the proof of the lemma above and is left to the reader. We just state the corresponding version of Lemma 6

Lemma 7. $\overline{\pi(C_{w_1})} = \overline{\pi(C_{w_2})}$.

Proof of Theorem 2. The direction $i) \Rightarrow ii)$ of the equivalence in Theorem 2b) follows from Lemma 6 and Lemma 7. It was proved in [4] (in a more general context) that if \mathcal{T} is a semistandard Young tableaux, then $\overline{\pi(C_{\gamma_{\mathcal{T}}})}$ is a MV-cycle in X_{ν^\vee} of coweight μ^\vee , where μ^\vee is the coweight of the gallery $\gamma_{\mathcal{T}}$, $\mathbf{d} = (d_1, \dots, d_r)$ is the shape of \mathcal{T} and $\nu^\vee = \sum_{j=1}^s \omega_{d_j}^\vee$. Moreover, the map $\mathcal{T} \rightarrow \overline{\pi(C_{\gamma_{\mathcal{T}}})}$ induces a bijection between the semistandard Young

tableaux of shape \mathbf{d} and coweight μ^\vee and the MV-cycles in X_{ν^\vee} of coweight μ^\vee . Since an arbitrary gallery is Knuth equivalent to a unique semistandard Young tableau, this proves the direction $i) \Leftarrow ii)$ of the equivalence and also part $a)$ of the theorem.

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