

# THE SQUARE ROOT PROBLEM FOR SECOND ORDER, DIVERGENCE FORM OPERATORS WITH MIXED BOUNDARY CONDITIONS ON $L^p$

PASCAL AUSCHER, NADINE BADR, ROBERT HALLER-DINTELMANN, AND JOACHIM REHBERG

ABSTRACT. We show that, under very general conditions on the domain  $\Omega$  and the Dirichlet part  $D$  of the boundary, the operator  $(-\nabla \cdot \mu \nabla + 1)^{1/2}$  with mixed boundary conditions provides a topological isomorphism between  $W_D^{1,p}(\Omega)$  and  $L^p(\Omega)$ , if  $p \in ]1, 2]$ .

## 1. INTRODUCTION

The main purpose of this paper is to identify the domain of the square root of a divergence form operator  $-\nabla \cdot \mu \nabla + 1$  on  $L^p(\Omega)$  as a Sobolev space  $W_D^{1,p}(\Omega)$  of differentiability order 1 for  $p \in ]1, 2]$ . (The subscript  $D$  indicates the subspace of  $W^{1,p}(\Omega)$  whose elements vanish on the boundary part  $D$ .) Our focus lies on non-smooth geometric situations in  $\mathbb{R}^d$ . So, we allow for mixed boundary conditions and, additionally, deviate from the Lipschitz property of the domain  $\Omega$  in the following spirit: the boundary  $\partial\Omega$  decomposes into a closed subset  $D$  (the Dirichlet part) and its complement, which may share a common frontier within  $\partial\Omega$ . Concerning  $D$ , we only demand that it satisfies the well-known Ahlfors-David condition (equivalently: is a  $(d-1)$ -set in the sense of Jonsson/Wallin [35, II.1]), and only for points from the complement we demand bi-Lipschitzian charts around. As special cases, the pure Dirichlet ( $D = \partial\Omega$ ) and pure Neumann case ( $D = \emptyset$ ) are also included in our considerations. Finally the coefficient function  $\mu$  is just supposed to be real, measurable, bounded and elliptic in general, cf. Assumption 4.2. Together, this setting should cover nearly everything that occurs in real-world problems – as long as the domain does not have irregularities like cracks meeting the Neumann boundary part  $\partial\Omega \setminus D$ .

The identification of the domain for fractional powers of elliptic operators, in particular that of square roots, has a long history. Concerning Kato's square root problem – in the Hilbert space  $L^2$  – see e.g. [9], [23], [6] (here only the non-selfadjoint case is of interest). Early efforts, devoted to the determination of domains for fractional powers in the non-Hilbert space case seem to culminate in [47]. In recent years the problem has been investigated in the case of  $L^p$  ( $p \neq 2$ ) for instance in [5], [7], [34], [32], [8]; but only the last two are dedicated to the case  $\Omega \neq \mathbb{R}^d$ . In [8] the domain is a strong Lipschitz domain and the boundary conditions are either pure Dirichlet or pure Neumann. Our result generalizes this to a large extent and, at the same time, gives a new proof for these special cases, using more 'global' arguments. Since, in the case of a non-symmetric coefficient function  $\mu$ , for the nonsmooth constellations described above no general condition is known that assures  $(-\nabla \cdot \mu \nabla + 1)^{1/2} : W_D^{1,2}(\Omega) \rightarrow L^2(\Omega)$  to be an isomorphism, this is supposed as one of our assumptions. This serves then as our starting point to show the corresponding isomorphism property of  $(-\nabla \cdot \mu \nabla + 1)^{1/2} : W_D^{1,p}(\Omega) \rightarrow L^p(\Omega)$  for  $p \in ]1, 2]$ .

While this is already interesting in itself, our original motivation comes from the applications: having the isomorphism  $(-\nabla \cdot \mu \nabla + 1)^{1/2} : W_D^{1,p}(\Omega) \rightarrow L^p(\Omega)$  at hand, the adjoint isomorphism  $((-\nabla \cdot \mu \nabla + 1)^{1/2})^* = (-\nabla \cdot \mu^T \nabla + 1)^{1/2} : L^q(\Omega) \rightarrow W_D^{-1,q}(\Omega)$  allows to carry over substantial properties of the operators  $-\nabla \cdot \mu \nabla$  on the  $L^p$ -scale to the scale of  $W_D^{-1,q}$ -spaces for  $q \in [2, \infty[$ .

---

1991 *Mathematics Subject Classification*. Primary: 35J15, 42B20, 47B44; Secondary: 26D15, 46B70, 35K20.

*Key words and phrases*. Kato's square root problem, Elliptic operators with bounded measurable coefficients, Interpolation in case of mixed boundary values, Hardy's inequality, Calderon-Zygmund decomposition.

In particular, this concerns the  $\mathcal{H}^\infty$ -calculus and maximal parabolic regularity, see Section 11, which in turn is a powerful tool for the treatment of linear and nonlinear parabolic equations, see e.g. [46] and [30].

The paper is organized as follows: after presenting some notation and general assumptions in Section 2, in Section 3 we introduce the Sobolev scale  $W_D^{1,p}(\Omega)$ ,  $1 \leq p \leq \infty$ , related to mixed boundary conditions and point out some of their properties. In Section 4 we define properly the elliptic operator under consideration and collect some known facts for it. The main result on the isomorphism property for the square root of the elliptic operator is precisely formulated in Section 5. The following sections contain preparatory material for the proof of the main result, which is finished at the end of Section 10. Some of these results have their own interest, such as Hardy's inequality for mixed boundary conditions that is proved in Section 6 and the results on real and complex interpolation for the spaces  $W_D^{1,p}(\Omega)$ ,  $1 \leq p \leq \infty$ , from Section 8, so we shortly want to comment on these.

Our proof of Hardy's inequality heavily rests on two things: first one uses an operator that extends functions from  $W_D^{1,p}(\Omega)$  to  $W_0^{1,p}(\Omega_\bullet)$ , where  $\Omega_\bullet$  is a domain containing  $\Omega$ . Then one is in a situation where the deep results of Ancona [2], Lewis [41] and Wannebo [51], combined with Lehrbäck's [40] ingenious characterization of  $p$ -fatness, may be applied.

The proof of the interpolation results, as well as other steps in the proof of the main result, are fundamentally based on an adapted Calderón-Zygmund decomposition for Sobolev functions. Such a decomposition was first introduced in [5] and has also successfully been used in [10], see also [11]. We have to modify it, since the main point here is, that the decomposition has to respect the boundary conditions. This is accomplished by incorporating Hardy's inequality into the controlling maximal operator. This result, which is at the heart of our considerations, is contained in Section 7.

All these preparations, together with off-diagonal estimates for the semigroup generated by our operator, cf. Section 9, lead to the proof of the main result in Section 10. Finally, in Section 11 we draw some consequences, as already sketched above.

**Acknowledgments.** The authors want to thank A. Ancona, P. Koskela, V. Maz'ya and W. Treibels for valuable discussions and hints on the topic.

## 2. NOTATION AND GENERAL ASSUMPTIONS

Throughout the paper we will use  $x, y, \dots$  for vectors in  $\mathbb{R}^d$  and the symbol  $B(x, r)$  stands for the ball in  $\mathbb{R}^d$  around  $x$  with radius  $r$ . For  $E, F \subseteq \mathbb{R}^d$  we denote by  $d(E, F)$  the distance between  $E$  and  $F$  and if  $E = \{x\}$ , then we write  $d(x, F)$  or  $d_F(x)$  instead.

Regarding our geometric setting, we suppose the following assumption throughout this work.

**Assumption 2.1.** (i)  $\Omega \subseteq \mathbb{R}^d$  is a bounded domain and  $D$  is a closed subset of the boundary  $\partial\Omega$  (to be understood as the Dirichlet boundary part). For every  $x \in \overline{\partial\Omega} \setminus D$  there exists an open neighbourhood  $U_x$  of  $x$  and a bi-Lipschitz map  $\phi_x$  from  $U_x$  onto the cube  $K := ]-1, 1[^d$ , such that the following three conditions are satisfied:

$$(2.1) \quad \begin{aligned} \phi_x(x) &= 0, \\ \phi_x(U_x \cap \Omega) &= \{x \in K : x_d < 0\} =: K_-, \\ \phi_x(U_x \cap \partial\Omega) &= \{x \in K : x_d = 0\} =: \Sigma. \end{aligned}$$

(ii) We suppose that  $D$  is either empty or satisfies the Ahlfors-David condition: There are constants  $c_0, c_1 > 0$  and  $r_{AD} > 0$ , such that for all  $x \in D$  and all  $r \in ]0, r_{AD}]$

$$(2.2) \quad c_0 r^{d-1} \leq \mathcal{H}_{d-1}(D \cap B(x, r)) \leq c_1 r^{d-1},$$

where  $\mathcal{H}_{d-1}$  denotes (here and in the sequel) the  $(d-1)$ -dimensional Hausdorff measure.

- Remark 2.2.** (i) Condition (2.2) means that  $D$  is a  $(d - 1)$ -set in the sense of Jons-son/Wallin [35, Ch. II].
- (ii) On the set  $\partial\Omega \cap (\bigcup_{x \in \partial\Omega \setminus D} U_x)$  the measure  $\mathcal{H}_{d-1}$  equals the surface measure  $\sigma$  which can be constructed via the bi-Lipschitzian charts  $\phi_x$  around these boundary points, compare [24, Section 3.3.4 C] or [31, Section 3]. In particular, (2.2) assures the property  $\sigma(\partial\Omega \cap (\bigcup_{x \in \partial\Omega \setminus D} U_x)) > 0$ .
- (iii) We emphasize that the cases  $D = \partial\Omega$  or  $D = \emptyset$  are not excluded.

**Assumption 2.3.** *In case of  $D \neq \partial\Omega$  we additionally require that there is a bounded domain  $\Omega_\star \supseteq \Omega$  and constants  $c_\star, r_\star > 0$ , such that*

- (i)  $\partial\Omega_\star$  satisfies the condition
- (2.3) 
$$c_\star r^{d-1} \leq \mathcal{H}_{d-1}(\partial\Omega_\star \cap B(x, r)), \quad x \in \partial\Omega_\star, r \in ]0, r_\star].$$
- (ii)  $\Omega_\bullet := \Omega_\star \setminus D$  is connected, and, hence, again a domain.
- (iii)  $\overline{\Omega}$  and  $\partial\Omega_\star \setminus D$  have positive distance to each other.

**Remark 2.4.** Let us comment on Assumption 2.3 since its context becomes clear only in Section 6.

- (i) It is established in order to get Hardy's inequality for elements from  $W_D^{1,p}(\Omega)$ . This is achieved via an extension operator from  $W_D^{1,p}(\Omega)$  to  $W_0^{1,p}(\Omega_\bullet)$ , and the validity of Hardy's inequality for functions from  $W_0^{1,p}(\Omega_\bullet)$  – which is to be proved.
- If there is an open ball  $B \supseteq \overline{\Omega}$ , such that  $B \setminus D$  is connected, then one can put  $\Omega_\star := B$ , and Assumption 2.3 is satisfied. But this is not always the case, as the following examples shows: take  $\Omega = \{x : 1 < |x| < 2\}$  and  $D = \{x : |x| = 1\} \cup \{x : |x| = 2, x_1 \geq 0\}$ . Obviously, if  $\Omega_\star$  is open and contains  $\Omega \cup D$ , then  $\Omega_\star \setminus D$  cannot be connected. In this case one can take  $\Omega_\star := \{x : 1 < |x| < 3\}$  instead. This suggest already the general procedure: if a connected component of  $\partial\Omega$  consists only of points from  $D$ , then  $\Omega$  should *not* be extended across this boundary part. In the opposite case it should.
- (ii) It is not hard to see that condition (iii) of the assumption implies  $\partial\Omega \setminus D \subseteq \Omega_\star$ . It seems that even every connected component of  $\partial\Omega$  which contains a point from  $\partial\Omega \setminus D$  must in total be contained in  $\Omega_\star$ . Since this is nowhere needed in our further considerations we consider this as a heuristics and do not prove it.
- (iii) We emphasize that – more or less – all constellations, relevant for applications, are covered by Assumptions 2.1 and 2.3.

If  $B$  is a closed operator on a Banach space  $X$ , then we denote by  $\text{dom}_X(B)$  the domain of this operator.  $\mathcal{L}(X, Y)$  denotes the space of linear, continuous operators from  $X$  into  $Y$ ; if  $X = Y$ , then we abbreviate  $\mathcal{L}(X)$ . Furthermore, we will write  $\langle \cdot, \cdot \rangle_{X'}$  for the pairing of elements of  $X$  and the dual space  $X'$  of  $X$ .

Finally, the letters  $c$  and  $C$  denote generic constants that may change value from occurrence to occurrence.

### 3. SOBOLEV SPACES RELATED TO BOUNDARY CONDITIONS

In this section we will introduce the Sobolev spaces related to mixed boundary conditions and prove some results related to them that will be needed later.

If  $\Upsilon$  is an open subset of  $\mathbb{R}^d$  and  $F$  a closed subset of  $\overline{\Upsilon}$ , e.g. the Dirichlet part  $D$  of  $\partial\Omega$ , for  $1 \leq q < \infty$  we define  $W_F^{1,q}(\Upsilon)$  as the completion of

$$(3.1) \quad C_F^\infty(\Upsilon) := \{\psi|_\Upsilon : \psi \in C^\infty(\mathbb{R}^d), \text{supp}(\psi) \cap F = \emptyset\}$$

with respect to the norm  $\psi \mapsto (\int_\Upsilon |\nabla\psi|^q + |\psi|^q \, dx)^{1/q}$ . For  $1 < q < \infty$  the dual of this space will be denoted by  $W_F^{-1,q'}(\Upsilon)$  with  $1/q + 1/q' = 1$ . Here, the dual is to be understood with respect

to the extended  $L^2$  scalar product, or, in other words:  $W_F^{-1,q'}(\Upsilon)$  is the space of continuous antilinear forms on  $W_F^{1,q}(\Upsilon)$ .

If misunderstandings are not to be expected, we drop the  $\Omega$  in the notation of spaces, i.e. function spaces without an explicitly given domain are to be understood as function spaces on  $\Omega$ .

**Remark 3.1.** The space  $W^{1,q}(\Omega)$  admits a continuous trace operator into the space  $L^q(D; \mathcal{H}_{d-1})$  for all  $1 \leq q < \infty$ , cf. [35, Ch. V]. Hence, the functions  $f \in W_D^{1,q}(\Omega)$  satisfy  $f|_D = 0$   $\mathcal{H}_{d-1}$ -a.e.

Finally, we define the respective spaces for the case  $q = \infty$ . We set  $W_F^{1,\infty}(\Upsilon) := \text{Lip}_{\infty,F}(\Upsilon)$  with

$$(3.2) \quad \text{Lip}_{\infty,F}(\Upsilon) := \{f|_{\Upsilon} : f \in (L^\infty \cap \text{Lip})(\mathbb{R}^d), f|_F = 0\} = \{f \in (L^\infty \cap \text{Lip})(\Upsilon), f|_F = 0\}.$$

The norm on this space is

$$\|f\|_{L^\infty(\Upsilon)} + \sup_{x,y \in \Upsilon, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

The last equality in (3.2) is a consequence of the Whitney extension theorem. We have  $\text{Lip}_{\infty,F}(\Upsilon) \subseteq \{f \in \mathcal{W}^{1,\infty}(\Upsilon) : f|_F = 0\}$  ( $\mathcal{W}^{1,\infty}(\Upsilon)$  is defined using distributions) and the converse holds iff  $\Omega$  is uniformly locally convex by [27, Theorem 7].

**Lemma 3.2.** *Let  $\Upsilon \subseteq \mathbb{R}^d$  be a bounded domain and  $F$  a (relatively) closed subset of  $\partial\Upsilon$ . Then  $W_F^{1,\infty}(\Upsilon) \subseteq W_F^{1,q}(\Upsilon)$  for  $1 \leq q < \infty$ .*

*Proof.* Let  $(\alpha_n)_n$  be the sequence of cut-off functions defined on  $\mathbb{R}^+$  by

$$\alpha_n(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1/n, \\ nt - 1, & \text{if } 1/n \leq t \leq 2/n, \\ 1, & \text{if } t > 2/n. \end{cases}$$

Remark that for  $t \neq 1$  the sequence  $\alpha_n(t)$  tends to 1 as  $n \rightarrow \infty$ . Furthermore, for all  $t \geq 0$  we have  $0 \leq t\alpha'_n(t) \leq 2$  and the sequence  $(t\alpha'_n(t))_n$  tends to 0.

For  $x \in \mathbb{R}^d$  we set  $w_n(x) := \alpha_n(d(x, F))$ . Then, by the above considerations,  $w_n \rightarrow 1$  almost everywhere as  $n \rightarrow \infty$  and

$$|\nabla w_n(x)| = |\alpha'_n(d(x, F))| |\nabla d(x, F)| \leq |\alpha'_n(d(x, F))|$$

almost everywhere. Thus  $d(x, F)|\nabla w_n(x)|$  is bounded and converges to 0 almost everywhere as  $n \rightarrow \infty$ .

Let  $g \in W_F^{1,\infty}(\Upsilon)$ , which we consider as defined on  $\mathbb{R}^d$ . Since  $\Upsilon$  is bounded, we may assume that  $g$  has compact support in some large ball  $B$ . Let  $g_n := gw_n$ . Then  $g_n$  is compactly supported in  $B$  and in  $\mathbb{R}^d \setminus F$ . We claim that  $g_n \rightarrow g$  in  $W^{1,q}(\mathbb{R}^d)$ . Indeed,  $g - g_n = g(1 - w_n)$  and, by the dominated convergence theorem,  $g(1 - w_n) \rightarrow 0$  in  $L^q(\mathbb{R}^d)$ , since  $w_n \rightarrow 1$ .

Now, for the gradient, we have

$$\nabla g_n - \nabla g = (1 - w_n)\nabla g + g\nabla w_n.$$

Again by the dominated convergence theorem, the first term converges to 0 in  $L^q(\mathbb{R}^d)$ .

It remains to prove that  $\|g\nabla w_n\|_{L^q(\mathbb{R}^d)}$  converges to 0. We have for  $x \in \mathbb{R}^d$

$$(3.3) \quad (g\nabla w_n)(x) = \frac{g(x)}{d(x, F)} d(x, F) \nabla w_n(x).$$

Since  $g$  is Lipschitz continuous on the whole of  $\mathbb{R}^d$  and satisfies  $g = 0$  on  $F$ , we find

$$\sup_{x \in \mathbb{R}^d} \left| \frac{g(x)}{d(x, F)} \right| = \sup_{x \in \mathbb{R}^d} \left| \frac{g(x) - g(x_*)}{x - x_*} \right| \leq C,$$

where  $x_* \in F$  denotes an element of  $F$  that realizes the distance of  $x$  to  $F$ . So both factors on the right hand side in (3.3) are bounded and  $d(x, F)\nabla w_n(x)$  goes to 0 almost everywhere as  $n \rightarrow \infty$ . Thus, since  $g$  has compact support, the dominated convergence theorem yields  $g\nabla w_n \rightarrow 0$  in  $L^q(\mathbb{R}^d)$ .

Finally, it suffices to convolve this approximation with a smooth mollifying function that has small support to conclude  $g \in W_F^{1,q}(\Upsilon)$ .  $\square$

Next, we establish the following extension property for function spaces on domains, satisfying just part (i) of Assumption 2.1. This has been proved in [22] for  $q = 2$ . For convenience of the reader we include a proof.

**Lemma 3.3.** *Let  $\Omega$  and  $D$  satisfy Assumption 2.1 (i). Then there is a continuous extension operator  $\mathfrak{E}$  which maps each space  $W_D^{1,q}(\Omega)$  continuously into  $W_D^{1,q}(\mathbb{R}^d)$ ,  $q \in [1, \infty]$ . Moreover,  $\mathfrak{E}$  maps  $L^q(\Omega)$  continuously into  $L^q(\mathbb{R}^d)$  for  $q \in [1, \infty]$ .*

*Proof.* Let, for every  $x \in \overline{\partial\Omega} \setminus \overline{D}$  the set  $U_x$  be an open neighbourhood that satisfies the condition from Assumption 2.1 (i). Let  $U_{x_1}, \dots, U_{x_\ell}$  be a finite subcovering of  $\overline{\partial\Omega} \setminus \overline{D}$  and let  $\eta \in C_0^\infty(\mathbb{R}^d)$  be a function that is identically one in a neighbourhood of  $\overline{\partial\Omega} \setminus \overline{D}$  and has its support in  $U := \bigcup_{j=1}^\ell U_{x_j}$ .

Assume  $\psi \in C_D^\infty(\Omega)$ ; then we can write  $\psi = \eta\psi + (1 - \eta)\psi$ . By the definition of  $C_D^\infty(\Omega)$  and  $\eta$  it is clear that the support of  $(1 - \eta)\psi$  is contained in  $\Omega$ , thus this function may be extended by 0 to the whole space  $\mathbb{R}^d$  – while its  $W^{1,q}$ -norm is preserved.

It remains to define the extension of the function  $\eta\psi$ , what we will do now. For this, let  $\eta_1, \dots, \eta_\ell$  be a partition of unity on  $\text{supp}(\eta)$ , subordinated to the covering  $U_{x_1}, \dots, U_{x_\ell}$ . Then we can write  $\eta\psi = \sum_{r=1}^\ell \eta_r \eta\psi$  and have to define an extension for every function  $\eta_r \eta\psi$ . For doing so, we first transform the corresponding function under the corresponding mapping  $\phi_{x_r}$  from Assumption 2.1 (i) to  $\widetilde{\eta_r \eta\psi} = (\eta_r \eta\psi) \circ \phi_{x_r}^{-1}$  on the half cube  $K_-$ . Afterwards, by even reflection, one obtains a function  $\widehat{\eta_r \eta\psi} \in W^{1,q}(K)$  on the cube  $K$ . It is clear by construction that  $\text{supp}(\widehat{\eta_r \eta\psi})$  has a positive distance to  $\partial K$ . Transforming back, one ends up with a function  $\underline{\eta_r \eta\psi} \in W^{1,q}(U_{x_r})$  whose support has a positive distance to  $\partial U_{x_r}$ . Thus, this function may also be extended by 0 to the whole of  $\mathbb{R}^d$ , preserving again the  $W^{1,q}$  norm.

Lastly, one observes that all the mappings  $W^{1,q}(U_{x_r} \cap \Omega) \ni \eta_r \eta\psi \mapsto \widetilde{\eta_r \eta\psi} \in W^{1,q}(K_-)$ ,  $W^{1,q}(K_-) \ni \widetilde{\eta_r \eta\psi} \mapsto \widehat{\eta_r \eta\psi} \in W^{1,q}(K)$  and  $W^{1,q}(K) \ni \widehat{\eta_r \eta\psi} \mapsto \underline{\eta_r \eta\psi} \in W^{1,q}(U_{x_r})$  are continuous. Thus, adding up, one arrives at an extension of  $\psi$  whose  $W^{1,q}(\mathbb{R}^d)$ -norm may be estimated by  $c\|\psi\|_{W^{1,q}(\Omega)}$  with  $c$  independent from  $\psi$ . Hence, the mapping  $\mathfrak{E}$ , up to now defined on  $C_D^\infty(\Omega)$ , continuously and uniquely extends to a mapping from  $W_D^{1,q}$  to  $W^{1,q}(\mathbb{R}^d)$ .

It remains to show that the images in fact even lie in  $W_D^{1,q}(\mathbb{R}^d)$ . For doing so, one first observes that, by construction of the extension operator, for any  $\psi \in C_D^\infty(\Omega)$ , the support of the extended function  $\mathfrak{E}\psi$  has a positive distance to  $D$  – but  $\mathfrak{E}\psi$  need not be smooth. Clearly, one may convolve  $\mathfrak{E}\psi$  suitably in order to obtain an appropriate approximation in the  $W^{1,q}(\mathbb{R}^d)$ -norm – maintaining a positive distance of the support to the set  $D$ . Thus,  $\mathfrak{E}$  maps  $C_D^\infty(\Omega)$  continuously into  $W_D^{1,q}(\mathbb{R}^d)$ , what is also true for its continuous extension to the whole space  $W_D^{1,q}(\Omega)$ .

It is not hard to see that the operator  $\mathfrak{E}$  extends to a continuous operator from  $L^q(\Omega)$  to  $L^q(\mathbb{R}^d)$ , where  $q \in [1, \infty]$ .  $\square$

**Remark 3.4.** (i) By construction, all extended functions  $\mathfrak{E}f$  have their support in  $\Omega \cup \bigcup_{j=1}^\ell U_{x_j}$ , and, hence, in a suitably large ball.  
(ii) Employing Lemma 3.3 in conjunction with (i), one can establish the corresponding Sobolev embeddings  $W_D^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  (compactness included) in a straightforward manner.

- (iii) When combining  $\mathfrak{E}$  with a multiplication operator that is induced by a function  $\eta_0 \in C_0^\infty(\mathbb{R}^d)$ ,  $\eta_0 \equiv 1$  on  $\Omega$ , one may achieve that the support of the extended functions shrinks to a set which is arbitrarily close to  $\Omega$ .

**Remark 3.5.** The geometric setting of Assumption 2.1 still allows for a Poincaré inequality for functions from  $W_D^{1,p}$ , as soon as  $D \neq \emptyset$ . This is proved in [31, Thm. 3.5], if  $\Omega$  is a Lipschitz domain. In fact, the proof only needs that a part of  $D$  admits positive boundary measure and this is guaranteed by Remark 2.2 (ii).

This Poincaré inequality entails that, whenever  $D \neq \emptyset$ , the norms given by  $\|f\|_{W_D^{1,p}}$  and  $\|\nabla f\|_{L^p}$  for  $f \in W_D^{1,p}$  are equivalent. So, in this case, in all subsequent considerations one may freely replace the one by the other.

#### 4. THE DIVERGENCE OPERATOR: DEFINITION AND ELEMENTARY PROPERTIES

We turn now to the definition of the elliptic divergence operator that will be investigated. Let us first introduce the ellipticity supposition on the coefficients.

**Assumption 4.1.** *The coefficient function  $\mu$  is a Lebesgue measurable, bounded function on  $\Omega$  taking its values in the set of real,  $d \times d$  matrices, satisfying for some  $\mu_\bullet > 0$  the usual ellipticity condition*

$$\xi^T \mu(x) \xi \geq \mu_\bullet |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \Omega.$$

The operator  $A : W_D^{1,2} \rightarrow W_D^{-1,2}$  is defined by

$$(4.1) \quad \langle A\psi, \varphi \rangle_{W_D^{-1,2}} := \mathfrak{t}(\psi, \varphi) := \int_{\Omega} \mu \nabla \psi \cdot \nabla \bar{\varphi} \, dx, \quad \psi, \varphi \in W_D^{1,2}.$$

Often we will write more suggestively  $-\nabla \cdot \mu \nabla$  instead of  $A$ .

The  $L^2$  realization of  $A$ , i.e. the maximal restriction of  $A$  to the space  $L^2$ , will be denoted by the same symbol  $A$ ; clearly this is identical with the operator that is induced by the sesquilinear form  $\mathfrak{t}$ . If  $B$  is a densely defined, closed operator on  $L^2$ , then by the  $L^p$  realization of  $B$  we mean its restriction to  $L^p$  if  $p > 2$  and the  $L^p$  closure of  $B$  if  $p \in [1, 2[$ . (For all operators we have in mind, this  $L^p$ -closure exists.)

As a starting point of our considerations we assume that the square root of our operator is well-behaved on  $L^2$ . This is true in many relevant cases, but seems not to be known under our assumptions in general.

**Assumption 4.2.** *The operator  $(-\nabla \cdot \mu \nabla + 1)^{1/2} : W_D^{1,2} \rightarrow L^2$  provides a topological isomorphism; in other words: the domain of  $(-\nabla \cdot \mu \nabla + 1)^{1/2}$  on  $L^2$  is the form domain  $W_D^{1,2}$ .*

**Remark 4.3.** (i) If this assumption is satisfied for a coefficient function  $\mu$ , then it is also true for the adjoint coefficient function, cf. [44, Thm. 8.2].

(ii) Assumption 4.2 is always fulfilled if the coefficient function  $\mu$  takes its values in the set of real *symmetric*  $d \times d$ -matrices.

(iii) In view of non-symmetric coefficient functions see [9] and [23].

Finally, we collect some facts on  $-\nabla \cdot \mu \nabla$  as an operator on the  $L^2$  and on the  $L^p$  scale.

**Proposition 4.4.** *Let  $\Omega \subseteq \mathbb{R}^d$  be a domain and let  $D \subseteq \partial\Omega$  (relatively) closed.*

- (i) *The restriction of  $-\nabla \cdot \mu \nabla$  to  $L^2$  is a densely defined sectorial operator.*
- (ii) *The operator  $\nabla \cdot \mu \nabla$  generates an analytic semigroup on  $L^2$ .*
- (iii) *The form domain  $W_D^{1,2}$  is invariant under multiplication with functions from  $W^{1,q}$ , if  $q > d$ .*



*Proof.* (i) It is not hard to see that the form  $t$  is closed and its numerical range lies in the sector  $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{\|\mu\|_{L^\infty}}{\mu_\bullet} \operatorname{Re} z\}$ . Thus, the assertion follows from a classical representation theorem for forms, see [37, Ch. VI.2.1].

(ii) This follows from (i) and [37, Ch. V.3.2].

(iii) First, for  $u \in C_D^\infty(\Omega)$  and  $v \in C^\infty(\Omega)$  the product  $uv$  is obviously in  $C_D^\infty(\Omega) \subseteq W_D^{1,2}$ . But, by definition of  $W_D^{1,2}$ , the set  $C_D^\infty(\Omega)$  (see (3.1)) is dense in  $W_D^{1,2}$  and  $C^\infty(\Omega)$  is dense in  $W^{1,q}$ . Thus, the assertion is implied by the continuity of the mapping

$$W_D^{1,2} \times W^{1,q} \ni (u, v) \mapsto uv \in W^{1,2},$$

because  $W_D^{1,2}$  is closed in  $W^{1,2}$ .  $\square$

**Proposition 4.5.** *Let  $\Omega$  and  $D$  satisfy Assumption 2.1 (i). Then the semigroup generated by  $\nabla \cdot \mu \nabla$  in  $L^2$  satisfies upper Gaussian estimates, precisely:*

$$(e^{t\nabla \cdot \mu \nabla} f)(x) = \int_{\Omega} K_t(x, y) f(y) \, dy, \quad \text{for a.a. } x \in \Omega, f \in L^2,$$

for some measurable function  $K_t : \Omega \times \Omega \rightarrow \mathbb{R}_+$  and for all  $\varepsilon > 0$  there exist constants  $C, c > 0$ , such that

$$(4.2) \quad 0 \leq K_t(x, y) \leq \frac{C}{t^{d/2}} e^{-c\frac{|x-y|^2}{t}} e^{\varepsilon t}, \quad t > 0, \text{ a.a. } x, y \in \Omega.$$

*Proof.* A proof is given in [22] – heavily resting on [4], compare also [44, Thm. 6.10].  $\square$

**Proposition 4.6.** *Let  $\Omega$  and  $D$  satisfy Assumption 2.1 (i).*

- (i) *For every  $p \in [1, \infty]$ , the operator  $\nabla \cdot \mu \nabla$  generates a semigroup of contractions on  $L^p$ .*
- (ii) *For all  $q \in ]1, \infty[$  the operator  $-\nabla \cdot \mu \nabla + 1$  admits a bounded  $\mathcal{H}^\infty$ -calculus on  $L^q$  with  $\mathcal{H}^\infty$ -angle  $\arctan \frac{\|\mu\|_{L^\infty}}{\mu_\bullet}$ . In particular, it admits bounded imaginary powers.*

*Proof.* (i) The operator  $\nabla \cdot \mu \nabla$  generates a semigroup of contractions on  $L^2$  (see [44, Thm 1.54]) as well as on  $L^\infty$  (see [44, Ch. 4.3.1]). By interpolation this carries over to every  $L^q$  with  $q \in ]2, \infty[$  and, by duality, to  $q \in [1, 2]$ .

(ii) Since the numerical range of  $-\nabla \cdot \mu \nabla$  is contained in the sector  $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \frac{\|\mu\|_{L^\infty}}{\mu_\bullet} \operatorname{Re} z\}$ , the assertion holds true for  $q = 2$ , see [26, Cor. 7.1.17]. Secondly, the semigroup generated by  $\nabla \cdot \mu \nabla - 1$  obeys the Gaussian estimate (4.2) with  $\varepsilon = 0$ . Thus, the first assertion follows from [20, Theorem 3.1]. The second claim is a consequence of the first, see [16, Section 2.4].  $\square$

## 5. THE MAIN RESULT: THE ISOMORPHISM PROPERTY OF THE SQUARE ROOT

We can now formulate our main goal, that is to prove that the mapping

$$(A + 1)^{1/2} = (-\nabla \cdot \mu \nabla + 1)^{1/2} : W_D^{1,q} \rightarrow L^q$$

is a topological isomorphism for  $q \in ]1, 2[$ . We abbreviate  $-\nabla \cdot \mu \nabla + 1$  by  $A_0$  throughout the rest of this work.

More precisely, we want to show the following main result of this paper.

**Theorem 5.1.** *Under Assumptions 2.1, 4.1 and 4.2 the following holds true:*

- (i) *For every  $q \in ]1, 2[$  the operator  $A_0^{-1/2}$  is a continuous operator from  $L^q$  into  $W_D^{1,q}$ . Hence, its adjoint continuously maps  $W_D^{-1,q}$  into  $L^q$  for any  $q \in [2, \infty[$ .*
- (ii) *If, additionally, Assumption 2.3 holds and  $q \in ]1, 2[$ , then  $A_0^{1/2}$  maps  $W_D^{1,q}$  continuously into  $L^q$ . Hence, its adjoint continuously maps  $L^q$  into  $W_D^{-1,q}$  for any  $q \in [2, \infty[$ .*

We can immediately give the proof of (i), i.e. the continuity of the operator  $A_0^{-1/2} : L^q \rightarrow W_D^{1,q}$ . We observe that this follows, whenever

1. The Riesz transform  $\nabla A_0^{-1/2}$  is a bounded operator on  $L^q$ , and, additionally,
2.  $A_0^{-1/2}$  maps  $L^q$  into  $W_D^{1,q}$ .

The first item is proved in [44, Thm. 7.26], compare also [19]. It remains to show 2. The first point makes clear that  $A_0^{-1/2}$  maps  $L^q$  continuously into  $W^{1,q}$ , thus one only has to verify the correct boundary behavior of the images. If  $f \in L^2 \hookrightarrow L^q$ , then one has  $A_0^{-1/2}f \in W_D^{1,2} \hookrightarrow W_D^{1,q}$ , due to Assumption 4.2. Thus, the assertion follows from 1. and the density of  $L^2$  in  $L^q$ .

**Remark 5.2.** Theorem 5.1 (i) is not true for other values of  $q$  in general, see [5, Ch. 4] for a further discussion.

The hard work is to prove the second part, that is the continuity of  $A_0^{1/2} : W_D^{1,q} \rightarrow L^q$ . The proof is inspired by [5], where this is shown in the case  $\Omega = \mathbb{R}^d$ , and will be developed in the following five sections.

## 6. HARDY'S INEQUALITY

A major tool in our considerations is an inequality of Hardy type for functions in  $W_D^{1,p}$ , so functions that vanish only on the part  $D$  of the boundary. Here the additional Assumption 2.3 comes into play.

We recall that, for a set  $F \subseteq \mathbb{R}^d$ , the symbol  $d_F$  denotes the function on  $\mathbb{R}^d$  that measures the distance to  $F$ . The result we want to show in this section, is the following.

**Theorem 6.1.** *Under Assumption 2.1 and Assumption 2.3, for every  $p \in ]1, \infty[$  there is a constant  $c_p$ , such that*

$$(6.1) \quad \int_{\Omega} \left| \frac{f}{d_D} \right|^p dx \leq c_p \int_{\Omega} |\nabla f|^p dx$$

holds for all  $f \in W_D^{1,p}$ .

Since the statement of this theorem is void for  $D = \emptyset$ , we exclude that case for this entire section.

Let us first quote the deep results on which the proof of Theorem 6.1 will base.

**Proposition 6.2** (see [41], [51], see also [38]). *Let  $\Xi \subseteq \mathbb{R}^d$  be a domain whose complement  $K := \mathbb{R}^d \setminus \Xi$  is uniformly  $p$ -fat (cf. [41] or [38]). Then Hardy's inequality*

$$(6.2) \quad \int_{\Xi} \left| \frac{g}{d_K} \right|^p dx = \int_{\Xi} \left| \frac{g}{d_{\partial\Xi}} \right|^p dx \leq c \int_{\Xi} |\nabla g|^p dx$$

holds for all  $g \in C_0^\infty(\Xi)$  (and extends to all  $g \in W_0^{1,p}(\Xi)$ ,  $p \in ]1, \infty[$  by density).

**Proposition 6.3** ([40, Theorem 1]). *Let  $\Xi \subseteq \mathbb{R}^d$  be a domain and let  $\mathcal{H}_{d-1}$  again denote the  $(d-1)$ -dimensional Hausdorff measure. If  $\Xi$  satisfies the inner boundary density condition, i.e.*

$$(6.3) \quad \mathcal{H}_{d-1}(\partial\Xi \cap B(x, 2d_{\partial\Xi}(x))) \geq c d_{\partial\Xi}(x)^{d-1}, \quad x \in \Xi,$$

for some constant  $c > 0$ , then the complement of  $\Xi$  in  $\mathbb{R}^d$  is uniformly  $p$ -fat for all  $p \in ]1, \infty[$ .

For the proof of Theorem 6.1 let us distinguish two cases, first assuming  $D = \partial\Omega$ . Then the Ahlfors-David condition (2.2) on  $D$  implies

$$(6.4) \quad \mathcal{H}_{d-1}(D \cap B(y, r)) \geq c \left( \frac{r_{AD}}{\text{diam}(\Omega)} \right)^{d-1} r^{d-1}, \quad y \in D, \quad r \in ]0, \text{diam}(\Omega)]$$



with  $c$  independent of  $y$  and  $r$ . But (6.4) implies the inner boundary density condition (6.3), compare [40]. Thus, one may apply Proposition 6.3 and Proposition 6.2 to obtain the claim of Theorem 6.1.

It remains to consider the case  $D \neq \partial\Omega$ . Let us start with some preparation that brings the assumption 2.3 into play.

**Lemma 6.4.** *Let  $\Omega_\star$  and  $\Omega_\bullet$  be defined as in Assumption 2.3. Then the following assertions hold.*

- (i)  $D \cup \partial\Omega_\star = \partial\Omega_\bullet$ .
- (ii) *There exists a modified, continuous extension operator  $\mathfrak{E}_\bullet : W_D^{1,q}(\Omega) \rightarrow W_0^{1,q}(\Omega_\bullet)$  consistently for all  $q \in [1, \infty]$ .*

*Proof.* (i) The inclusion  $\partial\Omega_\bullet \subseteq D \cup \partial\Omega_\star$  is clear; let us show the opposite inclusion. Obviously,  $D \cap \Omega_\bullet = \emptyset$ . Moreover, every  $x \in D$  is an accumulation point of  $\Omega_\bullet$  since it is an accumulation point of  $\Omega \subseteq \Omega_\bullet$ . Together, this shows  $D \subseteq \partial\Omega_\bullet$ . On the other hand, if  $x \in \partial\Omega_\star$ , then  $x \notin \Omega_\star \supseteq \Omega_\bullet$ . Finally, if  $x \in \partial\Omega_\star$ , then it is an accumulation point of the open set  $\Omega_\star$ , and, hence, also an accumulation point of  $\Omega_\bullet = \Omega_\star \setminus D$ , since  $D$  has Lebesgue measure 0, thanks to (2.2). Thus,  $\partial\Omega_\star \subseteq \partial\Omega_\bullet$ .

- (ii) Take a function  $\eta_0$  from  $C_0^\infty(\mathbb{R}^d)$  which is identically 1 on  $\bar{\Omega}$  and identically 0 on a neighbourhood of  $\partial\Omega_\star \setminus D$  – what is possible according to Assumption 2.3 (iii). Then we define  $\mathfrak{E}_\bullet \psi := (\eta_0 \mathfrak{E} \psi)|_{\Omega_\bullet}$ .

Let us show that  $\mathfrak{E}_\bullet$  satisfies the required properties: assume first  $\psi \in C_D^\infty(\Omega)$ . It is clear by the construction of  $\mathfrak{E}$  that the support of  $\mathfrak{E} \psi$  has a positive distance to  $D$ . Thus,  $\eta_0 \mathfrak{E} \psi$  has a positive distance to  $D \cup \partial\Omega_\star \supseteq \partial\Omega_\bullet$ . Mollifying with suitable kernels,  $\eta_0 \mathfrak{E} \psi$  can then be represented as the  $W^{1,p}(\mathbb{R}^d)$ -limit of smooth functions whose supports avoid  $\partial\Omega_\bullet$ . The continuity of  $\mathfrak{E}_\bullet : C_D^\infty(\Omega) \rightarrow W_0^{1,q}(\Omega)$  is obvious; hence  $\mathfrak{E}_\bullet$  extends continuously to  $W_D^{1,q}(\Omega)$ .  $\square$

Assuming for the moment that we have (6.2) in case of  $\Xi = \Omega_\bullet$ , we find with the help of Lemma 6.4 and Remark 3.5

$$(6.5) \quad \begin{aligned} \int_{\Omega} \left| \frac{f}{d_D} \right|^p dx &\leq \int_{\Omega} \left| \frac{f}{d_{\partial\Omega_\bullet}} \right|^p dx \leq \int_{\Omega_\bullet} \left| \frac{\mathfrak{E}_\bullet f}{d_{\partial\Omega_\bullet}} \right|^p dx \leq c \int_{\Omega_\bullet} |\nabla(\mathfrak{E}_\bullet f)|^p dx \\ &\leq c \|f\|_{W_D^{1,p}}^p \leq c \int_{\Omega} |\nabla f|^p dx. \end{aligned}$$

Thus Hardy's inequality (6.1) holds true, once we have shown (6.2) with  $\Omega_\bullet$  in the place of  $\Xi$  for all  $g \in W_0^{1,p}(\Omega_\bullet)$ .

In order to do so, we first recall that  $\Omega_\bullet$  is connected, and, hence, a domain, due to Assumption 2.3. If  $c_0, c_\star, r_{AD}, r_\star$  are the constants from Assumptions 2.1 and 2.3, we put  $c_\bullet := \min(c_0, c_\star)$  and  $r_\bullet := \min(r_{AD}, r_\star)$ . Thus, the condition (2.2) in conjunction with Lemma 6.4 implies for all  $y \in D$  the inequality

$$\mathcal{H}_{d-1}(\partial\Omega_\bullet \cap B(y, r)) \geq \mathcal{H}_{d-1}(D \cap B(y, r)) \geq c_0 r^{d-1} \geq c_\bullet r^{d-1}, \quad r \in ]0, r_\bullet],$$

what gives for  $r \in ]0, \text{diam}(\Omega_\bullet)]$

$$(6.6) \quad \mathcal{H}_{d-1}(\partial\Omega_\bullet \cap B(y, r)) \geq \begin{cases} c_\bullet r^{d-1}, & \text{if } r_\bullet \geq \text{diam}(\Omega_\bullet) \\ c_\bullet \left( \frac{r_\bullet}{\text{diam}(\Omega_\bullet)} \right)^{d-1} r^{d-1}, & \text{if } r_\bullet < \text{diam}(\Omega_\bullet). \end{cases}$$

Analogously, we obtain from Lemma 6.4 and Assumption 2.3, this time for all  $y \in \partial\Omega_\star$ ,

$$\mathcal{H}_{d-1}(\partial\Omega_\bullet \cap B(y, r)) \geq \mathcal{H}_{d-1}(\partial\Omega_\star \cap B(y, r)) \geq c_\star r^{d-1} \geq c_\bullet r^{d-1}, \quad r \in ]0, r_\bullet].$$

Thus, one obtains (6.6) also for  $y \in \partial\Omega_\star$ . Thanks to Lemma 6.4 (i), the estimate (6.6) is thus fulfilled for all  $y \in \partial\Omega_\bullet$ , what implies the inner boundary density condition (6.3), compare [40]. Applying Proposition 6.3 and Proposition 6.2, we get (6.2) for  $g \in W_0^{1,p}(\Omega_\bullet)$ . Thus the estimate (6.5) finishes the proof of Theorem 6.1.

**Remark 6.5.** There is another strategy of proof for Hardy's inequality (6.2), avoiding the concept of 'uniformly  $p$ -fat'. In [40] it is proved that the inner boundary density condition (6.3) implies the so-called  $p$ -pointwise Hardy inequality which implies Hardy's inequality, compare also [38].

## 7. AN ADAPTED CALDERÓN-ZYGMUND DECOMPOSITION

The proof of Theorem 5.1 heavily relies on a Calderón-Zygmund decomposition for  $W_D^{1,p}$  functions. The important point, which brings the mixed boundary conditions into play, is that we have to make sure that for  $f \in \text{dom}_{L^p}(A_0^{1/2})$  the good and the bad part of the decomposition are both also in this space. This is not guaranteed neither by the classical Calderón-Zygmund decomposition nor by the version for Sobolev functions in [5, Lemma 4.12]. This problem will be solved, by incorporating the Hardy inequality into the decomposition.

For the ease of notation, in the whole section we set  $1/d_0 = 0$  and we abbreviate for  $f \in W_D^{1,1}$  the extended function  $\mathfrak{E}f$  by  $\tilde{f}$ .

We denote by  $\mathcal{Q}$  the set of all closed axe-parallel cubes, i.e. all sets of the form  $\{x \in \mathbb{R}^d : |x - m|_\infty \leq \ell/2\}$  for some midpoint  $m \in \mathbb{R}^d$  and sidelength  $\ell > 0$ . In the following, for a given cube  $Q \in \mathcal{Q}$  we will often write  $sQ$  for some  $s > 0$ , meaning the cube with the same midpoint  $m$ , but sidelength  $s\ell$  instead of  $\ell$ .

Furthermore, for every  $x \in \mathbb{R}^d$  we set  $\mathcal{Q}_x := \{Q \in \mathcal{Q} : x \in Q^\circ\}$ . Now we may define the Hardy-Littlewood maximal operator  $M$  for all  $\varphi \in L^1(\mathbb{R}^d)$  by

$$(7.1) \quad (M\varphi)(x) = \sup_{Q \in \mathcal{Q}_x} \frac{1}{|Q|} \int_Q |\varphi|, \quad x \in \mathbb{R}^d.$$

It is well known (see [48, Ch. 1]) that  $M$  is of weak type  $(1, 1)$ , so there is some  $K > 0$ , such that for all  $p \geq 1$

$$(7.2) \quad |\{x \in \mathbb{R}^d : |[M(|\varphi|^p)](x)| > \alpha^p\}| \leq \frac{K}{\alpha^p} \|\varphi\|_{L^p(\mathbb{R}^d)}^p, \quad \text{for all } \alpha > 0 \text{ and } \varphi \in L^p(\mathbb{R}^d).$$

**Lemma 7.1.** *Let  $\Omega$  and  $D$  satisfy Assumptions 2.1 and 2.3. Let  $p \in ]1, \infty[$ ,  $f \in W_D^{1,p}$  and  $\alpha > 0$  be given. Then there exist an at most countable index set  $I$ , cubes  $Q_j \in \mathcal{Q}$ ,  $j \in I$ , and measurable functions  $g, b_j : \Omega \rightarrow \mathbb{R}$ ,  $j \in I$ , such that for some constant  $N \geq 0$  independent of  $\alpha$  and  $f$*

- (1)  $f = g + \sum_{j \in I} b_j$ ,
- (2)  $\|\nabla g\|_{L^\infty} + \|g\|_{L^\infty} + \|g/d_D\|_{L^\infty} \leq N\alpha$ ,
- (3)  $\text{supp}(b_j) \subseteq Q_j$ ,  $b_j \in W_D^{1,1} \cap W^{1,p}$  and  $\int_\Omega (|\nabla b_j| + |b_j| + \frac{|b_j|}{d_D}) \leq N\alpha|Q_j|$  for every  $j \in I$ ,
- (4)  $\sum_{j \in I} |Q_j| \leq \frac{N}{\alpha^p} \|f\|_{W_D^{1,p}}^p$ ,
- (5)  $\sum_{j \in I} \mathbf{1}_{Q_j}(x) \leq N$  for all  $x \in \mathbb{R}^d$ ,
- (6)  $\|g\|_{W_D^{1,p}} \leq N\|f\|_{W_D^{1,p}}$ .

If  $D \neq \emptyset$ , all the norms  $\|f\|_{W_D^{1,p}}$  may be replaced by  $\|\nabla f\|_{L^p}$ .

In order to verify the final statement, note that for  $D \neq \emptyset$  the Ahlfors-David condition guarantees that the surface measure of  $D$  is strictly positive, cf. Remark 2.2 (ii). Thus we can conclude by Remark 3.5.

We will subdivide the proof of Lemma 7.1 into six steps.

**Step 1: Adapted Maximal function.** Let  $f \in W_D^{1,p}$  and let  $\tilde{f} := \mathfrak{E}f \in W_D^{1,p}(\mathbb{R}^d)$  be the extended function according to Lemma 3.3. This means that for some fixed  $x_0 \in \Omega$  and  $R > 4 \operatorname{diam}(\Omega)$ , we have  $\operatorname{supp}(\tilde{f}) \subseteq B(x_0, R/3)$  and that  $\|\tilde{f}\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|f\|_{W_D^{1,p}}$  with a constant  $C$  that does not depend on  $f$ . Furthermore, Hardy's inequality

$$(7.3) \quad \|\tilde{f}/d_D\|_{L^p(\mathbb{R}^d)} \leq C\|\nabla\tilde{f}\|_{L^p(\mathbb{R}^d)}$$

holds, cf. Section 6.

**Remark 7.2.** Using  $\tilde{f}$ , we will construct the Calderón-Zygmund decomposition on all of  $\mathbb{R}^d$  and afterwards restrict again to  $\Omega$ . Admittedly, it would be more natural to stay inside  $\Omega$ , but this leads to several technical problems, since the regularity of the boundary of cubes in  $\Omega$ , i.e.  $\Omega \cap Q$  for some cube  $Q$  in  $\mathbb{R}^d$ , may be very low, so that for instance the validity of the Poincaré inequality is no longer obvious. If  $\Omega$  is more regular, say a strong Lipschitz domain, this extension can be omitted.

We consider the open set

$$E := \{x \in \mathbb{R}^d : [M(|\nabla\tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)](x) > \alpha\}.$$

The easiest case is that of  $E = \emptyset$ . Then we may take  $I = \emptyset$  and  $g = f$  and the only assertion we have to show is (2), the rest being trivial. So, let  $x \in \Omega$  be given. Since  $x$  is not in  $E$ , we have for almost all such  $x$ , by the fact that  $h(x) \leq (Mh)(x)$  for all Lebesgue points of an  $L^1(\mathbb{R}^d)$  function  $h$ ,

$$\begin{aligned} |\nabla g(x)| + |g(x)| + |g(x)|/d_D(x) &= |\nabla f(x)| + |f(x)| + |f(x)|/d_D(x) \\ &= |\nabla\tilde{f}(x)| + |\tilde{f}(x)| + |\tilde{f}(x)|/d_D(x) \\ &\leq [M(|\nabla\tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)](x) \leq \alpha. \end{aligned}$$

This implies (2).

So, we turn to the case  $E \neq \emptyset$ . By Jensen's inequality, (7.2), (7.3) and the continuity of the extension operator we obtain

$$(7.4) \quad \begin{aligned} |E| &\leq |\{x \in \mathbb{R}^d : [M((|\nabla\tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)^p)](x) > \alpha^p\}| \\ &\leq \frac{K}{\alpha^p} \| |\nabla\tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D \|_{L^p(\mathbb{R}^d)}^p \leq \frac{C}{\alpha^p} \|\tilde{f}\|_{W^{1,p}(\mathbb{R}^d)}^p \leq \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p. \end{aligned}$$

In particular this measure is finite, so  $F := \mathbb{R}^d \setminus E \neq \emptyset$ . This allows for choosing a Whitney decomposition of  $E$ , cf. [12, Lemmas 5.5.1 and 5.5.2], see also [48] and [49]. Thus, we get an at most countable index set  $I$  and a collection of cubes  $Q_j \in \mathcal{Q}$ ,  $j \in I$ , with sidelength  $\ell_j$  that fulfill the following properties for some  $c_1, c_2 \geq 1$

- (i)  $E = \bigcup_{j \in I} \frac{8}{9}Q_j$ .
- (ii)  $\frac{8}{9}Q_j \cap \frac{8}{9}Q_k = \emptyset$  for all  $j, k \in I$ ,  $j \neq k$ .
- (iii)  $Q_j \subseteq E$  for all  $j \in I$ .
- (iv)  $\sum_{j \in I} \mathbf{1}_{Q_j} \leq c_1$ .
- (v)  $\frac{1}{c_2}\ell_j \leq d(Q_j, F) \leq c_2\ell_j$  for all  $j \in I$ .

There are two immediate consequences of these properties that are important to observe. Firstly, the family  $Q_j^\circ$ ,  $j \in I$ , is an open covering of  $E$  and, secondly, (v) implies that for some  $\tilde{c} > 1$ , independent of  $j$ , we have

$$(7.5) \quad (\tilde{c}Q_j) \cap F \neq \emptyset \quad \text{for all } j \in I.$$

Now, (iv) immediately implies (5) and this, together with (7.4) allows to prove (4) due to

$$\sum_{j \in I} |Q_j| = \int_E \sum_{j \in I} \mathbf{1}_{Q_j} \leq c_1 |E| \leq \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p.$$

**Step 2: Definition of the good and bad functions.** Let  $(\varphi_j)_{j \in I}$  be a partition of unity on  $E$  with

- a)  $\varphi_j \in C^\infty(\mathbb{R}^d)$ ,  
b)  $\text{supp}(\varphi_j) \subseteq Q_j^\circ$ ,  
c)  $\varphi_j \equiv 1$  on  $\frac{8}{9}Q_j$ ,  
d)  $\|\varphi_j\|_{L^\infty} + \ell_j \|\nabla \varphi_j\|_{L^\infty} \leq c$ ,

for all  $j \in I$  and some  $c > 0$ . The construction of such a partition can be found e.g. in [12, Section 5.5].

Let us distinguish two types of cubes  $Q_j$ . We say that  $Q_j$  is a *usual* cube, if  $d(Q_j, D) \geq \ell_j$  and  $Q_j$  is a *special* cube, if  $d(Q_j, D) < \ell_j$  (In the case  $D = \emptyset$  all cubes are seen as usual ones). Then we define for every  $j \in I$ , using the notation  $h_Q := \frac{1}{|Q|} \int_Q h$ ,

$$\tilde{b}_j := \begin{cases} (\tilde{f} - \tilde{f}_{Q_j})\varphi_j, & \text{if } Q_j \text{ is usual,} \\ \tilde{f}\varphi_j, & \text{if } Q_j \text{ is special.} \end{cases}$$

Setting  $\tilde{g} := \tilde{f} - \sum_{j \in I} \tilde{b}_j$  as well as  $b_j := \tilde{b}_j|_\Omega$  and  $g := \tilde{g}|_\Omega$ , these functions automatically satisfy (1). Note that there is no problem of convergence in this sum, due to (5).

It is clear by construction that  $\text{supp}(b_j) \subseteq Q_j$  and  $b_j \in W^{1,p}(\Omega)$  for all  $j \in I$ . The next step is to show that  $b_j \in W_D^{1,1}$  and since  $W^{1,p} \hookrightarrow W^{1,1}$ , we only have to establish the right boundary behaviour of  $b_j$ .

We start with the case of a usual cube  $Q_j$ . Then  $b_j = ((\tilde{f} - \tilde{f}_{Q_j})\varphi_j)|_\Omega$ . Since  $\varphi_j$  has support in  $Q_j$  and  $d(Q_j, D) \geq \ell_j > 0$ , the function  $b_j$  can be approximated by  $C_c^\infty(\mathbb{R}^d \setminus D)$  functions in the norm of  $W^{1,1}$ . Thus  $b_j \in W_D^{1,1}$ .

If  $Q_j$  is a special cube, we have  $b_j = (\tilde{f}\varphi_j)|_\Omega$ . The fact that  $\tilde{f} \in W_D^{1,p}(\mathbb{R}^d)$  implies that there is a sequence  $(\tilde{f}_k)_k \subseteq C_c^\infty(\mathbb{R}^d \setminus D)$ , such that  $\tilde{f}_k \rightarrow \tilde{f}$  in  $W^{1,p}(\mathbb{R}^d)$ . Therefore,  $(\tilde{f}_k\varphi_j)_k$  is a sequence in  $C_c^\infty(\mathbb{R}^d \setminus D)$  and we show that it converges to  $\tilde{f}\varphi_j$  in  $W^{1,1}$ , so that we can conclude that  $b_j \in W_D^{1,1}$ . This convergence follows from  $\varphi_j \in W^{1,p'}(\mathbb{R}^d)$  by

$$\|\tilde{f}\varphi_j - \tilde{f}_k\varphi_j\|_{L^1} \leq \|\tilde{f} - \tilde{f}_k\|_{L^p} \|\varphi_j\|_{L^{p'}} \rightarrow 0 \quad (k \rightarrow \infty)$$

and the corresponding estimate for the gradient

$$\begin{aligned} \|\nabla(\tilde{f}\varphi_j) - \nabla(\tilde{f}_k\varphi_j)\|_{L^1} &\leq \|\nabla(\tilde{f} - \tilde{f}_k)\varphi_j\|_{L^1} + \|(\tilde{f} - \tilde{f}_k)\nabla\varphi_j\|_{L^1} \\ &\leq \|\nabla(\tilde{f} - \tilde{f}_k)\|_{L^p} \|\varphi_j\|_{L^{p'}} + \|\tilde{f} - \tilde{f}_k\|_{L^p} \|\nabla\varphi_j\|_{L^{p'}} \rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

**Step 3: Proof of (3).** After the above considerations, it remains to prove the estimate. We start again with the case of a usual cube and for later purposes we introduce some  $q \in [1, \infty]$ . On usual cubes it holds  $\nabla \tilde{b}_j = \nabla \tilde{f}\varphi_j + (\tilde{f} - \tilde{f}_{Q_j})\nabla\varphi_j$  and using d) we obtain

$$\begin{aligned} \int_{Q_j} |\nabla \tilde{b}_j|^q &\leq \int_{Q_j} (|\nabla \tilde{f}| |\varphi_j| + |\tilde{f} - \tilde{f}_{Q_j}| |\nabla \varphi_j|)^q \leq C \int_{Q_j} (|\nabla \tilde{f}|^q |\varphi_j|^q + |\tilde{f} - \tilde{f}_{Q_j}|^q |\nabla \varphi_j|^q) \\ (7.6) \quad &\leq C \left( \int_{Q_j} |\nabla \tilde{f}|^q + \frac{1}{\ell_j^q} \int_{Q_j} |\tilde{f} - \tilde{f}_{Q_j}|^q \right). \end{aligned}$$

In the second integral we may now apply the Poincaré inequality, since  $\tilde{f} - \tilde{f}_{Q_j}$  has zero mean on  $Q_j$ . This yields

$$(7.7) \quad \int_{Q_j} |\nabla \tilde{b}_j|^q \leq C \left( \int_{Q_j} |\nabla \tilde{f}|^q + \frac{1}{\ell_j^q} \text{diam}(Q_j)^q \int_{Q_j} |\nabla \tilde{f}|^q \right) \leq C \int_{Q_j} |\nabla \tilde{f}|^q.$$

We now specialize again to  $q = 1$  and, invoking (7.5), we pick some  $z \in \tilde{c}Q_j \cap F$ , and bring into play the maximal operator:

$$(7.8) \quad \int_{Q_j} |\nabla \tilde{b}_j| \leq C \int_{\tilde{c}Q_j} |\nabla \tilde{f}| \leq C|Q_j| \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} \left( |\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \\ \leq C|Q_j| \sup_{Q \in \tilde{\mathcal{Q}}_z} \frac{1}{|Q|} \int_Q \left( |\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) = C|Q_j| \left[ M \left( |\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \right](z).$$

Now, we capitalize that  $z \in F$  and obtain

$$(7.9) \quad \int_{\Omega} |\nabla b_j| \leq \int_{Q_j} |\nabla \tilde{b}_j| \leq C|Q_j|\alpha.$$

For the corresponding estimate for  $|b_j|$  we use again the Poincaré inequality for  $\tilde{f} - \tilde{f}_{Q_j}$  on  $Q_j$  to obtain for all  $q \in [1, \infty[$

$$(7.10) \quad \int_{\Omega} |b_j|^q \leq \int_{Q_j} |\tilde{b}_j|^q = \int_{Q_j} |\tilde{f} - \tilde{f}_{Q_j}|^q |\varphi_j|^q \leq C \int_{Q_j} |\tilde{f} - \tilde{f}_{Q_j}|^q \leq C \int_{Q_j} |\nabla \tilde{f}|^q.$$

Proceeding as in (7.8) and (7.9), we find, specialising to  $q = 1$ ,

$$(7.11) \quad \int_{\Omega} |b_j| \leq C|Q_j|\alpha.$$

For the third term  $|b_j|/d_D$  we note that on a usual cube  $Q_j$  we have  $d_D \geq \ell_j$ . Thus we get as before by the Poincaré inequality

$$\int_{\Omega} \frac{|b_j|}{d_D} \leq \int_{Q_j} \frac{|\tilde{b}_j|}{d_D} \leq \frac{C}{\ell_j} \int_{Q_j} |\tilde{f} - \tilde{f}_{Q_j}| \leq C \int_{Q_j} |\nabla \tilde{f}|$$

and we can again conclude as in (7.8) and (7.9).

So, we turn to the proof of the estimate in (3) for the case of a special cube. Then  $b_j = (\tilde{f}\varphi_j)|_{\Omega}$ , and we get with the help of d)

$$|\nabla \tilde{b}_j| \leq |\nabla \tilde{f}| |\varphi_j| + |\tilde{f}| |\nabla \varphi_j| \leq C \left( |\nabla \tilde{f}| + \frac{|\tilde{f}|}{\ell_j} \right).$$

Since  $Q_j$  is a special cube, we get for every  $x \in Q_j$

$$(7.12) \quad d_D(x) = d(x, D) \leq \text{diam}(Q_j) + d(Q_j, D) \leq C\ell_j + \ell_j \leq C\ell_j$$

and this in turn yields

$$(7.13) \quad |\nabla \tilde{b}_j| \leq C \left( |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right).$$

Since, obviously

$$(7.14) \quad |\tilde{b}_j| = |\tilde{f}\varphi_j| \leq C|\tilde{f}| \quad \text{and} \quad \frac{|\tilde{b}_j|}{d_D} = \frac{|\tilde{f}\varphi_j|}{d_D} \leq C \frac{|\tilde{f}|}{d_D}$$

hold, we find by one more repetition of the arguments in (7.8) and (7.9) with some  $z \in \tilde{c}Q_j \cap F$

$$(7.15) \quad \int_{\Omega} \left( |b_j| + |\nabla b_j| + \frac{|b_j|}{d_D} \right) \leq C \int_{Q_j} \left( |\tilde{f}| + |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \\ \leq \frac{C|Q_j|}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} \left( |\tilde{f}| + |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \leq C|Q_j|\alpha.$$

**Step 4: Proof of (2): Estimate of  $|g|$  and  $|g|/d_D$ .** The asserted bound for  $|g|$  and  $|g|/d_D$  is rather easy to obtain on  $F \cap \Omega$ , since on  $F$  all functions  $\tilde{b}_j$ ,  $j \in I$ , vanish, which means  $\tilde{g} = \tilde{f}$  on  $F$ . This implies for almost all  $x \in F \cap \Omega$  by the definition of  $F$

$$|g(x)| + \frac{|g(x)|}{d_D(x)} = |\tilde{f}(x)| + \frac{|\tilde{f}(x)|}{d_D(x)} \leq \left[ M \left( |\nabla \tilde{f}| + |\tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \right] (x) \leq \alpha.$$

So, for the estimate of these two terms we concentrate on the case  $x \in E$ . Setting  $I_u := \{j \in I : Q_j \text{ usual}\}$  and  $I_s := \{j \in I : Q_j \text{ special}\}$ , we obtain on  $E$

$$\begin{aligned} \tilde{g} &= \tilde{f} - \sum_{j \in I_u} \tilde{b}_j - \sum_{j \in I_s} \tilde{b}_j = \tilde{f} - \sum_{j \in I_u} (\tilde{f} - \tilde{f}_{Q_j}) \varphi_j - \sum_{j \in I_s} \tilde{f} \varphi_j = \tilde{f} - \tilde{f} \sum_{j \in I} \varphi_j + \sum_{j \in I_u} \tilde{f}_{Q_j} \varphi_j \\ &= \tilde{f} \mathbf{1}_F + \sum_{j \in I_u} \tilde{f}_{Q_j} \varphi_j = \sum_{j \in I_u} \tilde{f}_{Q_j} \varphi_j. \end{aligned}$$

Now, we fix some  $x \in E$ . Let  $I(x) := \{j \in I : x \in \text{supp}(\varphi_j)\}$ ,  $I_{u,x} := I_u \cap I(x)$  and  $I_{s,x} := I_s \cap I(x)$ . Then the above estimate yields together with d)

$$\begin{aligned} |\tilde{g}(x)| &\leq \sum_{j \in I_u} |\tilde{f}_{Q_j}| |\varphi_j(x)| \leq C \sum_{j \in I_{u,x}} |\tilde{f}_{Q_j}| = C \sum_{j \in I_{u,x}} \frac{1}{|Q_j|} \left| \int_{Q_j} \tilde{f}(y) \, dy \right| \\ (7.16) \quad &\leq C \sum_{j \in I_{u,x}} \frac{1}{|Q_j|} \int_{Q_j} |\tilde{f}(y)| \, dy. \end{aligned}$$

Picking again some  $z_j \in \tilde{c}Q_j \cap F$ ,  $j \in I$ , this yields with the argument that we used already several times and since  $I_{u,x}$  is finite

$$|\tilde{g}(x)| \leq C \sum_{j \in I_{u,x}} \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} |\tilde{f}(y)| \, dy \leq C \sum_{j \in I_{u,x}} [M(|\tilde{f}|)](z_j) \leq C \sum_{j \in I_{u,x}} \alpha \leq C\alpha.$$

In order to estimate  $\tilde{g}/d_D$  on  $E$ , we estimate as in (7.16) for  $x \in E$

$$\frac{|\tilde{g}(x)|}{d_D(x)} = \frac{|\sum_{j \in I_u} \tilde{f}_{Q_j} \varphi_j(x)|}{d_D(x)} \leq C \sum_{j \in I_{u,x}} \frac{|\tilde{f}_{Q_j}|}{d_D(x)} \leq C \sum_{j \in I_{u,x}} \frac{1}{|Q_j|} \int_{Q_j} \frac{|\tilde{f}(y)|}{d_D(x)} \, dy$$

Every cube in this sum is a usual one, so  $d(Q_j, D) \geq \ell_j$ . Furthermore, we have  $x \in Q_j$  for all  $j \in I_{u,x}$  by construction. This means that for every  $j \in I_{u,x}$  and all  $y \in Q_j$  the distance between  $x$  and  $y$  is less than  $C\ell_j$  for some constant  $C$  depending only on the dimension. Thus

$$d_D(y) = d(y, D) \leq d(y, x) + d(x, D) \leq C\ell_j + d_D(x) \leq Cd(Q_j, D) + d_D(x) \leq Cd_D(x).$$

Consequently, we get for some  $z_j \in \tilde{c}Q_j \cap F$  as before

$$\begin{aligned} \frac{|g(x)|}{d_D(x)} &\leq C \sum_{j \in I_{u,x}} \frac{1}{|Q_j|} \int_{Q_j} \frac{|\tilde{f}(y)|}{d_D(y)} \, dy \leq C \sum_{j \in I_{u,x}} \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} \frac{|\tilde{f}(y)|}{d_D(y)} \, dy \\ &\leq C \sum_{j \in I_{u,x}} [M(|\tilde{f}|/d_D)](z_j) \leq C\alpha. \end{aligned}$$

**Step 5: Proof of (2): Estimate of  $|\nabla g|$ .** In order to estimate  $|\nabla g|$ , it is not sufficient to know that  $\sum_{j \in I} \tilde{b}_j$  converges point-wise as before. At least we have to know some convergence in the sense of distributions to push the gradient through the sum. Let  $J \subseteq I$  be finite. Then we have, due to (7.11) for usual cubes and (7.15) for special cubes

$$\left\| \sum_{j \in J} \tilde{b}_j \right\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{j \in J} |\tilde{b}_j| = \sum_{j \in J} \int_{Q_j} |\tilde{b}_j| \leq C\alpha \sum_{j \in J} |Q_j|$$



with a constant  $C$  that is independent of the choice of  $J$ . Since  $\sum_{j \in I} |Q_j|$  is convergent due to (4), this implies that  $\sum_{j \in I} |\tilde{b}_j|$  is a Cauchy sequence in  $L^1(\mathbb{R}^d)$ .

In particular  $\sum_{j \in I} \tilde{b}_j$  converges in the sense of distributions, so we get  $\nabla \sum_{j \in I} \tilde{b}_j = \sum_{j \in I} \nabla \tilde{b}_j$  in the sense of distributions.

In a next step we show that the sum  $\sum_{j \in I} \nabla \tilde{b}_j$  converges absolutely in  $L^1$ . Investing the estimates in (7.7) and (7.13), respectively, we find

$$\int_{Q_j} |\nabla \tilde{b}_j| \leq C \int_{Q_j} \left( |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right).$$

Thus, we obtain by (5) and the fact that  $E$  has finite measure, cf. (7.4),

$$\begin{aligned} \sum_{j \in I} \|\nabla \tilde{b}_j\|_{L^1(\mathbb{R}^d)} &= \sum_{j \in I} \|\nabla \tilde{b}_j\|_{L^1(Q_j)} \leq C \sum_{j \in I} \int_{Q_j} \left( |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) = C \int_E \sum_{j \in I} \mathbf{1}_{Q_j} \left( |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right) \\ &\leq C \left\| |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right\|_{L^1(E)} \leq C \left\| |\nabla \tilde{f}| + \frac{|\tilde{f}|}{d_D} \right\|_{L^p(E)} \leq \|\nabla \tilde{f}\|_{L^p(\mathbb{R}^d)} + \left\| \frac{\tilde{f}}{d_D} \right\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Now, by Hardy's inequality (7.3) this last expression is finite and this yields the desired absolute convergence.

This allows us to calculate

$$\nabla \tilde{g} = \nabla \tilde{f} - \sum_{j \in I} \nabla \tilde{b}_j = \nabla \tilde{f} - \sum_{j \in I_u} (\nabla \tilde{f} \varphi_j + (\tilde{f} - \tilde{f}_{Q_j}) \nabla \varphi_j) - \sum_{j \in I_s} (\nabla \tilde{f} \varphi_j + \tilde{f} \nabla \varphi_j).$$

Note that the above argument also yields that the sums over  $\nabla \tilde{f} \varphi_j$ ,  $(\tilde{f} - \tilde{f}_{Q_j}) \nabla \varphi_j$  and  $\tilde{f} \nabla \varphi_j$  are absolutely convergent in  $L^1$ , so

$$\nabla \tilde{g} = \nabla \tilde{f} - \sum_{j \in I} \nabla \tilde{f} \varphi_j - \sum_{j \in I_u} (\tilde{f} - \tilde{f}_{Q_j}) \nabla \varphi_j - \sum_{j \in I_s} \tilde{f} \nabla \varphi_j = \nabla \tilde{f} \mathbf{1}_F - \sum_{j \in I_u} (\tilde{f} - \tilde{f}_{Q_j}) \nabla \varphi_j - \sum_{j \in I_s} \tilde{f} \nabla \varphi_j.$$

On  $F$  we know that every summand in the above two sums vanishes, so by the  $L^1$ -convergence shown above we see  $\nabla \tilde{g} = \nabla \tilde{f}$  on  $F$ . Thus on  $F$  we easily get the desired  $L^\infty$ -estimate for  $\nabla \tilde{g}$ , since for almost all  $x \in F$

$$|\nabla \tilde{g}(x)| = |\nabla \tilde{f}(x)| \leq M(|\nabla \tilde{f}|)(x) \leq M(|\nabla \tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)(x) \leq \alpha.$$

So, we concentrate on  $x \in E$ . Since  $E$  is open and by (5) all sums in

$$\nabla \tilde{g}(x) = - \sum_{j \in I_u} (\tilde{f}(x) - \tilde{f}_{Q_j}) \nabla \varphi_j(x) - \sum_{j \in I_s} \tilde{f}(x) \nabla \varphi_j(x)$$

are in fact finite and  $\sum_{j \in I} \varphi_j$  is constantly 1 in a neighbourhood of  $x$ . Thus, we may calculate for  $x \in E$

$$\nabla \tilde{g}(x) = \sum_{j \in I_u} \tilde{f}_{Q_j} \nabla \varphi_j(x) - \tilde{f}(x) \sum_{j \in I} \nabla \varphi_j(x) = \sum_{j \in I_u} \tilde{f}_{Q_j} \nabla \varphi_j(x).$$

We set on  $E$

$$h_u := \sum_{j \in I_u} \tilde{f}_{Q_j} \nabla \varphi_j \quad \text{and} \quad h_s := \sum_{j \in I_s} \tilde{f}_{Q_j} \nabla \varphi_j$$

and we will show in the following the estimates  $|h_s(x)| \leq C\alpha$  and  $|h_u(x) + h_s(x)| \leq C\alpha$  for all  $x \in E$ . Then we have the same bound for  $h_u$  and hence also for  $\nabla \tilde{g}$  on  $E$ .

In order to show the desired estimate for  $h_s$ , we recall that by (7.12) we have  $d_D(y) \leq C\ell_j$  for all  $y$  in a special cube  $Q_j$ . Using d) and this estimate we find for all  $x \in E$

$$\begin{aligned} |h_s(x)| &\leq \sum_{j \in I_s} |\tilde{f}_{Q_j}| |\nabla \varphi_j(x)| \leq \sum_{j \in I_{s,x}} \frac{C}{\ell_j} |\tilde{f}_{Q_j}| \leq C \sum_{j \in I_{s,x}} \frac{1}{|Q_j|} \int_{Q_j} \frac{|\tilde{f}(y)|}{\ell_j} dy \\ &\leq C \sum_{j \in I_{s,x}} \frac{1}{|Q_j|} \int_{Q_j} \frac{|\tilde{f}(y)|}{d_D(y)} dy. \end{aligned}$$

Now, we use again that the above sum is finite, uniformly in  $x$ , so it suffices to estimate each addend by  $C\alpha$ . In order to do so, we once more bring into play the maximal operator in some point  $z_j \in \tilde{c}Q_j \cap F$ :

$$\frac{1}{|Q_j|} \int_{Q_j} \frac{|\tilde{f}(y)|}{d_D(y)} dy \leq C \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} \frac{|\tilde{f}(y)|}{d_D(y)} dy \leq CM(|\nabla \tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)(z_j) \leq C\alpha.$$

We turn to the estimate of  $h_u + h_s$ . Since for every  $x \in E$  we have  $\sum_{j \in I} \nabla \varphi_j(x) = 0$ , one finds

$$(h_u + h_s)(x) = \sum_{j \in I} \tilde{f}_{Q_j} \nabla \varphi_j(x) = \sum_{j \in I} (\tilde{f}_{Q_j} - \tilde{f}_{\tilde{c}Q_j}) \nabla \varphi_j(x).$$

This implies thanks to d)

$$|(h_u + h_s)(x)| \leq \sum_{j \in I} |\tilde{f}_{Q_j} - \tilde{f}_{\tilde{c}Q_j}| |\nabla \varphi_j(x)| \leq \sum_{j \in I(x)} \frac{C}{\ell_j} |\tilde{f}_{Q_j} - \tilde{f}_{\tilde{c}Q_j}|.$$

For every  $j \in I(x)$  we have

$$\begin{aligned} |\tilde{f}_{Q_j} - \tilde{f}_{\tilde{c}Q_j}| &= \left| \frac{1}{|Q_j|} \int_{Q_j} \tilde{f}(y) dy - \tilde{f}_{\tilde{c}Q_j} \right| = \left| \frac{1}{|Q_j|} \int_{Q_j} (\tilde{f}(y) - \tilde{f}_{\tilde{c}Q_j}) dy \right| \\ &\leq \frac{1}{|Q_j|} \int_{Q_j} |\tilde{f}(y) - \tilde{f}_{\tilde{c}Q_j}| dy \leq C \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} |\tilde{f}(y) - \tilde{f}_{\tilde{c}Q_j}| dy. \end{aligned}$$

Applying the Poincaré inequality on  $\tilde{c}Q_j$ , we further estimate by

$$\leq C\ell_j \frac{1}{|\tilde{c}Q_j|} \int_{\tilde{c}Q_j} |\nabla \tilde{f}(y)| dy$$

and again continue as above to find for some point  $z_j \in \tilde{c}Q_j \cap F$

$$\leq C\ell_j M(|\nabla \tilde{f}| + |\tilde{f}| + |\tilde{f}|/d_D)(z_j) \leq C\ell_j \alpha.$$

Putting everything together and investing that  $I(x)$  is uniformly finite for every  $x \in E$ , we have achieved

$$|\nabla \tilde{g}(x)| \leq |h_s(x)| + |(h_u + h_s)(x)| \leq C\alpha$$

and have thus proved (2).

**Step 6: Proof of (6).** We first estimate

$$\|g\|_{W_D^{1,p}} \leq \|\tilde{g}\|_{W_D^{1,p}(\mathbb{R}^d)} = \left\| \tilde{f} - \sum_{j \in I} \tilde{b}_j \right\|_{W_D^{1,p}(\mathbb{R}^d)} \leq \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} + \left\| \sum_{j \in I} \tilde{b}_j \right\|_{W_D^{1,p}(\mathbb{R}^d)}.$$

By the continuity of the extension operator we have  $\|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} \leq C\|f\|_{W_D^{1,p}}$ , so we only have to estimate the sum of the  $\tilde{b}_j$ ,  $j \in I$ .

Here we stem again on (5) and the equivalence of norms in  $\mathbb{R}^N$  to obtain

$$(7.17) \quad \left\| \sum_{j \in I} \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left| \sum_{j \in I} \tilde{b}_j \right|^p \leq \int_{\mathbb{R}^d} \left( \sum_{j \in I} |\tilde{b}_j| \right)^p \leq C \int_{\mathbb{R}^d} \sum_{j \in I} |\tilde{b}_j|^p = C \sum_{j \in I} \int_{Q_j} |\tilde{b}_j|^p.$$

Investing the estimates in (7.10) for  $q = p$  and in (7.14) for usual and special cubes, respectively, we find

$$(7.18) \quad \int_{Q_j} |\tilde{b}_j|^p \leq C \int_{Q_j} (|\tilde{f}|^p + |\nabla \tilde{f}|^p).$$

Combining the two last estimates we thus have with the help of (5)

$$\left\| \sum_{j \in I} \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \sum_{j \in I} \int_{Q_j} (|\tilde{f}|^p + |\nabla \tilde{f}|^p) \leq C \int_{\mathbb{R}^d} \sum_{j \in I} \mathbf{1}_{Q_j} (|\tilde{f}|^p + |\nabla \tilde{f}|^p) \leq C \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)}.$$

For the estimate of the gradient, we first use that  $\nabla \sum_{j \in I} \tilde{b}_j = \sum_{j \in I} \nabla \tilde{b}_j$  holds in  $L^1$  and, hence, also in  $L^p$ . Arguing as in (7.17) and (7.18), we find thanks to the estimates in (7.7) for  $q = p$  and (7.13)

$$\left\| \sum_{j \in I} \nabla \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \sum_{j \in I} \int_{Q_j} |\nabla \tilde{b}_j|^p \leq C \sum_{j \in I} \int_{Q_j} \left( |\nabla \tilde{f}|^p + \frac{|\tilde{f}|^p}{d_D^p} \right).$$

Investing again (5) and the Hardy inequality in (7.3), we end up with

$$\left\| \sum_{j \in I} \nabla \tilde{b}_j \right\|_{L^p(\mathbb{R}^d)}^p \leq C \int_{\mathbb{R}^d} \left( |\nabla \tilde{f}|^p + \frac{|\tilde{f}|^p}{d_D^p} \right) \leq C \int_{\mathbb{R}^d} |\nabla \tilde{f}|^p \leq \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)}^p$$

and this finishes the proof, thanks to  $\|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} \leq C \|f\|_{W_D^{1,p}}$ .

Having the Calderón-Zygmund decomposition at hand, we can now show that it really respects the boundary condition on  $D$ .

**Corollary 7.3.** *Let  $f \in W_D^{1,p}$  be given. The functions  $g$  and  $b = \sum_{j \in I} b_j$  from Lemma 7.1 have the following properties:*

- (i)  $b \in W_D^{1,1}$  with  $\|b\|_{W^{1,1}} \leq C\alpha^{1-p} \|f\|_{W_D^{1,p}}^p$ ,
- (ii)  $g \in W_D^{1,\infty}$  with  $\|g\|_{W^{1,\infty}} \leq C\alpha$ ,
- (iii) If  $f \in W_D^{1,2}$ , then also  $g, b \in W_D^{1,2}$ .

*Proof.* (i) Thanks to (3) in Lemma 7.1 we have  $b_j \in W_D^{1,1}(\Omega)$  for all  $j \in I$ . Moreover, by the estimates in (3) and (4) of the same lemma,

$$(7.19) \quad \sum_{j \in I} \|b_j\|_{W^{1,1}} \leq C\alpha \sum_{j \in I} |Q_j| \leq C\alpha^{1-p} \|f\|_{W_D^{1,p}}^p < \infty.$$

Thus, the sum in  $b$  is absolutely convergent in  $W^{1,1}$ , which means that  $b$  satisfies the asserted norm estimate and lies in the closed subspace  $W_D^{1,1}$ . Thus, we have achieved (i). Note that, by Remark 3.1 this in particular means that  $b$  has trace zero on  $D$ .

- (ii) We first show that  $\tilde{g}$  has a Lipschitz continuous representative and that the Lipschitz constant is controlled by  $C\alpha$ . From the proof of Lemma 7.1 we have  $\tilde{g} \in W^{1,p}(\mathbb{R}^d)$  for all  $1 \leq p < \infty$ . So, from [28, Section 2] we can infer that for almost all  $x, y \in \mathbb{R}^d$

$$|\tilde{g}(x) - \tilde{g}(y)| \leq C|x - y| \left( (M(|\nabla \tilde{g}|^p))^{\frac{1}{p}}(x) + (M(|\nabla \tilde{g}|^p))^{\frac{1}{p}}(y) \right).$$

The Hardy-Littlewood maximal operator is bounded on  $L^\infty(\mathbb{R}^d)$ , so this implies

$$\sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|\tilde{g}(x) - \tilde{g}(y)|}{|x - y|} \leq C \|\nabla \tilde{g}\|_{L^\infty(\mathbb{R}^d)} \leq C\alpha$$

and we find  $\tilde{g} \in W^{1,\infty}(\mathbb{R}^d) = (L^\infty \cap \text{Lip})(\mathbb{R}^d)$ .

It remains to prove the right boundary behaviour of  $\tilde{g}$ , i.e.  $g|_D = 0$ . Since  $f, b \in W_D^{1,1}$ , by Remark 3.1 these two functions have zero trace on  $D$   $\sigma$ -almost everywhere, so the same is true for  $g$  and we only have to get rid of the ‘‘almost everywhere’’. Let  $x \in D$  be given.

Then for every  $\varepsilon > 0$ , by the Ahlfors-David condition (2.2), we have  $\sigma(B(x, \varepsilon) \cap D) > 0$ , so there must be points in this set, where  $\tilde{g}$  vanishes. But this means that  $x$  is an accumulation point of the set  $\{y \in D : \tilde{g}(y) = 0\}$ . By the continuity of  $g$  this implies  $g(x) = 0$ .

(iii) By (ii) and Lemma 3.2 we have  $g \in W_D^{1,\infty} \hookrightarrow W_D^{1,2}$ , so with  $f$  also  $b$  is in this space.  $\square$

## 8. REAL INTERPOLATION OF THE SPACES $W_D^{1,p}(\Omega)$

In this section we establish interpolation within the set of spaces  $\{W_D^{1,p}(\Omega)\}_{p \in [1, \infty]}$ . There already exist interpolation results for spaces of this scale which incorporate mixed boundary conditions (compare [43], [25]) but – to our knowledge – not of the required generality concerning the Dirichlet part. The key ingredient for this generalisation will be the Calderón-Zygmund decomposition proved in Section 7.

**8.1. The interpolation result.** The main result of this section is the following.

**Theorem 8.1.** *Let  $\Omega$  and  $D$  satisfy Assumptions 2.1 and 2.3. Then for all choices of  $1 \leq p_0 < p < p_1 \leq \infty$  we have for  $\alpha = \frac{(p-p_0)p_1}{(p_1-p_0)p}$*

$$W_D^{1,p}(\Omega) = (W_D^{1,p_0}(\Omega), W_D^{1,p_1}(\Omega))_{\alpha,p}$$

with equivalent norms.

We recall the following complex reiteration theorem:

**Theorem 8.2.** [13, 14] *For any compatible couple of Banach spaces  $(A_0, A_1)$  we have*

$$[(A_0, A_1)_{\lambda_0, p_0}, (A_0, A_1)_{\lambda_1, p_1}]_{\alpha} = (A_0, A_1)_{\beta, p}$$

for all  $\lambda_0, \lambda_1$  and  $\alpha$  in  $(0, 1)$  and all  $p_0, p_1$  in  $[1, \infty]$ , except for the case  $p_0 = p_1 = \infty$ . Here  $\beta$  and  $p$  are given by  $\beta = (1 - \alpha)\lambda_0 + \alpha\lambda_1$  and  $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$ .

From this theorem and our real interpolation Theorem 8.1 a complex interpolation result for Sobolev spaces  $W_D^{1,p}(\Omega)$  follows.

**Corollary 8.3.** *Let  $\Omega$  and  $D$  satisfy Assumptions 2.1 and 2.3. For  $1 < p_0 < p < p_1 < \infty$  and  $\alpha = \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}} = \frac{p_1(p-p_0)}{p(p_1-p_0)}$ , we have*

$$[W_D^{1,p_0}(\Omega), W_D^{1,p_1}(\Omega)]_{\alpha} = W_D^{1,p}(\Omega).$$

**8.2. The  $K$ -Method of real interpolation.** The reader can refer to [12], [13] for details on the development of this theory. Here we only recall the essentials to be used in the sequel.

Let  $A_0, A_1$  be two normed vector spaces embedded in a topological Hausdorff vector space  $V$ . For each  $a \in A_0 + A_1$  and  $t > 0$ , we define the  $K$ -functional of interpolation by

$$K(a, t, A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For  $0 < \theta < 1$ ,  $1 \leq q \leq \infty$ , the real interpolation space  $(A_0, A_1)_{\theta, q}$  between  $A_0$  and  $A_1$  is given by

$$(A_0, A_1)_{\theta, q} = \left\{ a \in A_0 + A_1 : \|a\|_{\theta, q} := \left( \int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

It is an exact interpolation space of exponent  $\theta$  between  $A_0$  and  $A_1$ , see [13, Chapter II].

The proof of the following reiteration theorem can be found in [13, Theorem 3.5.4, p. 51].

**Proposition 8.4.** *Let  $(A_0, A_1)$  be a compatible couple of Banach spaces and let  $1 \leq q_i \leq \infty$  and  $0 < \theta_i < 1$  for  $i = 0, 1$  with  $\theta_0 \neq \theta_1$  be given. Then*

$$((A_0, A_1)_{\theta_0, q_0}, (A_0, A_1)_{\theta_1, q_1})_{\eta, q} = (A_0, A_1)_{\theta, q},$$

where  $1 \leq q \leq \infty$ ,  $0 < \eta < 1$  and  $\theta = (1 - \eta)\theta_0 + \eta\theta_1$ .

**Definition 8.5.** Let  $f : X \rightarrow \mathbb{R}$  be a measurable function on a measure space  $(X, \mu)$ . The decreasing rearrangement of  $f$  is the function  $f^* : ]0, \infty[ \rightarrow \mathbb{R}$  defined by

$$f^*(t) = \inf\{\lambda : \mu(\{x : |f(x)| > \lambda\}) \leq t\}.$$

The maximal decreasing rearrangement of  $f$  is the function  $f^{**}$  defined for every  $t > 0$  by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.$$

**Remark 8.6.** It is well known that when  $X$  satisfies the doubling property, then  $(Mf)^* \leq Cf^{**}$ , where  $M$  is again the Hardy-Littlewood maximal operator from (7.1). This is an easy consequence of the fact that  $M$  is of weak type  $(1, 1)$  and of strong type  $(\infty, \infty)$ , see [12, Theorem 3.8, p. 122], and  $\mu(\{x : |f(x)| > f^*(t)\}) \leq t$  for all  $t > 0$ .

We refer to [12], [13] for other properties of  $f^*$  and  $f^{**}$ .

We conclude by quoting the following classical result ([13, p. 109]):

**Proposition 8.7.** *Let  $(X, \mu)$  be a measure space with a  $\sigma$ -finite positive measure  $\mu$ . Let  $f \in L^1(X) + L^\infty(X)$ . We then have*

- (i)  $K(f, t, L^1, L^\infty) = tf^{**}(t)$  and
- (ii) for  $1 \leq p_0 < p < p_1 \leq \infty$  it holds  $(L^{p_0}, L^{p_1})_{\theta, p} = L^p$  with equivalent norms, where  $1/p = (1 - \theta)/p_0 + \theta/p_1$  with  $0 < \theta < 1$ .

**8.3. Proof of the interpolation result.** The proof of Theorem 8.1 is based on the following estimates for the  $K$ -functional.

**Lemma 8.8.** *Let  $1 < p < \infty$ . We have for all  $t > 0$*

$$K(f, t, W_D^{1,1}, W_D^{1,\infty}) \geq C_1 t (|f|^{**}(t) + |\nabla f|^{**}(t)) \quad \text{for all } f \in W_D^{1,1} + W_D^{1,\infty}$$

and

$$K(f, t, W_D^{1,1}, W_D^{1,\infty}) \leq C_2 t \left( |\nabla \tilde{f}|^{**}(t) + |\tilde{f}|^{**}(t) + \left( \frac{|\tilde{f}|}{d_D} \right)^{**}(t) \right) \quad \text{for all } f \in W_D^{1,p}.$$

The constants  $C_1, C_2$  are independent of  $f$  and  $t$ , and  $\tilde{f} = \mathfrak{E}f$  is the Sobolev extension of  $f$  from Lemma 3.3.

*Proof.* For the lower bounds, let  $f \in W_D^{1,1} + W_D^{1,\infty}$  be given. Then due to Proposition 8.7 (i)

$$\begin{aligned} K(f, t, W_D^{1,1}, W_D^{1,\infty}) &\geq \left( \inf_{f=f_0+f_1} (\|f_0\|_{L^1} + t\|f_1\|_{L^\infty}) + \inf_{f=f_0+f_1} (\|\nabla f_0\|_{L^1} + t\|\nabla f_1\|_{L^\infty}) \right) \\ &= C(K(|f|, t, L^1, L^\infty) + K(|\nabla f|, t, L^1, L^\infty)) = Ct(|f|^{**}(t) + |\nabla f|^{**}(t)). \end{aligned}$$

Now, for the upper bound, we consider  $f \in W_D^{1,p}$ . For every  $t > 0$  we set

$$\alpha(t) := \left( M \left( |\nabla \tilde{f}| + |\tilde{f}| + \left| \frac{\tilde{f}}{d_D} \right| \right) \right)^*(t)$$

and we recall from the proof of Lemma 7.1 the notation

$$E = E_t = \left\{ x \in \mathbb{R}^d : M \left( |\nabla \tilde{f}| + |\tilde{f}| + \left| \frac{\tilde{f}}{d_D} \right| \right)(x) > \alpha(t) \right\}.$$

Remark that with this choice of  $\alpha(t)$ , we have  $|E_t| \leq t$  for all  $t > 0$ . Furthermore, due to Remark 8.6 applied with  $X = \mathbb{R}^d$

$$(8.1) \quad \alpha(t) \leq C \left( |\nabla \tilde{f}|^{**} + |\tilde{f}|^{**} + \left| \frac{\tilde{f}}{d_D} \right|^{**} \right) (t).$$

Now, we take the Calderón-Zygmund decomposition from Lemma 7.1 for  $f$  with this choice of  $\alpha(t)$ . This results in a decomposition of  $f \in W_D^{1,p}$  as  $f = g + b$  with  $b \in W_D^{1,1}$  and  $g \in W_D^{1,\infty}$ . Invoking, Corollary 7.3 (ii), we have  $\|g\|_{W_D^{1,\infty}} \leq C\alpha(t)$  and from (7.19) we deduce

$$\|b\|_{W_D^{1,1}} \leq C\alpha(t) \sum_{j \in I} |Q_j| \leq C\alpha(t) |E_t| = Ct\alpha(t).$$

Combining these estimates with (8.1), we find

$$K(f, t, W_D^{1,1}, W_D^{1,\infty}) \leq \|b\|_{W_D^{1,1}} + t\|g\|_{W_D^{1,\infty}} \leq Ct\alpha(t) \leq Ct \left( |\nabla \tilde{f}|^{**}(t) + |\tilde{f}|^{**}(t) + \left( \frac{|\tilde{f}|}{d_D} \right)^{**}(t) \right).$$

for all  $f \in W_D^{1,p}$  and for all  $t > 0$  and this was the claim.  $\square$

*Proof of Theorem 8.1.* By the reiteration Theorem 8.4, it suffices to establish the special case of  $p_0 = 1$  and  $p_1 = \infty$ , i.e.  $W_D^{1,p} = (W_D^{1,1}, W_D^{1,\infty})_{1-1/p,p}$  with equivalent norms for  $1 < p < \infty$ .

First, since  $\Omega$  is bounded we have  $W_D^{1,p} \hookrightarrow W_D^{1,1} \hookrightarrow W_D^{1,1} + W_D^{1,\infty}$ . Moreover, for  $f \in W_D^{1,p}$  we have due to Lemma 8.8

$$\begin{aligned} \|f\|_{1-1/p,p} &= \left( \int_0^\infty [t^{1/p-1} K(f, t, W_D^{1,1}, W_D^{1,\infty})]^p \frac{dt}{t} \right)^{1/p} \\ &\leq C \left( \int_0^\infty [t^{1/p} (|\nabla \tilde{f}|^{**}(t) + |\tilde{f}|^{**}(t) + \left( \frac{|\tilde{f}|}{d_D} \right)^{**}(t))]^p \frac{dt}{t} \right)^{1/p} \\ &= C \left\| |\nabla \tilde{f}|^{**} + |\tilde{f}|^{**} + \left( \frac{|\tilde{f}|}{d_D} \right)^{**} \right\|_{L^p(\mathbb{R}_+)}. \end{aligned}$$

Since  $\|g^{**}\|_{L^p(\mathbb{R}_+)} \sim \|g^*\|_{L^p(\mathbb{R}_+)} = \|g\|_{L^p}$ , this allows us to continue

$$\begin{aligned} &\leq C \left( \|\nabla \tilde{f}\|_{L^p(\mathbb{R}^d)} + \|\tilde{f}\|_{L^p(\mathbb{R}^d)} + \left\| \frac{\tilde{f}}{d_D} \right\|_{L^p(\mathbb{R}^d)} \right) \\ &\leq C \|\tilde{f}\|_{W_D^{1,p}(\mathbb{R}^d)} \leq C \|f\|_{W_D^{1,p}} \end{aligned}$$

thanks to the Hardy inequality in (7.3) and the continuity of the extension operator that assigns  $\tilde{f}$  to  $f$ .

Conversely, let  $f \in (W_D^{1,1}, W_D^{1,\infty})_{1-1/p,p}$ . Then, invoking the lower estimate in Lemma 8.8 we find as above and investing that  $g \mapsto g^{**}$  is sublinear

$$\begin{aligned} \|f\|_{1-1/p,p} &\geq C \left( \int_0^\infty [t^{1/p} (|f|^{**}(t) + |\nabla f|^{**}(t))]^p \frac{dt}{t} \right)^{1/p} = C \| |f|^{**} + |\nabla f|^{**} \|_{L^p(\mathbb{R}_+)} \\ &\geq C \| (|f| + |\nabla f|)^{**} \|_{L^p(\mathbb{R}_+)} \geq C \| |f| + |\nabla f| \|_{L^p} \geq C \|f\|_{W^{1,p}}. \end{aligned}$$

It remains to check the right boundary behaviour of  $f$ , i.e.  $f \in W_D^{1,p}$ . In order to do so, we use the fact that  $W_D^{1,1} \cap W_D^{1,\infty}$  is dense in  $(W_D^{1,1}, W_D^{1,\infty})_{1-1/p,p}$ , see [13, Theorem 3.4.2]. If  $f = \lim_{n \rightarrow \infty} f_n$  for some sequence  $(f_n)$  in  $W_D^{1,1} \cap W_D^{1,\infty}$ , then the limit is also in  $W^{1,p}(\Omega)$  by the above inequality. As  $W_D^{1,\infty} \subseteq W_D^{1,p}$  by Lemma 3.2, we have  $f_n \in W_D^{1,p}$  for every  $n \in \mathbb{N}$ . As this space is closed in  $W^{1,p}$ , this yields  $f \in W_D^{1,p}$  and we find

$$\|f\|_{W_D^{1,p}} = \|f\|_{W^{1,p}} \leq C \|f\|_{1-1/p,p}. \quad \square$$



## 9. OFF-DIAGONAL ESTIMATES

As a next preparatory step towards the proof of Theorem 5.1, we show that the Gaussian estimates imply  $L^p$ - $L^2$  off-diagonal estimates for the operators  $T(t) := e^{-tA_0}$  and  $tA_0T(t)$ .

**Lemma 9.1.** *Let  $p \in [1, 2]$  and let  $E, F \subseteq \Omega$  be relatively closed. Then there exist constants  $c, C \geq 0$ , such that for every  $h \in L^2 \cap L^p$  with  $\text{supp}(h) \subseteq E$  we have for all  $t > 0$*

- (i)  $\|T(t)h\|_{L^2(F)} \leq Ct^{(d/2-d/p)/2} e^{-c\frac{d(E,F)^2}{t}} \|h\|_{L^p}$  for  $p \geq 1$  and
- (ii)  $\|tA_0T(t)h\|_{L^2(F)} \leq Ct^{(d/2-d/p)/2} e^{-c\frac{d(E,F)^2}{t}} \|h\|_{L^p}$  for  $p > 1$ .

*Proof.* (i) We denote the kernel of  $T(t)$  by  $k_t$ . Since  $A_0 = -\nabla \cdot \mu \nabla + 1$ , using the notation of Proposition 4.5, we have  $k_t = e^{-t} K_t$ . Thus for  $k_t$  we have the Gaussian estimates

$$0 \leq k_t(x, y) \leq \frac{C}{t^{d/2}} e^{-c\frac{|x-y|^2}{t}}, \quad t > 0, \text{ a.a. } x, y \in \Omega,$$

without the term  $e^{\varepsilon t}$ . Using these, a straightforward calculation shows

$$\|T(t)h\|_{L^2(F)}^2 \leq \frac{C}{t^d} e^{-c\frac{d(E,F)^2}{t}} \|e^{-c\frac{|\cdot|^2}{2t}} * \tilde{h}\|_{L^2(\mathbb{R}^d)}^2,$$

where we denoted by  $\tilde{h}$  the extension by 0 of  $h$  to the whole of  $\mathbb{R}^d$ . Now, applying Young's inequality to bound the convolution one obtains the assertion.

- (ii) In a first step, we observe, that it is enough to show the assertion in the case  $p = 2$ . In fact, we have by the first part of the proof (set  $E = F = \Omega$  and  $p = 1$ )

$$\begin{aligned} \|tA_0T(t)h\|_{L^2(F)} &\leq \|T(t/2)tA_0T(t/2)h\|_{L^2} \leq Ct^{-d/4} \|tA_0T(t/2)h\|_{L^1} \\ &\leq Ct^{-d/4} \|h\|_{L^1}, \end{aligned}$$

since  $T(t)$  extrapolates to an analytic semigroup on  $L^1$  by the Gaussian estimates, cf. [33] or [3]. Admitting the assertion in the case  $p = 2$ :

$$\|tA_0T(t)h\|_{L^2(F)} \leq C e^{-c\frac{d(E,F)^2}{t}} \|h\|_{L^2},$$

the result then follows by interpolation using the Riesz-Thorin Theorem.

In order to prove the off-diagonal bounds in the case  $p = 2$ , we apply Davies' trick, following the proof of [5, Proposition 2.1]. Since this procedure is rather standard, we just give the major steps.

For some Lipschitz continuous function  $\varphi : \Omega \rightarrow \mathbb{R}$  with  $\|\nabla\varphi\|_{L^\infty} \leq 1$  and  $\varrho > 0$  we define the twisted form

$$a_\varrho(u, v) = \int_\Omega (\mu \nabla(e^{-\varrho\varphi} u) \cdot \nabla(e^{\varrho\varphi} \bar{v}) + u\bar{v}) \, dx, \quad u, v \in D(a_\varrho) := W_D^{1,2}.$$

Setting  $\kappa := 2\varrho^2\|\mu\|_{L^\infty}$  and estimating the real and imaginary part of the quadratic form  $a_\varrho + \kappa - 1$  one finds that the numerical range of  $a_\varrho + \kappa$  lies in the (shifted) sector  $\mathcal{S} + 1$ , where  $\mathcal{S} := \{\lambda \in \mathbb{C} : |\text{Im } \lambda| \leq \sqrt{\frac{\|\mu\|_{L^\infty}}{\mu_\bullet}} \text{Re } \lambda\}$  and  $\mu_\bullet$  is the ellipticity constant from Assumption 4.1.

In the following we denote by  $A_\varrho$  the operator associated to the form  $a_\varrho$  in  $L^2$ . Since  $A_\varrho + \kappa - 1$  is maximal accretive, cf. [37, Ch. VI.2], its negative generates an analytic  $C_0$ -semigroup  $e^{-tA_\varrho}$  on  $L^2$  and  $A_\varrho$  even admits a bounded  $\mathcal{H}^\infty$ -calculus there, cf. [16, Ch. 2.4]. Applying the functional calculus of  $A_\varrho$ , for every  $t \geq 0$  we find

$$\begin{aligned} (9.1) \quad \|tA_\varrho e^{-tA_\varrho}\| &\leq \|t(A_\varrho + \kappa) e^{-t(A_\varrho + \kappa)}\| e^{t\kappa} + \|e^{-t(A_\varrho + \kappa)}\| t\kappa e^{t\kappa} \\ &\leq C e^{t\kappa} + C e^{2t\kappa} \leq C e^{4\varrho^2 t \|\mu\|_{L^\infty}} \end{aligned}$$

Recalling, that the form domain  $W_D^{1,2}$  is invariant under multiplications with  $e^{\varphi}$  by Proposition 4.4 (iii), it is easy to verify, that for every  $f \in L^2$  with  $e^{-\varphi} f \in D(A_0)$ , we have  $A_\varrho f = -e^{\varphi} A_0 e^{-\varphi} f$ . From this we then deduce

$$R(\lambda, A_\varrho) = e^{\varphi} R(\lambda, A_0) e^{-\varphi}, \quad \text{for all } \lambda > \varrho^2 \|\mu\|_{L^\infty},$$

which finally yields for every  $f \in L^2$

$$e^{-tA_\varrho} f = \lim_{n \rightarrow \infty} \left[ \frac{n}{t} R(n/t, A_\varrho) \right]^n f = e^{\varphi} \lim_{n \rightarrow \infty} \left[ \frac{n}{t} R(n/t, A_0) \right]^n e^{-\varphi} f = e^{\varphi} T(t) e^{-\varphi} f.$$

Now we specify  $\varphi(x) = d(x, E)$  for  $x \in \Omega$ . Then for every  $h \in L^2$  with support in  $E$  and all  $\varrho, t > 0$  we get

$$tA_0 T(t)h = -t \frac{d}{dt} T(t)h = t e^{-\varphi} A_\varrho e^{-tA_\varrho} e^{\varphi} h = t e^{-\varphi} A_\varrho e^{-tA_\varrho} h,$$

as  $\varphi = 0$  on the support of  $h$ . This yields for all  $\varrho, t > 0$

$$\begin{aligned} \|tA_0 T(t)h\|_{L^2(F)} &= \|t e^{-\varphi} A_\varrho e^{-tA_\varrho} h\|_{L^2(F)} \leq e^{-\varrho d(E,F)} \|tA_\varrho e^{-tA_\varrho} h\|_{L^2} \\ &\leq C e^{4\varrho^2 \|\mu\|_{L^\infty} t - \varrho d(E,F)} \|h\|_{L^2}, \end{aligned}$$

thanks to (9.1). Minimizing over  $\varrho > 0$  finally yields the assertion with  $c = (8\|\mu\|_{L^\infty})^{-1}$ .  $\square$

## 10. PROOF OF THE MAIN RESULT

We now turn to the proof of Theorem 5.1. Building on the hypotheses that the assertion is true for  $p = 2$ , cf. Assumption 4.2, we will show the corresponding inequality in a weak  $(p, p)$  setting for all  $1 < p < 2$ . Then our result follows by interpolation. More precisely we want to show the following.

**Proposition 10.1.** *Let  $\Omega$  and  $D$  satisfy Assumptions 2.1 and 2.3 and let  $\mu$  be such that Assumptions 4.1 and 4.2 are true. Then there is a constant  $C \geq 0$ , such that for all  $p \in ]1, 2[$ , for every  $f \in C_D^\infty$  and all  $\alpha > 0$  we have*

$$(10.1) \quad \left| \left\{ x \in \Omega : |A_0^{1/2} f(x)| > \alpha \right\} \right| \leq \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p.$$

*Proof.* We follow the proof of [5, Lemma 4.13]. Let  $\alpha > 0$ ,  $p \in ]1, 2[$  and  $f \in C_D^\infty$  be given. We apply the Calderón-Zygmund decomposition from Lemma 7.1 to write  $f = g + \sum_{j \in I} b_j$ . In all what follows the references (1) – (6) will stand for the corresponding features in Lemma 7.1.

Since  $C_D^\infty \hookrightarrow W_D^{1,2} = \text{dom}_{L^2}(A_0^{1/2})$ , by Corollary 7.3 (iii) also the functions  $g$  and  $b = \sum_{j \in I} b_j$  are in the  $L^2$ -domain of  $A_0^{1/2}$ . Thus, we can estimate

$$(10.2) \quad \left| \left\{ x \in \Omega : |A_0^{1/2} f(x)| > \alpha \right\} \right| \leq \left| \left\{ x \in \Omega : |A_0^{1/2} g(x)| > \frac{\alpha}{2} \right\} \right| + \left| \left\{ x \in \Omega : \left| (A_0^{1/2} b)(x) \right| > \frac{\alpha}{2} \right\} \right|,$$

and our aim is to bound both terms on the right hand side by  $C \|f\|_{W_D^{1,p}}^p / \alpha^p$ .

The one containing  $g$  is as always the easy part. We first note, that thanks to (6) and (2) we know

$$\|g\|_{W_D^{1,p}} \leq C \|f\|_{W_D^{1,p}} \quad \text{and} \quad \|g\|_{W_D^{1,\infty}} \leq C \alpha.$$

By interpolation this yields

$$\|g\|_{W_D^{1,2}}^2 \leq C \|g\|_{W_D^{1,p}}^p \|g\|_{W_D^{1,\infty}}^{2-p} \leq C \alpha^{2-p} \|f\|_{W_D^{1,p}}^p.$$

This implies, using the Tchebychev inequality and the  $L^2$  result in Proposition 4.4 (iii)

$$\left| \left\{ x \in \Omega : |A_0^{1/2} g(x)| > \frac{\alpha}{2} \right\} \right| \leq \frac{C}{\alpha^2} \|A_0^{1/2} g\|_{L^2}^2 \leq \frac{C}{\alpha^2} \|g\|_{W_D^{1,2}}^2 \leq \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p.$$

Let's turn to the estimate of the second part in (10.2). We first recall the integral representation of the square root

$$A_0^{1/2}u = \frac{2}{\sqrt{\pi}} \int_0^\infty A_0 e^{-t^2 A_0} u \, dt \quad \text{for all } u \in \text{dom}_{L^2}(A_0^{1/2}),$$

which can be deduced straightforwardly from the well known formula (see [45, Ch. 2.6])

$$A_0^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t A_0}}{\sqrt{t}} \, dt.$$

This yields

$$\begin{aligned} \left| \left\{ x \in \Omega : \left| (A_0^{1/2}b)(x) \right| > \frac{\alpha}{2} \right\} \right| &= \left| \left\{ x \in \Omega : \left| \frac{2}{\sqrt{\pi}} \int_0^\infty (A_0 e^{-t^2 A_0} b)(x) \, dt \right| > \frac{\alpha}{2} \right\} \right| \\ &= \limsup_{m \rightarrow \infty} \left| \left\{ x \in \Omega : \left| \frac{2}{\sqrt{\pi}} \int_{2^{-m}}^\infty (A_0 e^{-t^2 A_0} b)(x) \, dt \right| > \frac{\alpha}{2} \right\} \right|. \end{aligned}$$

Now, it is interesting to note that this last integral converges for every  $L^p$  function at the place of  $b$  thanks to  $\|A_0 e^{-t^2 A_0} f\|_{L^p} \leq \frac{C}{t^2} \|f\|_{L^p}$ .

In the following we denote again by  $\ell_j$  the side-length of the cube  $Q_j$ ,  $j \in I$ , and we set  $r_j := 2^k$  for that value of  $k \in \mathbb{Z}$ , such that  $2^k \leq \ell_j < 2^{k+1}$ . With this notation we split the integral for every  $m \in \mathbb{N}$ :

$$\begin{aligned} &\left| \left\{ x \in \Omega : \left| \frac{2}{\sqrt{\pi}} \int_{2^{-m}}^\infty (A_0 e^{-t^2 A_0} \sum_{j \in I} b_j)(x) \, dt \right| > \frac{\alpha}{2} \right\} \right| \\ &\leq \left| \left\{ x \in \Omega : \left| \sum_{j \in I} \int_{2^{-m}}^{r_j \vee 2^{-m}} A_0 e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right| \\ (10.3) \quad &+ \left| \left\{ x \in \Omega : \left| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right|. \end{aligned}$$

For the estimate of the first integral we may restrict ourselves to the case  $r_j > 2^{-m}$ , since otherwise there is no contribution from this term. We do the usual trick to split off the union of the sets  $4Q_\iota$ ,  $\iota \in I$ , that does not produce any sort of problem due to

$$\left| \bigcup_{\iota \in I} 4Q_\iota \right| \leq \sum_{\iota \in I} |4Q_\iota| \leq C \sum_{\iota \in I} |Q_\iota| \stackrel{(4)}{\leq} \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p.$$

So, we only have to estimate

$$\begin{aligned} &\left| \left\{ x \in \Omega \setminus \bigcup_{\iota \in I} 4Q_\iota : \left| \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right| \\ &= \left| \left\{ x \in \Omega : \left| \mathbf{1}_{(\cup_{\iota \in I} 4Q_\iota)^c} \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi}\alpha}{8} \right\} \right|. \end{aligned}$$

By the Tchebychev inequality we get

$$(10.4) \quad \leq \frac{C}{\alpha^2} \left\| \mathbf{1}_{(\cup_{\iota \in I} 4Q_\iota)^c} \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right\|_{L^2}^2.$$

In order to estimate this norm we take  $u \in L^2(\Omega)$  with  $\|u\|_{L^2} = 1$ . Then

$$\left| \int_\Omega u \mathbf{1}_{(\cup_{\iota \in I} 4Q_\iota)^c} \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right| \leq \sum_{j \in I} \int_\Omega |u| \mathbf{1}_{(\cup_{\iota \in I} 4Q_\iota)^c} \left| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right|.$$

We now split the integration over  $\Omega$  into frame-like pieces and apply the Cauchy-Schwarz inequality. Note, that the characteristic function results in the sum over  $k$  starting only at  $k = 2$ .

$$(10.5) \quad \begin{aligned} &\leq \sum_{j \in I} \sum_{k=2}^{\infty} \int_{(2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega} |u| \left| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right| \\ &\leq \sum_{j \in I} \sum_{k=2}^{\infty} \|u\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)} \left\| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)}. \end{aligned}$$

In order to estimate the first factor of the last expression, we identify  $u$  with its trivial extension by zero to  $\mathbb{R}^d$ . Then we let appear the maximal operator to obtain for every  $y \in Q_j$

$$\|u\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)}^2 \leq \int_{2^{k+1}Q_j} |u|^2 \leq C 2^{d(k+1)} |Q_j| \frac{1}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j} |u|^2 \leq C 2^{dk} \ell_j^d [M(|u|^2)](y).$$

Applying the off-diagonal estimates for  $t^2 A_0 e^{-t^2 A_0}$  from Lemma 9.1 with the set  $Q_j \cap \Omega$  as  $E$ ,  $(2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega$  as  $F$ ,  $d/(d-1)$  as  $p$  and  $b_j$  as  $h$ , we get

$$\begin{aligned} \|A_0 e^{-t^2 A_0} b_j\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)} &\leq \frac{C}{t^2} t^{d/2-(d-1)} e^{-c \frac{d(E,F)^2}{t^2}} \|b_j\|_{L^{d/(d-1)}} \\ &\leq \frac{C}{t^{1+d/2}} e^{-c \frac{4^k r_j^2}{t^2}} \|b_j\|_{L^{d/(d-1)}}, \end{aligned}$$

since  $d(E, F) \geq d(Q_j, 2^{k+1}Q_j \setminus 2^kQ_j) \geq c(2^k \ell_j - \ell_j) \geq c(2^k - 1)r_j \geq c2^k r_j$  thanks to  $k \geq 2$ .

According to (3) the functions  $b_j$  are from  $W_D^{1,1}$ . Exploiting the Sobolev embedding  $W_D^{1,1} \hookrightarrow L^{d/(d-1)}$  (cf. Remark 3.4)

$$(10.6) \quad \|b_j\|_{L^{d/(d-1)}} \leq C \|b_j\|_{W^{1,1}} \leq C \alpha |Q_j| \leq C \alpha \ell_j^d.$$

Putting all this together we find for our second factor

$$\begin{aligned} \left\| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)} &\leq \int_{2^{-m}}^{r_j} \|A_0 e^{-t^2 A_0} b_j\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)} \, dt \\ &\leq C \alpha \ell_j^d \int_{2^{-m}}^{r_j} \frac{1}{t^{1+d/2}} e^{-c \frac{4^k r_j^2}{t^2}} \, dt \\ &= C \alpha \ell_j^d \int_{c4^k}^{c4^k r_j^2 4^m} \left( \frac{\sqrt{s}}{2^k r_j} \right)^{1+d/2} e^{-s} 2^k r_j s^{-3/2} \, ds \\ &\leq C \alpha \ell_j^d r_j^{-d/2} 2^{-kd/2} \int_{c4^k}^{\infty} s^{-1+d/4} e^{-s} \, ds, \end{aligned}$$

which is now independent of  $m \in \mathbb{N}$ . Since the integrand is positive and  $r_j \geq 2\ell_j$  we may continue

$$\begin{aligned} &\leq C \alpha \ell_j^{d/2} 2^{-kd/2} e^{-c4^k} \int_{c4^k}^{\infty} s^{-1+d/4} e^{-s+c4^k} \, ds \\ &= C \alpha \ell_j^{d/2} 2^{-kd/2} e^{-c4^k} \int_0^{\infty} (\sigma + c4^k)^{-1+d/4} e^{-\sigma} \, d\sigma \\ &= C \alpha \ell_j^{d/2} 4^{-k} e^{-c4^k} \int_0^{\infty} (\sigma 4^{-k} + c)^{-1+d/4} e^{-\sigma} \, d\sigma. \end{aligned}$$

This last integral is bounded uniformly in  $k \geq 2$ . In fact, if  $d > 4$ , then we estimate  $4^{-k} \leq 4^{-2}$  and if  $d \leq 4$ , we may just estimate by dropping out the whole  $\sigma 4^{-k}$ . So, estimating once more  $4^{-k} \leq 4^{-2}$ , we end up with

$$\left\| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right\|_{L^2((2^{k+1}Q_j \setminus 2^kQ_j) \cap \Omega)} \leq C \alpha \ell_j^{d/2} e^{-c4^k}.$$

Coming back to (10.5) we thus have

$$\int_{(2^{k+1}Q_j \setminus 2^k Q_j) \cap \Omega} |u| \left| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right| \leq C 2^{kd/2} \ell_j^{d/2} ([M(|u|^2)](y))^{1/2} \alpha \ell_j^{d/2} e^{-c4^k}$$

for every  $y \in Q_j$ . Averaging over  $y$  the inequality remains valid and we get

$$\begin{aligned} & \sum_{j \in I} \sum_{k=2}^{\infty} \int_{(2^{k+1}Q_j \setminus 2^k Q_j) \cap \Omega} |u| \left| \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right| \\ & \leq C \sum_{j \in I} \sum_{k=2}^{\infty} \frac{1}{|Q_j|} \int_{Q_j} \alpha 2^{kd/2} \ell_j^d e^{-c4^k} ([M(|u|^2)](y))^{1/2} \, dy \\ & \leq C \alpha \sum_{j \in I} \sum_{k=2}^{\infty} 2^{kd/2} e^{-c4^k} \int_{Q_j} ([M(|u|^2)](y))^{1/2} \, dy. \end{aligned}$$

The sum over  $k$  now turns out to be convergent, so we continue

$$\leq C \alpha \int_{\mathbb{R}^d} \sum_{j \in I} \mathbf{1}_{Q_j}(y) ([M(|u|^2)](y))^{1/2} \, dy \leq C \alpha \int_{\bigcup_{j \in I} Q_j} ([M(|u|^2)](y))^{1/2} \, dy,$$

where we used (5) in the last step. By the Kolmogorov inequality (cf. [49, IV.7.19]) we have

$$\int_{\bigcup_{j \in I} Q_j} ([M(|u|^2)](y))^{1/2} \, dy \leq C \left| \bigcup_{j \in I} Q_j \right|^{1/2} \| |u|^2 \|_{L^1(\mathbb{R}^d)}^{1/2} \leq C \left( \sum_{j \in I} |Q_j| \right)^{1/2} \|u\|_{L^2}.$$

Coming back to (10.4), we thus finally achieve (observe, that  $\|u\|_{L^2} = 1$ )

$$\begin{aligned} & \left| \left\{ x \in \Omega \setminus \bigcup_{\iota \in I} 4Q_\iota : \left| \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j(x) \, dt \right| > \frac{\sqrt{\pi} \alpha}{8} \right\} \right| \\ & \leq \frac{C}{\alpha^2} \left\| \mathbf{1}_{(\bigcup_{\iota \in I} 4Q_\iota)^c} \sum_{j \in I} \int_{2^{-m}}^{r_j} A_0 e^{-t^2 A_0} b_j \, dt \right\|_{L^2}^2 \leq C \sum_{j \in I} |Q_j| \leq \frac{C}{\alpha^p} \|\nabla f\|_{L^p}^p \end{aligned}$$

by (4).

We turn to the estimate of the second addend on the right hand side of (10.3). For this task, we will need the notion of a bounded  $\mathcal{H}^\infty$ -calculus. The definition and further information can be found in [16] or [26].

We define the function

$$\psi(z) := \int_1^\infty z e^{-t^2 z} \, dt, \quad \operatorname{Re}(z) > 0.$$

We show that

$$\psi \in \mathcal{H}_0^\infty(\Sigma_\mu) := \left\{ f : \Sigma_\mu \rightarrow \mathbb{C} \text{ analytic and } \exists \varepsilon > 0 \text{ s.t. } |f(z)| \leq C \frac{|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \text{ for all } z \in \Sigma_\mu \right\}$$

for every  $\mu \in ]0, \pi/2[$ , where  $\Sigma_\mu := \{z \in \mathbb{C} : |\arg(z)| < \mu\}$ . In fact we have substituting  $\tau = t^2 \operatorname{Re}(z) - \operatorname{Re}(z)$

$$\begin{aligned} \left| \frac{(1+|z|)^{2\varepsilon}}{|z|^\varepsilon} \psi(z) \right| & \leq \int_1^\infty |z|^{1-\varepsilon} (1+|z|)^{2\varepsilon} e^{-t^2 \operatorname{Re}(z)} \, dt \\ & = \int_0^\infty |z|^{1-\varepsilon} (1+|z|)^{2\varepsilon} e^{-\tau} e^{-\operatorname{Re}(z)} \frac{1}{2\sqrt{\operatorname{Re}(z)(\tau + \operatorname{Re}(z))}} \, d\tau \\ & \leq C |z|^{1/2-\varepsilon} (1+|z|)^{2\varepsilon} e^{-c|z|} \int_0^\infty \frac{e^{-\tau}}{\sqrt{\tau}} \, d\tau, \end{aligned}$$

since  $\operatorname{Re}(z) \sim |z|$ , thanks to  $|\arg(z)| < \mu < \pi/2$ . Thus, we may choose  $\varepsilon \in ]0, 1/2[$ .

Furthermore, we have for every  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) > 0$  and every  $r > 0$

$$\frac{1}{r} \psi(r^2 z) = \int_r^\infty z e^{-t^2 z} dt,$$

so since  $A_0$  has a bounded  $\mathcal{H}^\infty$ -calculus on  $L^q$ , see Proposition 4.6 (ii), we have the equality of operators

$$\int_r^\infty A_0 e^{-t^2 A_0} dt = \frac{1}{r} \psi(r^2 A_0)$$

in  $L^q$  for every  $1 < q < 2$ . Thus, denoting  $I_k := \{j \in I : r_j \vee 2^{-m} = 2^k\}$  for every  $k \in \mathbb{Z}$ , we get

$$\sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 e^{-t^2 A_0} b_j dt = \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \frac{1}{r_j \vee 2^{-m}} \psi((r_j \vee 2^{-m})^2 A_0) b_j = \sum_{k \in \mathbb{Z}} \psi(4^k A_0) \sum_{j \in I_k} \frac{b_j}{r_j \vee 2^{-m}}.$$

After these preparations we actually start the estimate. Let  $q := d/(d-1)$  be the Sobolev conjugated index to 1. Using the Tchebychev inequality for this  $q$ , we get

$$\begin{aligned} & \left| \left\{ x \in \Omega : \left| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 e^{-t^2 A_0} b_j(x) dt \right| > \frac{\sqrt{\pi} \alpha}{8} \right\} \right| \\ & \leq \frac{C}{\alpha^q} \left\| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 e^{-t^2 A_0} b_j dt \right\|_{L^q}^q = \frac{C}{\alpha^q} \left\| \sum_{k \in \mathbb{Z}} \psi(4^k A_0) \sum_{j \in I_k} \frac{b_j}{r_j \vee 2^{-m}} \right\|_{L^q}^q. \end{aligned}$$

Observe, that the sum over  $k$  is in fact a finite sum, since  $I_k$  is empty for  $k < -m$  by definition and for large  $k$  by the finite measure of  $E$ , cf. (7.4). Thus, there is no convergence problem in applying Lemma 10.2, which helps to estimate this expression further by

$$\leq \frac{C}{\alpha^q} \left\| \left( \sum_{k \in \mathbb{Z}} \left| \sum_{j \in I_k} \frac{b_j}{r_j \vee 2^{-m}} \right|^2 \right)^{1/2} \right\|_{L^q}^q = \frac{C}{\alpha^q} \int_\Omega \left( \sum_{k \in \mathbb{Z}} \left| \sum_{j \in I_k} \frac{b_j(x)}{r_j \vee 2^{-m}} \right|^2 \right)^{q/2} dx.$$

Now, by (5) the sum over  $k$  is finite for every  $x \in \Omega$  and the number of addends is even bounded uniformly in  $x$  and in  $m$ , so by the equivalence of norms in finite dimensional spaces, we may continue to estimate by

$$\leq \frac{C}{\alpha^q} \int_\Omega \left( \sum_{k \in \mathbb{Z}} \left| \sum_{j \in I_k} \frac{b_j(x)}{r_j \vee 2^{-m}} \right| \right)^q dx \leq \frac{C}{\alpha^q} \int_\Omega \left( \sum_{j \in I} \frac{|b_j(x)|}{r_j \vee 2^{-m}} \right)^q dx.$$

Next we estimate  $r_j \vee 2^{-m}$  by  $r_j$  and, using again the equivalence of norms in the finite sum over  $j$ , we get

$$\leq \frac{C}{\alpha^q} \int_\Omega \sum_{j \in I} \frac{|b_j(x)|^q}{r_j^q} dx \leq \frac{C}{\alpha^q} \sum_{j \in I} \ell_j^{-q} \int_\Omega |b_j(x)|^q dx,$$

since  $r_j \sim \ell_j$ . Using once more the Sobolev embedding  $W^{1,1} \hookrightarrow L^{d/(d-1)} = L^q$ , we see as in (10.6)

$$\int_\Omega |b_j(x)|^q dx = \|b_j\|_{L^q}^q \leq C(\alpha \ell_j^d)^q = C \alpha^q \ell_j^{dq}.$$

Summarizing we have shown

$$\begin{aligned} \left| \left\{ x \in \Omega : \left| \sum_{j \in I} \int_{r_j \vee 2^{-m}}^\infty A_0 e^{-t^2 A_0} b_j(x) dt \right| > \frac{\sqrt{\pi} \alpha}{8} \right\} \right| & \leq \frac{C}{\alpha^q} \sum_{j \in I} \ell_j^{-q} \alpha^q \ell_j^{dq} = C \sum_{j \in I} \ell_j^d \\ & \leq C \sum_{j \in I} |Q_j| \leq \frac{C}{\alpha^p} \|f\|_{W_D^{1,p}}^p, \end{aligned}$$

using one final time (4). □



It remains to prove Lemma 10.2, which serves as a substitute for Lemma 4.14 in [5]. We give a different proof, that instead of  $L^p$ - $L^2$  off-diagonal estimates relies on the  $\mathcal{H}^\infty$  functional calculus of the operator and gives the assertion for the full range of  $1 < q < \infty$ .

**Lemma 10.2.** *Let  $1 < q < \infty$ , let  $-B$  be the generator of a bounded analytic semigroup on  $L^q$ , such that  $B$  and  $B'$  admit bounded  $\mathcal{H}^\infty$ -calculus on  $L^q$  and  $L^{q'}$ , respectively and let  $\psi \in H_0^\infty(\Sigma_\phi)$  for some  $\phi \in ]\varphi_B^\infty, \pi]$ , where  $\varphi_B^\infty$  is the  $\mathcal{H}^\infty$ -angle of  $B$ . Then for every choice of functions  $f_k \in L^q$ ,  $k \in \mathbb{Z}$ , we have*

$$\left\| \sum_{k \in \mathbb{Z}} \psi(4^k B) f_k \right\|_{L^q} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2} \right\|_{L^q},$$

whenever the left hand side is convergent.

Before starting the proof, we observe, that thanks to [36, Theorem 5.3], the operator  $B$  even has an  $\mathcal{R}$ -bounded  $\mathcal{H}^\infty$ -calculus of angle  $\varphi_B^\infty$  on  $L^q$ , which means, that for every  $\phi > \varphi_B^\infty$  and every bounded set of functions  $\Xi \subseteq H^\infty(\Sigma_\phi)$  the set of operators  $\{\xi(A) : \xi \in \Xi\}$  is  $\mathcal{R}$ -bounded in  $\mathcal{L}(L^q)$ . Here a set  $\mathcal{T} \subseteq \mathcal{L}(L^q)$  is called  $\mathcal{R}$ -bounded, if there is a constant  $C \geq 0$ , such that for every  $N \in \mathbb{N}$ , for every choice of functions  $f_k \in L^q$ ,  $k = 1, \dots, N$ , operators  $T_k \in \mathcal{T}$ ,  $k = 1, \dots, N$ , and  $\{-1, 1\}$ -valued, symmetric and independent random variables  $\varepsilon_k$ ,  $k = 1, \dots, N$ , on some probability space  $S$ , we have

$$\left\| \sum_{k=1}^N \varepsilon_k T_k f_k \right\|_{L^2(S; L^q)} \leq C \left\| \sum_{k=1}^N \varepsilon_k f_k \right\|_{L^2(S; L^q)}.$$

In the proof of Lemma 10.2, we will use the following Lemma from [36, Lemma 4.1] (see also [15]).

**Lemma 10.3.** *Let  $1 < q < \infty$ , let  $-B$  be the generator of a bounded analytic semigroup on  $L^q$ , such that  $B$  admits a bounded  $\mathcal{H}^\infty$ -calculus on  $L^q$  and let  $\psi \in H_0^\infty(\Sigma_\phi)$  for some  $\phi \in ]\varphi_B^\infty, \pi]$ . Then there is a constant  $C \geq 0$ , such that for every bounded sequence  $(\alpha_k)_{k \in \mathbb{Z}} \subseteq \mathbb{C}$  and every  $t > 0$  we have*

$$\left\| \sum_{k \in \mathbb{Z}} \alpha_k \psi(2^k t B) \right\|_{\mathcal{L}(L^q)} \leq C \sup_{k \in \mathbb{Z}} |\alpha_k|.$$

*Proof of Lemma 10.2.* Since  $\psi \in H_0^\infty(\Sigma_\phi)$ , there exists an  $\varepsilon > 0$  with  $|\psi(z)| \leq C|z|^\varepsilon/(1+|z|)^{2\varepsilon}$  for all  $z \in \Sigma_\phi$ . Let  $\delta \in ]0, \varepsilon[$  and set

$$\psi_1(z) := \frac{z^\delta}{(1+z)^{2\delta}}, \quad \psi_2(z) := \frac{(1+z)^{2\delta}}{z^\delta} \psi(z), \quad z \in \Sigma_\phi.$$

Then we have  $\psi_1, \psi_2 \in H_0^\infty(\Sigma_\phi)$ ,  $\psi = \psi_1 \psi_2$  and  $(\psi_1(B))' = \overline{\psi_1(B')}$ .

Now, let  $N \in \mathbb{N}$  and let  $g \in L^{q'}$  with  $\|g\|_{L^{q'}} = 1$ , where  $1/q + 1/q' = 1$ . Then for every family of  $\{-1, 1\}$ -valued, symmetric and independent random variables  $\varepsilon_k$ ,  $k = -N, \dots, N$ , on some probability space  $S$ , we have

$$\left| \int_\Omega \sum_{k=-N}^N (\psi(4^k B) f_k)(x) g(x) \, dx \right| = \left| \int_S \sum_{k=-N}^N \varepsilon_k^2(\sigma) \int_\Omega (\psi_2(4^k B) f_k)(x) (\overline{\psi_1(4^k B')} g)(x) \, dx \, d\sigma \right|.$$

Since the random variables  $\varepsilon_k$ ,  $k = -N, \dots, N$ , are independent and thus orthogonal in  $L^2(S)$ , we may write this as

$$\begin{aligned} &= \left| \int_S \sum_{j,k=-N}^N \varepsilon_k(\sigma) \varepsilon_j(\sigma) \int_{\Omega} (\psi_2(4^k B) f_k)(x) (\overline{\psi_1}(4^j B') g)(x) \, dx \, d\sigma \right| \\ &\leq \int_S \left| \int_{\Omega} \sum_{k=-N}^N \varepsilon_k(\sigma) (\psi_2(4^k B) f_k)(x) \sum_{j=-N}^N \varepsilon_j(\sigma) (\overline{\psi_1}(4^j B') g)(x) \, dx \right| d\sigma \end{aligned}$$

and using twice the Hölder inequality we estimate by

$$\leq C \left\| \sum_{k=-N}^N \varepsilon_k \psi_2(4^k B) f_k \right\|_{L^2(S; L^q)} \left\| \sum_{j=-N}^N \varepsilon_j \overline{\psi_1}(4^j B') g \right\|_{L^2(S; L^{q'})}.$$

Now, in the first factor we use the  $\mathcal{R}$ -bounded  $\mathcal{H}^\infty$ -calculus of  $B$ . Since the set of functions  $\{\psi_2(4^k \cdot) : k \in \mathbb{Z}\}$  is bounded in  $H^\infty(\Sigma_\phi)$ , we get

$$\left\| \sum_{k=-N}^N \varepsilon_k \psi_2(4^k B) f_k \right\|_{L^2(S; L^q)} \leq C \left\| \sum_{k=-N}^N \varepsilon_k f_k \right\|_{L^2(S; L^q)} \leq C \left\| \left( \sum_{k=-N}^N |f_k|^2 \right)^{1/2} \right\|_{L^q},$$

where the last inequality follows from Khinchin's inequality (cf. [17, 1.10]).

In order to estimate the second factor, we apply Lemma 10.3 and get

$$\begin{aligned} \left\| \sum_{j=-N}^N \varepsilon_j \overline{\psi_1}(4^j B') g \right\|_{L^2(S; L^{q'})} &\leq \left( \int_S \left\| \sum_{j=-N}^N \varepsilon_j(\sigma) \overline{\psi_1}(2^{2j} B') \right\|_{\mathcal{L}(L^{q'})}^2 \|g\|_{L^{q'}}^2 \, d\sigma \right)^{1/2} \\ &\leq \left( \int_S \left( \sup_{j=-N}^N |\varepsilon_j(\sigma)| \right)^2 \, d\sigma \right)^{1/2} = 1. \end{aligned}$$

This implies

$$\left\| \sum_{k=-N}^N \psi(4^k B) f_k \right\|_{L^q} = \sup_{g \in L^{q'}; \|g\|_{L^{q'}}=1} \left| \int_{\Omega} \sum_{k=-N}^N (\psi(4^k B) f_k)(x) g(x) \, dx \right| \leq C \left\| \left( \sum_{k=-N}^N |f_k|^2 \right)^{1/2} \right\|_{L^q}$$

for every  $N \in \mathbb{N}$ . Letting  $N \rightarrow \infty$  the assertion follows.  $\square$

**Corollary 10.4.**  $A_0^{1/2}$  extends to a bounded operator from  $W_D^{1,p}$  to  $(L^\infty, L^1)_{\frac{1}{p}, \infty}$  for all  $1 < p < 2$ .

*Proof.* It is known that the weak Lebesgue space  $L_w^p(\Omega)$  is identical to the real interpolation space  $(L^\infty, L^1)_{\frac{1}{p}, \infty}$ , and the quasinorm  $f \mapsto \sup_{t \geq 0} t^p |\{x : |f(x)| > t\}|$  is equivalent to the  $(L^\infty, L^1)_{\frac{1}{p}, \infty}$ -norm (see [50, Ch. 1.18.6]). Thus, inequality (10.1) can be interpreted as follows:  $A_0^{1/2}$  is a continuous operator from  $C_D^\infty(\Omega)$  – equipped with the  $W^{1,p}$ -norm – into the Banach space  $(L^\infty, L^1)_{\frac{1}{p}, \infty}$ . Hence,  $A_0^{1/2}$  uniquely extends by density to a continuous operator from  $W_D^{1,p}$  into  $(L^\infty, L^1)_{\frac{1}{p}, \infty}$ .  $\square$

Let us now come to the final step of the proof of the second assertion of Theorem 5.1. Up to now, we have the two continuous mappings

$$A_0^{1/2} : W_D^{1,2} \rightarrow L^2 = [L^\infty, L^1]_{\frac{1}{2}}$$

and

$$A_0^{1/2} : W_D^{1,p} \rightarrow (L^\infty, L^1)_{\frac{1}{p}, \infty}$$

for all  $1 < p < 2$ . Let  $q \in ]1, 2[$  and choose  $p \in ]1, q[$ . Using real interpolation, this gives the continuous mapping

$$A_0^{1/2} : (W_D^{1,2}, W_D^{1,p})_{\theta,q} \rightarrow ([L^\infty, L^1]_{\frac{1}{2}}, (L^\infty, L^1)_{\frac{1}{p}, \infty})_{\theta,q} = (L^2, L_{p,\infty})_{\theta,q},$$

where  $L_{p,\infty}$  is a Lorentz space, see [13]. Setting  $\theta = \frac{p}{q} \frac{2-q}{2-p}$ , the left hand side is equal to  $W_D^{1,q}$  by Theorem 8.1 and the right hand side equals  $L^q$  according to [13, Thm. 5.3.1]. This finishes the proof.

**Corollary 10.5.** *Under the above assumptions one has for  $p \in ]1, 2]$  and  $\beta \in ]0, \frac{1}{2}[$*

$$(10.7) \quad \text{dom}_{L^p}(A_0^\beta) = [L^p, W_D^{1,p}]_{2\beta}.$$

*Proof.* The operator  $A_0$  admits bounded imaginary powers, according to Proposition 4.6 (ii). Hence, (10.7) follows from a classical result, see [50, Ch. 1.15.3].  $\square$

**Remark 10.6.** In view of this result it would be highly interesting to determine also the interpolation spaces in formula (10.7). We suggest the formula

$$(10.8) \quad [L^p, W_D^{1,p}]_\theta = \begin{cases} H^{\theta,p}, & \text{if } \theta < \frac{1}{p} \\ H_D^{\theta,p}, & \text{if } \theta > \frac{1}{p}, \end{cases}$$

$H^{\theta,p}$  being the space of Bessel potentials and  $H_D^{\theta,p}$  being the subspace which is defined via the trace-zero condition on  $D$ . Unfortunately, we are not able to prove this at present; but in the more restricted context of so called regular sets (10.8) is shown in [25]. Compare also [29, Section 5] for a simple characterization of regular sets in case of space-dimensions 2 and 3, and see also [43].

## 11. CONSEQUENCES

In this section we come back to the original motivation of our work, namely to carry over results which are known for divergence operators, when acting on  $L^p$  spaces, to the spaces from the scale  $W_D^{-1,q}$ ,  $q \in [2, \infty[$ , compare also [8], [21, Section 5], [30], [32]. In particular, this affects maximal parabolic regularity, which is an extremely powerful tool for the treatment of linear and nonlinear parabolic equations with nonsmooth data, see e.g. [46] or [30]. The crucial point is that here an explicit, discontinuous time-dependence of the right hand side is admissible – relevant for applications. Moreover, the spaces  $W_D^{-1,q}$  allow to include distributional right hand sides; the reader may think, e.g. of electric surface densities, concentrated on interfaces between different materials – even when these interfaces move in time.

**Definition 11.1.** Following [50, Ch.1.14], we call a densely defined operator  $B$  on a Banach space  $X$  positive, if it satisfies the resolvent estimate

$$(11.1) \quad \|(B + \lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{1 + \lambda}$$

for a constant  $c$  and all  $\lambda \in [0, \infty[$ . (Note that a positive operator is sectorial in the sense of [16, Ch. 1.1].)

Let us recall the notion of maximal parabolic regularity.

**Definition 11.2.** Let  $1 < s < \infty$ , let  $X$  be a Banach space and let  $J := ]T_0, T[ \subseteq \mathbb{R}$  be a bounded interval. Assume that  $B$  is a closed operator in  $X$  with dense domain  $D$  (in the sequel always equipped with the graph norm). We say that  $B$  satisfies *maximal parabolic  $L^s(J; X)$  regularity*, if for any  $f \in L^s(J; X)$  there exists a unique function  $u \in W^{1,s}(J; X) \cap L^s(J; D)$  satisfying

$$u' + Bu = f, \quad u(T_0) = 0,$$

where the time derivative is taken in the sense of  $X$ -valued distributions on  $J$  (see [1, Ch III.1]).

- Remark 11.3.** (i) It is well known that the property of maximal parabolic regularity of an operator  $B$  is independent of  $s \in ]1, \infty[$  and the specific choice of the interval  $J$  (cf. [18]). Thus, in the following we will say for short that  $B$  admits maximal parabolic regularity on  $X$ .
- (ii) If an operator satisfies maximal parabolic regularity on a Banach space  $X$ , then its negative generates an analytic semigroup on  $X$  (cf. [18]). In particular, a suitable left half plane belongs to its resolvent set.

**Lemma 11.4.** *Let  $X, Y$  be two Banach spaces, where  $X$  continuously and densely injects into  $Y$ . Assume that  $B$  is a positive operator on  $X$ , such that  $B^\beta : X \rightarrow Y$  is a topological isomorphism for some  $\beta \in ]0, 1[$ . Then the following holds true.*

- (i)  $B$  admits an extension  $\tilde{B}$  on  $Y$ , which also is a positive operator there.
- (ii) If  $B$  admits an  $\mathcal{H}^\infty$ -calculus, then  $\tilde{B}$  admits an  $\mathcal{H}^\infty$ -calculus with the same  $\mathcal{H}^\infty$ -angle.
- (iii) If  $B$  satisfies maximal parabolic regularity on  $X$ , then  $\tilde{B}$  satisfies maximal parabolic regularity on  $Y$ .

*Proof.* The well-known Balakrishnan formula  $B^{-\beta} = \frac{\sin \pi \beta}{\pi} \int_0^\infty t^{-\beta} (B+t)^{-1} dt$  (see [45, Ch. 2.6]) shows that the resolvent commutes with the fractional power  $B^{-\beta}$ . Hence, for  $\psi \in X$  and  $\lambda \geq 0$  one can estimate

$$\begin{aligned} \|(B + \lambda)^{-1} \psi\|_Y &= \|B^\beta (B + \lambda)^{-1} B^{-\beta} \psi\|_Y \\ &\leq \|B^\beta\|_{\mathcal{L}(X; Y)} \|(B + \lambda)^{-1}\|_{\mathcal{L}(X)} \|B^{-\beta}\|_{\mathcal{L}(Y; X)} \|\psi\|_Y \\ &\leq \|B^\beta\|_{\mathcal{L}(X; Y)} \|B^{-\beta}\|_{\mathcal{L}(Y; X)} \frac{c}{1 + \lambda} \|\psi\|_Y. \end{aligned}$$

This shows that the resolvent of  $B$  may be continuously extended to  $Y$  and that this extension admits the estimate  $\|(B + \lambda)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{c}{1 + \lambda}$ . Thus, one defines the extension  $\tilde{B}$  of  $B$  to  $Y$  as the inverse of  $\tilde{B}^{-1}$ . Since  $X \hookrightarrow Y$ ,  $\text{dom}_X(B) \hookrightarrow \text{dom}_Y(\tilde{B})$ . But  $\text{dom}_X(B)$  is dense in  $X$  by the definition of a positive operator and  $X$  was dense in  $Y$  by our assumption. Thus,  $\text{dom}_Y(\tilde{B}) \supset \text{dom}_X(B)$  is also dense in  $Y$ . For (ii) see [16, Prop. 2.11]. Finally, assertion (iii) is proved in [32, Lemma 5.12]. The main idea is again that the parabolic solution operator on  $L^r(J; X)$  commutes with the fractional power  $B^{-\beta}$ .  $\square$

**Theorem 11.5.** *Let  $\Omega$  and  $D$  satisfy Assumptions 2.1 and 2.3, let  $\mu$  satisfy Assumptions 4.1 and 4.2 and assume  $q \in [2, \infty[$ . Then the extension of  $-\nabla \cdot \mu \nabla + 1$  from  $L^q$  to  $W_D^{-1, q}$  (being identical with the restriction from  $W_D^{-1, 2}$ ) has the following properties:*

- (i) *It induces a positive operator.*
- (ii) *It admits a bounded  $\mathcal{H}^\infty$ -calculus with  $\mathcal{H}^\infty$ -angle  $\arctan \frac{\|\mu\|_{L^\infty}}{\mu_\bullet}$ ; in particular, it admits bounded imaginary powers.*
- (iii) *It satisfies maximal parabolic regularity; in particular, its negative generates an analytic semigroup.*

*Proof.* Thanks to Remark 4.3, the transposed coefficient function  $\mu^T$  also satisfies Assumption 4.2. Hence, the operator

$$(11.2) \quad (-\nabla \cdot \mu^T \nabla + 1)^{1/2} : W_D^{1, p} \rightarrow L^p$$

provides a topological isomorphism for all  $p \in ]1, 2]$ , according to Theorem 5.1. Clearly, the adjoint operator of (11.2), being identical with the operator  $(-\nabla \cdot \mu \nabla + 1)^{1/2} : L^q \rightarrow W_D^{-1, q}$ , with  $q = \frac{p}{p-1} \in [2, \infty[$ , is also a topological isomorphism. Consequently, we need to know the asserted properties only on the spaces  $L^q$  due to Lemma 11.4.

In order to see this for (i), it suffices to note that on every space  $L^q$ ,  $1 < q < \infty$ , the operator  $-\nabla \cdot \mu \nabla$  generates a strongly continuous semigroup of contractions (see Proposition 4.6), hence, the operator admits the required resolvent estimate by the Hille-Yosida theorem.

Assertion (ii) is discussed in Proposition 4.6 and, concerning (iii), the contraction property of the semigroup on all  $L^q$  spaces, provides maximal parabolic regularity on these spaces due to a deep result of Lamberton (see [39]).  $\square$

## REFERENCES

- [1] H. Amann: Linear and quasilinear parabolic problems. Birkhäuser, Basel-Boston-Berlin, 1995.
- [2] A. Ancona: *On strong barriers and an inequality of Hardy for domains in  $\mathbb{R}^n$* . J. London Math. Soc. (2) **34** (1986), no. 2, 274–290.
- [3] W. Arendt: *Semigroup properties by Gaussian estimates*. RIMS Kokyuroku **1009** (1997), 162–180.
- [4] W. Arendt and A. F. M. terElst: *Gaussian estimates for second order elliptic operators with boundary conditions*. J. Oper. Th. **38** (1997), 87–130.
- [5] P. Auscher: *On necessary and sufficient conditions for  $L^p$ -estimates of Riesz Transforms Associated to Elliptic Operators on  $\mathbb{R}^n$  and related estimates*. Mem. Amer. Math. Soc. **186** (2007), no. 871.
- [6] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and Ph. Tchamitchian: *The solution of the Kato square root problem for second order elliptic operators on  $\mathbb{R}^n$* . Ann. Math. (2) **156** (2002), no. 2, 633–654.
- [7] P. Auscher and Ph. Tchamitchian: *Square root problem for divergence operators and related topics*. Astérisque **249** (1998).
- [8] P. Auscher and Ph. Tchamitchian: *Square roots of elliptic second order divergence operators on strongly Lipschitz domains:  $L^p$  theory*. Math. Ann. **320** (2001), no. 3, 577–623.
- [9] A. Axelsson, S. Keith and A. McIntosh: *The Kato square root problem for mixed boundary value problems*. J. Lond. Math. Soc. (2) **74** (2006), no. 1, 113–130.
- [10] N. Badr: *Interpolation réelle des espaces de Sobolev sur les espaces métriques mesurés et applications aux inégalités fonctionnelles*. PhD Thesis, Université Paris-Sud, 2007. (<http://tel.archives-ouvertes.fr/tel-00736066>)
- [11] N. Badr: *Real interpolation of Sobolev spaces*. Math. Scand. **105** (2009), no. 2, 235–264.
- [12] C. Bennett and R. Sharpley: *Interpolation of operators*. Vol. 129 of Pure and Applied Mathematics, Academic Press, Inc., Boston, MA, 1988.
- [13] J. Bergh and J. Löfström: *Interpolation spaces, An introduction*. Springer, Berlin, 1976.
- [14] M. Cwikel: *Relations between real and complex interpolation spaces*. Indiana Univ. Math. J. **36** (1987), no. 4, 905–912.
- [15] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni: *New thoughts on old results of R. T. Seeley*. Math. Ann. **328** (2004), 545–583.
- [16] R. Denk, M. Hieber and J. Prüss:  *$\mathcal{R}$ -boundedness, Fourier multipliers and problems of elliptic and parabolic type*. Mem. Amer. Math. Soc. **166** (2003), no. 788.
- [17] J. Diestel, H. Jarchow and A. Tonge: *Absolutely summing operators*. Vol. 43 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.
- [18] G. Dore: *Maximal regularity in  $L^p$  spaces for an abstract Cauchy problem*. Adv. Differential Equations **5** (2000), 293–322.
- [19] X. T. Duong and A. McIntosh: *The  $L^p$  boundedness of Riesz transforms associated with divergence form operators*. Proceedings of the Centre for Mathematical Analysis, ANU, Canberra **37** (1999) 15–25.
- [20] X. T. Duong and D. Robinson: *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*. J. Funct. Anal. **142** (1996), no. 1, 89–128.
- [21] J. Elschner, J. Rehberg and G. Schmidt: *Optimal regularity for elliptic transmission problems including  $C^1$  interfaces*. Interfaces Free Bound. **9** (2007) 233–252.
- [22] A. F. M. ter Elst and J. Rehberg:  *$L^\infty$ -estimates for divergence operators on bad domains*. Anal. Appl. (Singap.) **10** (2012), no. 2, 207–214.
- [23] A. F. M. ter Elst and D. W. Robinson: *On Kato’s square root problem*. Hokkaido Math. J. **26** (1997), no. 2, 365–376.
- [24] L. C. Evans and R. F. Gariepy: *Measure theory and fine properties of functions*. Studies in advanced mathematics, CRC Press, Boca Raton-New York-London-Tokyo, 1992.
- [25] J. A. Griepentrog, K. Gröger, H. C. Kaiser and J. Rehberg: *Interpolation for function spaces related to mixed boundary value problems*. Math. Nachr. **241** (2002) 110–120.
- [26] M. Haase: *The functional calculus for sectorial operators*. Vol. 169 of Operator Theory: Advances and Applications, Birkhäuser, Basel, 2006.

- [27] P. Hajlasz, P. Koskela and H. Tuominen: *Sobolev embeddings, extensions and measure density condition*. J. Funct. Anal. **254** (2008) 1217–1234.
- [28] P. Hajlasz: Sobolev spaces on metric measure spaces (heat kernels and analysis on manifolds, graphs, and metric spaces). Vol. 338 of Contemp. Math., Amer. Math. Soc., 2003, pp. 173–218.
- [29] R. Haller-Dintelmann, C. Meyer, J. Rehberg and A. Schiela: *Hölder continuity and optimal control for nonsmooth elliptic problems*. Appl. Math. Optim. **60** (2009), no. 3, 397–428.
- [30] R. Haller-Dintelmann and J. Rehberg: *Maximal parabolic regularity for divergence operators including mixed boundary conditions*. J. Differential Equations **247** (2009), no. 5, 1354–1396.
- [31] R. Haller-Dintelmann and J. Rehberg: *Coercivity for elliptic operators and positivity of solutions on Lipschitz domains*. Arch. Math. **95** (2010) 457–468.
- [32] R. Haller-Dintelmann and J. Rehberg: *Maximal regularity for divergence operators on distribution spaces*. In: J. Escher, P. Guidotti, P. Mucha, J. Prüss, Y. Shibata, G. Simonett, C. Walker and W. Zajączkowski (Eds.), Parabolic Problems, The Herbert Amann Festschrift, vol. 80 of Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, 2011.
- [33] M. Hieber: *Gaussian estimates and holomorphy of semigroups on  $L^p$  spaces*. J. London Math. Soc. (2) **54** (1996), no. 1, 148–160.
- [34] T. Hytönen, A. McIntosh and P. Portal: *Kato’s square root problem in Banach spaces*. J. Funct. Anal. **254** (2008), no. 3, 675–726.
- [35] A. Jonsson and H. Wallin: *Function spaces on subsets of  $\mathbb{R}^n$* . Math. Rep. **2** (1984), no. 1.
- [36] N. Kalton and L. Weis: *The  $H^\infty$ -calculus and sums of closed operators*. Math. Ann. **321** (2001), 319–345.
- [37] T. Kato: Perturbation theory for linear operators. Reprint of the corr. print of the 2nd ed., Classics in Mathematics, Springer-Verlag, Berlin, 1980.
- [38] R. Korte, J. Lehrbäck and H. Tuominen: *The equivalence between pointwise Hardy inequalities and uniform fatness*. Math. Ann. **351** (2011), no. 3, 711–731.
- [39] D. Lamberton: *Equations d’évolution linéaires associées à des semi-groupes de contractions dans les espaces  $L^p$* . J. Funct. Anal. **72** (1987), 252–262.
- [40] J. Lehrbäck: *Pointwise Hardy inequalities and uniformly fat sets*. Proc. Amer. Math. Soc. **136** (2008), no. 6, 2193–2200.
- [41] J. L. Lewis: *Uniformly fat sets*. Trans. Amer. Math. Soc. **308** (1998), no. 1, 177–196.
- [42] V. G. Maz’ya: Sobolev spaces. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.
- [43] I. Mitrea and M. Mitrea: *The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains*. Trans. Amer. Math. Soc. **359** (2007), no. 9, 4143–4182.
- [44] E. Ouhabaz: Analysis of Heat Equations on Domains. Vol. 31 of London Mathematical Society Monographs Series, Princeton University Press, Princeton, 2005.
- [45] A. Pazy: Semigroups of linear operators and applications to partial differential equations. Springer, 1983.
- [46] J. Prüss: *Maximal regularity for evolution equations in  $L^p$ -spaces*. Conf. Semin. Mat. Univ. Bari **285** (2002) 1–39.
- [47] R. Seeley: *Fractional powers of boundary problems*. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Paris, 1971, pp. 795–801.
- [48] E. M. Stein: Singular integrals and differentiability properties of functions. Vol. 30 of Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1970.
- [49] A. Torchinsky: Real variable methods in harmonic analysis. Vol 123 of Pure and Applied Mathematics, Academic Press, Orlando, 1986.
- [50] H. Triebel: Interpolation theory, function spaces, differential operators. North Holland Publishing Company, Amsterdam-New York-Oxford, 1978.
- [51] A. Wannebo: *Hardy inequalities*, Proc. Amer. Math. Soc. **109** (1990) 85–95.

UNIVERSITE PARIS-SUD, BATIMENT 425, 91405 ORSAY CEDEX, FRANCE  
*E-mail address:* [Pascal.Auscher@math.u-psud.fr](mailto:Pascal.Auscher@math.u-psud.fr)

INSTITUT CAMILLE JORDAN, UNIVERSITÉ CLAUDE BERNARD LYON 1, 43, BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE  
*E-mail address:* [badr@math.univ-lyon1.fr](mailto:badr@math.univ-lyon1.fr)

TECHNISCHE UNIVERSITÄT DARMSTADT, FACHBEREICH MATHEMATIK, SCHLOSSGARTENSTR. 7, D-64298 DARMSTADT, GERMANY  
*E-mail address:* [haller@mathematik.tu-darmstadt.de](mailto:haller@mathematik.tu-darmstadt.de)

WEIERSTRASS INSTITUTE FOR APPLIED ANALYSIS AND STOCHASTICS, MOHRENSTR. 39, D-10117 BERLIN, GERMANY  
*E-mail address:* [rehberg@wias-berlin.de](mailto:rehberg@wias-berlin.de)