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# On the number of common sites visited by $N$ random walkers

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Majumdar and Tamm [Phys. Rev. E **86** 021135 (2012)] recently obtained analytical expressions for the mean number of *common* sites  $W_N(t)$  visited up to time  $t$  by  $N$  independent random walkers starting from the origin of a  $d$ -dimensional lattice. In this short note I show how the different regimes and the corresponding asymptotic power laws can be retrieved using the notion of fractal intersection.

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In their recent work [1], Majumdar and Tamm computed analytically the mean number of *common* sites  $W_N(t)$  visited at time  $t$  by  $N$  independent random walkers starting from the origin of a  $d$ -dimensional lattice at  $t = 0$ . Three distinct regimes were obtained for the large- $t$  behavior:

$$\begin{aligned} W_N(t) &\sim t^{d/2} \quad \text{for } d < 2 \\ &\sim t^{N-d(N-1)/2} \quad \text{for } 2 < d < d_c(N) \\ &\sim \text{const.} \quad \text{for } d > d_c(N) = \frac{2N}{N-1}. \end{aligned} \quad (1)$$

The exponent governing the asymptotic time-dependence of  $W_N(t)$  is continuously varying with  $N$  and  $d$  between the lower critical dimension  $d'_c = 2$  and the upper critical dimension  $d_c(N)$ . Logarithmic corrections appear exactly at the critical dimensions with  $W_N(t) \sim t/[\ln t]^N$  in  $d = d'_c = 2$  and  $W_N(t) \sim \ln t$  in  $d = d_c(N)$  (with  $N > 1$ ).

In this note I show how the long-time power-law behavior of  $W_N(t)$  given in Eq. (1) can be simply recovered using an heuristic argument based on the notion of fractal intersection.

In the problem at hand the common sites visited by the  $N$  random walkers belong to a fractal object which is the intersection of the  $N$  independent random walks. Let  $d_f=2$  denote the fractal dimension of a single random walk and  $\overline{d}_f = d - d_f \geq 0$  its fractal codimension. According to Mandelbrot [2] the codimension of the intersection of  $N$  independent random fractals is given by

$$\overline{d}_f(N) = \min(d, N \overline{d}_f) \geq 0. \quad (2)$$

Below the lower critical dimension  $d'_c = d_f$ ,  $\overline{d}_f$  and  $\overline{d}_f(N)$  vanish, giving

$$d_f(N) = d - \overline{d}_f(N) = d, \quad d < d'_c = d_f. \quad (3)$$

The upper critical dimension  $d_c(N)$  corresponds to the equality of the two terms on the right in Eq. (2) so that

$$d_c(N) = \frac{Nd_f}{N-1}. \quad (4)$$

Above this value  $\overline{d}_f(N) = d$ , leading to

$$d_f(N) = 0, \quad d > d_c(N). \quad (5)$$

Between the two critical dimensions  $\overline{d}_f(N) = N \overline{d}_f$  applies and one finds

$$d_f(N) = Nd_f - d(N-1), \quad d_f < d < d_c(N). \quad (6)$$

The radius of a walk,  $R(t)$ , typically grows as  $t^{1/d_f}$  such that

$$W_N(t) \sim R(t)^{d_f(N)} \sim t^{d_f(N)/d_f}. \quad (7)$$

Collecting these results, one finally obtains

$$\begin{aligned} W_N(t) &\sim t^{d/d_f} \quad \text{for } d < d_f \\ &\sim t^{N-(N-1)d/d_f} \quad \text{for } d_f < d < d_c(N) \\ &\sim \text{const.} \quad \text{for } d > d_c(N) = \frac{d_f N}{N-1}, \end{aligned} \quad (8)$$

in complete agreement with Eq. (1) for the random walks with  $d_f = 2$ .

Note that the logarithmic growth of  $W_N(t)$  at the upper critical dimension  $d_c(N)$  can be obtained by working with the fractal density of the intersection [3]

$$\rho_N(r) \sim \frac{d[r^{d_f(N)}]}{d[r^d]} \sim r^{-\overline{d}_f(N)}, \quad r \leq R(t). \quad (9)$$

Then:

$$W_N(t) \sim \int_a^{R(t)} \rho_N(r) r^{d-1} dr \sim \int_a^{R(t)} r^{d_f(N)-1} dr. \quad (10)$$

At the upper critical dimension  $d_f(N)$  vanishes so that:

$$W_N(t) \sim \ln \frac{R(t)}{a} \sim \ln t. \quad (11)$$

These results are expected to apply as well in the case of subdiffusive or superdiffusive diffusion processes [4] with the appropriate value for the fractal dimension of the  $N$  walks,  $d_f \neq 2$ . For directed walks, an extension of the rules of fractal intersection to *anisotropic* fractals is needed.

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- [1] S. N. Majumdar and M. V. Tamm, Phys. Rev. E **86**, 021135 (2012).
- [2] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982) p. 329–330 and 365.
- [3] L. Turban, *Mean number of encounters of  $N$  random walkers and intersection of strongly anisotropic fractals*, arXiv:1304.4106.
- [4] See for example J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990); R. Metzler and J. Klafter, *ibid.* **339**, 1 (2000).