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# Minimax properties of Fréchet means of discretely sampled curves

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## Abstract

We study the problem of estimating a mean pattern from a set of similar curves in the setting where the variability in the data is due to random geometric deformations and additive noise. This problem requires to define non-Euclidean distances by using the action of a Lie group on an infinite dimensional space of curves. This approach leads to the construction of estimators based on the notion of Fréchet mean that is a generalization of the standard notion of averaging to non-Euclidean spaces. A recent research direction in nonparametric statistics is the study of the properties of the Fréchet mean in deformable models, and the development of consistent estimators of a mean pattern. Using such models, we show the links that exist between minimax theory in nonparametric statistics and the problem of estimating a mean pattern from a sequence of curves.

*Keywords:* Fréchet mean, Non-Euclidean metric, Deformable models, Lie group action, Curve registration, Minimax rate of convergence, Sobolev balls, Functional data analysis.

*AMS classifications:* Primary 62G08; secondary 62G20.

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## 1 Introduction

### 1.1 Fréchet means

The Fréchet mean [Fré48] is an extension of the usual Euclidean mean to non-linear spaces endowed with non-Euclidean metrics. If  $\mathbf{Y}_1, \dots, \mathbf{Y}_J$  denote i.i.d. random variables with values in a metric space  $\mathcal{M}$  with metric  $d_{\mathcal{M}}$ , then the empirical Fréchet mean  $\bar{\mathbf{Y}}_{\mathcal{M}}$  of the sample

$\mathbf{Y}_1, \dots, \mathbf{Y}_J$  is defined as a minimizer (not necessarily unique) of

$$\bar{\mathbf{Y}}_{\mathcal{M}} \in \operatorname{argmin}_{y \in \mathcal{M}} \frac{1}{J} \sum_{j=1}^J d_{\mathcal{M}}^2(y, \mathbf{Y}_j).$$

For random variables belonging to a nonlinear manifold, a well-known example is the computation of the mean of a set of planar shapes in the Kendall's shape space [Ken84] that leads to the Procrustean means studied in [Goo91]. A detailed study of some properties of the Fréchet mean in finite dimensional Riemannian manifolds (such as consistency and uniqueness) has been performed in [Afs11, BP03, BP05, Huc11]. However, there is not so much work on the properties of the Fréchet mean in infinite dimensional and non-Euclidean spaces of curves or images. In this paper, we are concerned by the nonparametric estimation of a mean pattern (belonging to a non-linear space) from a set of similar curves in the setting where the variability in the data is due to random geometric deformations and additive noise.

More precisely, let us consider noisy realizations of  $J$  curves  $f_1, \dots, f_J : [0, 1] \rightarrow \mathbb{R}$  sampled at  $n$  equispaced points  $t_\ell = \frac{\ell}{n}, \ell = 1, \dots, n$ ,

$$Y_{\ell,j} = f_j(t_\ell) + \epsilon_{\ell,j}, \ell = 1, \dots, n, \text{ and } j = 1, \dots, J, \quad (1.1)$$

where the  $\epsilon_{\ell,j}$ 's are independent and identically distributed (i.i.d) Gaussian variables with zero expectation and variance  $\sigma^2 > 0$ . In many applications, the observed curves have a similar structure that may lead to the assumption that the  $f_j$ 's are random elements varying around the same mean pattern  $f : [0, 1] \rightarrow \mathbb{R}$  (also called reference template). However, due to additive noise and geometric variability in the data, this mean pattern is typically unknown and has to be estimated. In this setting, a widely used approach is Grenander's pattern theory [Gre93, GM07, TY05, TY11] that models geometric variability by the action of a Lie group on an infinite dimensional space of curves (or images).

When the curves  $f_j$  in (1.1) exhibit a large source of geometric variations in time, this may significantly complicates the construction of a consistent estimator of a mean pattern. In what follows, we consider the simple model of randomly shifted curves that is commonly used in many applied areas such as neurosciences [TIR11] or biology [Røn01]. In such a framework, we have

$$f_j(t) = f(t - \theta_j^*), \text{ for all } t \in [0, 1] \text{ and } j = 1, \dots, J, \quad (1.2)$$

where  $f : [0, 1] \rightarrow \mathbb{R}$  is an unknown curve that can be extended outside  $[0, 1]$  by 1-periodicity. The shifts  $\theta_j^*$ 's are supposed to be i.i.d. real random variables (independent of the  $\epsilon_{\ell,j}$ 's) that are sampled from an unknown distribution  $\mathbb{P}_*$  on  $\mathbb{R}$ . In model (1.2), the shifts  $\theta_j^*$  represent a source of geometric variability in time.

## 1.2 Model and objectives

The main goal of this paper is to construct nonparametric estimators of the mean pattern  $f$  from the data

$$Y_{\ell,j} = f(t_\ell - \theta_j^*) + \epsilon_{\ell,j}, \ell = 1, \dots, n, \text{ and } j = 1, \dots, J, \quad (1.3)$$

in the setting where both the number  $J$  of curves and the number  $n$  of design points are allowed to vary and to tend towards infinity.

In the sequel of the paper, it will be assumed that the random shifts are sampled from an unknown density  $g$  with respect to the Lebesgue measure  $d\theta$  (namely  $d\mathbb{P}_*(\theta) = g(\theta)d\theta$ ). Note that since  $f$  is assumed to be 1-periodic, one may restrict to the case where the density  $g$  has a compact support included in the interval  $[-\frac{1}{2}, \frac{1}{2}]$ . Under the assumption (1.2), the (standard) Euclidean mean  $\bar{Y}_\ell = \frac{1}{J} \sum_{j=1}^J Y_{\ell,j}$  of the data is generally not a consistent estimator of the mean pattern  $f$  at  $t = t_\ell$ . Indeed, the law of large numbers implies that

$$\lim_{J \rightarrow \infty} \bar{Y}_\ell = \lim_{J \rightarrow \infty} \frac{1}{J} \sum_{j=1}^J f(t_\ell - \theta_j^*) = \int_{\mathbb{R}} f(t_\ell - \theta)g(\theta)d\theta \quad a.s..$$

Thus, under mild assumptions on  $f$ , one has  $\lim_{J \rightarrow \infty} \bar{Y}_\ell \neq f(t_\ell)$  *a.s.*

To build a consistent estimator of  $f$  in model (1.3), we propose to use a notion of empirical Fréchet mean in an infinite dimensional space. Recently, some properties of Fréchet means in randomly shifted curves models have been investigated in [BG10] and [BC11]. However, studying the rate of convergence and the minimax properties of such estimators in the double asymptotic setting  $\min(n, J) \rightarrow +\infty$  has not been considered so far.

Note that model (1.3) is clearly not identifiable as for any  $\tilde{\theta} \in [-\frac{1}{2}, \frac{1}{2}]$ , one may replace  $f(\cdot)$  by  $\tilde{f}(\cdot) = f(\cdot - \tilde{\theta})$  and  $\theta_j^*$  by  $\tilde{\theta}_j = \theta_j^* - \tilde{\theta}$  without changing model (1.3). Therefore, estimation of  $f$  is only feasible up to a time shift. Thus, we propose to consider the problem of estimating its equivalence class  $[f]$  (or orbit) under the action of shifts. More precisely, let  $L_{per}^2([0, 1])$  be the space of squared integrable functions on  $[0, 1]$  that can be extended outside  $[0, 1]$  by 1-periodicity. Let  $\mathbb{S}_1$  be the one-dimensional torus. We recall that any element  $\tau = \tau(\theta) \in \mathbb{S}_1$  can be identified with an element  $\theta$  in the interval  $[-\frac{1}{2}, \frac{1}{2}[$ . For  $f \in L_{per}^2([0, 1])$ , we define its equivalence class by the action of a time shift as

$$[f] := \{f^\tau, \tau \in \mathbb{S}_1\},$$

where for  $\tau = \tau(\theta) \in \mathbb{S}_1$  (with  $\theta \in [-\frac{1}{2}, \frac{1}{2}[$ ),  $f^\tau(t) = f(t - \theta)$  for all  $t \in [0, 1]$ . Let  $f, h \in L_{per}^2([0, 1])$ , we define the distance between  $[f], [h] \in L_{per}^2([0, 1])/\mathbb{S}_1$  as

$$d([f], [h]) = \inf_{\theta \in [-\frac{1}{2}, \frac{1}{2}]} \left( \int_0^1 |f(t - \theta) - h(t)|^2 dt \right)^{1/2}. \quad (1.4)$$

In the setting of Grenander's pattern theory,  $(L_{per}^2([0, 1])/\mathbb{S}_1, d)$  represents an infinite dimensional and non-linear set of curves, and  $\mathbb{S}_1$  is a Lie group modeling geometric variability in the data.

### 1.3 Main contributions

Let us assume that  $\mathcal{F} \subset L_{per}^2([0, 1])$  represents some smoothness class of functions (e.g a Sobolev ball). Suppose also that the unknown density  $g$  of the random shifts in (1.2) belongs to some set  $\mathcal{G}$  of probability density functions on  $[-\frac{1}{2}, \frac{1}{2}]$ . Let  $\hat{f}_{n,J}$  be some estimator of  $f$  based on the random variables  $Y_{\ell,j}$  given by (1.3) taking its values in  $L_{per}^2([0, 1])$ . For some  $f \in \mathcal{F}$ , the risk of

the estimator  $\hat{f}_{n,J}$  is defined by

$$\mathcal{R}_g(\hat{f}_{n,J}, f) = \mathbb{E}^g \left( d^2([\hat{f}_{n,J}], [f]) \right),$$

where the above expectation  $\mathbb{E}^g$  is taken with respect to the distribution of the  $Y_{\ell,j}$ 's in (1.3) and under the assumption that the shifts are i.i.d. random variables sampled from the density  $g$ . We propose to investigate the optimality of an estimator by introducing the following minimax risk

$$\mathcal{R}_{n,J}(\mathcal{F}, \mathcal{G}) = \inf_{\hat{f}_{n,J}} \sup_{g \in \mathcal{G}} \sup_{f \in \mathcal{F}} \mathcal{R}_g(\hat{f}_{n,J}, f),$$

where the above infimum is taken over the set of all possible estimators in model (1.3).

For  $f \in L^2_{per}([0, 1])$ , let us denote its Fourier coefficients by  $c_k = \int_0^1 f(t) e^{-i2\pi kt} dt$ ,  $k \in \mathbb{Z}$ . Suppose that  $\mathcal{F}$  is a Sobolev ball of non-constant functions with degree of smoothness  $s > 1/2$ , namely

$$\mathcal{F} = \tilde{W}_s(A, c_*) = \left\{ f \in L^2_{per}([0, 1]); \sum_{k \in \mathbb{Z}} (1 + |k|^{2s}) |c_k|^2 \leq A^2, \text{ with } |c_1| \geq c_* \right\}$$

for some positive reals  $A$  and  $c_*$ .

Moreover, let  $\mathcal{G}^\kappa$  be a set of probability densities having a compact support of size smaller than  $\kappa$  with  $0 < \kappa < 1/8$  defined as

$$\mathcal{G}^\kappa := \left\{ g : \left[ -\frac{1}{2}, \frac{1}{2} \right] \rightarrow \mathbb{R}^+ ; \int_{-1/2}^{1/2} g(\theta) d\theta = 1, \text{ and } \text{supp}(g) = [-\kappa/2, \kappa/2] \right\}.$$

Suppose also that the following condition holds

$$J \asymp n^\alpha, \text{ for some } 0 < \alpha < 1/6,$$

where the notation  $J \asymp n^\alpha$  means that there exists two positive constants  $c_2 \geq c_1 > 0$  such that  $c_1 n^\alpha \leq J \leq c_2 n^\alpha$  for any choices of  $J$  and  $n$ .

Then, under such assumptions, the main contribution of the paper is to show that one can construct an estimator  $\hat{f}_{n,J}$  based on a smoothed Fréchet mean of discretely sampled curves that satisfies

$$\limsup_{\min(n,J) \rightarrow +\infty} r_{n,J}^{-1} \sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \tilde{W}_s(A, c_*)} \mathcal{R}_g(\hat{f}_{n,J}, f) \leq C_0$$

where  $C_0 > 0$  is a constant that only depends on  $A$ ,  $s$ ,  $\kappa$ ,  $c_*$  and  $\sigma^2$ . The rate of convergence  $r_{n,J}$  is given by

$$r_{n,J} = n^{-1} + (nJ)^{-\frac{2s}{2s+1}}.$$

The two terms in the rate  $r_{n,J}$  have different interpretations. The second term  $(nJ)^{-\frac{2s}{2s+1}}$  is the usual nonparametric rate for estimating the function  $f$  (over a Sobolev ball) in model (1.3) that we would obtain if the true shifts  $\theta_1^*, \dots, \theta_J^*$  were known.

The first term  $n^{-1}$  in the rate  $r_{n,J}$  can be interpreted as follows. As shown later in the paper, the computation of a Fréchet mean of curves is a two step procedure. It consists in

building estimators  $\hat{\theta}_j^0$  of the unknown shifts and, then, in aligning the observed curves. For  $\theta = (\theta_1, \dots, \theta_J) \in \mathbb{R}^J$ , let us define the Euclidean norm  $\|\theta\| = \left(\sum_{j=1}^J |\theta_j|^2\right)^{1/2}$ . One of the contributions of this paper is to show that an estimation of the vector

$$\theta^0 = (\theta_1^* - \bar{\theta}_J, \dots, \theta_J^* - \bar{\theta}_J)' \in \mathbb{R}^J, \text{ where } \bar{\theta}_J = \frac{1}{J} \sum_{j=1}^J \theta_j^*,$$

is feasible at the rate  $n^{-1}$  for the normalized quadratic risk  $\frac{1}{J}\mathbb{E}\|\hat{\theta}^0 - \theta^0\|^2$  and that this allows to build a consistent Fréchet mean. If the number  $J$  of curves was fixed,  $n^{-1}$  would correspond to the usual semi-parametric rate for estimating the shifts in model (1.3) in the setting where the  $\theta_j^*$  are non-random parameters, see [BLV10, GLM07, Vim10] for further details. Here, this rate of convergence has been obtained in the double asymptotic setting  $\min(n, J) \rightarrow +\infty$ . This setting significantly complicates the estimation of the vector  $\theta^0 \in \mathbb{R}^J$  since its dimension  $J$  is increasing with the sample size  $nJ$ . Hence, in the case of  $\min(n, J) \rightarrow +\infty$ , estimating the shifts at the rate  $n^{-1}$  is not a standard semi-parametric problem and we had to impose the constraint  $J \asymp n^\alpha$  (with  $0 < \alpha < 1/6$ ) to obtain this result. The term  $n^{-1}$  in the rate  $r_{n,J}$  is thus the price to pay for not knowing the random shifts in (1.3) that need to be estimated to compute a Fréchet mean.

## 1.4 Organization of the paper

In Section 2, we define a notion of smoothed Fréchet means of curves. We also discuss the connection between this approach and the well known problems of curve registration and image warping. In Section 3, we discuss the rate of converge of the estimators of the shifts. We also build a Fréchet mean using model selection techniques and we derive an upper bound on its rate of convergence. In Section 4, we derive a lower bound on the minimax risk and we give some sufficient conditions to obtain a smoothed Fréchet mean converging at an optimal rate in the minimax sense. In Section 5, we discuss the main results of the paper and their connections with the nonparametric literature on deformable models. Some numerical experiments on simulated data are presented in Section 6. The proof of the main results are gathered in a technical Appendix.

## 2 Smoothed Fréchet means of curves

Let  $f_1, \dots, f_J$  be a set of functions in  $L^2_{per}([0, 1])$ . We define the Fréchet mean  $[\bar{f}]$  of  $[f_1], \dots, [f_J]$  as

$$[\bar{f}] \in \underset{[f] \in L^2_{per}([0, 1])/\mathbb{S}_1}{\operatorname{argmin}} \frac{1}{J} \sum_{j=1}^J d^2([f], [f_j]).$$

It can be easily checked that a representant  $\bar{f} \in L^2_{per}([0, 1])$  of the class  $[\bar{f}]$  is given by the following two steps procedure:

1. Computation of shifts to align the curves

$$(\tilde{\theta}_1, \dots, \tilde{\theta}_J) \in \underset{(\theta_1, \dots, \theta_J) \in [-\frac{1}{2}, \frac{1}{2}]^J}{\operatorname{argmin}} \frac{1}{J} \sum_{j=1}^J \int_0^1 \left| f_j(t + \theta_j) - \frac{1}{J} \sum_{j'=1}^J f_{j'}(t + \theta_{j'}) \right|^2 dt. \quad (2.1)$$

2. Averaging after an alignment step :  $\bar{f}(t) = \frac{1}{J} \sum_{j=1}^J f_j(t + \tilde{\theta}_j)$ ,  $t \in [0, 1]$ .

Let us now explain how the above two steps procedure can be used to define an estimator of  $f$  in model (1.3). Let

$$\left\{ \phi_k(t) = e^{i2\pi kt}, t \in [0, 1] \right\}_{k \in \mathbb{Z}}$$

be the standard Fourier basis. For legibility, we assume that  $n = 2N \geq 4$  is even and we split the data into two samples as follows,

$$Y_{q,j}^{(0)} = Y_{2q,j} \quad \text{and} \quad Y_{q,j}^{(1)} = Y_{2q-1,j}, \quad q = 1, \dots, N,$$

for  $j = 1, \dots, J$ , and

$$t_q^{(0)} = t_{2q} \quad \text{and} \quad t_q^{(1)} = t_{2q-1}, \quad q = 1, \dots, N.$$

For any  $z \in \mathbb{C}$ , we denote by  $\bar{z}$  its complex conjugate. Then, we define the following empirical Fourier coefficients

$$\begin{aligned} \hat{c}_{k,j}^{(0)} &= \frac{1}{N} \sum_{q=1}^N Y_{q,j}^{(0)} \overline{\phi_k(t_q^{(0)})} = \bar{c}_{k,j}^{(0)} + \frac{1}{\sqrt{N}} z_{k,j}^{(0)}, \quad -\frac{N}{2} \leq k < \frac{N}{2}, \quad j = 1, \dots, J, \\ \hat{c}_{k,j}^{(1)} &= \frac{1}{N} \sum_{q=1}^N Y_{q,j}^{(1)} \overline{\phi_k(t_q^{(1)})} = \bar{c}_{k,j}^{(1)} + \frac{1}{\sqrt{N}} z_{k,j}^{(1)}, \quad -\frac{N}{2} \leq k < \frac{N}{2}, \quad j = 1, \dots, J, \end{aligned}$$

where

$$\bar{c}_{k,j}^{(p)} = \frac{1}{N} \sum_{q=1}^N f(t_q^{(p)} - \theta_j^*) \overline{\phi_k(t_q^{(p)})}, \quad p \in \{0, 1\},$$

and the  $z_{k,j}^{(p)}$ 's are i.i.d. complex Gaussian variables with zero expectation and variance  $\sigma^2$ .

Then, we define estimators of the unknown random shifts  $\theta_j^*$  as

$$(\hat{\theta}_1, \dots, \hat{\theta}_J) \in \underset{(\theta_1, \dots, \theta_J) \in [-\frac{1}{2}, \frac{1}{2}]^J}{\operatorname{argmin}} M_n(\theta_1, \dots, \theta_J) \quad (2.2)$$

where

$$M_n(\theta_1, \dots, \theta_J) = \frac{1}{J} \sum_{j=1}^J \sum_{|k| \leq k_0} \left| \hat{c}_{k,j}^{(0)} e^{i2\pi k \theta_j} - \frac{1}{J} \sum_{j'=1}^J \hat{c}_{k,j'}^{(0)} e^{i2\pi k \theta_{j'}} \right|^2, \quad (2.3)$$

with some positive integer  $k_0$  that will be discussed later. The smoothed Fréchet mean of  $f$  is then defined as

$$\hat{f}_{n,J}^{(m)}(t) = \sum_{|k| \leq m} \left( \frac{1}{J} \sum_{j=1}^J \hat{c}_{k,j}^{(1)} e^{i2\pi k \hat{\theta}_j} \right) \phi_k(t) = \frac{1}{J} \sum_{j=1}^J \hat{f}_j^{(m)}(t + \hat{\theta}_j), \quad t \in [0, 1],$$

where the integer  $m \in \{1, \dots, N/2\}$  is a frequency cut-off parameter that will be discussed later and  $\hat{f}_j^{(m)}(t) = \sum_{|k| \leq m} \hat{c}_{k,j}^{(1)} \phi_k(t)$ . Note that the estimators  $\hat{\theta}_j$  of the shifts have been computed using only half of the data and that the curves  $\hat{f}_j^{(m)}$  are calculated using the other half of the data. By splitting the data in such a way, the random variables  $\hat{\theta}_j$  and  $\hat{f}_j^{(m)}$  are independent conditionally to  $(\theta_1^*, \dots, \theta_J^*)$ .

Note also that this two step procedure does not require the use of a reference template to compute estimators  $\hat{\theta}_1, \dots, \hat{\theta}_J$  of the random shifts. Indeed, one can interpret the term  $\frac{1}{J} \sum_{j'=1}^J \hat{f}_{j'}(t + \theta_{j'})$  in (2.1) as a template that is automatically estimated. In statistics, estimating a mean pattern from set of curves that differ by a time transformation is usually referred to as the curve registration problem. It has received a lot of attention over the last two decades, see e.g. [Big06, Røn01, TIR11] and references therein. Hence, there exists a connection between our approach and the well known problems of curve registration and its generalization to higher dimensions (image warping), see e.g. [GM01]. However, studying the minimax properties of an estimator of a mean pattern in curve registration models has not been investigated so far.

### 3 Upper bound on the risk

#### 3.1 Consistent estimation of the unknown shifts

Note that, due to identifiability issues in model (1.3), the minimization (2.2) is not well defined. Indeed, for any  $(\hat{\theta}_1, \dots, \hat{\theta}_J)$  that minimizes (2.3), one has that for any  $\tilde{\theta}$  such that  $(\hat{\theta}_1 + \tilde{\theta}, \dots, \hat{\theta}_J + \tilde{\theta}) \in [-\frac{1}{2}, \frac{1}{2}]^J$ , this vector is also a minimizer of  $M_n$ . Choosing identifiability conditions amounts to impose constraints on the minimization of the criterion

$$M(\theta_1, \dots, \theta_J) = \frac{1}{J} \sum_{j=1}^J \sum_{|k| \leq k_0} \left| c_k e^{i2\pi k(\theta_j - \theta_j^*)} - \frac{1}{J} \sum_{j'=1}^J c_k e^{i2\pi k(\theta_{j'} - \theta_{j'}^*)} \right|^2, \quad (3.1)$$

where  $c_k$ ,  $k \in \mathbb{Z}$ , are the Fourier coefficients of the mean pattern  $f$ . The criterion (3.1) can be interpreted as a version without noise of the criterion (2.3) when replacing  $\hat{c}_{k,j}^{(0)}$  by  $c_k e^{-i2\pi k \theta_j^*}$ . Obviously, the criterion (3.1) admits a minimum at  $\theta^* = (\theta_1^*, \dots, \theta_J^*)$  such that  $M(\theta^*) = 0$ . However, this minimizer over  $[-\frac{1}{2}, \frac{1}{2}]^J$  is clearly not unique. To impose uniqueness of some minimum of  $M$  over a restricted set, let us introduce the following identifiability conditions:

**Assumption 1.** *The distribution  $g$  of the random shifts has a compact support included in  $[-\kappa/2, \kappa/2]$  for some  $0 < \kappa < 1/8$ .*

**Assumption 2.** *The mean pattern  $f$  in model (1.3) is such that  $c_1 = \int_0^1 f(x) e^{-i2\pi x} dx \neq 0$ .*

Assumption 1 means that the support of the density  $g$  of the random shifts should be sufficiently small. This implies that the shifted curves  $f(t - \theta_j^*)$  are somehow concentrated around the unknown mean pattern  $f$ . Such an assumption of concentration of the data around a reference shape has been used in various papers to prove the uniqueness and the consistency of Fréchet means for random variables lying in a Riemannian manifold, see [Afs11, BP03, BP05, Huc11].



Assumption 2 is an identifiability condition to avoid the case where the function  $f$  is constant over  $[0, 1]$  which would make impossible the estimation of the unobserved random shifts.

For  $0 < \kappa < 1/8$ , let us define the constrained set

$$\Theta_\kappa = \left\{ (\theta_1, \dots, \theta_J) \in [-\kappa/2, \kappa/2]^J, \sum_{j=1}^J \theta_j = 0 \right\}.$$

Let

$$\theta_j^0 = \theta_j^* - \frac{1}{J} \sum_{j'=1}^J \theta_{j'}^*, \quad j = 1, \dots, J, \quad \text{and } \theta^0 = (\theta_1^0, \dots, \theta_J^0).$$

Thanks to Proposition 4.1 in [BC11], we have

**Proposition 3.1.** *Suppose that Assumptions 1 and 2 hold. Then, for any  $(\theta_1, \dots, \theta_J) \in \Theta_\kappa$ ,*

$$M(\theta_1, \dots, \theta_J) - M(\theta_1^0, \dots, \theta_J^0) \geq C(f, \kappa) \frac{1}{J} \sum_{j=1}^J |\theta_j - \theta_j^0|^2,$$

where  $C(f, \kappa) = 4\pi^2 |c_1|^2 \cos(4\pi\kappa) > 0$ .

Therefore, over the constrained set  $\Theta_\kappa$ , the criterion (3.1) has a unique minimum at  $\theta^0$  such that  $M(\theta^0) = 0$ . Let us now consider the estimators

$$\hat{\theta}^0 = (\hat{\theta}_1^0, \dots, \hat{\theta}_J^0) \in \underset{(\theta_1, \dots, \theta_J) \in \Theta_\kappa}{\operatorname{argmin}} M_n(\theta_1, \dots, \theta_J). \quad (3.2)$$

The following Theorem shows that, under appropriate assumptions, the vector  $\hat{\theta}^0$  is a consistent estimator of  $\theta^0$ .

**Theorem 3.1.** *Suppose that Assumptions 1 and 2 hold. Let  $J \geq 2$  and  $s \geq 2$ , then there exists a constant  $C > 0$  that only depends on  $A, s, \kappa, c_*$  and  $\sigma^2$  such that, for any  $f \in \tilde{W}_s(A, c_*)$ , we have*

$$\frac{1}{J} \mathbb{E}^g \|\hat{\theta}^0 - \theta^0\|^2 \leq \frac{C}{n} \left( 1 + \frac{k_0^5}{n^{1/2}} \right) \left( 1 + \frac{k_0^{3/2} J^3}{n^{1/2}} \right). \quad (3.3)$$

Inequality (3.3) shows that the quality of the estimation of the random shifts depends on the ratio between  $n$  and  $J$ . In particular, it suggests that the quality of this estimation should deteriorate if the number  $J$  of curves increases and  $n$  remains fixed. This shows that estimating the vector  $\hat{\theta}^0 \in \mathbb{R}^J$  is not a standard parametric problem, since the dimension  $J$  is allowed to grow to infinity in our setting. To the contrary, if  $J$  is not too large with respect to  $n$ , then an estimation of the shifts is feasible at the usual parametric rate  $n^{-1}$ . More precisely, by Theorem 3.1, we immediately have the following result:

**Corollary 3.1.** *Suppose that the assumptions of Theorem 3.1 are satisfied. If  $k_0 \geq 1$  is a fixed integer and  $J \asymp n^\alpha$  for some  $0 < \alpha \leq 1/6$ , then there exists  $C_1 > 0$  that only depends on  $A, s, \sigma^2, \kappa$  and  $k_0$  such that*

$$\frac{1}{J} \mathbb{E}^g \|\hat{\theta}^0 - \theta^0\|^2 \leq \frac{C_1}{n}.$$

Therefore, under the additional assumption that  $J \asymp n^\alpha$ , for some  $0 < \alpha \leq 1/6$ , the vector  $\hat{\theta}^0$  converges to  $\theta^0$  at the rate  $n^{-1}$  for the normalized Euclidean norm.

### 3.2 Estimation of the mean pattern

For  $p \in \{0, 1\}$ , let  $Y^{(p)} = (Y_{q,j}^{(p)})_{1 \leq q \leq N, 1 \leq j \leq J}$ . Thanks to the estimator  $\hat{\theta}^0$  of the random shifts, we can align the data  $Y^{(1)}$  in order to estimate the mean pattern  $f$  in (1.3). Let  $m_1 < N/2$  be some positive integer. For any  $m \in \{1, \dots, m_1\}$ , we recall that the estimator  $\hat{f}_{n,J}^{(m)}$  is given by

$$\hat{f}_{n,J}^{(m)}(t) = \frac{1}{J} \sum_{j=1}^J \hat{f}_j^{(m)}(t + \hat{\theta}_j^0), \quad t \in [0, 1].$$

To simplify the notations, we omit the dependency on  $k_0$ ,  $n$  and  $J$  of the above estimators and we write  $\hat{f}^{(m)} = \hat{f}_{n,J}^{(m)}$ . We denote by  $\mathbb{E}^{(1)}$  the expectation according to the distribution of  $Y^{(1)}$ . By construction, we recall that  $Y^{(0)}$  and  $Y^{(1)}$  are independent. Thus, note that  $\hat{f}^{(m)}$  is a biased estimator of  $f$  with respect to  $\mathbb{E}^{(1)}$  since

$$\mathbb{E}^{(1)} \left[ \hat{f}^{(m)}(t) \right] = \bar{f}^{(m)}(t) = \frac{1}{J} \sum_{j=1}^J \bar{f}_j^{(m)}(t + \hat{\theta}_j^0), \quad t \in [0, 1],$$

where we have set

$$\bar{f}_j^{(m)}(t) = \sum_{|k| \leq m} \bar{c}_{k,j}^{(1)} \phi_k(t), \quad j \in \{1, \dots, J\}.$$

The idea of the procedure is that if the estimators  $\hat{\theta}_j^0$  of the shifts behave well then  $d^2([f], [\bar{f}^{(m_1)}])$  is small and estimating  $f$  amounts to estimate  $\bar{f}^{(m_1)}$ .

To choose an estimator of  $\bar{f}^{(m_1)}$  among the  $\hat{f}^{(m)}$ 's, we take a model selection approach. Before describing the procedure, let us compute the quadratic risk of an estimator  $\hat{f}^{(m)}$ ,

$$\mathbb{E}^{(1)} \left[ \int_0^1 \left| \bar{f}^{(m_1)}(t) - \hat{f}^{(m)}(t) \right|^2 dt \right] = \int_0^1 \left| \bar{f}^{(m_1)}(t) - \bar{f}^{(m)}(t) \right|^2 dt + \frac{(2m+1)\sigma^2}{NJ}.$$

This risk is a sum of two non-negative terms. The first one is a bias term that is small when  $m$  is close to  $m_1$  while the second one is a variance term that is small when  $m$  is close to zero. The aim is to find a trade-off between these two terms thanks to the data only. More precisely, we choose some  $\hat{m} \in \{1, \dots, m_1\}$  such that

$$\hat{m} \in \operatorname{argmin}_{m \in \{1, \dots, m_1\}} \left\{ \int_0^1 \left| \bar{f}^{(m_1)}(t) - \hat{f}^{(m)}(t) \right|^2 dt + \eta \frac{(2m+1)\sigma^2}{NJ} \right\} \quad (3.4)$$

where  $\eta > 1$  is some constant. In the sequel, the estimator that we finally consider is  $\hat{f}_{n,J} = \hat{f}^{(\hat{m})}$ .

Such a procedure is well known and we refer to Chapter 4 of [Mas07] for more details. In particular, the estimator  $\hat{f}_{n,J}$  satisfies the following inequality

$$\begin{aligned} & \mathbb{E}^{(1)} \left[ \int_0^1 \left| \bar{f}^{(m_1)}(t) - \hat{f}_{n,J}(t) \right|^2 dt \right] \\ & \leq C(\eta) \left\{ \min_{m \in \{1, \dots, m_1\}} \mathbb{E}^{(1)} \left[ \int_0^1 \left| \bar{f}^{(m_1)}(t) - \hat{f}^{(m)}(t) \right|^2 dt \right] + \frac{\sigma^2}{NJ} \right\} \\ & \leq C(\eta) \min_{m \in \{1, \dots, m_1\}} \left\{ \int_0^1 \left| \bar{f}^{(m_1)}(t) - \bar{f}^{(m)}(t) \right|^2 dt + \frac{2(m+1)\sigma^2}{NJ} \right\} \end{aligned} \quad (3.5)$$

where  $C(\eta) > 0$  only depends on  $\eta$ . It is known that an optimal choice for  $\eta$  is a difficult problem from a theoretical point of view. However, in practice, taking some  $\eta$  slightly greater than 2 leads to a procedure that behaves well as we discuss in Section 6.

### 3.3 Convergence rates over Sobolev balls

Let us denote by  $\lfloor x \rfloor$  the largest integer smaller than  $x \in \mathbb{R}$ . We now focus on the performances of our estimation procedure from the minimax point of view and with respect to the distance  $d$  defined in (1.4). Note that, in the subsection 3.2, we only use truncated Fourier series expansion for building the estimators  $\hat{f}^{(m)}$ . In practice, we could use other bases of  $L^2_{per}([0, 1])$  as we would still have a result like (3.5). In particular, the following theorem would remain true by combining model selection techniques with bases like piecewise polynomials or orthonormal wavelets to approximate a function.

**Theorem 3.2.** *Assume that  $nJ \geq \max\{21J, (4\sigma^2)^{2s+1}/c^{2s}\}$  where  $0 < c < 1$  is such that  $J \leq cn^\alpha$  for some  $\alpha > 0$ . Take  $m_1 = \lfloor N/2 \rfloor - 1$  and let  $s > 3/2$  and  $A > 0$ . Then, the estimator  $\hat{f}_{n,J}$  defined by the procedure (3.4) is such that*

$$\sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \tilde{W}_s(A, c_*)} \mathcal{R}_g(\hat{f}_{n,J}, f) \leq C (|m_1|^{-2s} + m_1 n^{-2s+1} + \frac{1}{J} \mathbb{E}^g (\|\hat{\theta}^0 - \theta^0\|^2) + (nJ)^{-\frac{2s}{2s+1}})$$

for some  $C > 0$  that only depends on  $A, s, \sigma^2, \kappa, k_0, \eta$  and  $c$ .

Therefore, using the results of Corollary 3.1 on the convergence rate of  $\hat{\theta}^0$  to  $\theta^0$ , we finally obtain the following result.

**Corollary 3.2.** *Suppose that the assumptions of Theorem 3.1 and Theorem 3.2 are satisfied. If  $k_0 \geq 1$  is a fixed integer and  $J \asymp n^\alpha$  for some  $0 < \alpha \leq 1/6$ , then there exists  $C' > 0$  that only depends on  $A, s, \sigma^2, \kappa, k_0, \eta, c_*$  and  $c$  such that*

$$\sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \tilde{W}_s(A, c_*)} \mathcal{R}_g(\hat{f}_{n,J}, f) \leq C' \left( n^{-1} + (nJ)^{-\frac{2s}{2s+1}} \right).$$

## 4 A lower bound on the risk

The following theorem gives a lower bound on the risk over the Sobolev ball  $\tilde{W}_s(A, c_*)$ .

**Theorem 4.1.** *Let us recall that*

$$\mathcal{R}_{n,J}(\tilde{W}_s(A, c_*), \mathcal{G}^\kappa) = \inf_{\hat{f}_{n,J}} \sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \tilde{W}_s(A, c_*)} \mathcal{R}_g(\hat{f}_{n,J}, f).$$

*There exists a constant  $C > 0$  that only depends on  $A, s, c_*$  and  $\sigma^2$  such that*

$$\liminf_{\min(n,J) \rightarrow +\infty} (nJ)^{\frac{2s}{2s+1}} \mathcal{R}_{n,J}(\tilde{W}_s(A, c_*), \mathcal{G}^\kappa) \geq C.$$

Therefore, under the assumption that  $2\alpha s \leq 1$ , the smoothed Fréchet mean converges at the optimal rate  $(nJ)^{-\frac{2s}{2s+1}}$  as shown by the following corollary.

**Corollary 4.1.** *Suppose that the assumptions of Corollary 3.2 hold and assume that  $2\alpha s \leq 1$ . Then, there exists a constant  $C' > 0$  that only depends on  $A, s, \sigma^2, \kappa, k_0, \eta, c_*$  and  $c$  such that*

$$\sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \tilde{W}_s(A, c_*)} \mathbb{E}^g \left( d^2([\hat{f}_{n,J}], [f]) \right) \leq C' (nJ)^{-\frac{2s}{2s+1}}. \quad (4.1)$$

Note that inequality (4.1) is a direct consequence of Corollary 3.2 and the fact that  $n^{-1} = O\left((nJ)^{-\frac{2s}{2s+1}}\right)$  in the settings  $2\alpha s \leq 1$  and  $J \asymp n^\alpha$ .

## 5 Discussion

As explained previously, the rate of convergence  $r_{n,J} = n^{-1} + (nJ)^{-\frac{2s}{2s+1}}$  of the estimator  $\hat{f}_{n,J}$  is the sum of two terms having different interpretations. The term  $(nJ)^{-\frac{2s}{2s+1}}$  is the usual nonparametric rate that would be obtained if the random shifts  $\theta_1^*, \dots, \theta_J^*$  were known. To interpret the second term  $n^{-1}$ , let us mention the following result that has been obtained in [BC11].

**Proposition 5.1.** *Suppose that the function  $f$  is continuously differentiable. Assume that the density  $g \in \mathcal{G}^\kappa$  with  $\lim_{\theta \rightarrow -\kappa/2} g(\theta) = \lim_{\theta \rightarrow \kappa/2} g(\theta) = 0$  and that  $\int_{-1/2}^{1/2} \left(\frac{\partial}{\partial \theta} \log g(\theta)\right)^2 g(\theta) d\theta < +\infty$ . Let  $(\hat{\theta}_1, \dots, \hat{\theta}_J)$  denote any estimator of the true shifts  $(\theta_1^*, \dots, \theta_J^*)$  computed from the  $Y_{\ell,j}$ 's in model (1.3). Then,*

$$\mathbb{E}^g \left( \frac{1}{J} \sum_{j=1}^J (\hat{\theta}_j - \theta_j^*)^2 \right) \geq \frac{\sigma^2}{n \int_0^1 |f'(t)|^2 dt + \sigma^2 \int_{-1/2}^{1/2} \left(\frac{\partial}{\partial \theta} \log g(\theta)\right)^2 g(\theta) d\theta}. \quad (5.1)$$

Proposition (5.1) shows that it is not possible to build consistent estimators of the shifts by considering only the asymptotic setting where the number of curves  $J$  tends towards infinity. Indeed Inequality (5.1) implies that  $\liminf_{J \rightarrow +\infty} \mathbb{E}^g \left( \frac{1}{J} \sum_{j=1}^J (\hat{\theta}_j - \theta_j^*)^2 \right) > 0$  for any estimators  $(\hat{\theta}_1, \dots, \hat{\theta}_J)$ . We recall that, under the assumptions of Corollary 3.1, one has

$$\mathbb{E}^g \left( \frac{1}{J} \sum_{j=1}^J |\hat{\theta}_j^0 - \theta_j^0|^2 \right) \leq \frac{C_1}{n}.$$

The above inequality shows that, in the setting where  $n$  and  $J$  are both allowed to increase, the estimation of the unknown shifts  $\hat{\theta}_j^0 = \hat{\theta}_j^* - \frac{1}{J} \sum_{m=1}^J \hat{\theta}_m^*$  is feasible at the rate  $n^{-1}$ . By Proposition 5.1, this rate of convergence cannot be improved. We thus interpret the term  $n^{-1}$  appearing in the rate  $r_{n,J}$  of the smoothed Fréchet mean  $[\hat{f}_{n,J}]$  as the price to pay for having to estimate the shifts to compute such estimators.

To conclude this discussion, we would like to mention the results that have been obtained in [BG10] in an asymptotic setting where only the number  $J$  of curves is let going to infinity. Consider the following model of randomly shifted curves with additive white noise

$$dY_j(t) = f(t - \theta_j^*)dt + \epsilon dW_j(t), \quad t \in [0, 1], \quad j = 1, \dots, J \text{ with } \theta_j^* \sim_{iid} g, \quad (5.2)$$

where the  $W_j$ 's are independent Brownian motions with  $\epsilon > 0$  being the level of additive noise. In model (5.2), the expectation of each observed curve  $dY_j$  is equal to the convolution of  $f$  by the density  $g$  since

$$\mathbb{E}^g [f(t - \theta_j^*)] = \int f(t - \theta)g(\theta)d\theta = f \star g(t).$$

Therefore, in the ideal situation where  $g$  is assumed to be known, it has been shown in [BG10] that estimating  $f$  in the asymptotic setting  $J \rightarrow +\infty$  (with  $\epsilon > 0$  being fixed) is a deconvolution problem. Indeed, suppose that, for some  $\nu > 1/2$ ,

$$\gamma_k = \int_0^1 g(\theta)e^{-i2\pi k\theta}d\theta \asymp |k|^{-\nu}, \quad k \in \mathbb{Z},$$

with  $g$  being known. Then, one can construct an estimator  $\hat{f}_J^*$  by a deconvolution procedure such that

$$\sup_{f \in W_s(A)} \mathbb{E}^g \int_0^1 |\hat{f}_J^*(t) - f(t)|^2 dt \leq C J^{-\frac{2s}{2s+\nu+1}},$$

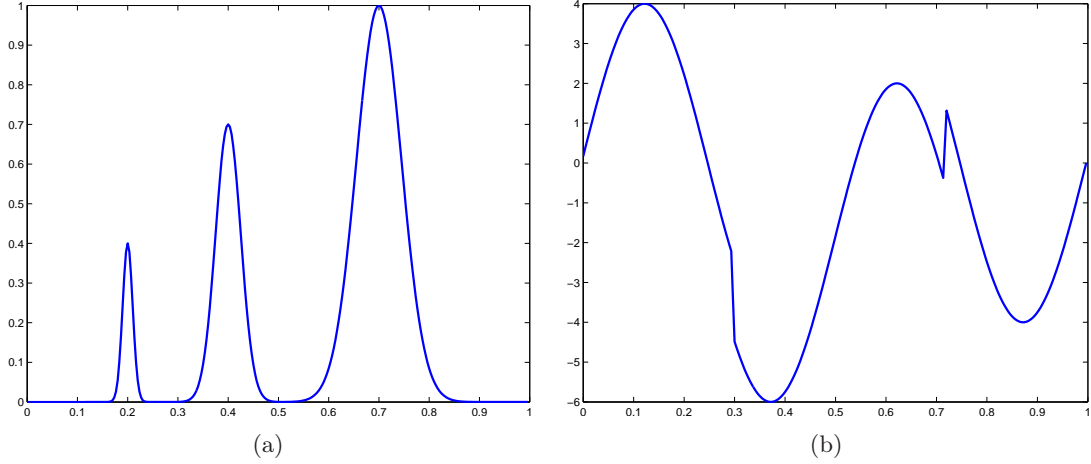
for some  $C > 0$  that only depends on  $A$ ,  $s$  and  $\epsilon$  and where  $W_s(A)$  is Sobolev ball of degree  $s > 1/2$ . Moreover, this rate of convergence is optimal since the results in [BG10] show that if  $s > 2\nu + 1$ , then there exists a constant  $C' > 0$  that only depends on  $A$ ,  $s$  and  $\epsilon$  such that

$$\liminf_{J \rightarrow +\infty} J^{\frac{2s}{2s+2\nu+1}} \inf_{\hat{f}_J} \sup_{f \in W_s(A)} \mathcal{R}(\hat{f}_J, f) \geq C'$$

where the above infimum is taken over the set of all estimators  $\hat{f}_J$  of  $f$  in model (5.2). Hence,  $r_J = J^{-\frac{2s}{2s+2\nu+1}}$  is the minimax rate of convergence over Sobolev balls in model (5.2) in the case of known  $g$ . This rate is of polynomial order of the number of curves  $J$ , and it deteriorates as the smoothness  $\nu$  of the convolution kernel  $g$  increases. This phenomenon is a well-known fact in deconvolution problems, see e.g [Fan91, PV99]. Hence, depending on  $g$  being known or not and the choice of the asymptotic setting, there exists a significant difference in the rates of convergence that can be achieved in a randomly shifted curves model. Our setting yields to the rate  $r_{n,J} = n^{-1} + (nJ)^{-\frac{2s}{2s+1}}$  (in the case where  $J \asymp n^\alpha$  with  $\alpha < 1/6$ ) that is clearly faster than the rate  $r_J = J^{-\frac{2s}{2s+2\nu+1}}$ . Nevertheless, the arguments in [BG10] also suggest that a smoothed Fréchet mean in (5.2) is not a consistent estimator of  $f$  if one only lets  $J$  going to infinity. Therefore, the number  $n$  of design points is clearly of primary importance to obtain consistent estimators of a mean pattern when using Fréchet means of curves.

## 6 Numerical experiments

The goal of this section is to study the performances of the estimator  $\hat{f}_{n,J}$ . The factors in the simulations are the number  $J$  of curves and the number  $n$  of design points. As a mean pattern  $f$  to recover, we consider the two test functions displayed in Figure 1. Then, for each combination



**Figure 1:** Two test functions  $f$ . (a) MixtGauss: a mixture of three Gaussians. (b) HeaviSine: a piecewise smooth curve with a discontinuity.

of  $n$  and  $J$ , we generate  $M = 100$  repetitions of model (1.3) of  $J$  curves with shifts sampled from the uniform distribution on  $[-\kappa, \kappa]$  with  $\kappa = 1/16$ . The level of the additive Gaussian noise is measured as the root of the signal-to-noise ratio ( $rsnr$ ) defined as

$$rsnr = \left( \frac{1}{\sigma^2} \int_0^1 (f(t) - \bar{f})^2 dt \right)^{1/2} \quad \text{where } \bar{f} = \int_0^1 f(t) dt,$$

that is fixed to  $rsnr = 0.5$  in all the simulations. For each repetition  $p \in \{1, \dots, M\}$ , we compute the estimator  $\hat{f}_{n,J,p}$  using a gradient descent algorithm to minimize the criterion (3.2) for estimating the shifts. For all values of  $n$  and  $J$ , we took  $k_0 = 5$  in (2.3). The frequency cut-off  $\hat{m}$  is chosen using (3.4) with  $\eta = 2.5$ .

To analyze the numerical performances of this estimator, we have considered the following ideal estimator that uses the knowledge of the true random shifts  $\theta_{j,p}^*$  (sampled from the  $p$ -th replication)

$$\tilde{f}_{n,J,p}^{(m)}(t) = \sum_{|k| \leq m} \left( \frac{1}{J} \sum_{j=1}^J \hat{c}_{k,j,p}^{(1)} e^{i2\pi k \theta_{j,p}^*} \right) \phi_k(t), \quad t \in [0, 1].$$

The frequency cut-off  $\hat{m}_*$  for the above ideal estimator is chosen using a model selection procedure based on the knowledge of the true shifts i.e.

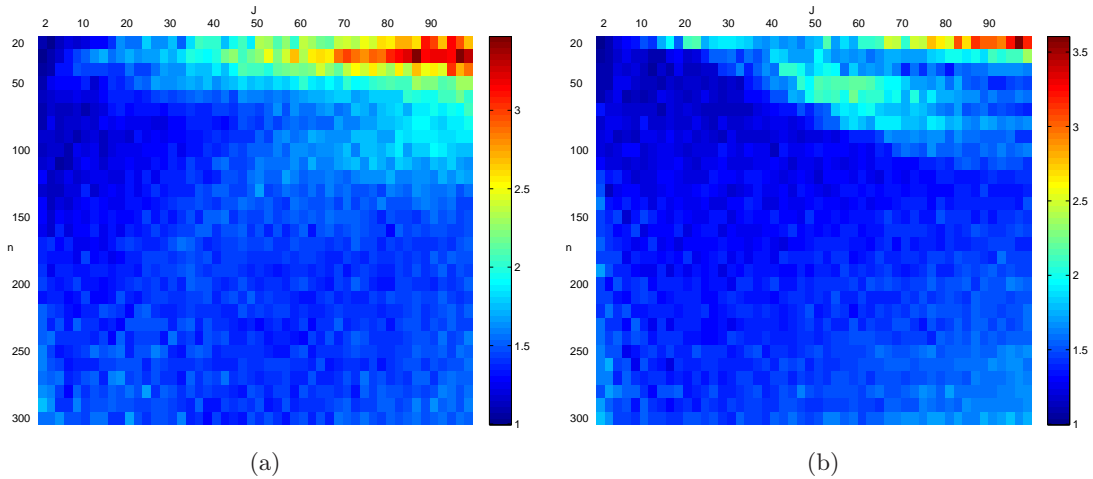
$$\hat{m}_* \in \underset{m \in \{1, \dots, m_1\}}{\operatorname{argmin}} \left\{ \int_0^1 \left| \tilde{f}_{n,J,p}^{(m_1)}(t) - \tilde{f}_{n,J,p}^{(m)}(t) \right|^2 dt + \eta \frac{(2m+1)\sigma^2}{NJ} \right\}$$

with  $\eta = 2.5$ .

Then, we define the relative empirical error between the two estimators as

$$R(n, J) = \frac{\frac{1}{M} \sum_{p=1}^M d^2([\hat{f}_{n,J,p}], [f])}{\frac{1}{M} \sum_{p=1}^M d^2([\tilde{f}_{n,J,p}^{(\hat{m}^*)}], [f])}$$

In Figure 2, we display the ratio  $R(n, J)$  for various values of  $n$  and  $J$  and for the two test functions displayed in Figure 1. It can be seen that the function  $J \mapsto R(n, J)$  is increasing. This means that the numerical performances of the estimator  $\hat{f}_{n,J}$  deteriorate as the number  $J$  of curves increases and the number  $n$  remains fixed. This is clearly due to the fact that the estimation of the shifts becomes less precise when the dimension  $J$  increases. These numerical results are thus consistent with inequality (3.3) in Theorem 3.1 and our discussion on the rate of convergence of  $\hat{f}_{n,J}$  in Section 3. On the opposite, the function  $n \mapsto R(n, J)$  is decreasing and it confirms that the number  $n$  of design points is a key parameter to obtain consistent estimators of a mean pattern  $f$  with Fréchet means of curves.



**Figure 2:** Relative empirical error  $R(n, J)$  for various values of  $n$  (vertical axis) and  $J$  (horizontal axis) over  $M = 100$  replications: (a) MixtGauss, (b) HeaviSine.

## A Proof of the main results

Throughout the proofs, we repeatedly use the following lemma which follows immediately from Lemma 1.10 in [Tsy09].

**Lemma A.1.** *If  $f \in \tilde{W}_s(A, c_*)$  then there exists a constant  $A_0 > 0$  only depending on  $A$  and  $s$  such that*

$$\max_{-\frac{N}{2} \leq k < \frac{N}{2}} |\bar{c}_{k,j}^{(p)} - c_k e^{-i2\pi k \theta_j^*}| \leq A_0 N^{-s+1/2}, \quad p \in \{0, 1\},$$

for all  $1 \leq j \leq J$ .

### A.1 Proof of Theorem 3.1

For legibility, we will write  $\mathbb{E} = \mathbb{E}^g$ , i.e. we omit the dependency on  $g$  of the expectation. The proof is divided in several lemmas. Let  $\|\cdot\|$  denote the standard Euclidean norm in  $\mathbb{R}^J$ . First, we derive upper bounds on the second, fourth and sixth moments of  $\|\hat{\theta}^0 - \theta^0\|$ .

**Lemma A.2.** *Let  $N \geq 2$ ,  $J \geq 1$  and  $1 \leq k_0 \leq N/2$ . We assume that Assumptions 1 and 2 are satisfied and we suppose that  $s > 3/2$ . Then, we have the following upper bounds, for any  $f \in \tilde{W}_s(A, c_*)$ ,*

$$\frac{1}{J} \mathbb{E} \left( \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq C_1 \frac{k_0^{1/2}}{n^{1/2}}, \quad (\text{A.1})$$

$$\frac{1}{J} \mathbb{E} \left( \|\hat{\theta}^0 - \theta^0\|^4 \right) \leq C_2 \frac{k_0 J}{n} \quad (\text{A.2})$$

and

$$\frac{1}{J} \mathbb{E} \left( \|\hat{\theta}^0 - \theta^0\|^6 \right) \leq C_3 \frac{k_0^{3/2} J^2}{n^{3/2}}, \quad (\text{A.3})$$

where  $C_1$ ,  $C_2$  and  $C_3$  are positive constants that only depend on  $A$ ,  $s$ ,  $c_*$ ,  $\sigma^2$  and  $\kappa$ .

*Proof.* Let  $f \in \tilde{W}_s(A, c_*)$ . Since  $\hat{\theta}^0 = (\hat{\theta}_1^0, \dots, \hat{\theta}_J^0)$  is a minimizer of  $M_n$ , it follows that

$$M(\hat{\theta}^0) - M(\theta^0) \leq 2 \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)|.$$

Therefore, by Proposition 3.1, we get

$$\frac{1}{J} \mathbb{E} \|\hat{\theta}^0 - \theta^0\|^2 \leq 2C^{-1}(c_*, \kappa) \mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)| \right), \quad (\text{A.4})$$

$$\frac{1}{J} \mathbb{E} \|\hat{\theta}^0 - \theta^0\|^4 \leq 4C^{-2}(c_*, \kappa) J \mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)|^2 \right), \quad (\text{A.5})$$

and

$$\frac{1}{J} \mathbb{E} \|\hat{\theta}^0 - \theta^0\|^6 \leq 8C^{-3}(c_*, \kappa) J^2 \mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)|^3 \right), \quad (\text{A.6})$$

where we have set  $C(c_*, \kappa) = 4\pi^2 c_*^2 \cos(8\pi\kappa)$ .

Let  $\theta \in \Theta_\kappa$  and note that  $M_n(\theta)$  can be decomposed as

$$M_n(\theta) = \bar{M}(\theta) + Q(\theta) + L(\theta), \quad (\text{A.7})$$

where

$$\bar{M}(\theta) = \frac{1}{J} \sum_{j=1}^J \sum_{|k| \leq k_0} \left| \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} - \frac{1}{J} \sum_{j'=1}^J \bar{c}_{k,j'}^{(0)} e^{2ik\pi\theta_{j'}} \right|^2,$$



$$Q(\theta) = \frac{1}{NJ} \sum_{j=1}^J \sum_{|k| \leq k_0} \left| z_{k,j}^{(0)} e^{2ik\pi\theta_j} - \frac{1}{J} \sum_{j'=1}^J z_{k,j'}^{(0)} e^{2ik\pi\theta_{j'}} \right|^2$$

and

$$L(\theta) = \frac{2}{J\sqrt{N}} \sum_{j=1}^J \sum_{|k| \leq k_0} \Re \left[ \left( \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} - \frac{1}{J} \sum_{j'=1}^J \bar{c}_{k,j'}^{(0)} e^{2ik\pi\theta_{j'}} \right) \overline{\left( z_{k,j}^{(0)} e^{2ik\pi\theta_j} - \frac{1}{J} \sum_{j'=1}^J z_{k,j'}^{(0)} e^{2ik\pi\theta_{j'}} \right)} \right].$$

Using Lemma A.1, it follows that, for any  $\theta \in \Theta_\kappa$ ,

$$\begin{aligned} |\bar{M}(\theta) - M(\theta)| &\leq \frac{1}{J} \sum_{j=1}^J \sum_{|k| \leq k_0} \left| (\bar{c}_{k,j}^{(0)} - c_k e^{-2ik\pi\theta_j^*}) e^{2ik\pi\theta_j} - \frac{1}{J} \sum_{j'=1}^J (\bar{c}_{k,j'}^{(0)} - c_k e^{-2ik\pi\theta_{j'}^*}) e^{2ik\pi\theta_{j'}} \right| \\ &\quad \times \left| |\bar{c}_{k,j}^{(0)}| + \frac{1}{J} \sum_{j'=1}^J |\bar{c}_{k,j'}^{(0)}| + 2|c_k| \right| \\ &\leq 2A_0 N^{-s+1/2} \sum_{|k| \leq k_0} \left( 4|c_k| + 2A_0 N^{-s+1/2} \right) \\ &\leq 8A_0 N^{-s+1/2} (2k_0 + 1)^{1/2} \sqrt{\sum_{|k| \leq k_0} |c_k|^2} + 4A_0^2 (2k_0 + 1) N^{-2s+1}. \end{aligned}$$

Hence, there exists a positive constant  $C$  that only depends on  $A$  and  $s$  such that

$$\sup_{\theta \in \Theta_\kappa} |\bar{M}(\theta) - M(\theta)| \leq C k_0^{1/2} N^{-s+1/2}. \quad (\text{A.8})$$

Now, note that  $Q(\theta) \leq \frac{\sigma^2}{NJ} Z$  with  $Z = \sum_{|k| \leq k_0} \sum_{j=1}^J |z_{k,j}^{(0)} / \sigma|^2$  for any  $\theta \in \Theta_\kappa$ . Thus, it follows that

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |Q(\theta)| \leq \sigma^2 (2k_0 + 1) N^{-1}. \quad (\text{A.9})$$

and

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |Q(\theta)|^2 \leq 2\sigma^4 (2k_0 + 1)^2 N^{-2}. \quad (\text{A.10})$$

By Jensen's Inequality, we get  $\mathbb{E} Z^{3/2} \leq (\mathbb{E} Z^2)^{3/4}$  and, since  $\mathbb{E} Z^2 \leq 2J^2 (2k_0 + 1)^2$ , we obtain

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |Q(\theta)|^{3/2} \leq 8^{1/4} \frac{\sigma^3}{N^{3/2}} (2k_0 + 1)^{3/2}. \quad (\text{A.11})$$

Finally, using  $\mathbb{E} Z^3 \leq 6J^3 (2k_0 + 1)^3$ , we have

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |Q(\theta)|^3 \leq 6 \frac{\sigma^6}{N^3} (2k_0 + 1)^3. \quad (\text{A.12})$$

By Cauchy-Schwarz's Inequality

$$L(\theta) \leq 2\sqrt{\bar{M}(\theta)}\sqrt{Q(\theta)}. \quad (\text{A.13})$$

Thanks to Lemma A.1, we get

$$\bar{M}(\theta) \leq \frac{1}{J} \sum_{|k| \leq k_0} \sum_{j=1}^J \left| \bar{c}_{k,j}^{(0)} \right|^2 \leq \sum_{|k| \leq k_0} |c_k|^2 + A_0^2 N^{-2s+1} (2k_0 + 1).$$

Thus, it follows from (A.9), (A.10), (A.11) and (A.13) that there exists a positive constant  $C'$ , only depending on  $A$ ,  $s$  and  $\sigma^2$ , such that

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |L(\theta)| \leq C' k_0^{1/2} N^{-1/2}, \quad (\text{A.14})$$

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |L(\theta)|^2 \leq C'^2 k_0 N^{-1} \quad (\text{A.15})$$

and

$$\mathbb{E} \sup_{\theta \in \Theta_\kappa} |L(\theta)|^3 \leq C'^3 k_0^{3/2} N^{-3/2}. \quad (\text{A.16})$$

Since  $s > 3/2$ , we obtain, by Inequalities (A.8), (A.9) and (A.14),

$$\mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)| \right) \leq C'_1 k_0^{1/2} N^{-1/2},$$

by Inequalities (A.8), (A.10) and (A.15),

$$\mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)|^2 \right) \leq C'_2 k_0 N^{-1},$$

and by Inequalities (A.8), (A.12) and (A.16),

$$\mathbb{E} \left( \sup_{\theta \in \Theta_\kappa} |M_n(\theta) - M(\theta)|^3 \right) \leq C'_3 k_0^{3/2} N^{-3/2},$$

where  $C'_1$ ,  $C'_2$  and  $C'_3$  are positive constants that only depend on  $A$ ,  $s$  and  $\sigma^2$ . Combined with Inequalities (A.4), (A.5) and (A.6), the announced result follows from the above upper bounds.  $\square$

In order to prove Theorem 3.1, we divide the rest of the proof in the three following steps. In the sequel of this section, we always assume that the hypotheses of Theorem 3.1 are satisfied and we use the decomposition of  $M_n(\theta)$  as defined in (A.7).

**Step 1 :** there exists some positive constant  $C_1$  that only depends on  $c_*$  such that

$$\frac{n}{J} \|\hat{\theta}^0 - \theta^0\|^2 \leq C_1 \left( nJ \|\nabla M_n(\theta^0)\|^2 + nJ \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 M_n(\theta) - \nabla^2 M(\theta^0)\|_{op}^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \quad (\text{A.17})$$

where  $\nabla$  and  $\nabla^2$  denote the gradient and the Hessian operators, respectively, and where we have set

$$\mathcal{U}_\kappa = \left\{ \theta \in \Theta_\kappa \text{ such that } \|\theta - \theta^0\| \leq \|\hat{\theta}^0 - \theta^0\| \right\}$$

and, for any  $J \times J$  matrix  $B$ , the operator norm  $\|B\|_{op}$  is defined by

$$\|B\|_{op} = \sup_{\theta \in \mathbb{R}^J \setminus \{0\}} \frac{\|B\theta\|}{\|\theta\|}.$$

**Step 2 :** there exists some positive constant  $C_2$  that only depends on  $A$ ,  $s$  and  $\sigma^2$  such that

$$nJ\mathbb{E}\|\nabla M_n(\theta^0)\|^2 \leq C_2 \left( 1 + \frac{k_0^3}{n} \right). \quad (\text{A.18})$$

**Step 3 :** there exists some positive constant  $C_3$  that only depends on  $A$ ,  $s$ ,  $\kappa$ ,  $c_*$  and  $\sigma^2$  such that

$$nJ\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 M_n(\theta) - \nabla^2 M(\theta^0)\|_{op}^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq C_3 \left( 1 + \frac{k_0^5}{n^{1/2}} \right) \frac{k_0^{3/2} J^3}{n^{1/2}}. \quad (\text{A.19})$$

The result announced in Theorem 3.1 follows from Inequalities (A.17), (A.18) and (A.19).

### A.1.1 Proof of Step 1

The gradients of  $\bar{M}(\theta)$ ,  $Q(\theta)$  and  $L(\theta)$  follows from easy computations. We have, for any  $1 \leq \ell \leq J$ ,

$$\frac{\partial}{\partial \theta_\ell} \bar{M}(\theta) = \frac{4\pi}{J^2} \sum_{|k| \leq k_0} k \Re \left[ \overline{i \bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right], \quad (\text{A.20})$$

$$\frac{\partial}{\partial \theta_\ell} Q(\theta) = \frac{4\pi}{NJ^2} \sum_{|k| \leq k_0} k \Re \left[ \overline{i z_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right] \quad (\text{A.21})$$

and

$$\frac{\partial}{\partial \theta_\ell} L(\theta) = -\frac{4\pi}{J^2 \sqrt{N}} \sum_{|k| \leq k_0} k \Im \left[ \overline{z_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1, j \neq \ell}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) - \bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right]. \quad (\text{A.22})$$

Similarly, we can compute the Hessians of these functions as follows, for  $1 \leq \ell, \ell' \leq J$ , if  $\ell \neq \ell'$ ,

$$\frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_\ell} \bar{M}(\theta) = -\frac{8\pi^2}{J^2} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{\bar{c}_{k,\ell}^{(0)} \bar{c}_{k,\ell'}^{(0)} e^{2ik\pi(\theta_{\ell'} - \theta_\ell)}} \right], \quad (\text{A.23})$$

$$\frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_\ell} Q(\theta) = -\frac{8\pi^2}{NJ^2} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{z_{k,\ell}^{(0)} z_{k,\ell'}^{(0)} e^{2ik\pi(\theta_{\ell'} - \theta_\ell)}} \right] \quad (\text{A.24})$$

and

$$\frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_{\ell}} L(\theta) = -\frac{8\pi^2}{J^2 \sqrt{N}} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{\bar{c}_{k,\ell'}^{(0)} z_{k,\ell}^{(0)}} e^{2ik\pi(\theta_{\ell'} - \theta_{\ell})} + \overline{\bar{c}_{k,\ell}^{(0)} z_{k,\ell'}^{(0)}} e^{2ik\pi(\theta_{\ell} - \theta_{\ell'})} \right], \quad (\text{A.25})$$

and, if  $\ell = \ell'$ ,

$$\frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} \bar{M}(\theta) = \frac{8\pi^2}{J^2} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{\bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_{\ell}}} \left( \sum_{j=1, j \neq \ell}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right], \quad (\text{A.26})$$

$$\frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} Q(\theta) = \frac{8\pi^2}{N J^2} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{z_{k,\ell}^{(0)} e^{2ik\pi\theta_{\ell}}} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right] \quad (\text{A.27})$$

and

$$\begin{aligned} \frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} L(\theta) &= \frac{8\pi^2}{J^2 \sqrt{N}} \sum_{|k| \leq k_0} k^2 \Re \left[ \overline{\bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_{\ell}}} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right. \\ &\quad \left. + \overline{z_{k,\ell}^{(0)} e^{2ik\pi\theta_{\ell}}} \left( \sum_{j=1, j \neq \ell}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right]. \end{aligned} \quad (\text{A.28})$$

Using the fact that  $\hat{\theta}^0 \in \Theta_{\kappa}$  is a minimizer of  $M_n$ , so  $\nabla M_n(\hat{\theta}^0) = 0$ , a Taylor expansion of  $\theta \mapsto \nabla M_n(\theta)$  with an integral form of the remainder term leads to

$$0 = \nabla M_n(\theta^0) + \int_0^1 \nabla^2 M_n(\bar{\theta}(t)) (\hat{\theta}^0 - \theta^0) dt \quad (\text{A.29})$$

where, for any  $t \in [0, 1]$ , we have set

$$\bar{\theta}(t) = \theta^0 + t(\hat{\theta}^0 - \theta^0) \in \mathcal{U}_{\kappa}.$$

Thus, we have

$$\nabla^2 M(\theta^0) (\hat{\theta}^0 - \theta^0) = -\nabla M_n(\theta^0) - \int_0^1 (\nabla^2 M_n(\bar{\theta}(t)) - \nabla^2 M(\theta^0)) (\hat{\theta}^0 - \theta^0) dt. \quad (\text{A.30})$$

It follows from similar computations as we did for  $\bar{M}$  that

$$\nabla^2 M(\theta^0) = \frac{8\pi^2}{J} \sum_{|k| \leq k_0} k^2 |c_k|^2 \left( I_J - \frac{1}{J} \mathbb{1}_J \right)$$

where  $I_J$  is the  $J \times J$  identity matrix and  $\mathbb{1}_J$  denotes the  $J \times J$  matrix with all entries equal to one. Therefore, using the fact that  $\sum_{j=1}^J (\hat{\theta}_j^0 - \theta_j^0) = 0$ , we obtain

$$\|\nabla^2 M(\theta^0) (\hat{\theta}^0 - \theta^0)\|^2 = \frac{64\pi^4}{J^2} \left( \sum_{|k| \leq k_0} k^2 |c_k|^2 \right)^2 \|\hat{\theta}^0 - \theta^0\|^2$$

and it shows that there exists a constant  $C > 0$  that only depends on  $c_*$  such that

$$\|\nabla^2 M(\theta^0)(\hat{\theta}^0 - \theta^0)\|^2 \geq C \frac{1}{J^2} \|\hat{\theta}^0 - \theta^0\|^2. \quad (\text{A.31})$$

Then, Inequality (A.17) follows from (A.30) and (A.31).

### A.1.2 Proof of Step 2

By using Lemma A.1, for any  $1 \leq k \leq k_0$  and  $1 \leq \ell \leq J$ , we can expand

$$\bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell^0} = c_k e^{2ik\pi(\theta_\ell^0 - \theta_\ell^*)} + \alpha_{k,\ell} e^{2ik\pi\theta_\ell^0}$$

with  $|\alpha_{k,\ell}| \leq A_0 N^{-s+1/2}$ . Because, for any  $j$ ,  $\theta_j^0 - \theta_j^* = \bar{\theta}_J$  does not depend on  $j$ , we have

$$\left| \Re \left[ \overline{i\bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right] \right| \leq A_0 J N^{-s+1/2} (2|c_k| + A_0 N^{-s+1/2}).$$

Thus, by Equation (A.20) and using Cauchy-Schwarz's Inequality, we obtain

$$\begin{aligned} \left| \frac{\partial}{\partial \theta_\ell} \bar{M}(\theta^0) \right|^2 &\leq \frac{16\pi^2}{J^4} (2k_0 + 1) \sum_{|k| \leq k_0} k^2 \left| \Re \left[ \overline{i\bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right] \right|^2 \\ &\leq \frac{32\pi^2}{J^2} (2k_0 + 1) A_0^2 N^{-2s+1} \sum_{|k| \leq k_0} k^2 (4|c_k|^2 + A_0^2 N^{-2s+1}) \\ &\leq \frac{64\pi^2}{J^2} (2k_0 + 1) A_0^2 N^{-2s+1} (A^2 + A_0^2 k_0^3 N^{-2s+1}). \end{aligned}$$

Thus, there exists a positive constant  $C$  that only depends on  $A$  and  $s$  such that

$$nJ \|\nabla \bar{M}(\theta^0)\|^2 \leq C k_0 (1 + k_0^3 n^{-2s+1}) n^{-2s+2}. \quad (\text{A.32})$$

We now focus on  $\nabla Q$  and, by (A.21), we can obtain

$$\begin{aligned} \sum_{\ell=1}^J \mathbb{E} \left| \frac{\partial}{\partial \theta_\ell} Q(\theta^0) \right|^2 &= \frac{16\pi^2}{N^2 J^4} \sum_{\ell=1}^J \sum_{|k| \leq k_0} k^2 \mathbb{E} \left| \Re \left[ \overline{i z_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell^0}} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j^0} \right) \right] \right|^2 \\ &\leq \frac{16\pi^2 (J-1)^2}{N^2 J^4} \sum_{\ell=1}^J \sum_{|k| \leq k_0} k^2 \frac{1}{J-1} \sum_{j=1, j \neq \ell}^J \mathbb{E} |z_{k,\ell}^{(0)}|^2 |z_{k,j}^{(0)}|^2 \\ &\leq \frac{32\pi^2 \sigma^4 k_0^3}{N^2 J}. \end{aligned}$$

Hence, we get

$$nJ \mathbb{E} \|\nabla Q(\theta^0)\|^2 \leq \frac{128\pi^2 \sigma^4 k_0^3}{n}. \quad (\text{A.33})$$

Finally, we deal with  $\nabla L$ . Equation (A.22) and Lemma A.1 imply

$$\begin{aligned}
\mathbb{E} \left| \frac{\partial}{\partial \theta_\ell} L(\theta) \right|^2 &= \frac{16\pi^2}{J^4 N} \sum_{|k| \leq k_0} k^2 \mathbb{E} \left| \overline{z_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell}} \left( \sum_{j=1, j \neq \ell}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) - \bar{c}_{k,\ell}^{(0)} e^{2ik\pi\theta_\ell} \left( \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2ik\pi\theta_j} \right) \right|^2 \\
&\leq \frac{16\pi^2 \sigma^2}{J^4 N} \sum_{|k| \leq k_0} k^2 \left( \left| \sum_{j=1, j \neq \ell}^J \bar{c}_{k,j}^{(0)} e^{2ik\pi\theta_j} \right|^2 + J \left| \bar{c}_{k,\ell}^{(0)} \right|^2 \right) \\
&\leq \frac{16\pi^2 \sigma^2}{J^4 N} \sum_{|k| \leq k_0} k^2 \left( (J-1) \sum_{j=1, j \neq \ell}^J \left| \bar{c}_{k,j}^{(0)} \right|^2 + J \left| \bar{c}_{k,\ell}^{(0)} \right|^2 \right) \\
&\leq \frac{16\pi^2 \sigma^2}{J^2 N} \sum_{|k| \leq k_0} k^2 \left( \frac{1}{J} \sum_{j=1}^J \left| \bar{c}_{k,j}^{(0)} \right|^2 \right) \leq \frac{32\pi^2 \sigma^2}{J^2 N} \sum_{|k| \leq k_0} k^2 \left( |c_k|^2 + A_0^2 N^{-2s+1} \right).
\end{aligned}$$

This last inequality leads to

$$nJ\mathbb{E} \|\nabla L(\theta^0)\|^2 \leq 64\pi^2 \sigma^2 (A^2 + 2A_0^2) (1 + k_0^3 n^{-2s+1}). \quad (\text{A.34})$$

By combining Inequalities (A.32), (A.33) and (A.34), we then obtain Inequality (A.18).

### A.1.3 Proof of Step 3

We introduce the Frobenius norm  $\|B\|_F$  defined, for any  $J \times J$  matrix  $B = [B_{\ell,\ell'}]_{1 \leq \ell, \ell' \leq J}$ , as

$$\|B\|_F = \sqrt{\sum_{\ell, \ell'=1}^J B_{\ell,\ell'}^2}.$$

We also recall the classical inequalities

$$\|B\|_{op} \leq \|B\|_F \quad \text{and} \quad \|B\|_{op} \leq \max_{1 \leq \ell' \leq J} \sum_{\ell=1}^J |B_{\ell,\ell'}|. \quad (\text{A.35})$$

In order to prove Inequality (A.19), we use the following decomposition

$$\begin{aligned}
\|\nabla^2 M_n(\theta) - \nabla^2 M(\theta^0)\|_{op}^2 &\leq 4 \left( \|\nabla^2 \bar{M}(\theta) - \nabla^2 M(\theta)\|_F^2 \right. \\
&\quad \left. + \|\nabla^2 M(\theta) - \nabla^2 M(\theta^0)\|_{op}^2 \right. \\
&\quad \left. + \|\nabla^2 Q(\theta)\|_F^2 + \|\nabla^2 L(\theta)\|_F^2 \right). \quad (\text{A.36})
\end{aligned}$$

We now deal with each term in the above inequality. Hereafter,  $\ell$  and  $\ell'$  always denote two integers in  $\{1, \dots, J\}$ .

First, let us consider  $\ell \neq \ell'$ , by (A.23) and Lemma A.1, we get

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_{\ell}} \bar{M}(\theta) - \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_{\ell}} M(\theta) \right|^2 &= \frac{64\pi^4}{J^4} \left( \sum_{|k| \leq k_0} k^2 \Re \left[ \left( \bar{c}_{k,\ell}^{(0)} \overline{\bar{c}_{k,\ell'}^{(0)}} - |c_k|^2 e^{2ik\pi(\theta_{\ell'}^* - \theta_{\ell}^*)} \right) e^{2ik\pi(\theta_{\ell} - \theta_{\ell'})} \right] \right)^2 \\ &\leq \frac{64\pi^4}{J^4} \left( \sum_{|k| \leq k_0} k^2 \left| \bar{c}_{k,\ell}^{(0)} \overline{\bar{c}_{k,\ell'}^{(0)}} - |c_k|^2 e^{2ik\pi(\theta_{\ell'}^* - \theta_{\ell}^*)} \right| \right)^2 \\ &\leq \frac{256\pi^4}{J^4} \left( \sum_{|k| \leq k_0} k^2 \left( A_0 |c_k| N^{-s+1/2} + A_0^2 N^{-2s+1} \right) \right)^2. \end{aligned}$$

In the case of  $\ell = \ell'$ , (A.26) and Lemma A.1 lead to

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} \bar{M}(\theta) - \frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} M(\theta) \right|^2 &\leq \frac{64\pi^4}{J^4} \left( \sum_{|k| \leq k_0} k^2 \sum_{j=1, j \neq \ell}^J \left| \bar{c}_{k,\ell}^{(0)} \overline{\bar{c}_{k,j}^{(0)}} - |c_k|^2 e^{2ik\pi(\theta_j^* - \theta_{\ell}^*)} \right| \right)^2 \\ &\leq \frac{256\pi^4}{J^2} \left( \sum_{|k| \leq k_0} k^2 \left( A_0 |c_k| N^{-s+1/2} + A_0^2 N^{-2s+1} \right) \right)^2. \end{aligned}$$

Therefore, the above inequalities and (A.1) imply that there exists some positive constant  $C_M$  that only depends on  $A$ ,  $s$ ,  $\kappa$  and  $\sigma^2$  such that

$$nJ\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_{\kappa}} \|\nabla^2 \bar{M}(\theta) - \nabla^2 M(\theta)\|_F^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq C_M \frac{k_0^{6+1/2} J}{n^{2s-3/2}}. \quad (\text{A.37})$$

Secondly, using the fact that  $2(1 - \cos(t)) \leq t^2$  for any  $t \in \mathbb{R}$ , if  $\ell \neq \ell'$ , then we have

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_{\ell}} M(\theta) - \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_{\ell}} M(\theta^0) \right| &= \frac{8\pi^2}{J^2} \left| \sum_{|k| \leq k_0} k^2 |c_k|^2 \Re \left[ e^{2ik\pi(\theta_{\ell} - \theta_{\ell}^0 + \theta_{\ell'}^0 - \theta_{\ell'})} - 1 \right] \right| \\ &\leq \frac{16\pi^4}{J^2} \left( \sum_{|k| \leq k_0} k^4 |c_k|^2 \right) |\theta_{\ell} - \theta_{\ell}^0 + \theta_{\ell'}^0 - \theta_{\ell'}|^2 \end{aligned}$$

and if  $\ell = \ell'$  then we obtain

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} M(\theta) - \frac{\partial^2}{\partial \theta_{\ell} \partial \theta_{\ell}} M(\theta^0) \right| &= \frac{8\pi^2}{J^2} \left| \sum_{|k| \leq k_0} k^2 |c_k|^2 \Re \left[ \left( \sum_{j=1, j \neq \ell}^J \left( e^{2ik\pi(\theta_{\ell} - \theta_{\ell}^0 - \theta_j + \theta_j^0)} - 1 \right) \right) \right] \right| \\ &\leq \frac{16\pi^4}{J^2} \left( \sum_{|k| \leq k_0} k^4 |c_k|^2 \right) \sum_{j=1, j \neq \ell}^J |\theta_{\ell} - \theta_{\ell}^0 - \theta_j + \theta_j^0|^2. \end{aligned}$$

Therefore, by (A.35) and under the condition  $s \geq 2$ , we get

$$\begin{aligned} \|\nabla^2 M(\theta) - \nabla^2 M(\theta^0)\|_{op} &\leq \frac{32\pi^4 A^2}{J^2} \max_{1 \leq \ell \leq J} \sum_{j=1, j \neq \ell}^J |\theta_\ell - \theta_\ell^0 - \theta_j + \theta_j^0|^2 \\ &\leq \frac{64\pi^4 A^2}{J} \|\theta - \theta^0\|^2. \end{aligned}$$

Thus, by definition of  $\mathcal{U}_\kappa$  and by (A.3), we know that there exists some  $C'_M > 0$  that only depends on  $A, s, \sigma^2$  and  $\kappa$  such that

$$\begin{aligned} nJ \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 M(\theta) - \nabla^2 M(\theta^0)\|_{op}^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) &\leq \frac{n(64\pi^4 A^2)^2}{J} \mathbb{E} \left( \|\hat{\theta}^0 - \theta^0\|^6 \right) \\ &\leq C'_M \frac{k_0^{3/2} J^2}{n^{1/2}}. \end{aligned} \quad (\text{A.38})$$

Next, we deal with the term relative to  $\|\nabla^2 Q(\theta)\|_F^2$ . Let us begin by noting that  $\|\hat{\theta}^0 - \theta^0\|^2 \leq 4J\kappa^2$ . Thus, we have

$$\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 Q(\theta)\|_F^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq 4J\kappa^2 \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 Q(\theta)\|_F^2 \right). \quad (\text{A.39})$$

If we take  $\ell \neq \ell'$  then, using (A.24), we get

$$\left| \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_\ell} Q(\theta) \right|^2 \leq \frac{64\pi^4}{N^2 J^4} \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell}^{(0)}| |z_{k,\ell'}^{(0)}| \right)^2$$

and if  $\ell = \ell'$  then, by (A.27), we have

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_\ell^2} Q(\theta) \right|^2 &\leq \frac{64\pi^4}{N^2 J^4} \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell}^{(0)}| \left| \sum_{j=1, j \neq \ell}^J z_{k,j}^{(0)} e^{2i\pi k \theta_j} \right| \right)^2 \\ &\leq \frac{64\pi^4}{N^2 J^3} \sum_{j=1, j \neq \ell}^J \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell}^{(0)}| |z_{k,j}^{(0)}| \right)^2. \end{aligned}$$

Hence, Cauchy-Schwarz Inequality leads to the following upper bound

$$\begin{aligned} \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 Q(\theta)\|_F^2 \right) &\leq \frac{128\pi^4}{N^2 J^3} \mathbb{E} \sum_{\ell=1}^J \sum_{\ell'=1, \ell' \neq \ell}^J \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell}^{(0)}| |z_{k,\ell'}^{(0)}| \right)^2 \\ &\leq \frac{128\pi^4}{N^2 J^3} \sum_{\ell=1}^J \sum_{\ell'=1, \ell' \neq \ell}^J \mathbb{E} \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell}^{(0)}|^2 \right) \mathbb{E} \left( \sum_{|k| \leq k_0} k^2 |z_{k,\ell'}^{(0)}|^2 \right) \\ &\leq \frac{512\pi^4 \sigma^4 k_0^6}{N^2 J}. \end{aligned}$$



Combining this bound with (A.39) gives us some  $C_Q > 0$  that only depends on  $\kappa$  and  $\sigma^2$  such that

$$nJ\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 Q(\theta)\|_F^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq C_Q \frac{k_0^6 J}{n}. \quad (\text{A.40})$$

Finally, we focus on the term concerning  $\|\nabla^2 L(\theta)\|_F^2$ . By Cauchy-Schwarz Inequality and (A.2), we have

$$\begin{aligned} \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) &\leq \sqrt{\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^4 \right)} \sqrt{\mathbb{E} \left( \|\hat{\theta}^0 - \theta^0\|^4 \right)} \\ &\leq \sqrt{\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^4 \right)} \sqrt{C_2 \frac{k_0 J^2}{n}}. \end{aligned} \quad (\text{A.41})$$

Using Lemma A.1 and (A.25), if  $\ell \neq \ell'$  we obtain

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_{\ell'} \partial \theta_\ell} L(\theta) \right|^2 &\leq \frac{64\pi^4}{J^4 N} \left( \sum_{|k| \leq k_0} k^2 \left( \left| \bar{c}_{k,\ell}^{(0)} \overline{z_{k,\ell'}} \right| + \left| \bar{c}_{k,\ell'}^{(0)} \overline{z_{k,\ell}} \right| \right) \right)^2 \\ &\leq \frac{64\pi^4}{J^4 N} \left( \sum_{|k| \leq k_0} k^2 \left( |c_k| + A_0 N^{-s+1/2} \right) \left( |z_{k,\ell}^{(0)}| + |z_{k,\ell'}^{(0)}| \right) \right)^2 \end{aligned}$$

and, by (A.28), if  $\ell = \ell'$ , we get

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_\ell \partial \theta_\ell} L(\theta) \right|^2 &\leq \frac{64\pi^4}{J^4 N} \left( \sum_{|k| \leq k_0} k^2 \left( \left| \bar{c}_{k,\ell}^{(0)} \left( \sum_{j=1, j \neq \ell}^J |z_{k,j}^{(0)}| \right) + |z_{k,\ell}^{(0)}| \left( \sum_{j=1, j \neq \ell}^J |\bar{c}_{k,j}^{(0)}| \right) \right) \right) \right)^2 \\ &\leq \frac{64\pi^4}{J^4 N} \left( \sum_{|k| \leq k_0} k^2 \left( \left( |c_k| + A_0 N^{-s+1/2} \right) \left( J |z_{k,\ell}^{(0)}| + \sum_{j=1, j \neq \ell}^J |z_{k,j}^{(0)}| \right) \right) \right)^2. \end{aligned}$$

Hence, we upper bound the expectation in (A.41),

$$\begin{aligned}
& \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^4 \right) \\
& \leq \frac{4096\pi^8}{J^8 N^2} \mathbb{E} \left( \left[ \sum_{\ell=1}^J \left( \sum_{|k| \leq k_0} k^2 \left( (|c_k| + A_0 N^{-s+1/2}) \left( J|z_{k,\ell}^{(0)}| + \sum_{j=1, j \neq \ell}^J |z_{k,j}^{(0)}| \right) \right) \right) \right]^2 \right. \\
& \quad \left. + \sum_{\ell'=1, \ell' \neq \ell}^J \left( \sum_{|k| \leq k_0} k^2 \left( |c_k| + A_0 N^{-s+1/2} \right) \left( |z_{k,\ell}^{(0)}| + |z_{k,\ell'}^{(0)}| \right) \right) \right]^2 \right) \\
& \leq \frac{4096\pi^8}{J^4 N^2} \mathbb{E} \left( \left[ J \left( \sum_{|k| \leq k_0} k^2 \left( |c_k| + A_0 N^{-s+1/2} \right) \sum_{j=1}^J |z_{k,j}^{(0)}| \right) \right]^2 \right. \\
& \quad \left. + \frac{4}{J} \sum_{\ell=1}^J \left( \sum_{|k| \leq k_0} k^2 \left( |c_k| + A_0 N^{-s+1/2} \right) |z_{k,\ell}^{(0)}| \right) \right]^2 \right) \\
& \leq \frac{4 \times 4096\pi^8}{J^2 N^2} \mathbb{E} \left( \left( \sum_{|k| \leq k_0} k^2 \left( |c_k| + A_0 N^{-s+1/2} \right) \sum_{j=1}^J |z_{k,j}^{(0)}| \right)^4 \right) \\
& \leq \frac{4 \times 4096\pi^8}{N^2} \left( \sum_{|k| \leq k_0} k^4 \left( |c_k| + A_0 N^{-s+1/2} \right)^2 \right)^2 \mathbb{E} \left( \left( \sum_{|k| \leq k_0} \sum_{j=1}^J |z_{k,j}^{(0)}|^2 \right)^2 \right) \\
& \leq \frac{168 \times 4096\pi^8 \sigma^4 J^2 k_0^2}{N^2} \left( \sum_{|k| \leq k_0} k^4 \left( |c_k| + A_0 N^{-s+1/2} \right)^2 \right)^2.
\end{aligned}$$

Therefore, there exists some constant  $C'_L > 0$  that only depends on  $A$ ,  $s$  and  $\sigma^2$  such that

$$\mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^4 \right) \leq C'_L \frac{k_0^2 J^2}{n^2} \left( 1 + \frac{k_0^5}{n^{2s-1}} \right)^2. \quad (\text{A.42})$$

Using (A.41) and (A.42), we know that there exists some constant  $C_L > 0$  that only depends on  $c_*$ ,  $\kappa$ ,  $A$ ,  $s$  and  $\sigma^2$  such that

$$nJ \mathbb{E} \left( \sup_{\theta \in \mathcal{U}_\kappa} \|\nabla^2 L(\theta)\|_F^2 \|\hat{\theta}^0 - \theta^0\|^2 \right) \leq C_L \frac{k_0^{3/2} J^3}{\sqrt{n}} \left( 1 + \frac{k_0^5}{n^{2s-1}} \right). \quad (\text{A.43})$$

Finally, we use (A.37), (A.38), (A.40) and (A.43) with (A.36) to get (A.19).

## A.2 Proof of Theorem 3.2

Let us assume that  $f \in \tilde{W}_s(A, c_*)$ . We upper bound the distance between  $f$  and  $\hat{f}_{n,J}$ ,

$$\begin{aligned} d^2([f], [\hat{f}_{n,J}]) &= \inf_{\theta \in [-1/2, 1/2]} \int_0^1 |f(t - \theta) - \hat{f}_{n,J}(t)|^2 dt \\ &\leq 2d^2([f], [\bar{f}^{(m_1)}]) + 2 \int_0^1 |\bar{f}^{(m_1)}(t) - \hat{f}_{n,J}(t)|^2 dt . \end{aligned}$$

Taking the expectation according to the distribution of  $Y^{(1)}$  on both side and using (3.5) leads to

$$\begin{aligned} &\mathbb{E}^{(1)} \left[ d^2([f], [\hat{f}_{n,J}]) \right] \\ &\leq 2d^2([f], [\bar{f}^{(m_1)}]) + 2C(\eta) \min_{m \in \{1, \dots, m_1\}} \left\{ \int_0^1 |\bar{f}^{(m_1)}(t) - \bar{f}^{(m)}(t)|^2 dt + \frac{2(m+1)\sigma^2}{NJ} \right\} \\ &\leq 2d^2([f], [\bar{f}^{(m_1)}]) + 2C(\eta) \min_{m \in \{1, \dots, m_1\}} \left\{ \sum_{m < |k| \leq m_1} \left| \frac{1}{J} \sum_{j=1}^J \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 + \frac{2(m+1)\sigma^2}{NJ} \right\} \end{aligned} \quad (\text{A.44})$$

Let  $\bar{\theta}_J = (\theta_1^* + \dots + \theta_J^*)/J$ , we recall that  $\theta^0 = \theta^* - \bar{\theta}_J$ . We begin by upper bounding the first term. Thanks to Jensen's inequality, we obtain

$$\begin{aligned} d^2([f], [\bar{f}^{(m_1)}]) &\leq \int_0^1 |f(t - \bar{\theta}_J) - \bar{f}^{(m_1)}(t)|^2 dt \\ &\leq \sum_{|k| > m_1} |c_k|^2 + \sum_{|k| \leq m_1} \left| c_k e^{-2i\pi k \bar{\theta}_J} - \frac{1}{J} \sum_{j=1}^J \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 \\ &\leq \sum_{|k| > m_1} |c_k|^2 + \sum_{|k| \leq m_1} \frac{1}{J} \sum_{j=1}^J \left| c_k e^{-2i\pi k \bar{\theta}_J} - \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 . \end{aligned} \quad (\text{A.45})$$

Since  $f \in \tilde{W}_s(A, c_*)$ , we easily upper bound the bias part

$$\sum_{|k| > m_1} |c_k|^2 \leq A |m_1|^{-2s} . \quad (\text{A.46})$$

To deal with the other part, we split it into two sums,

$$\begin{aligned}
& \frac{1}{J} \sum_{j=1}^J \left| c_k e^{-2i\pi k \bar{\theta}_j} - \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 \\
& \leq \frac{2}{J} \sum_{j=1}^J \left| c_k e^{-2i\pi k \bar{\theta}_j} - c_k e^{2i\pi k (\hat{\theta}_j^0 - \theta_j^*)} \right|^2 + \frac{2}{J} \sum_{j=1}^J \left| c_k e^{2i\pi k (\hat{\theta}_j^0 - \theta_j^*)} - \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 \\
& \leq \frac{2|c_k|^2}{J} \sum_{j=1}^J \left| 1 - e^{2i\pi k (\hat{\theta}_j^0 - \theta_j^*)} \right|^2 + \frac{2}{J} \sum_{j=1}^J \left| \bar{c}_{k,j}^{(1)} - c_k e^{-2i\pi k \theta_j^*} \right|^2 \\
& \leq \frac{8\pi^2 k^2 |c_k|^2}{J} \sum_{j=1}^J \left( \hat{\theta}_j^0 - \theta_j^0 \right)^2 + 2A_0^2 N^{-2s+1} \tag{A.47}
\end{aligned}$$

where the last inequality follows from Lemma A.1 and from  $2(1 - \cos t) \leq t^2$ ,  $t \in \mathbb{R}$ . Combining (A.45), (A.46) and (A.47), we get, for any  $g \in \mathcal{G}^\kappa$ ,

$$\begin{aligned}
\mathbb{E}^g \left[ d^2([f], [\bar{f}^{(m_1)}]) \right] & \leq A|m_1|^{-2s} + 2A_0^2(2m_1 + 1)N^{-2s+1} \\
& \quad + 8\pi^2 \left( \sum_{|k| \leq m_1} k^2 |c_k|^2 \right) \mathbb{E}^g \left[ \frac{1}{J} \sum_{j=1}^J \left( \hat{\theta}_j^0 - \theta_j^0 \right)^2 \right]. \tag{A.48}
\end{aligned}$$

We now focus on the second term in (A.44). Let  $\alpha_{k,j} = c_k e^{-2i\pi k \theta_j^*} - \bar{c}_{k,j}^{(1)}$ . Using Jensen's inequality and Lemma A.1, for any  $m \in \{1, \dots, m_1\}$ , we have

$$\begin{aligned}
\sum_{m < |k| \leq m_1} \left| \frac{1}{J} \sum_{j=1}^J \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 & \leq \sum_{m < |k| \leq m_1} \frac{1}{J} \sum_{j=1}^J \left| c_k e^{-2i\pi k \theta_j^*} + \alpha_{k,j} \right|^2 \\
& \leq 2 \sum_{|k| > m} |c_k|^2 + \frac{2}{J} \sum_{m < |k| \leq m_1} \sum_{j=1}^J |\alpha_{k,j}|^2 \\
& \leq 2Am^{-2s} + 4A_0^2 m_1 N^{-2s+1}.
\end{aligned}$$

Let us consider  $m_*$  such that

$$m_* = \left\lfloor \left( \frac{nJ}{c} \right)^{1/(2s+1)} \right\rfloor$$

where  $c$  is the constant such that  $J \leq cn^\alpha$ . Note that such a choice is allowed because it is such that  $m_* \in \{1, \dots, m_1\}$  since  $n \geq 21$ ,  $s > 3/2$ ,  $\alpha \in (0, 1/6]$  and  $c \in (0, 1)$ . In particular, such a

choice leads to the following upper bound

$$\begin{aligned}
& \min_{m \in \{1, \dots, m_1\}} \left\{ \sum_{m < |k| \leq m_1} \left| \frac{1}{J} \sum_{j=1}^J \bar{c}_{k,j}^{(1)} e^{2i\pi k \hat{\theta}_j^0} \right|^2 + \frac{2(m+1)\sigma^2}{NJ} \right\} \\
& \leq 4A_0^2 m_1 N^{-2s+1} + \frac{2\sigma^2}{NJ} + 2 \min_{m \in \{1, \dots, m_1\}} \left\{ Am^{-2s} + \frac{m\sigma^2}{NJ} \right\} \\
& \leq 4A_0^2 m_1 N^{-2s+1} + \frac{2\sigma^2}{NJ} + 2 \left( \frac{A}{2} c^{2s/(2s+1)} + 2\sigma^2 c^{2s/(2s+1)} \right) (nJ)^{-2s/(2s+1)} \\
& \leq 4A_0^2 m_1 N^{-2s+1} + \left( 1 + (A + 4\sigma^2) c^{2s/(2s+1)} \right) (nJ)^{-2s/(2s+1)}. \tag{A.49}
\end{aligned}$$

Putting (A.48) and (A.49) in (A.44) leads to, for any  $g \in \mathcal{G}^\kappa$ ,

$$\begin{aligned}
\mathcal{R}_g(\hat{f}_{n,J}, f) & \leq A|m_1|^{-2s} + 4A_0^2((1 + C(\eta))m_1 + 1)N^{-2s+1} \\
& \quad + 8\pi^2 \left( \sum_{|k| \leq m_1} k^2 |c_k|^2 \right) \mathbb{E}^g \left[ \frac{1}{J} \sum_{j=1}^J (\hat{\theta}_j^0 - \theta_j^0)^2 \right] \\
& \quad + C(\eta) \left( 1 + (A + 4\sigma^2) c^{2s/(2s+1)} \right) (nJ)^{-2s/(2s+1)},
\end{aligned}$$

that completes the proof using the fact that  $m_1 = \lfloor N/2 \rfloor - 1$ .

### A.3 Proof of Theorem 4.1

The arguments that we use to derive this result are based on Assouad's cube Lemma (see e.g. [Tsy09]). This Lemma is classically used in nonparametric statistics to derive lower bounds on a risk. We will show that one can construct a set of functions  $\mathcal{F}_0 \subset \tilde{W}_s(A, c_*)$  such that there exists a constant  $C > 0$  (only depending on  $A$ ,  $s$ ,  $c_*$  and  $\sigma^2$ ) such that, for any large enough  $n$  and  $J$ ,

$$\mathcal{R}_{n,J}(\tilde{W}_s(A, c_*), \mathcal{G}^\kappa) \geq \inf_{\hat{f}_{n,J}} \sup_{g \in \mathcal{G}^\kappa} \sup_{f \in \mathcal{F}_0} \mathcal{R}_{P_0}(\hat{f}_{n,J}, f) \geq C(nJ)^{-\frac{2s}{2s+1}}, \tag{A.50}$$

where  $\hat{f}_{n,J}$  denote some estimator of  $f$ . For the sake of legibility, we assume in the sequel that  $c_* = 1$ . Let

$$\mathcal{F}_0 = \left\{ f_w : [0, 1] \rightarrow \mathbb{R}; f_w(t) = \sqrt{\mu_{n,J}} \sum_{k \in K_{n,J}} w_k \phi_k(t), w_k \in \{-1, 1\}, w_{-k} = w_k \right\},$$

where  $K_{n,J} = \{k \in \mathbb{Z}, 0 < |k| \leq D_{n,J}\}$ ,  $\mu_{n,J}$  is a positive real and  $D_{n,J}$  is a positive integer that will be specified below. Let us introduce the notation  $\Omega = \{-1, 1\}^{D_{n,J}}$  and note that any function  $f_w \in \mathcal{F}_0$  is parametrized by a unique element  $w \in \Omega$ . Under the condition

$$\mu_{n,J} = cD_{n,J}^{-2s-1}, \text{ with } c \leq A, \tag{A.51}$$

it can easily be checked that  $\mathcal{F}_0 \subset \tilde{W}_s(A, c_*)$ . In what follows,  $D_{n,J}$  is chosen as the largest integer smaller than  $(nJ)^{\frac{1}{2s+1}}$ . Hereafter,  $\mathbb{E}_w^g$  will denote the expectation with respect to the distribution

$\mathbb{P}_w^g$  of the random vector  $(Y_{\ell,j})_{1 \leq \ell \leq n, 1 \leq j \leq J} \in \mathbb{R}^{nJ}$  in model (1.3) under the hypothesis that  $f = f_w$  and the assumption that the shifts are i.i.d. random variables with density  $g \in \mathcal{G}^\kappa$ . Note that for any  $g \in \mathcal{G}^\kappa$

$$\begin{aligned} \sup_{f \in \mathcal{F}_0} \mathcal{R}_g(\hat{f}_{n,J}, f) &= \sup_{f \in \mathcal{F}_0} \mathbb{E} \left[ \inf_{\theta \in [0,1]} \left( \int_0^1 |\hat{f}_{n,J}(t - \theta) - f(t)|^2 dt \right) \right] \\ &\geq \frac{1}{|\Omega|} \sum_{w \in \Omega} \mathbb{E}_w^g \left[ \inf_{\theta \in [0,1]} \left( \int_0^1 |\hat{f}_{n,J}(t - \theta) - f_w(t)|^2 dt \right) \right] \\ &\geq \frac{1}{|\Omega|} \sum_{w \in \Omega} \mathbb{E}_w^g \left[ \inf_{\theta \in [0,1]} \left( \sum_{k \in K_{n,J}} \left| \hat{c}_k e^{-2ik\pi\theta} - \sqrt{\mu_{n,J}} w_k \right|^2 \right) \right], \end{aligned}$$

where  $\hat{c}_k = \int_0^1 \hat{f}_{n,J}(t) \overline{\phi_k(t)} dt$  is the  $k$ -th Fourier coefficient of  $\hat{f}_{n,J}$ . Now, we consider, for  $k \in K_{n,J}$  and  $\theta \in [0, 1]$ ,

$$\hat{w}_{k,\theta} \in \operatorname{argmin}_{v \in \{-1,1\}} |\hat{c}_k e^{-2ik\pi\theta} - \sqrt{\mu_{n,J}} v|.$$

We have the inequality

$$\begin{aligned} |\sqrt{\mu_{n,J}} \hat{w}_{k,\theta} - \sqrt{\mu_{n,J}} w_k| &\leq |\hat{c}_k e^{-2ik\pi\theta} - \sqrt{\mu_{n,J}} \hat{w}_{k,\theta}| + |\hat{c}_k e^{-2ik\pi\theta} - \sqrt{\mu_{n,J}} w_k| \\ &\leq 2|\hat{c}_k e^{-2ik\pi\theta} - \sqrt{\mu_{n,J}} w_k| \end{aligned}$$

that implies

$$\sup_{f \in \mathcal{F}_0} \mathcal{R}_g(\hat{f}_{n,J}, f) \geq \frac{\mu_{n,J}}{4|\Omega|} \sum_{k \in K_{n,J}} \sum_{w \in \Omega} \mathbb{E}_w^g \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} - w_k|^2) \right].$$

For  $w \in \Omega$  and  $k \in K_{n,J}$ , we define  $w^{(k)} \in \Omega$  such that, for any  $\ell \neq k$ ,  $w_\ell^{(k)} = w_\ell$  and  $w_k^{(k)} = -w_k$ . Then, it follows that

$$\sup_{f \in \mathcal{F}_0} \mathcal{R}_g(\hat{f}_{n,J}, f) \geq \frac{\mu_{n,J}}{4|\Omega|} \sum_{k \in K_{n,J}} \sum_{w \in \Omega | w_k=1} R_k \quad (\text{A.52})$$

where we have set

$$R_k = \mathbb{E}_w^g \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} - w_k|^2) \right] + \mathbb{E}_{w^{(k)}}^g \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} + w_k|^2) \right].$$

Let  $\theta^* = (\theta_1^*, \dots, \theta_J^*)$ , we introduce the notation  $\mathbb{E}_w^{\theta^*}$  to denote expectation with respect to the distribution  $\mathbb{P}_w^{\theta^*}$  of the random vector  $(Y_{\ell,j})_{1 \leq \ell \leq n, 1 \leq j \leq J} \in \mathbb{R}^{nJ}$  in model (1.1) conditionally to  $\theta_1^*, \dots, \theta_J^*$ . Hence, using these notations, we have

$$R_k = \int_{[-\frac{1}{2}, \frac{1}{2}]^J} R_k(\theta^*) g(\theta_1^*) \dots g(\theta_J^*) d\theta_1^* \dots d\theta_J^* \quad (\text{A.53})$$

where

$$R_k(\theta^*) = \mathbb{E}_w^{\theta^*} \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} - w_k|^2) \right] + \mathbb{E}_{w^{(k)}}^{\theta^*} \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} + w_k|^2) \right].$$

Now, note that for any  $0 < \delta < 1$ ,

$$\begin{aligned} R_k(\theta^*) &= \mathbb{E}_w^{\theta^*} \left[ \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} - w_k|^2) + \inf_{\theta \in [0,1]} (|\hat{w}_{k,\theta} + w_k|^2) \frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \right] \\ &\geq 4\mathbb{E}_w^g \min \left( 1, \frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \right) \geq 4\delta \mathbb{P}_w^{\theta^*} \left( \frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \geq \delta \right), \end{aligned} \quad (\text{A.54})$$

where  $Y \in \mathbb{R}^{nJ}$  is the random vector obtained from the concatenation of the observations from model (1.3) under the hypothesis  $f = f_w$  and conditionally to  $\theta_1^*, \dots, \theta_J^*$ . Because  $w_k = 1$ , we know that

$$\begin{aligned} &\frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \\ &= \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^J \sum_{\ell=1}^n |f_{w^{(k)}}(t_\ell - \theta_j^*)|^2 - |f_w(t_\ell - \theta_j^*)|^2 + 2Y_{\ell,j} (f_w(t_\ell - \theta_j^*) - f_{w^{(k)}}(t_\ell - \theta_j^*)) \right) \\ &= \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^J \sum_{\ell=1}^n |f_{w^{(k)}}(t_\ell - \theta_j^*) - f_w(t_\ell - \theta_j^*)|^2 + 2\epsilon_{\ell,j} (f_w(t_\ell - \theta_j^*) - f_{w^{(k)}}(t_\ell - \theta_j^*)) \right) \\ &= \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^J \sum_{\ell=1}^n 4\mu_{n,J} |\phi_k(t_\ell - \theta_j^*)|^2 + 4\sqrt{\mu_{n,J}} \epsilon_{\ell,j} \phi_k(t_\ell - \theta_j^*) \right) \end{aligned}$$

Therefore,  $\log \left( \frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \right)$  is a random variable that is normally distributed with mean  $-\frac{2}{\sigma^2} nJ\mu_{n,J}$  and variance  $\frac{4}{\sigma^2} nJ\mu_{n,J}$ . Now, since  $D_{n,J}$  is the largest integer smaller than  $(nJ)^{\frac{1}{2s+1}}$ , it follows from equation (A.51) that  $0 \leq nJ\mu_{n,J} \leq 2A$ , for any large enough  $n$  and  $J$ . Thus, there exists  $0 < \delta < 1$  and a constant  $c_\delta > 0$  (only depending on  $A$ ,  $\sigma^2$  and  $\delta$ ) such that

$$\mathbb{P}_w^{\theta^*} \left( \frac{d\mathbb{P}_w^{\theta^*}}{d\mathbb{P}_w^{\theta^*}}(Y) \geq \delta \right) \geq c_\delta.$$

Combining this inequality with (A.52), (A.53) and (A.54) lead to

$$\sup_{f \in \mathcal{F}_0} \mathcal{R}_g(\hat{f}_{n,J}, f) \geq \frac{4\delta\mu_{n,J}}{|\Omega|} \sum_{k \in K_{n,J}} \sum_{w \in \Omega | w_k=1} c_\delta \geq \delta c_\delta \mu_{n,J} D_{n,J}. \quad (\text{A.55})$$

Since  $\mu_{n,J} = cD_{n,J}^{-2s-1}$  and  $D_{n,J} \leq (nJ)^{\frac{1}{2s+1}}$ , it follows that

$$\mu_{n,J} D_{n,J} = cD_{n,J}^{-2s} \geq c(nJ)^{-\frac{2s}{2s+1}},$$

which combined with (A.55) proves inequality (A.50) and completes the proof of Theorem 4.1.

## References

- [Afs11] Bijan Afsari. Riemannian  $l^p$  center of mass: existence, uniqueness, and convexity. *Proceedings of the American Mathematical Society*, 139(2):655–673, 2011.
- [BC11] J. Bigot and B. Charlier. On the consistency of fréchet means in deformable models for curve and image analysis. *Electronic Journal of Statistics*, 5:1054–1089, 2011.
- [BG10] Jérémie Bigot and Sébastien Gadat. A deconvolution approach to estimation of a common shape in a shifted curves model. *Ann. Statist.*, 38(4):2422–2464, 2010.
- [Big06] J. Bigot. Landmark-based registration of curves via the continuous wavelet transform. *J. Comput. Graph. Statist.*, 15(3):542–564, 2006.
- [BLV10] J. Bigot, J.M. Loubes, and M. Vimond. Semiparametric estimation of shifts on compact Lie groups for image registration. *Probability Theory and Related Fields*, 148:1–49, 2010.
- [BP03] R. Bhattacharya and V. Patrangenaru. Large sample theory of intrinsic and extrinsic sample means on manifolds (i). *Annals of statistics*, 31(1):1–29, 2003.
- [BP05] R. Bhattacharya and V. Patrangenaru. Large sample theory of intrinsic and extrinsic sample means on manifolds (ii). *Annals of statistics*, 33:1225–1259, 2005.
- [Fan91] Jianqing Fan. On the optimal rate of convergence for nonparametric deconvolution problems. *Ann. Statist.*, 19:1257–1272, 1991.
- [Fré48] M. Fréchet. Les éléments aléatoires de nature quelconque dans un espace distancié. *Ann. Inst. H.Poincaré, Sect. B, Prob. et Stat.*, 10:235–310, 1948.
- [GLM07] Fabrice Gamboa, Jean-Michel Loubes, and Elie Maza. Semi-parametric estimation of shifts. *Electron. J. Stat.*, 1:616–640, 2007.
- [GM01] C. A. Glasbey and K. V. Mardia. A penalized likelihood approach to image warping. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63(3):465–514, 2001.
- [GM07] U. Grenander and M. Miller. *Pattern Theory: From Representation to Inference*. Oxford Univ. Press, Oxford, 2007.
- [Goo91] C. Goodall. Procrustes methods in the statistical analysis of shape. *J. Roy. Statist. Soc. Ser. B*, 53(2):285–339, 1991.
- [Gre93] U. Grenander. *General pattern theory - A mathematical study of regular structures*. Clarendon Press, Oxford, 1993.
- [Huc11] Stephan F. Huckemann. Intrinsic inference on the mean geodesic of planar shapes and tree discrimination by leaf growth. *Ann. Statist.*, 39(2):1098–1124, 2011.
- [Ken84] D.G. Kendall. Shape manifolds, procrustean metrics, and complex projective spaces. *Bull. London Math Soc.*, 16:81–121, 1984.



- [Mas07] Pascal Massart. *Concentration inequalities and model selection*, volume 1896 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 33rd Summer School on Probability Theory held in Saint-Flour, July 6–23, 2003, With a foreword by Jean Picard.
- [PV99] Marianna Pensky and Brani Vidakovic. Adaptive wavelet estimator for nonparametric density deconvolution. *Ann. Statist.*, 27:2033–2053, 1999.
- [Røn01] B. B. Rønn. Nonparametric maximum likelihood estimation for shifted curves. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63(2):243–259, 2001.
- [TIR11] Thomas Trigano, Uri Isserles, and Ya’acov Ritov. Semiparametric curve alignment and shift density estimation for biological data. *IEEE Trans. Signal Process.*, 59(5):1970–1984, 2011.
- [Tsy09] Alexandre B. Tsybakov. *Introduction to nonparametric estimation*. Springer Series in Statistics. Springer, New York, 2009. Revised and extended from the 2004 French original, Translated by Vladimir Zaiats.
- [TY05] A. Trouvé and L. Younes. Local geometry of deformable templates. *SIAM J. Math. Anal.*, 37(1):17–59 (electronic), 2005.
- [TY11] A. Trouvé and L. Younes. Shape spaces. In *Handbook of Mathematical Methods in Imaging*. Springer, 2011.
- [Vim10] Myriam Vimond. Efficient estimation for a subclass of shape invariant models. *Ann. Statist.*, 38(3):1885–1912, 2010.