Generalised Graetz problem: analytical solutions for concentric or parallel configurations

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GENERALISED GRAETZ PROBLEM: ANALYTICAL SOLUTIONS FOR CONCENTRIC OR PARALLEL CONFIGURATIONS

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Abstract. In this paper are presented analytical solutions for the generalised Graetz problem for symmetric domains (either concentric or parallel). Such configurations are considered as protoypes for heat exchanger devices and appear and numerous applications involving heat or mass transfer. Solutions for the generalised Graetz problem are derived as analytical series. The computation of the series coefficients (the closure functions) is recursive. This computation can be done using standard formal calculus software.

1. Introduction

2. Setting the problem

2.1. Physical problem.

We study stationnary convection diffusion for an axi-symmetric configuration.

The domain is set to $\Omega \times (a,b)$ with $(a,b) \subset \mathbb{R}$ an interval and Ω the circle $C(0,R)$ with centre 0 and radius R. The longitudinal coordinate is denoted z and cylindrical coordinates (r,ϕ) are used in the transverse plane.

The transverse section Ω is splitted into m different compartments either fluid or solid as follows. Considee a sequence $0 = r_0 < \cdots < r_m = R$ with $m \geq 2$. Consider $\Omega_1, \ldots, \Omega_m$ the annuli $\Omega_j = \{M = (r,\phi) \in \mathbb{R}^2,\ r_{j-1} \leq r \leq r_j\}$. The interface between Ω_j and Ω_{j+1} is denoted $\Gamma_j = C(0, r_{j+1})$.

The following axi-symmetric configuration is adopted throughout this paper (see Fig. 1):

(1) Velocity: $\mathbf{v}(r,\phi,z) = v(r)\ \mathbf{e}_z$ with $\mathbf{e}_z$ the unit vector along the z direction.
In case the region Ω_j is associated with a solid we will have $v_{|\Omega_j} = 0$.
We will assume that $r \mapsto v(r)$ is analytical on each compartment $[r_{j-1}, r_j]$ (but possibly discontinuous at each interface r_j).

(2) Conductivity: $k(r,\phi,z) = k(r)$ and moreover $k_{|\Omega_j} = k_j > 0$ is a constant.

The general equation for the transfer reads

$$\operatorname{div}_{(r,\phi,z)}(k\nabla_{(r,\phi,z)}T) = \mathbf{v} \cdot \nabla_{(r,\phi,z)}T \quad \text{in } \Omega. \tag{1}$$

From now on we denote $\operatorname{div} = \operatorname{div}_{(r,\phi)}$ and $\nabla = \nabla_{(r,\phi)}$ the gradient and divergence operators restricted to the transverse plane. The physical

Date: February, 2011.
Key words and phrases. Heat and mass transfer, convection diffusion,...

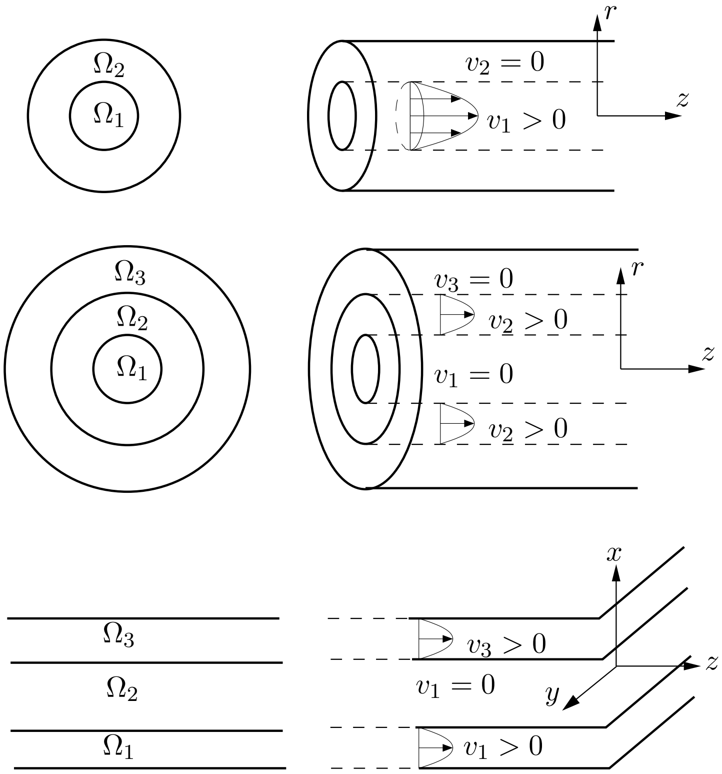

FIGURE 1. Illustration of three possible configurations. Above: fluid flowing inside a circular tube with a solid wall. Middle: fluid flowing inside an annulus between a solid core and a solid external wall. Below : planar configuration.

and geometrical assumptions above induce the following simplification of problem (1)

$$\operatorname{div}(k\nabla T) + k\partial_z^2 T = v\partial_z T \quad \text{in } \Omega \times (a,b). \tag{2}$$

A homogeneous Dirichlet boundary condition is fixed on the wall boundary

$$T = 0 \quad \text{on } \partial\Omega \times (a,b). \tag{3}$$

Later on this boundary condition will be supposed to be homogeneous either of Dirichlet, Neumann or Robin type. Homogeneous Dirichlet being firstly fixed for more simplicity in the problematic exposition.

2.2. **Resolution background.** Equations (2) (3) have been reformulated in [4] into a system of two order one equations, adding a supplementary vectorial unknown $\mathbf{p}: \ \Omega \to \mathbb{R}^2$, which reads

$$\partial_z \Psi = A\Psi \quad \text{with} \quad \Psi = (T, \mathbf{p}) \ , \quad A = \begin{pmatrix} vk^{-1} & -k^{-1}\operatorname{div}(\cdot) \\ k\nabla\cdot & 0 \end{pmatrix}.$$

The differential operator $A: \ D(A) \subset L^2(\Omega) \times [L^2(\Omega)]^2 \mapsto L^2(\Omega) \times [L^2(\Omega)]^2$ has for domain $D(A) = H_0^1(\Omega) \times H_{\text{div}}(\Omega)$. The operator A has been showed to be self adjoint with compact resolvant. Its spectrum Λ

(excepting the value 0) is then composed of a set of eigenvalues (i.e with finite order) $\Lambda = (\lambda_i)_{i\in\mathbb{Z}^\star}$ that moreover satisfies

$$-\infty \leftarrow \leq \lambda_n \leq \cdots \leq \lambda_1 < 0 < \lambda_{-1} \leq \cdots \leq \lambda_{-n} \to +\infty. \quad (4)$$

The spectrum Λ splits into a double set of eigenvalues. The $\{\lambda_i,\ i<0\}$ are positive and referred to as upstream modes, whereas the $\{\lambda_i,\ i>0\}$ are negative and referred to as downstream modes. The associated eigenfunctions denoted $(\Psi_i)_{i\in\mathbb{Z}^\star}$ form a complete orthogonal system in $\mathrm{Ker}(A)^\perp$. We decompose $\Psi_i = (\Theta_i, \mathbf{p}_i)$, the vectorial component moreover satisfies $\mathbf{p}_i = k\nabla\Theta_i/\lambda_i$.
It is important to notice that the $\Theta_i:\ \Omega \mapsto \mathbb{R}$ only encode the scalar component of the associated eigenfunctions Ψ_i: as a result the $(\Theta_i)_{i\in\mathbb{Z}^\star}$ neither form an orthogonal system nor a complete one in $L^2(\Omega)$.

It has been proven in [2] that both the $\{\Theta_i,\ i\in\mathbb{Z}^+\}$ and the $\{\Theta_i,\ i\in\mathbb{Z}^-\}$ form a complete (Hilbert) basis for $L^2(\Omega)$. Solutions to (2)-(3) are sought under the following series expansion

$$T(r,\phi,z) = \sum_{i\in\mathbb{Z}^+} c_i\Theta_i(r,\phi)\, e^{\lambda_i z} \quad \text{or} \quad T(r,\phi,z) = \sum_{i\in\mathbb{Z}^-} c_i\Theta_i(r,\phi)\, e^{\lambda_i z}, \quad (5)$$

for a set of constants c_i to be determined. Analytical solution to (2)-(3) under the form (5) have already been derived for various configurations in [4, 2]:

- Infinite cylinder $\Omega\times\mathbb{R}$, using two series: one on $(0,+\infty)$ (resp. $(-\infty,0)$) considering the upstream (resp. downstream) eigenmodes only.
- Semi infinite cylinder $\Omega\times(0,+\infty)$, considering one series on the upstream modes only together with an L^2 entry condition,
- Finite cylinder $\Omega\times(0,L)$, considering L^2 entry and exit conditions and both the upstream/downstream modes,

including non-axisymmetric configurations using finite element computation. In this contribution we focus our attention to axisymmetrical concentric configurations for which we which to search for optimal configurations.

Hence, in the following we focus on the definition and computation of the eigenvalues/eigenfunctions λ_i and Θ_i in the particular above described axi symmetrical configuration.

In a second part, we will explain how to use those definition to implement and analyse optimal configurations.

Corollary 2.1. *The spectrum $\Lambda = (\lambda_i)_{i\in\mathbb{Z}^\star}$ for the operator A exactly is the spectrum for the generalised eigenproblem:*

$$\mathrm{div}(k(r)\nabla\Theta) + \lambda^2 k(r)\Theta = \lambda v(r)\Theta \quad \text{on} \quad \Omega, \quad (6)$$

$$\Theta = 0 \quad \text{on} \quad \partial\Omega. \quad (7)$$

More precisely, $\Psi = (\Theta,\mathbf{p}) \in D(A)$ is an eigenfunction associated to $\lambda\in\Lambda$ if and only if there exists $\Theta\in H^1_0(\Omega)$ and a scalar λ solution of (6) (distribution sense).

The regularity assumptions on the parameters v and k per compartment imply by elliptic regularity (see e.g. [3]) that the solution Θ to (6) is smooth on each Ω_j.

Moreover the usual transmission conditions on the interfaces Γ_j (continuity of the temperature Θ and of the normal heat flux $k\nabla\Theta \cdot \mathbf{n}$ across Γ_j) also naturally are satisfied by a solution Θ to (6).

J'ai essayé de mettre l'emphase dans cette section sur (a) la problématique qui est ici restreinte au calcul des λ_i et des Θ_i, (b) l'importance de ce calcul au vu du cadre théorique et pratique que l'on a développé, (c) qu'on a des facilités ici exploitées pour le calculs des λ_i, Θ_i en axi-symétrique.

3. Developments enduced by axi-symmetry

In this section is introduced a set of functions $T_{N,\lambda} : [0, R] \mapsto \mathbb{R}$, referred to as base functions, for $N \in \mathbb{N}$ and for $\lambda \in \mathbb{C}$. The base functions are defined assuming that Θ in (6) has shape $\Theta(r, \phi) = \theta(r)e^{iN\phi}$. In other words we here study the Fourier decomposition (in ϕ) of the eigenfunctions $\Theta_i(r, \phi)$. This allows a reduction of the original PDE problem into ODEs.

The spectrum Λ and the eigenfunctions Θ_i are reinterpreted in terms of base functions in lemma 3.2. Lemma 3.2 defines the general resolution strategy for the computation of the eigenvalues/eigenfunctions λ_i and Θ_i in (5). Firstly the original spectrum Λ is decomposed into a sequence of secondary spectra Λ_N following the Fourier decomposition of the eigenfunctions. Secondly the secondary spectra Λ_N have the implicit definition (12) involving only the base functions $T_{N,\lambda}$.

The computation of the $T_{N,\lambda}$ and the resolution of this implicit equation (12) are the following steps: they are presented in section 4.

3.1. Base functions.

Let $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}$. We search for a function $\theta : [0, R] \mapsto \mathbb{C}$ so that for all $j = 1, \dots, m$

$$\lambda^2 k_j \theta + k_j \Delta_N \theta = \lambda v(r)\theta \quad \text{on} \quad (r_{j-1}, r_j), \tag{8}$$

together with the transmission conditions: for all $j = 1, \dots, m-1$,

$$\theta(r_j^-) = \theta(r_j^+) \;, \quad k_j \frac{d}{dr}\theta(r_j^-) = k_{j+1}\frac{d}{dr}\theta(r_j^+), \tag{9}$$

and where the operator Δ_N is defined by,

$$\begin{aligned} \Delta_N \theta &= \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}\theta) - \frac{N^2}{r^2}\theta \\ &= \frac{1}{r^{N+1}}\frac{d}{dr}\left(r^{2N+1}\frac{d}{dr}\left(\frac{\theta}{r^N}\right)\right). \end{aligned} \tag{10}$$

Lemma 3.1. *Assume that $v(r)$ is analytical at $r = 0$.*

For all $\lambda \in \mathbb{C}$ and all $N \in \mathbb{N}$, there exists a unique function $T_{N,\lambda} : [0, R] \mapsto \mathbb{C}$ satisfying:

- *equations (8) for $j = 1, \dots, m$,*
- *the transmission conditions (9) for $j = 1, \dots, m-1$,*
- *the normalisation condition*

$$\lim_{r \to 0} \frac{T_{N,\lambda}(r)}{r^N} = 1. \tag{11}$$

We will refer to $T_{N,\lambda}$ as base function. In case $\lambda \in \mathbb{R}$ then $T_{N,\lambda}$ is real valued.

Proof. Let us fix $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}$.

Assume first that $T_{N,\lambda}(r_j)$ and $\partial_r T_{N,\lambda}(r_j)$ are given for some $j \geq 1$. Then $T_{N,\lambda}(r)$ is uniquely determided on $[r_j, r_{j+1}]$ by these two initial conditions together with the order 2 regular ODE (8). By induction, existence and unicity for $T_{N,\lambda}$ only need to be proven on $[0, r_1]$.

On this interval $[0, r_1]$, the order 2 ODE (8) is singular at $r = 0$. This equation can be studied using the Frobenius method (see e.g. [5]) under the assumption *"v(r) analytical at $r = 0$"*. It has a space of solutions of dimension 2. Two base functions are given by the Frobenius method. The first one is unbounded at the origin. The second one, denoted here θ, is bounded at the origin and moreover has the following behaviour $\theta(r) \sim Cr^N$ at the origin $r = 0$ with $C \neq 0$. As a result, condition (11) uniquely determines $T_{N,\lambda}$ in $[0, r_1]$. □

3.2. **Fourier decomposition of Λ.**

Definition 3.1. For $\lambda \in \Lambda$ we denote by $\mathcal{E}_\lambda$ the eigenspace associated to $\lambda \in \Lambda$ (that is the space of all Θ solution of (6)).
With the notations in (4) (5), in case $\lambda \in \Lambda$ is of order p we have $\lambda = \lambda_i = \cdots = \lambda_{i+p}$ for a given $i \in \mathbb{Z}^\star$ and the functions $\Theta_i, \ldots, \Theta_{i+p}$ form a basis of $\mathcal{E}_\lambda$.

For $N \in \mathbb{N}$ let us define

$$\Lambda_N = \{\lambda \in \mathbb{C}\ , \quad T_{N,\lambda}(R) = 0\}\,. \tag{12}$$

For $\lambda \in \Lambda_N$ we introduce the vector space $E_{N,\lambda}$

$$E_{N,\lambda} = \mathrm{Span}\left(T_{N,\lambda}(r)\cos(N\phi), T_{N,\lambda}(r)\sin(N\phi)\right), \tag{13}$$

that is of dimension 1 if $N = 0$ or of dimension 2 otherwise.
We add the convention $E_{N,\lambda} = \{0\}$ in case $\lambda \notin \Lambda_N$.

Lemma 3.2. *We have,*

$$\Lambda = \bigcup_{N \in \mathbb{N}} \Lambda_N \quad \textit{and for all} \ \ \lambda \in \Lambda : \quad \mathcal{E}_\lambda = \bigoplus_{N \in \mathbb{N}} E_{N,\lambda}. \tag{14}$$

For the equality on the right: $E_{N,\lambda} \neq \{0\}$ only for a finite number of $N \in \mathbb{N}$ (since $\mathcal{E}_\lambda$ has a finite dimension and because the sum obviously is direct).
Consequently

- *for $\lambda \in \Lambda$ there exists a finite number of $N \in \mathbb{N}$ so that $\lambda \in \Lambda_N$,*
- *the eigenfunctions Θ_i in (5) can be chosen on the following form*

$$\Theta_i(r, \phi) = \begin{cases} T_{\lambda_i, N}(r)\cos(N\phi) \\ \textit{or} \\ T_{\lambda_i, N}(r)\sin(N\phi) \end{cases}$$

Proof. Let us fix $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}$. By construction $\Theta \in E_{N,\lambda}$ is a solution of (6). If additionnaly $\lambda \in \Lambda_N$ then it also satisfies (7). Then $\Lambda_N \subset \Lambda$ and $E_{N,\lambda} \subset \mathcal{E}_{N,\lambda}$.

Consider now $\lambda \in \Lambda$ and set $\Theta \in \mathcal{E}_\lambda$ (with $\Theta \neq 0$). Let us perform its Fourier decomposition:

$$\Theta(r, \phi) = \sum_{N \in \mathbb{N}} \theta_N^1(r)\cos(2N\phi) + \theta_N^2(r)\sin(2N\phi).$$

By regularity of Θ over each compartment Ω_j (see corollary 2.1), the sum can be differentiated term by term. Embedding this decomposition into equation (6) imply that the $\theta^j(r)$, $j = 1, 2$, are solutions of (8). The transmission conditions on Θ (see also corollary 2.1) imply that the $\theta^j(r)$, $j = 1, 2$, satisfy conditions (9) as well. As a result of lemma 3.1 we have $\theta^j = c^j T_{N,\lambda}$ for two constant c^j. In case one of these two constants is non zero (which may occur at least for one value of $N \in \mathbb{N}$ since $\Theta \neq 0$), equation (7) implies that $T_{N,\lambda}(R) = 0$ and so $\lambda \in \Lambda_N$.
This makes sure that $\Lambda \subset \bigcup_{N\in\mathbb{N}} \Lambda_N$ and that $\mathcal{E}_{N,\lambda} \subset \bigoplus_{N\in\mathbb{N}} E_{N,\lambda}$. $\square$

A propos de la régularité. Dans la preuve ci dessus le point essentiel est de pouvoir dériver à l'ordre 2 sous le signe somme dans l'expansion en série de Fourier de Θ.
Pour cela il faut que les séries de Fourier des dérivées secondes de Θ soient normalement convergentesm ce qui est le cas si elles sont C^1 (par compartiment) donc si Θ elle même est C^3 (par compartiment). Pour que cela soit vrai il suffit (par régularité elliptique) que les données k et v soient C^1 sur chaque Ω_j.

4. λ-ANALICITY OF THE BASE FUNCTIONS

In this section we define a set of *closure functions* $t_{N,p}: [0, R] \mapsto \mathbb{R}$ for $N, p \in \mathbb{N}$. They are recursively defined with p. This recursive definition moreover is explicit, allowing their computation.

Our main result is that the base function $T_{N,\lambda}$ are analytical in the parameter λ. Precisely, a perturbation analysis around $T_{N,0}$ provide the following series expansion

$$T_{N,\lambda}(r) = T_{N,0}(r) + \sum_{p\geq 1} t_{N,p}(r)\lambda^p. \tag{15}$$

With this equality the spectra Λ_N are given by the zero set of the analytical series $T_{N,0}(R) + \sum_{p\in\mathbb{N}} t_{N,p}(R)\lambda^p$. The explicit definition of the closure functions thus provides a simple way to approximate both the eigenvalues and the eigenfunctions in (5).

Eventually the closure function and the base function definitions being independent of the boundary condition on $\partial\Omega$, the method extends to Neumann or Robin boundary conditions in a straightforward way.

For more simplicity in the sequel, and without ambiguity, the closure functions $t_{N,p}$ also will be denoted t_p (N being fixed).

4.1. **Closure functions.**

Let $N \in \mathbb{N}$ be fixed. We denote $t_p^j = t_{p|[r_{j-1},r_j]}$ the restriction of t_p to $[r_{j-1}, r_j]$.

The closure function definition is obtained by embedding the serie expansion (15) into constitutive equations for base function (8), (9) and (11). This provides a recursive definition starting with $t_{-1} = 0$ and $t_0 = T_{N,0}$.

1- We recover (8) on the $T_{N,\lambda}$ by imposing for all $p \geq 1$ and all $j = 1 \dots m$ the order 2 ODE

$$k_j \Delta_N t_p + k_j t_{p-2} = v(r) t_{p-1}. \quad \text{on} \quad (r_{j-1}, r_j). \tag{16}$$

2- We recover the normalisation condition (11) on the $T_{N,\lambda}$ by imposing for all $p \geq 1$:

$$\lim_{r\to 0} \frac{t_p(r)}{r^N} = 0. \tag{17}$$

3- Furthermore, transmission conditions (9) on the $T_{N,\lambda}$ are found by imposing for all $j > 1$ and all $p \geq 1$:

$$t_p^j(r_{j-1}) = t_p^{j-1}(r_{j-1}) \quad \text{and} \quad k_j \frac{d}{dr} t_p^j(r_{j-1}) = k_{j-1} \frac{d}{dr} t_p^{j-1}(r_{j-1}). \tag{18}$$

The following inverse F for the operator Δ_N is considered

$$F \cdot h(r) := r^N \int_0^r \frac{1}{x^{2N+1}} \int_0^x y^{N+1} h(y) dy dx, \tag{19}$$

that is well defined for $h(r) = O(r^{N-2+\varepsilon})$ at the origin $r = 0$ and for $\varepsilon > 0$.

Lemma 4.1. *The closure functions $(t_p)_{p\in\mathbb{N}}$ are well defined by the above recursive scheme.*

Precisely, denoting $f_p(r) = v(r)t_{p-1}(r)/k(r) - t_{p-2}(r)$, they are explicitely given by

- *If $j = 1$: $t_0^1(r) = r^N$ and for $p \geq 1$,*

$$t_p^1 = F \cdot f_p. \tag{20}$$

- *If $j > 1$: for all $p \geq 0$,*

$$t_p^j(r) = C_1\psi_1(r) + C_2\psi_2(r) + r^N \int_{r_{j-1}}^r \frac{1}{x^{2N+1}} \int_{r_{j-1}}^x y^{N+1} f_p(y) dy dx, \tag{21}$$

where $\psi_1(r) = r^N$, where $\psi_2(r) = ln(r)$ in case $N = 0$ and $\psi_2(r) = 1/r^N$ otherwise and where the two constants C_1 and C_2 are uniquely determined by the transmission conditions (18). *Here these transmission conditions moreover reduce to*

$$C_1\psi_1(r_{j-1}) + C_2\psi_2(r_{j-1}) = t_p^{j-1}(r_{j-1})$$
$$C_1 \frac{d}{dr}\psi_1(r_{j-1}) + C_2 \frac{d}{dr}\psi_2(r_{j-1}) = \frac{k_{j-1}}{k_j} \frac{d}{dr} t_p^{j-1}(r_{j-1}).$$

Proof. The main argument here is that the general soultion to (16) simply reads (21).

In case $j > 1$, the definition of C_1 and C_2 is fixed by the transmission conditions (18). This definition reduces to the simple system in lemma 4.1 because the integral in (21) and its derivative equal zero at r_{j-1}.

In case $j = 1$, we have

$$t_p^1 = C_1\psi_1 + C_2\psi_2 + F_p,$$

with $F_p := F.f_p$. One has to prove that $t_p^1 = F_p$, i.e. $C_1 = C_2 = 0$. We use equation (17)

$$\frac{t_p(r)}{r^N} = C_1 + C_2 \frac{\psi_2(r)}{r^N} + \frac{F_p(r)}{r^N} \underset{r\to 0}{\longrightarrow} 0.$$

To prove that $C_1 = C_2 = 0$ it suffices to prove that $F_p(r) = o(r^N)$ at $r = 0$. We proceed by induction by showing that $t_p(r) = O(r^N)$ at $r = 0$ for all $p \geq 0$.

The argument for this is that $h(r) = O(r^N)$ implies $F \cdot h(r) = O(r^{N+2})$ at $r = 0$ which is easy to check.

Assume that at $r = 0$ we both have $t_{p-2}(r) = O(r^N)$ and $t_{p-1}(r) = O(r^N)$. Then the analycity of $v(r)$ near 0 also ensures that $f_p(r) = O(r^N)$ and so $F_p(r) = O(r^{N+2}) = o(r^N)$. Thus $t_p = F_p$ ensuring in turn $t_p(r) = O(r^N)$.

Finally the induction assumption is obviously true for $p = -1$ and $p = 0$.

□

4.2. **Analycity in λ.**

Theorem 4.2. *Let $N \in \mathbb{N}$. The base functions are given for all $\lambda \in \mathbb{C}$ by equation* (15), *i.e*

$$T_{N,\lambda}(r) = \sum_{p \in \mathbb{N}} t_{N,p}(r) \lambda^p.$$

Moreover, on each interval $[r_{j-1}, r_j]$ we have

$$\frac{d}{dr} T_{N,\lambda} = \sum_{p \in \mathbb{N}} \frac{d}{dr} t_{N,p}(r) \lambda^p. \tag{22}$$

Let us stress practical consequences of theorem 4.2:

(1) It firstly provides a method to approximate the base functions $T_{N,\lambda}$, by truncating (15) at order q. The computation of the closure functions $t_{N,p}$ itself is recursive and explicit.
(2) It secondly permits the computation of Λ_N

$$\Lambda_N = \{\lambda \ , \quad \sum_{p \in \mathbb{N}} t_{N,p}(R) \lambda^p = 0\}. \tag{23}$$

(3) It is a mesh-free, mode truncated numerical computation. It eventually allows to compute the eigenvalues λ_i and the eigenfunctions Θ_i in (5).
(4) Last but not least, the method is also independent on the imposed boundary condition. For example, the eigenfunctions Θ_i in (5) for a Neumann boundary condition can be similarly computed by defining:

$$\Lambda_N^{\text{Neumann}} = \{\lambda \ , \quad \sum_{p \in \mathbb{N}} \frac{d}{dr} t_{N,p}(R) \lambda^p = 0\}, \tag{24}$$

which is justified by equation (22).

Proof. Let N be fixed. Assume that the three functions $(\lambda, r) \to T_{N,\lambda}(r)$, $(\lambda, r) \to \partial_r T_{N,\lambda}(r)$ and $(\lambda, r) \to \Delta_N T_{N,\lambda}(r)$ are analytical in λ on one of the compartment $[r_{j-1}, r_j]$. Then, using the integration theorem one gets that the coefficients in the analytical series expansion necessarily satisfy the closure problems (16), (17) and (18). One then has to prove the analicity in λ of these three functions. From equation (8), it remains to prove the analicity of $T_{N,\lambda}$ and $\partial_r T_{N,\lambda}$ on each compartment by induction.

Assume then that $T_{N,\lambda}$ and $\partial_r T_{N,\lambda}$ are analytical on $[r_{j-1}, r_j]$, so that $T_{N,\lambda}(r_j)$ and $\partial_r T_{N,\lambda}(r_j)$ are analytical in λ. Now $T_{N,\lambda}$ is the solution of a regular ODE on $[r_j, r_{j+1}]$ analytically depending on the parameter λ and on the initial conditions at r_j. Then, we can apply classical results on ODEs (see eg [1], corollary 4 p. 52 and extensions in chapter 32.5), that ensures the analicity of $T_{N,\lambda}$ and $\partial_r T_{N,\lambda}$ on $[r_j, r_{j+1}]$.

Proving the analicity on $[r_0, r_1]$ is made harder because of the singularity at $r = 0$: we proceed by hand.
Let us start with some preliminaries. We can assume $r_1 \leq 1$: the problem induced by the singularity is local. We recall that $t_0 = r^N$ here. We consider succesive iteration for the functional F in (19),

$$F^{(i)} \cdot t_0 = F \cdot \ \dots \ \cdot F \cdot t_0 = \frac{r^{N+2i}}{2^{2i} K_i}, \quad K_i = i!(N+1)\dots(N+i).$$

We use $F^{(j)} \cdot t_0 \leq F^{(i)} \cdot t_0$ for $j \geq i$ since $0 \leq r \leq 1$ here. We also use the positivity of F:

$$h_1 \leq h_2 \ \Rightarrow \ F \cdot h_1 \leq F \cdot h_2, \quad |F \cdot h| \leq F \cdot |h|. \tag{25}$$

We denote $\bar{t}_i = |t_i|$ and $M = \max(1, \|v\|_\infty / k_1) \geq 1$ with $\|v\|_\infty = \sup_{[0,r_1]} |v(r)|$.

We prove that, for $j = 2i$ or $j = 2i+1$ and $i \in \mathbb{N}$:

$$\bar{t}_j \leq 2^{2i} M^{2i} F^{(i)} \cdot t_0 \leq M^{2i} / K_i. \tag{26}$$

This last inequality clearly ensures that $S(\lambda, r) := \sum_{p\geq 0} t_p(r)\lambda^p$ is normally converging. As we will see, it also proves that $\sum_{p\geq 0} dt_p/dr(r)\lambda^p$ is normally converging. With equation (16) it finally implies that $\sum_{p\geq 0} \Delta_N t_p(r)\lambda^p$ is normally converging. Then, the integration theorem ensures that $\partial_r S(\lambda, r) = \sum_{p\geq 0} dt_p/dr(r)\lambda^p$ and $\Delta_N S(\lambda, r) = \sum_{p\geq 0} \Delta_N t_p(r)\lambda^p$. Together with (16), (17) this ensures that $S(\lambda, \cdot)$ satisfies (8) and (11). It follows that $T_{N,\lambda}(r) = S(\lambda, r)$. We exactly proved that $T_{N,\lambda}$ and $\partial_r T_{N,\lambda}$ are analytical in λ which ends the proof for theorem 4.2.

Let us then prove that equation (26) ensures that $\sum_{p\geq 0} dt_p/dr(r)\lambda^p$ is normally converging. We have (with $f_p = v t_{p-1}/k - t_{p-2}$):

$$\begin{aligned} \frac{d}{dr} t_p(r) &= \frac{d}{dr} \left(F \cdot f_p\right)(r) \\ &= N r^{N-1} \int_0^r \frac{1}{x^{2N+1}} \int_0^x y^{N+1} f_p(y) dy dx + \frac{1}{r^N} \int_0^r y^{N+1} f_p(y) dy \\ &= \frac{N}{r} t_p(r) + \frac{1}{r^N} \int_0^r y^{N+1} f_p(y) dy. \end{aligned}$$

In this equality, the first term is zero in case $N = 0$. In case $N > 0$, the series $\sum_{p\geq 0} N t_p(r)/r \lambda^p$ is normally converging because of the upper bound (26) and from the expression for $F^{(i)} \cdot t_0$. It thus only remains to bound the term on the right. Using that $|f_p| \leq M(\bar{t}_{p-2} + \bar{t}_{p-1})$, the upper bounds (26) and equality

$$\frac{1}{r^N} \int_0^r y^{N+1} F^{(i)} \cdot t_0(y) dy = \frac{1}{2^{2i} K_i (2N + i + 1)} r^{N+2i+1},$$

eventually imply that $\sum_{p\geq 0} dt_p/dr\lambda^p$ is normally converging.

It eventually remains to prove (26).
This is quite clear for $i = 0$ (i.e. $j = 0,\ 1$) in which case (26) indeed is an equality.
Assume that (26) holds at order $i \in \mathbb{N}$, *i.e.* for $j = 2i$ and $j = 2i + 1$.
Equation (20) together with (25) ensures that,

$$\bar{t}_{2(i+1)} = |F \cdot (vt_{2i+1}/k_1 - t_{2i})| \leq M \left(F \cdot \bar{t}_{2i+1} + F \cdot \bar{t}_{2i} \right).$$

Thus assumption (26) together with (25) gives

$$\bar{t}_{2(i+1)} \leq 2^{2i+1} M^{2i+1} F^{(i+1)} \cdot t_0,$$

which implies (26) for $j = 2(i + 1)$ since $M \geq 1$.
Again, for $j = 2i + 3$:

$$\bar{t}_{2i+3} \leq M \left(F \cdot \bar{t}_{2(i+1)} + F \cdot \bar{t}_{2i+1} \right).$$

With the upper bound above on $\bar{t}_{2(i+1)}$, equation (26) at order $2i+1$ and property (25) we get:

$$\begin{aligned} \bar{t}_{2i+3} &\leq 2^{2i+1} M^{2(i+1)} F^{(i+2)} \cdot t_0 + 2^{2i} M^{2i+1} F^{(i+1)} \cdot t_0 \\ &\leq 2^{2(i+1)} \left(\frac{1}{2} + \frac{1}{4} \right) M^{2(i+1)} F^{(i+1)} \cdot t_0, \end{aligned}$$

because $M \geq 1$ and $F^{(i+2)} \cdot t_0 \leq F^{(i+1)} \cdot t_0$, which inequality gives (26) at order $2(i + 1) + 1$. This ends the proof for (26).

4.3. **Extension to planar configurations.** Few remarks to transpose the results in the planar case where the Fourier decomposition applies in the asymutal direction y and the Laplacien is no more singular when expressed with the transverse variable x.

□

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