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MATCHING ASYMPTOTIC METHOD AND NUCLEATION OF A DEFECT AT A NOTCH

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ABSTRACT

In the present study, we use matching asymptotic expansion to treat the elastic problem associated with a small defect located at the tip of a notch. The use of such asymptotic methods is necessary because the presence both of small parameters and singularities leads to inaccurate computations by classical finite element methods. The method is applied here in an antiplane setting.

Key Words: *Brittle fracture, cohesive model, asymptotic methods, singularities.*

1. INTRODUCTION

A major issue in fracture mechanics is how to model the initiation of a crack in a sound material, see [1]. In particular, if one uses Griffith criterion like in [4], the main difficulty is to compute with a good accuracy the energy release rate associated with a crack of small length which appears at the tip of a notch, see [5]. The classical finite element method leads to inaccurate results because of the overlap of two singularities which cannot be correctly captured by this method: one due to the tip of the notch, the other due to the tip of the crack. A specific method of approximation based on asymptotic expansions is preferable because the singularities are then obtained explicitly. The present paper is devoted to the presentation of this asymptotic method in the more general case of a defect (and not only of a crack) located at the tip of a notch in the simplified context of antiplane linear elasticity. It turns out that many works have been devoted to the study of elliptic problems in corner domains, see for example [3] and [2]. But no proposes a comprehensive method to compute with a precision as large as it is needed the mechanical fields and all related physical quantities in such a situation. Accordingly, we present here the method of construction of the matched asymptotic expansions first in the general case and then to the particular case of a non cohesive crack. The numerical results are finally compared with those given by the classical finite element method.

2. THE REAL PROBLEM

Here, we are interested to the case where a small geometrical defect of size ℓ (like a crack or a void) is located near the corner of a notch. The geometry of the notch is characterized by the parameter $\epsilon = \tan(\pi - \frac{\omega}{2})$, ω being the angle of the notch, see Figure 1. The tip of the notch is taken as the origin of the space and we will consider two scales of coordinates: the "macroscopic" coordinates $\mathbf{x} = (x_1, x_2)$ which are used in the outer domain and the "microscopic" coordinates $\mathbf{y} = \mathbf{x}/\ell = (y_1, y_2)$ which are used in the neighborhood of the tip of the notch where the defect is located. In the case of a crack, the
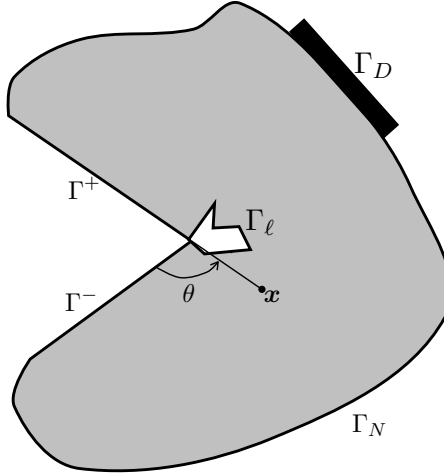
FIGURE 1. The domain Ω_ℓ for the real problem

axis x_1 is chosen in such a way that the crack corresponds to the segment $(0, \ell) \times \{0\}$. The unit vector orthogonal to the (x_1, x_2) plane is denoted $\mathbf{e}_3$.

The natural reference configuration of the *sound* two-dimensional body is denoted by Ω_0 while the associated body which contains a defect of size ℓ is denoted by Ω_ℓ. Accordingly one has

$$\Omega_\ell = \Omega_0 \backslash \Gamma_\ell \tag{1}$$

where Γ_ℓ denotes the boundary of the defect of size ℓ. (Γ_ℓ is contained in the disk of center $(0, 0)$ and radius ℓ.) In the case of a crack, one has $\Gamma_\ell = (0, \ell) \times \{0\}$. The two edges of the notch are denoted by Γ^+ and Γ^-, see Figure 1. When one uses polar coordinates (r, θ), the pole is the tip of the notch and the origin of the polar angle is the edge Γ^-. Accordingly, we have

$$r = |\mathbf{x}|, \quad \Gamma^- = \{(r, \theta), 0 < r < r^*, \quad \theta = 0\}, \qquad \Gamma^+ = \{(r, \theta), 0 < r < r^*, \quad \theta = \omega\}. \tag{2}$$

This body is made of an elastic isotropic material whose shear modulus is $\mu > 0$. It is submitted to a loading such that the displacement field at equilibrium $\mathbf{u}$ be antiplane, *i.e.*

$$\mathbf{u}(\mathbf{x}) = u^\ell(x_1, x_2)\mathbf{e}_3$$

where the superscript ℓ is used in order to recall that the real displacement depends on the size of the defect. We assume that the body forces are zero and then u^ℓ must be an harmonic function in order to satisfy the equilibrium equations in the bulk:

$$\Delta u^\ell = 0 \quad \text{in} \quad \Omega_\ell. \tag{3}$$

The edges of the notch are free while Γ_ℓ is submitted to a density of (antiplane) surface forces. Accordingly, the boundary conditions on Γ_ℓ and $\Gamma^\pm$ read as

$$\frac{\partial u^\ell}{\partial \nu} = 0 \quad \text{on} \quad \Gamma^\pm, \qquad \frac{\partial u^\ell}{\partial \nu} = \ell g(\mathbf{y}) \quad \text{on} \quad \Gamma_\ell. \tag{4}$$

In (4), ν denotes the unit outer normal vector to the domain and we assume that the density of (antiplane) surface forces depends on the microscopic variable $\mathbf{y}$ and has a magnitude of the order of ℓ.

The remaining part of the boundary of Ω_ℓ is divided into two parts: Γ_D where the displacement is prescribed and Γ_N where (antiplane) surface forces are prescribed. Specifically, we have

$$u^\ell = f(\mathbf{x}) \quad \text{on} \quad \Gamma_D, \qquad \frac{\partial u^\ell}{\partial \nu} = h(\mathbf{x}) \quad \text{on} \quad \Gamma_N. \tag{5}$$

The following Proposition is a characterization of functions which are harmonic in an angular sector and whose normal derivatives vanish on the edges of the sector. It is of constant use thereafter.

Proposition 1. *Let r_1 and r_2 be such that $0 \leq r_1 < r_2 \leq +\infty$ and let $\mathcal{D}_{r_1}^{r_2}$ be the angular sector*

$$\mathcal{D}_{r_1}^{r_2} = \{(r,\theta) \; : \; r \in (r_1, r_2), \quad \theta \in (0, \omega)\}.$$

Then any function u which is harmonic in $\mathcal{D}_{r_1}^{r_2}$ and which satisfies the Neumann condition $\partial u / \partial \theta = 0$ on the sides $\theta = 0$ and $\theta = \omega$ can be read as

$$u(r, \theta) = b_0 \ln(r) + c_0 + \sum_{n \in \mathbb{N}_*} \left(b_n r^{-n\lambda} + c_n r^{n\lambda} \right) \cos(n\lambda\theta) \tag{6}$$

with $\lambda = \pi/\omega$, whereas the b_n's and the c_n's constitute two sequences of real numbers which are characteristic of u.

Proof. Since the normal derivative vanishes at $\theta = 0$ and $\theta = \omega$, $u(r, \theta)$ can be read as the following Fourier series:

$$u(r, \theta) = \sum_{n \in \mathbb{N}} f_n(r) \cos(n\lambda\theta).$$

In order that u be harmonic, the functions f_n must satisfy $r^2 f_n'' + r f_n' - n^2 \lambda^2 f_n = 0$, for each n. One easily deduces that $f_0(r) = b_0 \ln(r) + c_0$ and $f_n(r) = b_n r^{-n\lambda} + c_n r^{n\lambda}$ for $n \geq 1$. □

3. THE MATCHING ASYMPTOTIC METHOD

When the length ℓ of the defect is small by comparison with the characteristic length of the body (here, this characteristic length is equal to 1), then it is necessary to make an asymptotic analysis of the problem rather than to try to obtain directly an approximation by classical finite element methods. In the case of a crack for instance, because of the overlap of two singularities (one at the tip of the notch and the other at the tip of the crack), it is difficult and even impossible to obtain accurate results without using a relevant asymptotic method. Here we will use the matched asymptotic expansion technique which consists in making two asymptotic expansions of the field u^ℓ in terms of the small parameter ℓ. The first one, called the inner expansion, is valid in the neighborhood of the tip of the notch, while the other, called the outer expansion, is valid far from this tip. These two expansions are matched in an intermediate zone.
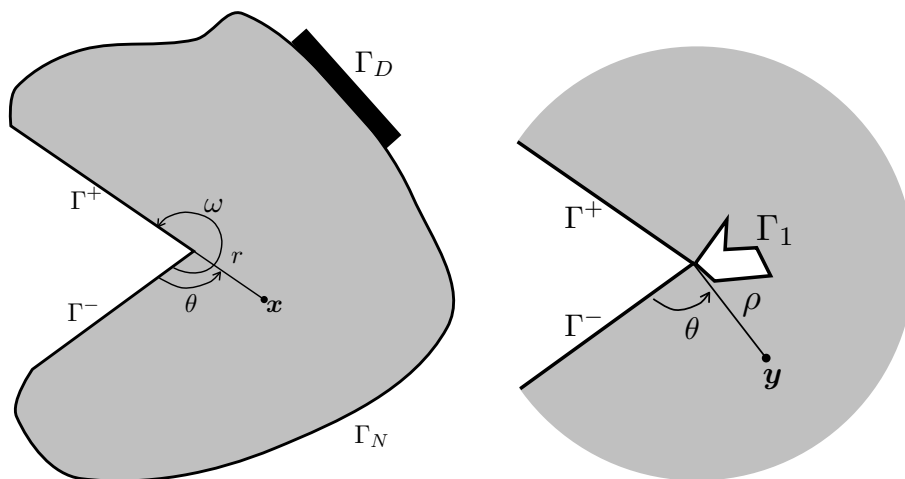

FIGURE 2. The domains Ω_0 and Ω^∞ for, respectively, the outer (left) and the inner (right) problems

3.1. The outer expansion.

Far from the tip of the notch, *i.e.* for $r \gg \ell$, we assume that the real displacement field u^ℓ can be expanded as follows

$$u^\ell(\mathbf{x}) = \sum_{i \in \mathbb{N}} \ell^{i\lambda} U^i(\mathbf{x}). \tag{7}$$

In (7), even if this expansion is valid far enough from $r = 0$ only, the fields U^i must be defined in the whole outer domain Ω_0 which corresponds to the sound body, see Figure 2-left. Inserting this expansion into the set of equations constituting the real problem, one obtains the following equations that the U^i's must satisfy:

The first outer problem $i = 0$

$$\begin{cases} \Delta U^0 = 0 & \text{in } \Omega^0 \\ \dfrac{\partial U^0}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial U^0}{\partial \nu} = h(\mathbf{x}) & \text{on } \Gamma_N \\ U^0 = f(\mathbf{x}) & \text{on } \Gamma_D \end{cases} \tag{8}$$

The other outer problems $i \geq 1$

$$\begin{cases} \Delta U^i = 0 & \text{in } \Omega^0 \\ \dfrac{\partial U^i}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial U^i}{\partial \nu} = 0 & \text{on } \Gamma_N \\ U^i = 0 & \text{on } \Gamma_D \end{cases} \tag{9}$$

Moreover, the behavior of U^i in the neighborhood of $r = 0$ is singular and the singularity will be given by the matching conditions.

3.2. The inner expansion.

Near the tip of the notch, *i.e.* for $r \ll 1$, we assume that the real displacement field u^ℓ can be expanded as follows

$$u^\ell(\mathbf{x}) = \ln(\ell) \sum_{i \in \mathbb{N}} \ell^{i\lambda} W^i(\mathbf{y}) + \sum_{i \in \mathbb{N}} \ell^{i\lambda} V^i(\mathbf{y}). \tag{10}$$

In (10), even if this expansion is valid only in the neighborhood of $r = 0$, the fields V^i and W^i must be defined in the infinite inner domain Ω^∞. The domain Ω^∞ is the infinite angular sector $\mathcal{D}_0^\infty$ of the (y_1, y_2) plane from which one removes the rescaled defect of size 1, see Figure 2-right:

$$\Omega^\infty = \mathcal{D}_0^\infty \backslash \Gamma_1. \tag{11}$$

(In the case of a crack, one has $\Gamma_1 = (0,1) \times \{0\}$.) Inserting this expansion into the set of equations constituting the real problem, one obtains the following equations that the V^i's must satisfy:

The first inner problem $i = 0$

$$\begin{cases} \Delta V^0 = 0 & \text{in } \Omega^\infty \\ \dfrac{\partial V^0}{\partial \theta} = 0 & \text{on } \theta = 0 \text{ and } \theta = \omega \\ \dfrac{\partial V^0}{\partial \nu} = g(\mathbf{y}) & \text{on } \Gamma_1 \end{cases} \tag{12}$$

The other inner problems $i \geq 1$

$$\begin{cases} \Delta V^i = 0 & \text{in } \Omega^\infty \\ \dfrac{\partial V^i}{\partial \theta} = 0 & \text{on } \theta = 0 \text{ and } \theta = \omega \\ \dfrac{\partial V^i}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases} \tag{13}$$

The W^i's must satisfy, for every $i \geq 0$ the same equations as the V^i's for $i \geq 1$. To complete the set of equations one must add the behavior at infinity of the V^i's and the W^i's. This behavior will be given by the matching conditions with the outer problems.

3.3. **Matching conditions.** Since all the displacement fields U^i are harmonic in the sector $\mathcal{D}_0^{r_2}$ and satisfy an homogeneous Neumann boundary condition on the edges of this angular sector, we can use Proposition 1. Accordingly, in $\mathcal{D}_0^{r_2}$ the field U^i can read as

$$U^i(\mathbf{x}) = b_0^i \ln(r) + c_0^i + \sum_{n \in \mathbb{N}_*} \left(b_n^i r^{-n\lambda} + c_n^i r^{n\lambda} \right) \cos(n\lambda\theta). \tag{14}$$

In the same way for the inner expansion, since all the displacement fields V^i and W^i are harmonic in the sector $\mathcal{D}_1^\infty$ of the $\mathbf{y}$ plane and satisfy an homogeneous Neumann boundary condition on the edges of this angular sector, we can use Proposition 1 with the macroscopic coordinates $\mathbf{x}$ and r replaced by the microscopic coordinates $\mathbf{y}$ and $\rho = |\mathbf{y}| = r/\ell$. Accordingly, in $\mathcal{D}_1^\infty$ the fields V^i and W^i can read as

$$V^i(\mathbf{y}) = B_0^i \ln(\rho) + C_0^i + \sum_{n \in \mathbb{N}_*} \left(B_n^i \rho^{-n\lambda} + C_n^i \rho^{n\lambda} \right) \cos(n\lambda\theta), \tag{15}$$

$$W^i(\mathbf{y}) = E_0^i \ln(\rho) + F_0^i + \sum_{n \in \mathbb{N}_*} \left(E_n^i \rho^{-n\lambda} + F_n^i \rho^{n\lambda} \right) \cos(n\lambda\theta). \tag{16}$$

The outer expansion and the inner expansion are both valid in an intermediate zone $\mathcal{D}_{r_1}^{r_2}$ with $\ell \ll r_1 < r_2 \ll 1$. Inserting (14) into the outer expansion (7) with $r = \ell\rho$ leads to

$$u^\ell(\mathbf{x}) = \sum_{i \in \mathbb{N}} \ln(\ell)\ell^{i\lambda} b_0^i + \sum_{i \in \mathbb{N}} \ell^{i\lambda} \left(b_0^i \ln(\rho) + c_0^i + \sum_{n \in \mathbb{N}_*} \left(b_n^{i+n} \rho^{-n\lambda} + c_n^{i-n} \rho^{n\lambda} \right) \cos(n\lambda\theta) \right) \tag{17}$$

with the convention that $c_n^{i-n} = 0$ when $n > i$. Inserting (15) and (16) into the inner expansion (10) leads to

$$\begin{aligned} u^\ell(\mathbf{x}) &= \sum_{i \in \mathbb{N}} \ln(\ell)\ell^{i\lambda} \left(E_0^i \ln(\rho) + F_0^i + \sum_{n \in \mathbb{N}_*} \left(E_n^i \rho^{-n\lambda} + F_n^i \rho^{n\lambda} \right) \cos(n\lambda\theta) \right) \\ &+ \sum_{i \in \mathbb{N}} \ell^{i\lambda} \left(B_0^i \ln(\rho) + C_0^i + \sum_{n \in \mathbb{N}_*} \left(B_n^i \rho^{-n\lambda} + C_n^i \rho^{n\lambda} \right) \cos(n\lambda\theta) \right). \end{aligned} \tag{18}$$

Both expansions (17) and (18) are valid provided that $1 \ll \rho \ll 1/\ell$. By identification one gets the following properties for the coefficients of the inner and outer expansions, see Table 1.

$E_n^i = 0$	$i \geq 0, n \geq 0$
$F_0^i = b_0^i$	$i \geq 0$
$F_n^i = 0$	$i \geq 0, n \geq 1$
$b_n^i = 0$	$n > i \geq 0$
$B_n^i = b_n^{i+n}$	$i \geq 0, n \geq 0$
$C_n^i = 0$	$n > i \geq 0$
$c_n^i = C_n^{i+n}$	$i \geq 0, n \geq 0$

TABLE 1. The relations between the coefficients of the inner and outer expansions given by the matching conditions

Remark 1. *One deduces from Table 1 that the fields W^i are constant in the whole inner domain:*

$$W^i(\mathbf{y}) = b_0^i, \qquad \forall \mathbf{y} \in \Omega^\infty, \quad \forall i \geq 0. \tag{19}$$

Therefore, these fields will be determined once the constants b_0^i will be known.

3.4. The singular behavior of the U^i's and the V^i's. We deduce from the matching conditions the behavior of U^i in the neighborhood of $r = 0$ and the behavior of V^i in the neighborhood of $\rho = \infty$. In particular, one obtains the form of their singularities. Let us first precise what one means by singularity.

Definition 1. *A field u defined in Ω_0 is said* regular *in Ω_0 if $u \in H^1(\Omega_0)$, i.e. $u \in L^2(\Omega_0)$ and $\nabla u \in L^2(\Omega_0)^2$. It is said* singular *otherwise.*

A field u defined in the unbounded sector Ω^∞ is said regular *in Ω^∞ if $\nabla u \in L^2(\Omega^\infty)^2$ and $\lim_{\rho \to \infty} u(\rho, \theta) = 0$. It is said* singular *otherwise.*

By virtue of the analysis of the previous subsection, the field U^0 can be read in a neighborhood of the tip of the notch as

$$U^0(\mathbf{x}) = b_0^0 \ln(r) + \sum_{n \in \mathbb{N}} C_n^n r^{n\lambda} \cos(n\lambda\theta). \tag{20}$$

Since $\ln(r)$ is singular in Ω_0 whereas $r^{n\lambda} \cos(n\lambda\theta)$ is regular (for $n \geq 0$) in Ω_0 in the sense of Definition 1, $b_0^0 \ln(r)$ can be considered as the singular part of the field U^0. Accordingly, one can decompose U^0 into its singular and its regular part as follows

$$U^0(\mathbf{x}) = U_S^0(\mathbf{x}) + \bar{U}^0(\mathbf{x}), \tag{21}$$

$$U_S^0(\mathbf{x}) = b_0^0 \ln(r), \quad \bar{U}^0 \in H^1(\Omega_0). \tag{22}$$

In the same way, for $i \geq 1$, the field U^i can be read in a neighborhood of the tip of the notch as

$$U^i(\mathbf{x}) = b_0^i \ln(r) + \sum_{n=1}^{i} b_n^i r^{-n\lambda} \cos(n\lambda\theta) + \sum_{n \in \mathbb{N}} C_n^{i+n} r^{n\lambda} \cos(n\lambda\theta). \tag{23}$$

Since $r^{-n\lambda} \cos(n\lambda\theta)$ is singular (for $n \geq 0$) in the sense of Definition 1, one can decompose U^i into its singular and its regular part as follows

$$U^i(\mathbf{x}) = U_S^i(\mathbf{x}) + \bar{U}^i(\mathbf{x}), \tag{24}$$

$$U_S^i(\mathbf{x}) = b_0^i \ln(r) + \sum_{n=1}^{i} b_n^i r^{-n\lambda} \cos(n\lambda\theta), \quad \bar{U}^i \in H^1(\Omega_0). \tag{25}$$

For the fields V^i of the inner expansion, one has to study their behavior at infinity. By virtue of the analysis of the previous subsection, the field V^i for $i \geq 0$ can be read in a neighborhood of $\rho = \infty$ as

$$V^i(\mathbf{y}) = b_0^i \ln(\rho) + \sum_{n=0}^{i} C_n^i \rho^{n\lambda} \cos(n\lambda\theta) + \sum_{n \in \mathbb{N}_*} b_n^{i+n} \rho^{-n\lambda} \cos(n\lambda\theta). \tag{26}$$

The field $\ln(\rho)$ as well as the fields $\rho^{n\lambda} \cos(n\lambda\theta)$, for $n \geq 0$, are singular in Ω^∞ in the sense of Definition 1 (even the constant field 1 corresponding to $n = 0$ is singular). Since the fields $\rho^{-n\lambda} \cos(n\lambda\theta)$ are regular when $n \geq 1$, $b_0^i \ln(\rho) + \sum_{n=0}^{i} C_n^i \rho^{n\lambda} \cos(n\lambda\theta)$ can be considered as the singular part of the field V^i. Accordingly, one can decompose V^i into its singular and its regular part as follows

$$V^i(\mathbf{y}) = V_S^i(\mathbf{y}) + \bar{V}^i(\mathbf{y}), \tag{27}$$

$$V_S^i(\mathbf{y}) = b_0^i \ln(\rho) + \sum_{n=0}^{i} C_n^i \rho^{n\lambda} \cos(n\lambda\theta), \quad \nabla \bar{V}^i \in L^2(\Omega^\infty), \quad \lim_{|\mathbf{y}| \to \infty} \bar{V}^i(\mathbf{y}) = 0. \tag{28}$$

Remark 2. *This analysis of the singularities shows that the singular parts of the fields U^i and V^i will be known once the coefficients b_n^i and C_n^i will be determined for $0 \leq n \leq i$.*

3.5. The problems giving the regular parts $\bar{U}^i$ and $\bar{V}^i$. We are now in position to set the inner and outer problems giving the fields V^i and U^i. Since, by construction, the singular parts of these fields are harmonic and satisfy the homogeneous Neumann boundary conditions on the edges of the notch, their regular parts must verify the following boundary value problems.

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The first outer problem, $i = 0$
Find $\bar{U}^0$ regular in Ω_0 such that

$$\begin{cases} \Delta \bar{U}^0 = 0 & \text{in } \Omega^0 \\ \dfrac{\partial \bar{U}^0}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial \bar{U}^0}{\partial \nu} = h - \dfrac{\partial U_S^0}{\partial \nu} & \text{on } \Gamma_N \\ \bar{U}^0 = f - U_S^0 & \text{on } \Gamma_D \end{cases} \tag{29}$$

The other outer problems, $i \geq 1$
Find $\bar{U}^i$ regular in Ω_0 such that

$$\begin{cases} \Delta \bar{U}^i = 0 & \text{in } \Omega^0 \\ \dfrac{\partial \bar{U}^i}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial \bar{U}^i}{\partial \nu} = -\dfrac{\partial U_S^i}{\partial \nu} & \text{on } \Gamma_N \\ \bar{U}^i = -U_S^i & \text{on } \Gamma_D \end{cases} \tag{30}$$

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<div style="display:flex;justify-content:space-between;">

The first inner problem, $i = 0$
Find $\bar{V}^0$ regular in Ω^∞ such that

$$\begin{cases} \Delta \bar{V}^0 = 0 & \text{in } \Omega^\infty \\ \dfrac{\partial \bar{V}^0}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial \bar{V}^0}{\partial \nu} = g - \dfrac{\partial V_S^0}{\partial \nu} & \text{on } \Gamma_1 \end{cases} \tag{31}$$

The other inner problems, $i \geq 1$
Find $\bar{V}^i$ regular in Ω^∞ such that

$$\begin{cases} \Delta \bar{V}^i = 0 & \text{in } \Omega^\infty \\ \dfrac{\partial \bar{V}^i}{\partial \nu} = 0 & \text{on } \Gamma^+ \cup \Gamma^- \\ \dfrac{\partial \bar{V}^i}{\partial \nu} = -\dfrac{\partial V_S^i}{\partial \nu} & \text{on } \Gamma_1 \end{cases} \tag{32}$$

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Let us study first the outer problems. We have the following Proposition which is a direct consequence of classical results for the Laplace equation:

Proposition 2. *Let $i \geq 0$. For a given singular part U_S^i, i.e. if the coefficients b_n^i are known for all n such that $0 \leq n \leq i$, then there exists a unique solution $\bar{U}^i$ of (30) (or of (29) when $i = 0$). Consequently, the coefficients C_n^{i+n} are then determined for all $n \geq 0$.*

Let us consider now the inner problems. We obtain the following Proposition

Proposition 3. *Let $i \geq 0$. For given C_n^i with $0 \leq n \leq i$, there exists a regular solution V^i for the i-th inner problem if and only is the coefficient b_0^i is such that*

$$b_0^0 = -\frac{1}{\omega} \int_{\Gamma_1} g(s)ds, \qquad b_0^i = 0 \quad \text{for} \quad i \geq 1. \tag{33}$$

Moreover, if this condition is satisfied, then the solution is unique and therefore the coefficients b_n^{i+n} are determined for all $n \geq 0$.

Proof. The inner problems are pure Neumann problems in which no Dirichlet boundary conditions are imposed to the V^i's. Consequently, they admit a solution (if and) only if the Neumann data satisfy a global compatibility condition. Let us re-establish that condition. Let Ω^R be the part of Ω^∞ included in the ball of radius $R > 1$, i.e. $\Omega^R = \Omega^\infty \cap \{\mathbf{y} : |\mathbf{y}| < R\}$. Let us consider first the case $i = 0$. Integrating the equation $\Delta V^0 = 0$ over Ω^R and using the boundary conditions leads to

$$0 = \int_{\partial \Omega^R} \frac{\partial V^0}{\partial \nu} ds = \int_0^\omega \frac{\partial V^0}{\partial \rho}(R, \theta) R \, d\theta + \int_{\Gamma_1} g(s)ds. \tag{34}$$

Using (26), one gets $R\dfrac{\partial V^0}{\partial \rho}(R, \theta) = b_0^0 + \sum_{n \in \mathbb{N}_*} n\lambda\big(-B_n^0 R^{-n\lambda} + C_n^0 R^{n\lambda}\big) \cos(n\lambda\theta)$. Since $\int_0^\omega \cos(n\lambda\theta)d\theta = 0$ for all $n \geq 1$, after inserting in (34) one obtains the desired condition for b_0^0.

One proceeds exactly in the same manner for $i \geq 1$ and one obtains the desired condition because the integral over Γ_1 vanishes.

If the compatibility condition (33) is satisfied, then one proves the existence of a regular solution for $\bar{V}^i$ by standard arguments. Note however that, since $\nabla \bar{V}^i$ belongs to $L^2(\Omega^\infty)$, $\bar{V}^i$ tends to a constant at infinity and this constant is fixed to 0 by the additional regularity condition. As far as the uniqueness is concerned, the solution of this pure Neumann problem is unique up to a constant and the constant is fixed by the condition that $\bar{V}^i$ vanishes at infinity.

Once V^i is determined, one obtains the coefficients b_n^{i+n} by virtue of Proposition 1 and (26). $\qquad \square$

Remark 3. *If the forces applied to the boundary of the defect are equilibrated, i.e. if $\int_{\Gamma_1} g(s)ds = 0$, then all the coefficients b_0^i vanish and hence the terms in $\ln(\ell)$ disappear in the inner expansion. There is no more logarithmic singularities in the U^i's and the V^i's.*

3.6. **The construction of the outer and inner expansions.** Equipped with the previous results, we are in position to explain how one can determine the different terms of the two expansions. Let us explain first how one obtains the first terms.

(1) One obtains b_0^0 by (34) and hence one knows U_S^0.
(2) Knowing U_S^0, one determines $\bar{U}^0$ and hence U^0 by solving (29), see Proposition 2.
(3) Knowing U^0, one calculates C_n^n for $n \geq 0$ as a regular part of U^0, see the next subsection for the practical method. Hence, one knows V_S^0.
(4) Knowing V_S^0, one determines $\bar{V}^0$ and hence V^0 by solving (31), see Proposition 3.
(5) Knowing V^0, one calculates b_n^n for $n \geq 1$ as a regular part of V^0, see the next subsection for the practical method. Hence, since $b_0^1 = 0$, one knows U_S^1.

Then one proceeds by induction. Let $i \geq 1$. Assuming that the following properties hold true,

H1 U^j and V^j have been determined for $0 \leq j \leq i - 1$
H2 C_n^j is known for $0 \leq n \leq j \leq i - 1$
H3 b_n^{j+n} and C_n^{j+n} are known for $0 \leq j \leq i - 1$ and $n \geq 0$
H4 b_n^j is known for $0 \leq n \leq j \leq i$

let us prove that they remain true for $i + 1$.

C1 Knowing b_n^i for $0 \leq n \leq i$, one knows U_S^i. Knowing U_S^i, one determines $\bar{U}^i$ and hence U^i by solving (30), see Proposition 2.
C2 Knowing U^i, one calculates C_n^{i+n} for $n \geq 0$ as a regular part of U^i, see the next subsection for the practical method. Hence, one knows V_S^i.
C3 Since C_0^i is known and since $C_n^i = C_n^{j+n}$ with $j = i - n$, one knows C_n^i for $0 \leq n \leq i$.
C4 Knowing V_S^i, one determines $\bar{V}^i$ and hence V^i by solving (32), see Proposition 3.
C5 One knows that $b_0^i = 0$. Knowing V^i, one calculates b_n^{i+n} for $n \geq 1$ as a regular part of V^i, see the next subsection for the practical method.
C6 Since $b_0^{i+1} = 0$ and since $b_n^{i+1} = b_n^{j+n}$ with $j = i + 1 - n$, one knows b_n^{i+1} for $0 \leq n \leq i + 1$.

This iterative method is summarized in the Table 2.

3.7. **The practical method for determining the coefficients b_n^i and C_n^i for $0 \leq n \leq i$.** The coefficients b_n^i and C_n^i can be obtained by path integrals (which are path independent) as it is proved in the following Proposition. Throughout this section, $\mathcal{C}_r$ denotes the arc of circle of radius r starting on Γ^- and ending on Γ^+:

$$\mathcal{C}_r = \{(r, \theta) \ : \ 0 \leq \theta \leq \omega\}.$$

b_n^i / C_n^i	i=0	i=1	i=2	i=3	i=4
n=0	(33) /Outer 0	0 / Outer 1	0 / Outer 2	0 / Outer 3	0 / Outer 4
n=1	0	Inner 0 / Outer 0	Inner 1 / Outer 1	Inner 2 / Outer 2	Inner 3 / Outer 3
n=2	0	0	Inner 0 / Outer 0	Inner 1/ Outer 1	Inner 2 / Outer 2
n=3	0	0	0	Inner 0 / Outer 0	Inner 1 / Outer 1
n=4	0	0	0	0	Inner 0 / Outer 0

TABLE 2. Summary of the inductive method to obtain the coefficients b_n^i and C_n^i: in the corresponding cell is indicated the problem which must be solved

Proposition 4. *Let $i \geq 0$ and let us assume that the ith inner and outer problems are solved and thus that $\bar{V}^i$ and $\bar{U}^i$ are known. Then*

(1) *For $n \geq 1$, b_n^{i+n} is given by the following path integral over $\mathcal{C}_\rho$ which is independent of ρ provided that $\rho > 1$:*

$$b_n^{i+n} = \frac{2\rho^{n\lambda}}{\omega} \int_0^\omega \bar{V}^i(\rho, \theta) \cos(n\lambda\theta) d\theta \tag{35}$$

(2) *For $n \geq 0$, C_n^{i+n} is given by the following path integral over $\mathcal{C}_r$ which is independent of r provided that $0 < r < r^*$:*

$$C_0^i = \frac{1}{\omega} \int_0^\omega \bar{U}^i(r, \theta) d\theta, \qquad C_n^{i+n} = \frac{2r^{-n\lambda}}{\omega} \int_0^\omega \bar{U}^i(r, \theta) \cos(n\lambda\theta) d\theta \quad for \quad n \geq 1 \tag{36}$$

Proof. The proofs are identical for the two families of coefficients and then one gives only the proof for C_n^{i+n}. By virtue of (23), the regular part $\bar{U}^i$ of U^i is given by

$$\bar{U}^i(r, \theta) = \sum_{p\in\mathbb{N}} C_p^{i+p} r^{p\lambda} \cos(p\lambda\theta)$$

for $0 < r < r^*$. Since $\int_0^\omega \cos(p\lambda\theta) d\theta$ is equal to ω if $p = 0$ and is equal to 0 otherwise, one obtains the expression for C_0^i. For $n \geq 1$, since $\int_0^\omega \cos(p\lambda\theta) \cos(n\lambda\theta) d\theta$ is equal to $\omega/2$ if $p = n$ and is equal to 0 otherwise, one obtains the expression for C_n^{i+n}. $\qquad\square$

4. CASE OF A CRACK

In this section, the method is applied to the case where the defect is a non cohesive crack. Specifically, let Ω be the rectangle $(-H, L) \times (-H, +H)$. We remove from Ω the following sector $\mathcal{N}_\epsilon$ where $0 < \epsilon < 1$:

$$\mathcal{N}_\epsilon = \{\mathbf{x} = (x_1, x_2 : -H < x_1 \leq 0, |x_2| \leq \epsilon|x_1|)\}.$$

We obtains so the notch-shaped body $\Omega_0 = \Omega \backslash \mathcal{N}_\epsilon$. (Note that the case $\epsilon = 0$ corresponds to a body with a crack of length H.) Finally we remove from Ω_0 the segment line $\Gamma_\ell = (0, \ell) \times \{0\}$ and obtains the cracked body Ω_ℓ, see Figure 3.

The boundary Γ_D where the displacement are prescribed corresponds to the sides $D_\epsilon^\pm$ and Γ_L. Specifically, the boundary conditions read as

$$u^\ell(\mathbf{x}) = \begin{cases} +1 & \text{on} \quad D_\epsilon^+ = \{-H\} \times (\epsilon H, H) \\ -1 & \text{on} \quad D_\epsilon^- = \{-H\} \times (-H, -\epsilon H) \\ 0 & \text{on} \quad \Gamma_L = \{L\} \times (-H, H) \end{cases}$$

whereas the remaining parts of the boundary (including the lips of the crack) are free. Accordingly, we have

$$\frac{\partial u^\ell}{\partial x_2} = 0 \quad \text{on} \quad \Gamma_\ell. \tag{37}$$

FIGURE 3. The cracked notch-shaped body Ω_ℓ

We are in the case where $g = 0$ on Γ_ℓ. Therefore, by virtue of Proposition 3, all the coefficients b_0^i vanish and there is no logarithmic singularities. Accordingly, the solution can be expanded as follows:

Outer expansion $\quad u^\ell(\mathbf{x}) = U^0(\mathbf{x}) + \ell^\lambda U^1(\mathbf{x}) + \ell^{2\lambda} U^2(\mathbf{x}) + \ell^{3\lambda} U^3(\mathbf{x}) + \ldots \tag{38}$

Inner expansion $\quad u^\ell(\mathbf{x}) = V^0(\mathbf{y}) + \ell^\lambda V^1(\mathbf{y}) + \ell^{2\lambda} V^2(\mathbf{y}) + \ell^{3\lambda} V^3(\mathbf{y}) + \ldots \tag{39}$

with

$$\lambda = \frac{\pi}{\omega}, \qquad \omega = 2\pi - \arctan(2\epsilon).$$

Remark 4. *By symmetry of the geometry and the loading, the real field u^ℓ is an odd function of x_2, i.e.*

$$u^\ell(x_1, -x_2) = -u^\ell(x_1, x_2), \qquad u^\ell(r, \omega - \theta) = -u^\ell(r, \theta).$$

Therefore, all the fields U^i, $\bar{U}^i$, V^i, $\bar{V}^i$ admit the same symmetry. One deduces from Proposition 4 that all the coefficients C_{2n}^{i+2n} and b_{2n}^{i+2n} vanish. We can also prove (but the proof is too long to be reproduced here) that all the odd terms of the outer expansion and all the even terms of the inner expansions vanish, i.e. $U^{2i+1} = 0$ and $V^{2i} = 0$ for all $i \in \mathbb{N}$.

The main goal of this section is to obtain an accurate value of the elastic energy stored in the cracked body, say $\mathcal{P}^\ell$, for small values of ℓ by using the matched asymptotic method because it is difficult to evaluate the accuracy of its values obtained by a classical finite element method. By definition, the elastic energy is given by

$$\mathcal{P}^\ell = \frac{1}{2} \int_{\Omega_\ell} \mu \nabla u^\ell \cdot \nabla u^\ell dx.$$

By virtue of Clapeyron's formula and owing to the symmetry of u^ℓ, the elastic energy can also read as the following integral over D_ϵ^+:

$$\mathcal{P}^\ell = -\int_{\epsilon H}^{H} \mu \frac{\partial u^\ell}{\partial x_1}(-H, x_2) dx_2$$

which involves only the displacement field far from the tip of the notch. Accordingly, one can expand $\mathcal{P}^\ell$ by using the outer expansion of u^ℓ. That leads to

$$\mathcal{P}^\ell = \sum_{i \in \mathbb{N}} P^{2i} \ell^{2i\lambda} \qquad \text{with} \qquad P^{2i} = -\int_{\epsilon H}^{H} \mu \frac{\partial U^{2i}}{\partial x_1}(-H, x_2) dx_2 \tag{40}$$

An important quantity for the study of the propagation of the crack is the *energy release rate* $\mathcal{G}^\ell$ which is the opposite of the derivative of the elastic energy with respect to the length of the crack: $\mathcal{G}^\ell = -\frac{d\mathcal{P}^\ell}{d\ell}$, see [1, 4]. Its expansion can be immediately deduce from that of the energy:

$$\mathcal{G}^\ell = -\sum_{i \in \mathbb{N}_*} 2i\lambda P^{2i}\ell^{2i\lambda - 1} \tag{41}$$

To obtain the ith term of the expansion of $\mathcal{P}^\ell$ and $\mathcal{G}^\ell$, one must determine both the singular part U_S^i and the regular part $\bar{U}^i$ of U^i. The singular part involves the coefficients b_n^i for $1 \leq n \leq i$ which are obtained as the regular parts of the V^j's for $j \leq i$, see Section 3.6. Therefore, one must also solve the inner problems and hence determine the coefficients C_n^i for $0 \leq n \leq i$. In practise, these coefficients are obtained by using Proposition 4 after the inner and the outer problems have been solved with a finite element method. The advantage is that those problems do not contain a small defect and the accuracy is guaranteed.

In the next tables are given the computed values of the first coefficients of the inner and outer expansions ($H = 1$, $L = 5$, $\mu = 1$). They are necessary to compute the first terms of the expansion of the energy and of the energy release rate. The graphs of $\ell \mapsto \mathcal{P}^\ell$ and $\ell \mapsto \mathcal{G}^\ell$ obtained from these expansions are plotted on Figure 4. They are compared with the values obtained directly by the finite element code Comsol, see [5]. It turns out that Comsol is not able to give accurate results for small values of ℓ (here when $\ell < 0.01$) whereas the matching asymptotic expansion is valid in this range of values (and even the smaller ℓ, the more accurate the asymptotic method). The two methods give the same results in the intermediate range of values $(0.01, 0.1)$ of ℓ. Note however that the asymptotic method diverges from the finite element method for large values of ℓ. It is due to the fact that one has only computed the first terms of the expansion and one should compute more terms to obtain accurate results.

ϵ	C_0^0	C_1^1	C_2^2	C_3^3	C_4^4	P^0	C_0^2	C_1^3	C_2^4	C_3^5	C_4^6	P^2
0	0	-0.7834	0	-0.2059	0	0.6940	0	0.2384	0	0.1058	0	-0.4836
0.1	0	-0.7482	0	-0.2085	0	0.6606	0	0.2091	0	0.0992	0	-0.4413
0.2	0	-0.7089	0	-0.2081	0	0.6238	0	0.1777	0	0.0905	0	-0.3957
0.3	0	-0.6657	0	-0.2045	0	0.5847	0	0.1451	0	0.0800	0	-0.3486
0.4	0	-0.6187	0	-0.1977	0	0.5420	0	0.1125	0	0.0683	0	-0.3005

TABLE 3. The computed values of the coefficients C_n^n and C_n^{n+2} for $0 \leq n \leq 4$ and of the leading terms P^0 and P^2 of the expansion of the potential energy for several values of the angle of the notch

ϵ	b_1^2	b_2^3	b_3^4	b_4^5	b_5^6
0	-0.3930	0	0.0987	0	-0.0494
0.1	-0.3756	0	0.0943	0	-0.0472
0.2	-0.3559	0	0.0893	0	-0.0446
0.3	-0.3342	0	0.0838	0	-0.0418
0.4	-0.3106	0	0.0778	0	-0.0389

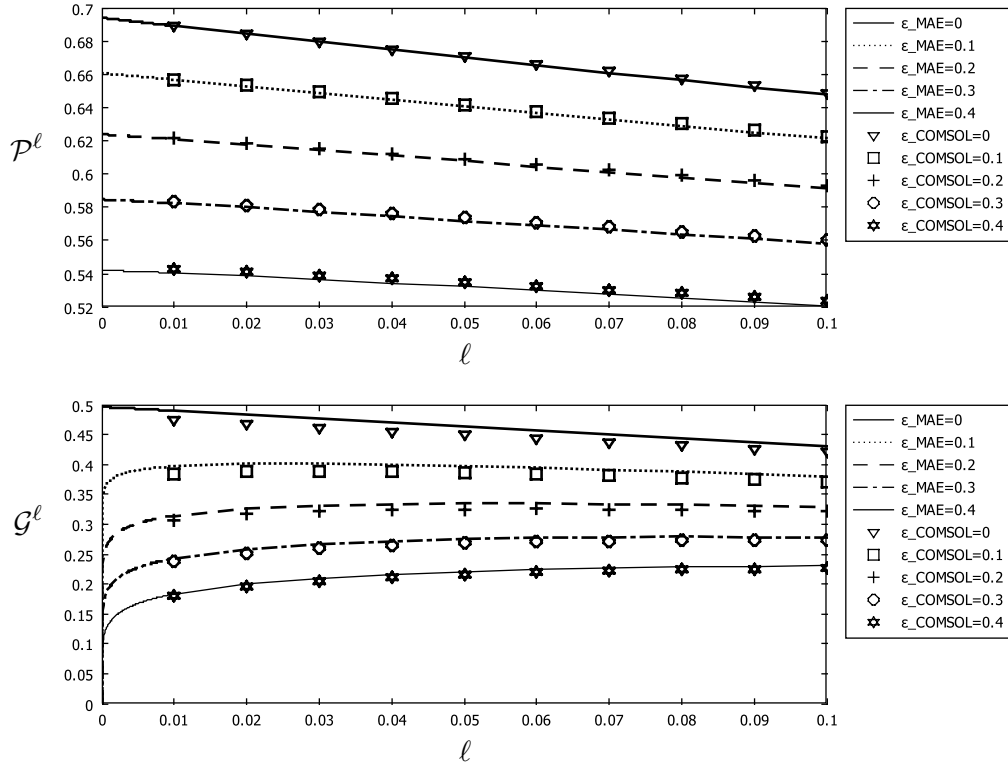TABLE 4. The computed values of the coefficients b_n^{n+1} for $1 \leq n \leq 5$

FIGURE 4. Comparisons of the graphs of $\mathcal{P}^\ell$ and $\mathcal{G}^\ell$ near $\ell = 0$ obtained by Matching Asymptotic Expansion or by the finite element code COMSOL for five values of the angle of the notch

5. CONCLUSION

We have presented here a general method based on matched asymptotic expansions which can be applied to determine the mechanical fields and all related mechanical quantities in the case of a defect located at the tip of a notch. Applying this method to the case of a non cohesive crack, it turns out that it is sufficient to solve few inner and outer problems to obtain with a very good accuracy the dependence of the energy and the energy release rate on the length of the crack. Moreover, this approximation can be used for very small values of the length of the crack and hence for determining the onset of the cracking whereas a classical finite element method gives rise to inaccurate results. The next step will be to apply this method in the case of a cohesive crack.

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